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Whitney equisingularity and invariants of map germs from $\mathbb{C}^n$ to $\mathbb{C}^3$, $n > 3$

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We study how to minimize the number of invariants that is sufficient for the Whitney equisingularity of a one parameter deformation of corank one finitely determined holomorphic germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^3, 0$, with $n > 3$. As we can see in the work of Gaffney, the invariants needed to control the Whitney equisingularity are the 0-stable invariants and the polar multiplicities which appear in the stable types of a stable deformation of the germ. First we give a description of all stable types which appear in these dimensions, then we use the relationship between the invariants in the stable types in the source and the target to reduce the number using these relations. October, 2004 ICMC-USP

1. INTRODUCTION

Gaffney describes in [2] the following problem: "Given a 1-parameter family of map germs $F\colon \mathbb{C} \times \mathbb{C}^n, (0, 0) \to \mathbb{C} \times \mathbb{C}^p, (0, 0)$, find analytic invariants whose constancy in the family implies the family is Whitney equisingular." He shows that for the class of finitely determined map germs of discrete stable type, the Whitney equisingularity (hence the topological triviality) of such a family is guaranteed by the invariance of the zero stable types and the polar multiplicities associated to all stable types.

The number of invariants depends on the dimensions (n, p) and it can be very big according to n and p are big. Then a natural question arises: "For a fixed pair of dimensions (n, p), what is the minimum number of invariants in Gaffney's theorem that are necessary to guarantee the Whitney equisingularity of the family?"

In the case of corank one map germs, Vohra in [19] used Gaffney's approach to study map germs from n-space ($n \geq 3$), to the plane. Recently the case $n < p$ was described by V.H.J.Perez and M.J.Saia and the case $n = p$ by D. Levcovitz, V.H.J.Perez and M.J.Saia.

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In this paper we also consider germs of corank one and investigate the case $(n,3)$ with $n > 3$. We reduce the number of invariants needed by finding relations among them and using the fact that they are upper semi-continuous. To obtain these relations we apply a result of Lê-Greuel to all strata which are complete intersection with isolated singularity, ICIS for short. We remark that in these dimensions there are some stable types which appear in the source which, possibly are with non isolated singularities, for these sets we apply results of Lê and Teissier which show how the Lê numbers control the invariants of these strata.

The plan of this paper is as follows. In section 2 we recall the basic definitions. An explicit description of the stable types which appear in dimensions $(n,3)$ is given in section 3. In section 4 we describe the invariants, polar multiplicities, for the stable types in the target. I n sections 5 and 6 the invariants, polar multiplicities and Lê numbers, for the stable types in the source are given. The main result of this paper is shown in section 7.

2. NOTATION AND PRELIMINARIES

We follow the notation used by Gaffney in [2] and denote by $\mathcal{O}(n,p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$. Let $\mathcal{O}_e(n,p)$ denotes the set of germs at the origin but not necessarily origin preserving.

For a germ $f \in \mathcal{O}_e(n,p)$, $J(f)$ denotes the ideal generated by the set of $p \times p$ minors of the derivative of f. The critical set of f, denoted by $\Sigma(f)$, is the set of points $x \in \mathbb{C}^n$ such that $J(f)(x) = 0$. The discriminant of f, denoted by $\Delta(f)$, is the image of the critical set by f. The determinant of the derivative of $f \in \mathcal{O}_e(n,n)$ is denoted by $J[f]$.

Our interest is in $\mathcal{A}$-finitely determined map-germs, where $\mathcal{A}$ denotes the usual Mather group of germs of holomorphic difeomorphisms in the source and in the target. We denote by $F : (\mathbb{C}^s \times \mathbb{C}^n, (0,0)) \to (\mathbb{C}^s \times \mathbb{C}^p, (0,0))$ a versal unfolding of such an f.

Definition 2.1. We say that a stable type $\mathcal{Q}$ appears in F if for any representative $F = (id, f_u(x))$ of F, there exists a point $(u,y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the germ $f_u : \mathbb{C}^n, S \to \mathbb{C}^p, y$ is a stable germ of type $\mathcal{Q}$, where $S = f^{-1}(y) \cap \Sigma(f_u)$. The points (u,y) and (u,x) with $x \in S$ are called points of stable type $\mathcal{Q}$ in the target and in the source, respectively.

If f is stable, we denote the set of points in $\mathbb{C}^s \times \mathbb{C}^p$ of type $\mathcal{Q}$ by $\mathcal{Q}(f)$ and call $\mathcal{Q}_S(f) = f^{-1}(\mathcal{Q}(f)) - \mathcal{Q}_\Sigma(f)$, where $\mathcal{Q}_\Sigma(f)$ denotes $f^{-1}(\mathcal{Q}(f)) \cap \Sigma(f)$.

If f is finitely determined, we denote by $\overline{\mathcal{Q}(f)} = (\{0\} \times \mathbb{C}^p) \cap \overline{\mathcal{Q}(F)}$ and $\overline{\mathcal{Q}_S(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_S(F)}$, $\overline{\mathcal{Q}_\Sigma(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_\Sigma(F)}$, where the bar over a set means the closure of this set.

Definition 2.2. We say that $\mathcal{Q}$ is a zero-dimensional stable type for the pair (n,p) if $\mathcal{Q}(f)$ has dimension 0 where f is a representative of the stable type $\mathcal{Q}$.

We observe that the set $\overline{\mathcal{Q}(F)} = \cap F(j^{(p+1)}F^{-1}(\overline{\mathcal{A}z_i}))$ is closed and analytic, where z_i is the $p+1$ jet of the stable type $\mathcal{Q}$ and $\mathcal{A}z_i$ is the $\mathcal{A}$-orbit of z_i.

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A finitely determined germ f has discrete stable type if there exist a versal unfolding F of f in which appears only a finite number of stable types. If (n,p) is in the nice range of dimensions or in this boundary, then any finitely determined germ f has a discrete stable type.

Suppose that $\mathcal{Q}(F)=\{p_1,\cdots,p_r\}$ is the set of points of zero-dimensional type, where F is a versal unfolding of f. The 0-stable invariant of type $\mathcal{Q}$ of f, denoted by $m(f;\mathcal{Q})$ is the multiplicity of the ideal $m_s\mathcal{O}_{\overline{\mathcal{Q}(F)},(0,0)}$ in $\mathcal{O}_{\overline{\mathcal{Q}(F)},(0,0)}$, where m_s denotes the ideal generated by the coordinates of the space of parameters $\mathbb{C}^s$.

Let $F:(\mathbb{C}\times\mathbb{C}^n,(0,0))\to(\mathbb{C}\times\mathbb{C}^p,(0,0))$, $F=(t,\overline{f}(t,x))$, be a 1-parameter unfolding of a finitely determined germ f, such that $\overline{f}(t,-)$ preserves the origin for all t. Let $T:=\mathbb{C}\times\{0\}$.

The unfolding F is a *good unfolding* of f if there exist neighborhoods U, W of the origin in $\mathbb{C}\times\mathbb{C}^n$ and $\mathbb{C}\times\mathbb{C}^p$ respectively such that $F^{-1}(W)=U$, F maps $U\cap\Sigma(F)-T$ to $W-T$ and if $(t_0,y_0)\in W-T$, then the germ $f_{t_0}:\mathbb{C}^n,S\to\mathbb{C}^p,y_0$ is stable, where $S=F^{-1}(t_0,y_0)\cap\Sigma(F)$.

A good unfolding is said to be *excellent* if all the 0-stable invariants are constant in the unfolding and f is of discrete type. In the equidimensional case $n=p$, it is also assumed that the degree of f, $\delta(f)=\dim_{\mathbb{C}}\frac{\mathcal{O}_n}{f^*(m_n)\mathcal{O}_n}$, is constant in the unfolding.

We say that an unfolding F of f is Whitney equisingular along the parameter space T if there exists a regular stratification of the source and the target, with T a stratunm of the source and the target and these stratifications are Whitney equisingular along T, i.e. satisfy the Whitney conditions **a** and **b**.

It is shown in [2] p. 208, that if f has discrete stable type, and F is a versal unfolding which only a finite number of stable types, then the family F is Whitney equisingular.

One of the questions of main interest is to show when an excellent unfolding $F:(\mathbb{C}\times\mathbb{C}^n,(0,0))\to(\mathbb{C}\times\mathbb{C}^p,(0,0))$ of a finitely determined germ $f\in\mathcal{O}(n,p)$ is Whitney equisingular.

Using the polar invariants, i.e, polar multiplicities of the polar varieties of the stable types (defined by Teissier in [18]) and Thom's isotopy lemmas, Gaffney showed the following principal result.

THEOREM 2.1. ([2], p. 207) *Suppose that $F:(\mathbb{C}\times\mathbb{C}^n,(0,0))\to(\mathbb{C}\times\mathbb{C}^p,(0,0))$ is an excellent unfolding of a finitely determined germ $f\in\mathcal{O}(n,p)$. Also suppose that the polar invariants of all stable types defined in:*

1.the discriminant $\Delta(f_t)=f_t(\Sigma(f_t))$,
2.the singular set $\Sigma(f_t)$ and also in the set
3.$X(f_t)=\overline{\left(f_t^{-1}(\Delta(f_t))-\Sigma(f_t)\right)}$,

are constant at the origin for all t. Then the unfolding is Whitney equisingular.

Remark 2. 1. The theorem remains valid if we replace the term "an excellent unfolding" in the hypothesis by "a 1-parameter unfolding which, when stratified by stable types and by the parameter axis T, has only the parameter axis T as 1-dimensional stratum at the origin" ([19]). We shall apply this version of the Theorem.

Here we reduce the number of invariants needed by finding relations among them and using the fact that they are upper semi-continuous.

We remark that in the case of corank one map germs, the stable types which appear in the set $\Sigma(f_t)$ are ICIS and the stable types which appear in $\Delta(f_t)$ are also related to ICIS which are in $\mathbb{C}^n$. For these sets we shall apply the following results.

Theorem 2.2. (Lê-Greuel, [9], page 47) *Let X_1 be an ICIS, with a singularity at $0 \in \mathbb{C}^n$. Let X be an ICIS defined in X_1 by $f_k = 0$, and let $f_1, ..., f_{k-1}$ be the generators of the ideal that defines X_1 at 0 in $\mathbb{C}^n$. Then*

$$\mu(X_1, 0) + \mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(f_1, ..., f_{k-1}, J(f_1, ..., f_k))}.$$

In the case of a zero-dimensional ICIS we can use the following simpler formula. Let $f : \mathbb{C}^k, 0 \to \mathbb{C}^k, 0$ be a germ such that $X = f^{-1}(0)$ is an ICIS Then $\mu(X, 0) = \delta(f) - 1$ (see [12] p. 78). Another elementary result that we use here is the following. Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ be a finitely determined germ. Then $f : \Sigma(f) \subset \mathbb{C}^n, 0 \to \Delta(f) \subset \mathbb{C}^n, 0$ is bimeromorphic (see [3] p.154, or [12]).

When $n > 3$, the stable types which appear in $X(F) = \overline{(F^{-1}(\Delta(F)) - \Sigma(F))}$ possibly are not ICIS. In this case, we use the associate Lê numbers to control all invariants needed to show the Whitney equissingularity of these stable types along T. We apply the results of Lê and Teissier given in [11] and of Gaffney and Gassler in [5].

3. THE STABLE TYPES IN $\mathcal{O}(N, 3)$

According to the Theorem of Gaffney, the constancy of the polar invariants of all stable types defined in $\Delta(f_t)$, $\Sigma(f_t)$ and $X(f_t)$ is the condition for the Whitney equisingularity.

Therefore, the first step in the strategy to minimize the number of invariants is to give a full description of all stable types which appear in these sets.

We remark that, as $(n, 3)$ is in the range of the nice dimensions of Mather, any finitely determined map germ $f \in \mathcal{O}(n, 3)$ is of discrete type, hence the stratification has a finite number of strata.

After that we establish relations among the invariants on all strata. As these invariants are upper semi-continuous, the relations will allow us to reduce the number of invariants required in Gaffney's theorem.

In general, for any pair of dimensions (n, p) the description of the stable types can be done in terms of subschemes of multiple points of a germ f, as we can see in [16] for the case $n = p$ or in [17] for the case $n > p$.

Here we use the Thom-Boardman stratification to describe the stable types which appear in the singular set $\Sigma(f)$, then we show the stable types in the discriminant $\Delta(f)$ of f and finally the stable types of $X(f) = \overline{(f^{-1}(\Delta(f)) - \Sigma(f))}$.

Stratification of $\Sigma(f) \subset \mathbb{C}^n$

The strata in $\Sigma(f)$ are the smooth parts of the following sets:

1. The 2-dimensional set of points of type $\Sigma^{n-2}(f) = \Sigma(f)$;

2. The 1-dimensional set of points of type $\Sigma^{n-2,1}(f)$;

3. The 1-dimensional set of double points $D^2_1(f|\Sigma(f))$, which we describe below.

Let $f_{|\Sigma(f)}$ be the restriction of f to $\Sigma(f)$, the set of double points of $f_{|\Sigma(f)}$, denoted by $D^2(f|\Sigma(f))$, is the set of points

$$\big\{(p_1, p_2) \in \mathbb{C}^{2n},\ p_1 \neq p_2,\ \text{such that}\ p_1, p_2 \in \Sigma(f),\ \text{and}\ f(p_1) = f(p_2).\big\}$$

If f is of corank 1, we can write $f(x_1, \dots, x_{n-1}, z)) = (x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1}, z))$, and consider $D^2(f|\Sigma(f))$ (see [8] for details) as the subset $(x_1, \dots, x_{n-1}, z, z_1) \in \mathbb{C}^{n+1}$ such that

$$\frac{g(x_1, \dots, x_{n-1}, z) - g(x_1, \dots, x_{n-1}, z_1)}{z - z_1} = \frac{z_1 \frac{\partial g(x_1,\dots,x_{n-1},z)}{\partial z} - z \frac{\partial g(x_1,\dots,x_{n-1},z_1)}{\partial z_1}}{z - z_1} =$$

$$= \frac{\frac{\partial g(x_1,\dots,x_{n-1},z)}{\partial z} - \frac{\partial g(x_1,\dots,x_{n-1},z_1)}{\partial z_1}}{z - z_1} = 0.$$

Then we denote by $D^2_1(f|\Sigma(f))$ the projection of $D^2(f|\Sigma(f))$ to the $(x_1, \dots, x_{n-1}, z_1)$-space. We remember that the set $D^2_1(f|\Sigma(f))$ is part of the singular set of f.

Stratification of $\Delta(f) \subset \mathbb{C}^3$.

The stratification in the **target** is done in the discriminant set $\Delta(f) = f(\Sigma(f)) \subset \mathbb{C}^3$ and is the smooth parts of the following sets.

1. The discriminant $\Delta(f) = f(\Sigma(f))$, which is 2-dimensional;
2. The 1-dimensional set $f(\Sigma^{(n-2,1)}(f))$;
3. The image of the double points of f, 1-dimensional and denoted by $f(D^2_1(f|\Sigma(f)))$.

3. Stratification of $X(f) = \overline{(f^{-1}(\Delta(f)) - \Sigma(f))} \subset \mathbb{C}^n$

The stratification of the set $X(f)$ is obtained by the inverse image of f of the stable types in the target and also the inverse image of the multiple points.

Therefore it is done by the smooth parts of the following sets.

1. The (n-1)-dimensional set $X(f) = \overline{(f^{-1}(\Delta(f)) - \Sigma(f))}$;

2. The (n-2)-dimensional set $X_1(f) = \overline{(f^{-1}(f(\Sigma^{n-2,1}(f))) - \Sigma(f) \cap \Sigma^{n-2,1}(f))}$;

3. The (n-3)-dimensional set $X_2(f) = \overline{(f^{-1}(f(\Sigma^{n-2,1,1}(f))) - \Sigma(f) \cap \Sigma^{n-2,1,1}(f))}$;

4. The (n-2)-dimensional set $X_3(f) = \overline{(f^{-1}(f(D_1^2(f|\Sigma(f)))) - \Sigma(f) \cap D_1^2(f|\Sigma(f)))}$;

5. The (n-3)-dimensional set $X_4(f) = \overline{(f^{-1}(f(A_1A_2)) - \Sigma(f) \cap A_1A_2)}$;

6. The (n-3)-dimensional set $X_5(f) = \overline{(f^{-1}(f(D_1^3(f|\Sigma(f)))) - \Sigma(f) \cap D_1^3(f|\Sigma(f)))}$.

To a k-dimensional variety are associated $k+1$ polar invariants. Since $\Sigma(f)$ and $\Delta(f)$ are of dimension 2 and the dimension of $D_1^2(f|\Sigma(f))$, $\Sigma^{n-2,1}(f)$, $f(\Sigma^{n-2,1}(f))$ and $f(D_1^2(f|\Sigma(f)))$ is 1, there are 14 polar invariants defined on these sets. We also have 3 multiplicities of the zero-dimensional stable types and $6n-8$ polar multiplicities of the sets $X(f)$, $X_1(f)$, $X_2(f)$, $X_3(f)$, $X_4(f)$ and $X_5(f)$.

Therefore to apply Theorem 2.1 to germs in $\mathcal{O}(n,3)$ we need the constancy of $6n+9$ invariants. In the following sections we show how they are related. We remark that as the set $X(f)$ is possibly with non isolated singularities, we shall use the relationship between its polar multiplicities and Lê numbers.

4. POLAR INVARIANTS OF THE STABLE TYPES IN THE DISCRIMINANT

To show how the stable types are related in the discriminant we also follow the method developed by Gaffney in the cases $n=p=2$ and $n=2,\ p=3$. The main idea is to compute the polar multiplicities associated to the stable types. The fact that $\Sigma(f)$, $\Sigma^{n-2,1}(f)$ and $D_1^2(f|\Sigma(f))$ are ICIS is strongly used here. From this we can apply the theorem 2.2 and also the results shown in the final part of the section 2.

The first relation is for the polar multiplicities of the discriminant, as it is 2-dimensional, there are 3 polar multiplicities, that we describe here:

To compute these multiplicities we follow the definition done by Teissier in [18]. In order to compute $m_1(\Delta(f))$ we choose a generic projection $p_1 : \mathbb{C}^3 \to \mathbb{C}^2$ such that the degree of $p_{1|\Delta(f)}$ is the the multiplicity of $\Delta(f)$ at 0 and also the polar variety $P_1(\Delta(f))$ is $\overline{\Sigma(p_{1|\Delta^0(f)})}$; we denote by $m_1(\Delta(f))$ its multiplicity.

To compute $m_2(\Delta(f))$ we also choose another linear generic projection $p : \mathbb{C}^2 \to \mathbb{C}$ such that the degree of $(p \circ p_1)_{|P_1(\Delta(f))}$ is $m_1(\Delta(f))$ and the projection $(p \circ p_1) := p_2$ is also generic and defines the polar variety $P_2(\Delta(f))$, or $P_{2(\Delta(f))} = \overline{\Sigma(p_{2|\Delta^0(f)})}$; we denote by $m_2(\Delta(f))$ its multiplicity.

To obtain multiplicity $m_0(\Delta(f))$, we consider the following diagram:

$$\begin{array}{ccc} \Sigma(f) \subset \mathbb{C}^n & \xrightarrow{f} & \Delta(f) \subset \mathbb{C}^3 \\ & \searrow^{p_1 \circ f} \quad \swarrow_{p_1} & \\ & (\mathbb{C}^2, 0) & \end{array}$$

From the choice of p_1, we obtain $m_0(\Delta(f)) = deg(p_{1|\Delta(f)})$.

Next we give a relation between the polar multiplicities of $\Delta(f)$ in terms of the Milnor number of the singular set.

THEOREM 4.1. *Let $f \in \mathcal{O}(n, 3), n > 3$ be a finitely determined map germ. Then:*

$$m_2(\Delta(f)) - m_1(\Delta(f)) + m_0(\Delta(f) = \mu(\Sigma(f)) + 1 \qquad (I)$$

Proof: We have the following diagram:

$$\Sigma(f) \subset \mathbb{C}^n \xrightarrow{f} \Delta(f) \subset \mathbb{C}^3 \xrightarrow{p_1} \mathbb{C}^2 \xrightarrow{p_2} \mathbb{C}$$

Now call $X_2 = V(p_2 \circ f, J[f])$ and $X_1 = V(p_1 \circ f, J[f])$. As X_1 and X_2 are ICIS, since they are subsets of $V(J[f]) = \Sigma(f)$, we apply the Theorem 2.2 to obtain:

$$\mu(X_2) + \mu(X_1) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle p_2 \circ f, J[f], J[p_1 \circ f, J[f]] \rangle} \qquad (1)$$

Since $\Sigma(f)$ is also an ICIS we apply again the Theorem 2.2 to $\Sigma(f) = V(J[f])$ and X_2 to get

$$\mu(\Sigma(f)) + \mu(X_2) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle J[f], J[p_2 \circ f, J[f]] \rangle}$$

Then

$$\mu(X_2) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle J[f], J[p_2 \circ f, J[f]] \rangle} - \mu(\Sigma(f)) \qquad (2)$$

and

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle J[f], J[p_2 \circ f, J[f]] \rangle} - \mu(\Sigma(f)) + \mu(X_1) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle p_2 \circ f, J[f], J[p_1 \circ f, J[f]] \rangle} \qquad (3)$$

But, X_1 is 0-dimensional then

$$\mu(X_1) = deg(p_1 \circ f, J[f]) - 1 \qquad (4)$$

and we obtain

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle J[f], J[p_2 \circ f, J[f]] \rangle} - \mu(\Sigma(f)) + deg(p_1 \circ f, J[f]) - 1 = dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle p_2 \circ f, J[f], J[p_1 \circ f, J[f]] \rangle} \tag{5}$$

From the equation (5) we shall obtain the equation (I) which gives the relationship between the polar multiplicities of the discriminant. In fact these multiplicities are implicitly described in the equation (5), as we shall see now.

Since $f : \Sigma(f) \to \Delta(f)$ is finite and bimeromorphic, $deg(p_1|\Delta(f)) = deg(p_1 \circ f|\Sigma(f)) = deg(p_1 \circ f, J[f])$.

Therefore

$$deg(p_1 \circ f, J[f]) = m_0(\Delta(f)) \tag{i}$$

Now we find $m_1(\Delta(f))$. Let $V' = V(J[f], J[p_1 \circ f, J[f]])$ and call $g : \mathbb{C}^3 \to \mathbb{C}$ the defining equation of $\Delta(f)$, that is $g^{-1}(0) = \Delta(f)$. As $f_{|V'}$ is finite and bimeromorphic we can also define $P_1(\Delta(f))$ in the following way:

Define the map $H : \mathbb{C}^3 \to \mathbb{C}^3$, $x \mapsto (g(x), p_1(x))$. Then $\overline{V(H)_{|\Delta(f)}} = P_1(\Delta(f))$, or $\overline{V(J[g, p_1], J[f])} = P_1(\Delta(f))$.

From this definition of $P_1(\Delta(f))$, since $V' = V(J[f], J[p_1 \circ f, J[f]])$ we have that the projection of $p_2 \circ f(V')$ in $\mathbb{C}$ gives the same image than the projection of $p_2(P_1(\Delta(f)))$ in $\mathbb{C}$.

$$\begin{array}{ccccc} V' \subset \Sigma(f) & \xrightarrow{f} & \Delta(f) \supset P_1(\Delta(f)) \\ & \searrow^{p_2 \circ f} & & \swarrow_{p_2} \\ & & \mathbb{C} & & \end{array}$$

Then $deg(p_2 \circ f_{|V'}) = deg(p_{2|P_1(\Delta(f))})$, and from the choice of p_2 we obtain,

$$deg(p_{2|P_1(\Delta(f))}) = m_1(\Delta(f)).$$

Then $m_1(\Delta(f)) = deg(p_2 \circ f_{|V'})$.

Since V' is I.C.I.S, the ring $\mathcal{O}_{V'}$ is Cohen-Macaulay,then

$$m_1(\Delta(f)) = deg(p_2 \circ f_{|V'}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle p_2 \circ f, J[f], J[p_1 \circ f, J[f]] \rangle} \tag{ii}$$

Now we shall show that

$$m_2(\Delta(f)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle J[f], J[p_2 \circ f, J[f]] \rangle}$$

Since this multiplicity involves the stable types, choose an s parameters versal unfolding F of f, to get

$$\begin{aligned} F : \Sigma(F) \subset \mathbb{C}^s \times \mathbb{C}^n &\longrightarrow \Delta(F) \subset \mathbb{C}^s \times \mathbb{C}^3 \\ (x, u) &\longmapsto (u, f_u(x)) \end{aligned}$$

From the fact that p_2 is generic and linear, we have

$$\Sigma(((\pi_s, p_2) \circ F)|\Sigma(F)) = V(J[F], J[(\pi_s, p_2) \circ F, J[F]]) = V \subset \mathbb{C}^n \times \mathbb{C}^s$$

We remark that $m_2(\Delta(f))$ is controlled by the degree of the projection $(\pi_s, p_2)_{|V}$, or in other words, by the length $e_J(f)$ of the maximal ideal m_s in $\mathcal{O}_s$. Then

$$e_J(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle J[f], J[p_2 \circ f, J[f]]\rangle}$$

The possible components of V are the closure of the sets: $F^{-1}(P_2(\Delta(F), \pi_s))$, $F^{-1}(A_3)$, $F^{-1}(A_{(1,2)})$ and $F^{-1}(A_{(1,1,1)})$.

Now we need to count the contribution for the degree of the projection (π_s, p_2) restrict to each one of these components.

To do this we choose a generic parameter u and neighbourhoods $U_2 \subset \mathbb{C}^s \times \mathbb{C}^n, U_1 \subset \mathbb{C}^s$ such that for all point in U_1 there exist $e_J(f)$ pré-images in $V \cap U_2$, counting its multiplicities.

Then we obtain

$$e_J(f) = \sum_{x \in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{s+n,x}}{\langle m_s, J[f], J((\pi_s, p_2) \circ f, J[f])\rangle} = \sum_{x \in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\langle J[f_u], J(p_2 \circ f_u, J[f_u])\rangle}$$

where $S = {\pi_s}^{-1}(0) \cap V$.

Since the parameter u is generic we can consider that f_u is stable. Then to count the contribution of these components we need to use the normal forms of the stable types which appear in the dimensions $(n, 3)$.

We present here an explicit description of the normal forms of all corank one stable map germs in $\mathcal{O}(n, 3), n > 3$, we remember that they are suspensions of the stable map germs which appear in $\mathcal{O}(3, 3)$.

First we describe the mono germs.

Stable Germs:

1. Submersion Σ^0, (A_0): $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, x_n)$

2. Fold Σ^{n-2}, (A_1): $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \pm x_3^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^2)$

3. Cuspidal edge $\Sigma^{n-2,1}$, (A_2): $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \pm x_3^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^3 + x_1 x_n)$

4. Swallowtail $\Sigma^{n-2,1,1}$, (A_3): $f(x_1,x_2,\ldots,x_n)=(x_1,x_2,\pm x_3^2\pm\cdots\pm x_{n-1}^2\pm x_n^4\pm x_1x_n\pm x_2x_n^2)$

To describe the stable multigerms we consider the normal crossing between the stable germs.

Stable multigerms:

1. Double points, $(A_{(1,1)})$: $\big\{(x_1,x_2,\pm x_3^2\pm\cdots\pm x_{n-1}^2\pm x_n^2);$
$(x_1,\pm x_2^2\pm x_4^2\pm\cdots\pm x_{n-1}^2\pm x_n^2,x_3)\big\}$

2. Plane with a cuspidal edge, $(A_{(1,2)})$:

$\big\{(x_1,\pm x_2^2\pm x_4^2\pm\cdots\pm x_{n-1}^2\pm x_n^2,x_3);\ (x_1,x_2,\pm x_3^2\pm\cdots\pm x_{n-1}^2\pm x_n^3+x_1x_n)\big\}$

3. Triple points, $(A_{(1,1,1)})$: $\big\{(x_1,x_2,\pm x_3^2\pm\cdots\pm x_{n-1}^2\pm x_n^2);$
$(x_1,\pm x_2^2\pm x_4^2\pm\cdots\pm x_{n-1}^2\pm x_n^2,x_3);\ (\pm x_1^2\pm x_4^2\pm\cdots\pm x_{n-1}^2\pm x_n^2,x_2,x_3)\big\}$

Now we return to count **the contribution of the stable types:** we remark that these points can or not appear in V, depending of the type of the singularity.

• Singularity of type A_3.

Here $\langle J[f_u],J(p_2\circ f_u,J[f_u])\rangle=\big\langle x_3,\ldots,x_{n-1},4{x_n}^3+2x_2x_n+x_1,\ldots,\mathbf{1}\big\rangle$. Hence

$$\dim_{\mathbb{C}}\frac{\mathcal{O}_{n,x}}{\langle J[f_u],J(p_2\circ f_u,J[f_u])\rangle}=\dim_{\mathbb{C}}\frac{\mathcal{O}_{n,x}}{\langle x_3,\ldots,x_{n-1},4{x_n}^3+2x_2x_n+x_1,\ldots,\mathbf{1}\rangle}=0,$$

And we obtain that there is no contribution of this stable type.

• Singularity of type $A_{(1,2)}$. Here the ideal $J[f_u]$ is generated by the generators of the jacobian ideals of the germs which define f_u, then $J[f_u]=\big\langle x_2,x_3,\ldots,x_{n-1},x_n,3{x_n}^2+x_1\big\rangle=\langle x_1,x_2,x_3,\ldots,x_{n-1},x_n\rangle$.

Hence $\langle J[f_u],J(p_2\circ f_u,J[f_u])\rangle=\langle x_1,x_2,x_3,\ldots,x_{n-1},x_n,\mathbf{1}\rangle$ and

$$\dim_{\mathbb{C}}\frac{\mathcal{O}_{n,x}}{\langle J[f_u],J(p_2\circ f_u,J[f_u])\rangle}=\dim_{\mathbb{C}}\frac{\mathcal{O}_{n,x}}{\langle x_3,\ldots,x_1,x_2,x_3,\ldots,x_{n-1},x_n,\mathbf{1}\rangle}=0$$

And we obtain again that there is no contribution of the stable type $A_{(1,2)}$.

• Singularity of type $A_{(1,1,1)}$. Again we have $J[f_u]=\langle x_1,x_2,x_3,\ldots,x_{n-1},x_n\rangle$ and

$$\langle J[f_u],J(p_2\circ f_u,J[f_u])\rangle=\langle x_1,x_2,x_3,\ldots,x_{n-1},x_n,\mathbf{1}\rangle\,.$$

Therefore

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\langle J[f_u], J(p_2 \circ f_u, J[f_u])\rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\langle x_3, \ldots, x_1, x_2, x_3, \ldots, x_{n-1}, x_n, \mathbf{1}\rangle} = 0.$$

Again we obtain that there is no contribution of the stable type $A_{(1,1,1)}$.
We conclude that the components of V are in the closure of $F^{-1}(P_2(\Delta(F), \pi_s))$, then

$$e_J(f) = \sum_{x \in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\langle J[f_u], J(p_2 \circ f_u, J[f_u])\rangle} = deg(((\pi_s, p_2) \circ F)|V)$$

Since $F_{|V}$ is finite and bimeromorphic we get

$$deg(((\pi_s, p_2) \circ F)|V) = deg((\pi_s, p_2)|P_2(\Delta(F), \pi_s)) = m_2(\Delta(f)).$$

On the other side,

$$m_2(\Delta(f)) = e_J(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle J[f], J[p_2 \circ f, J[f]]\rangle} \qquad (iii)$$

Now we use (i), (ii) and (iii) in the equation (5) to obtain (I):

$$m_2(\Delta(f)) - m_1(\Delta(f)) + m_0(\Delta(f) = \mu(\Sigma(f)) + 1.$$

■

Next we give the relation for the polar multiplicities of $f(\Sigma^{n-2,1}(f))$, which are are 2. The relation between them is given in terms of the Milnor number of the set $\Sigma^{n-2,1}(f)$ and also of the number of singularities of type A_3.

Remark 4. 1. The ideal that defines $\Sigma^{n-2,1}(f)$ is $J_{(n-2,1)}(f) = I_n(d(f, I_3(d(f))))$, where $d(h)$ denotes the Jacobian matrix of a map germ h and $I_s(M)$ denotes the ideal generated by the s minors of some matrix M.

Then, for any corank one map germ $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, g(x_1, x_2, \ldots, x_n))$ we obtain $J_{(n-2,1)}(f) = \langle g_{x_3}, g_{x_4}, \ldots, g_{x_n}, M\rangle$, where g_{x_i} denotes the parcial derivative of g in the variable x_i, M is the determinant $\left| g_{x_3^2} \cdots g_{x_3 x_n} g_{x_4 x_3} \cdots g_{x_4 x_n} \vdots \ddots \vdots g_{x_n x_3} \cdots g_{x_n^2} \right|$ and $g_{x_i x_j}$ denotes the partial derivative of g_{x_i} in the variable x_j.

Theorem 4.2. *For any corank on map germ f from C^n to $\mathbb{C}^3$:*

$$m_0(f(\Sigma^{n-2,1}(f))) - m_1(f(\Sigma^{n-2,1}(f))) = \sharp A_3 - \mu(\Sigma^{n-2,1}(f)) + 1 \qquad (II)$$

Proof: Since f is finitely determined, $\Sigma^{n-2,1}(f)$ has reduced structure, from the fact that f is of corank one $\Sigma^{n-2,1}(f) = V(J_{(n-2,1)}(f))$ is an ICIS, then to get the equation (II) we apply again the Theorem 2.2.

Choose a generic linear projection $p_1 : \mathbb{C}^3 \to \mathbb{C}$ such that $X := \Sigma^{n-2,1}(f) \cap (p_1 \circ f^{-1}(0))$ is an ICIS and $m_0(f(\Sigma^{n-2,1}(f))) = deg(p_1|(f(\Sigma^{n-2,1}(f)))) = V(J_{(n-2,1)}(f), p_1 \circ f)$

We apply the Theorem 2.2 for the sets $\Sigma^{n-2,1}(f)$ and X to obtain

$$\mu(\Sigma^{n-2,1}(f)) + \mu(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle J_{(n-2,1)}(f), J[J_{(n-2,1)}(f), p_1 \circ f]\right\rangle} \tag{1}$$

From this equation we obtain the equation (II).

First we remark that X is an 0-dimensional ICIS, then

$$\mu(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle J_{(n-2,1)}(f), p_1 \circ f\right\rangle} - 1 = deg((p_1 \circ f)|\Sigma^{n-2,1}(f))$$

Since $f|\Sigma^{n-2,1}(f)$ is bimeromorphic and finite,

$$deg((p_1 \circ f)|\Sigma^{n-2,1}(f)) = deg(p_1|f(\Sigma^{n-2,1}(f))).$$

On the other side, from the choice of p_1, $deg(p_1|f(\Sigma^{n-2,1}(f))) = m_0(f(\Sigma^{n-2,1}(f)))$.

Therefore $deg((p_1 \circ f)|\Sigma^{n-2,1}(f)) = m_0(f(\Sigma^{n-2,1}(f)))$ and

$$\mu(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle J_{(n-2,1)}(f), p_1 \circ f\right\rangle} - 1 = m_0(f(\Sigma^{n-2,1}(f))) - 1.$$

Then we have $m_0(f(\Sigma^{n-2,1}(f))) = \dim_{\mathbb{C}} \dfrac{\mathcal{O}_n}{\left\langle J_{(n-2,1)}(f), p_1 \circ f\right\rangle}$. $\quad (i)$

Our next step is to work with $m_1(f(\Sigma^{n-2,1}(f)))$ to get the equation (II).

Since $\dim(f(\Sigma^{n-2,1}(f))) = 1$ we need to consider all stable types which appear here, then we need to count the contribution of each one of the 0-stable types in $m_1(f(\Sigma^{n-2,1}(f)))$.

To obtain this, we consider a s-parameters versal unfolding F of f

$$\begin{aligned} F : \Sigma^{n-2,1}(F) \subset \mathbb{C}^s \times \mathbb{C}^n &\longrightarrow F(\Sigma^{n-2,1}(F)) \subset \mathbb{C}^s \times \mathbb{C}^3 \\ (u, x) &\longmapsto (u, f_u(x)) \end{aligned}$$

From the linearity of the generic projection p_1,

$$\Sigma(((\pi_s, p_1) \circ F)|\Sigma^{(n-2,1)}(F)) = V(J_{(n-2,1)}(F), J((\pi_s, p_1) \circ F, J_{(n-2,1)}(F))) = V \subset \mathbb{C}^s \times \mathbb{C}^n$$

and we conclude that the $m_1(f(\Sigma^{n-2,1}(f)))$ is controlled by the degree of the projection π_s restrict to V, that is, by the length $e_J(f)$ of the maximal ideal m_s in the source $\mathcal{O}_s$. Then

$$e_J(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle J_{(n-2,1)}(f), J(p_1 \circ f, J_{(n-2,1)}(f))\rangle}.$$

Since the possible components of V are the closure of the sets $F^{-1}(A_3)$, $F^{-1}(A_{(1,2)})$, $F^{-1}(A_{(1,1,1)})$ and $F^{-1}(P_1(F(\Sigma^{(n-2,1)}(F)), \pi_s))$, we need to count the contribution for the degree of the projection (π_s, p_1) restrict to each one of these components.

To do this we choose a generic parameter u and neighborhoods $U_2 \subset \mathbb{C}^s \times \mathbb{C}^n, U_1 \subset \mathbb{C}^s$ such that for each point in U_1 there exist $e_J(f)$ pre images in $V \cap U_2$, counting its multiplicities.

Therefore, for $S = {\pi_s}^{-1}(0) \cap V$:

$$e_J(f) = \sum_{x \in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{s+n,x}}{\langle m_s, J_{(n-2,1)}(F), J((\pi_s, p_1) \circ F, J_{(n-2,1)}(F))\rangle} =$$

$$\sum_{x \in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\langle J_{(n-2,1)}(f_u), J(p_1 \circ f_u, J_{(n-2,1)}(f_u))\rangle}$$

From the genericity of the parameter u we suppose that f_u is stable and to count the contribution of the components we use the normal forms.

Contribution of the stable types:

For the type A_3, $J_{(n-2,1)}(f_u) = \langle x_3, x_4, \ldots, x_{n-1}, 4{x_n}^3 + 2x_2x_n + x_1, 12{x_n}^2 + 2x_2\rangle$ and $\langle J_{(n-2,1)}(f_u), J(p_1 \circ f_u, J_{(n-2,1)}(f_u))\rangle = \langle x_3, \ldots, x_{n-1}, 4{x_n}^3 + 2x_2x_n + x_1, 12{x_n}^2 + 2x_2,$

$J(x_2, x_3, \ldots, x_{n-1}, 4{x_n}^3 + 2x_2x_n{+}x_1{,}12{x_n}^2{+}2x_2))\rangle$.

But $J(x_2, x_3, \ldots, x_{n-1}, 4{x_n}^3 + 2x_2x_n + x_1, 12{x_n}^2 + 2x_2) = 24x_n$ and

$$\langle J_{(n-2,1)}(f_u){,}J(x_2{,}J_{(n-2,1)}(f_u))\rangle = \langle x_3, x_4, \ldots, x_{n-1}, 4{x_n}^3 2{+}2x_2x_n 2{+}x_1{,}12{x_n}^2 2{+}2x_2{,}24x_n\rangle$$

Therefore $\dim_{\mathbb{C}} \dfrac{\mathcal{O}_{n,x}}{\langle J_{(n-2,1)}(f_u), J(x_2, J_{(n-2,1)}(f_u))\rangle} = 1$ and we conclude that, here the contribution of the stable type A_3 is 1.

For the type $A_{(1,2)}$. Since f_u is a multi germ, the ideal $J_{(n-2,1)}(f_u)$ is generated by the generators of the iterated jacobean ideals $J_{(n-2,1)}(f_1)$ and $J_{(n-2,1)}(f_2)$ of the germs which define f_u. Consider $f_1(x_1, \ldots, x_n) = (x_1, \pm x_2^2 \pm x_4^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^2, x_3)$, and $f_2(x_1, \ldots, x_n) = (x_1, x_2, \pm x_3^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^3 + x_1x_n)$.

Here $J_{(n-2,1)}(f_1) = \langle x_2, x_4, \ldots, x_{n-1}, x_n, 1\rangle$ and $J_{(n-2,1)}(f_2) = \langle x_2, \ldots, x_n, \mathbf{1}\rangle$ then there is no contribution of $A_{(1,2)}$.

Now we consider the type $A_{(1,1,1)}$, let $f_1(x_1, \ldots, x_n) = (x_1, x_2, \pm x_3^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^2)$, $f_2(x_1, \ldots, x_n) = (x_1, \pm x_2^2 \pm x_4^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^2, x_3)$ and $f_3(x_1, \ldots, x_n) = (\pm x_1^2 \pm x_4^2 \pm \cdots \pm x_{n-1}^2 \pm x_n^2, x_2, x_3)$.

Here use the results done for the ideal $J_{(n-2,1)}(f_1)$ in the case $A_{(1,2)}$ to conclude also that there is no contribution of the stable type $A_{(1,1,1)}$.

Therefore

$$e_J(f) = \sum_{x\in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\left\langle J_{(n-2,1)}(f_u), J(p_1 \circ f_u, J_{(n-2,1)}(f_u))\right\rangle} = deg(((\pi_s, p_1) \circ F)|V)$$

Since $F|V$ is finite and bimeromorphic, we get

$$deg(((\pi_s, p_1) \circ F)|V) = deg((\pi_s, p_1)|P_1(F(\Sigma^{(n-2,1)}(F), \pi_s) = m_1(f(\Sigma^{(n-2,1)}(f)) + \sharp A_3$$

On the other side

$$e_J(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle J_{(n-2,1)}(f), J(p_1 \circ f, J_{(n-2,1)}(f))\right\rangle}$$

Then

$$m_1(f(\Sigma^{(n-2,1)}(f)) + \sharp A_3 = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle J_{(n-2,1)}(f), J(p_1 \circ f, J_{(n-2,1)}(f))\right\rangle} \tag{ii}$$

Using (i) and (ii) in the equation (1) we obtain the equation (II):

$$m_0(f(\Sigma^{n-2,1}(f))) - m_1(f(\Sigma^{n-2,1}(f))) = \sharp A_3 - \mu(\Sigma^{n-2,1}(f)) + 1$$

■

Remark 4. 2. From these results we obtain the following equality:

$$m_1(\Delta(f)) = m_0(f(\Sigma^{n-2,1}(f))) \tag{III}$$

Proof: Since $f \in \mathcal{O}(n, 3), n > 3$ is finitely determined and of corank one, we have $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, g(x_1, x_2, \ldots, x_n))$ and $J_{(n-2,1)}(f) = \langle J[f], M\rangle$, where M is the determinant $\left| g_{x_3^2} \ldots g_{x_3 x_n} g_{x_4 x_3} \ldots g_{x_4 x_n} \vdots \ddots \vdots g_{x_n x_3} \ldots g_{x_n^2} \right|$.

In the proof of the Theorems 4.1 e 4.2 we see that

$$m_1(\Delta(f)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle p_2 \circ f, J[f], J[p_1 \circ f, J[f]]\rangle}$$

and

$$m_0(f(\Sigma^{n-2,1}(f))) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle J_{(n-2,1)}(f), p_1 \circ f\right\rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\left\langle J_{(n-2,1)}(f), p_1 \circ f\right\rangle}$$

To obtain the equality $m_1(\Delta(f)) = m_0(f(\Sigma^{n-2,1}(f)))$, or

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle p_2 \circ f, J[f], J[p_1 \circ f, J[f]]\rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle p_1 \circ f, J[f], M\rangle}$$

we need to show, from the genericity of the projections, that

$$J[p_1 \circ f, J[f]] = \langle M\rangle$$

Since f is of corank one, we choose $p_1(y_1, y_2, y_3) = (y_1, y_2)$, then $(p_1 \circ f)(x_1, x_2, \ldots, x_n) = (x_1, x_2)$ and $J[p_1 \circ f, J[f]] = J[x_1, x_2, J[f]] = \langle M\rangle$ ■

Remark 4. 3. Since the polar multiplicities are upper semi continuous, from the Theorem 4.1 we see that if $m_1(\Delta(f_t))$ and $\mu(\Sigma(f_t))$ are constants in the family, then we obtain that $m_2(\Delta(f_t))$ and $m_0(\Delta(f_t))$ are also constants in the family.

We have the following equalities

$$m_2(\Delta(f_t)) - m_1(\Delta(f_t)) + m_0(\Delta(f_t) = \mu(\Sigma(f_t)) + 1$$

$$m_2(\Delta(f_0)) - m_1(\Delta(f_0)) + m_0(\Delta(f_0) = \mu(\Sigma(f_0)) + 1$$

Therefore,

$$(m_2(\Delta(f_t)) - m_2(\Delta(f_0))) - (m_1(\Delta(f_t)) - m_1(\Delta(f_0))) \ +$$

$$+ \ (m_0(\Delta(f_t)) - m_0(\Delta(f_0))) \ = \ (\mu(\Sigma(f_t)) - \mu(\Sigma(f_0))) \qquad (*)$$

If $m_1(\Delta(f_t))$ and $\mu(\Sigma(f_t))$ are constants in the family, the equation (*) gives

$$(m_2(\Delta(f_t)) - m_2(\Delta(f_0))) + (m_0(\Delta(f_t)) - m_0(\Delta(f_0))) = 0$$

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From the upper semi continuity of both multiplicities, we have

$$(m_2(\Delta(f_t)) - m_2(\Delta(f_0))) > 0$$

$$(m_0(\Delta(f_t)) - m_0(\Delta(f_0))) > 0$$

Since the sum of positive parts implies that each part is zero, we get $m_2(\Delta(f_t) = m_2(\Delta(f_0))$, and $m_0(\Delta(f_t) = m_0(\Delta(f_0)$, for all t.

Now we show how the polar multiplicities of $f(D_1^2(f|\Sigma(f)))$ are related, there are here also two polar multiplicities and the relation between them is given in terms of the Milnor number of the set $f(D_1^2(f|\Sigma(f)))$ and also of the number of singularities of type A_3.

Theorem 4.3. *Let $f(\mathbb{C}^n, 0) \to (\mathbb{C}^3, 0)$ be a finitely determined map germ of corank one. Then:*

$$2m_0(f(D_1^2(f|\Sigma(f)))) - 2m_1(f(D_1^2(f|\Sigma(f)))) + \mu(D_1^2(f|\Sigma(f))) = 3\sharp A_{(1,2)} + 3\sharp A_3 + 6\sharp A_{(1,1,1)} + 1 \qquad (IV)$$

Proof: Call $\mathcal{I} = f^*(m_3)\mathcal{O}_{D_1^2(f|\Sigma(f))}$, from 8.1 of ([2], p. 209) and ([18], p. 294), we choose a linear generic projection $p : \mathbb{C}^3 \to \mathbb{C}$ in such a way that

$$e(\mathcal{I}) = deg(p \circ (f|_{(D_1^2(f|\Sigma(f)))}) \text{ and } deg(p|_{(f((D_1^2(f|\Sigma(f)))))}) = m_0(f(D_1^2(f|\Sigma(f))))$$

Since $f|_{(D_1^2(f|\Sigma(f)))} - \{0\}$ is a two sheets recovering of $f(D_1^2(f|\Sigma(f))) - \{0\}$ we know that $e(\mathcal{I}) = 2deg(p|_{(f((D_1^2(f|\Sigma(f)))))}) = 2m_0(f(D_1^2(f|\Sigma(f))))$.

Therefore

$$2m_0(f(D_1^2(f|\Sigma(f)))) = deg(p \circ (f|_{(D_1^2(f|\Sigma(f)))}) = deg(p \circ f, I_1^2(f|\Sigma(f)))$$

where $I_1^2(f|\Sigma(f))$ is the defining ideal of $(D_1^2(f|\Sigma(f)))$.

Call $X_2 = V(I_1^2(f|\Sigma(f))) = D_1^2(f|\Sigma(f))$ and $X_1 = V(I_1^2(f|\Sigma(f)), p \circ f) = D_1^2(f|\Sigma(f)) \cap (p \circ f)^{-1}(0)$.

As $D_1^2(f|\Sigma(f))$ is an ICIS, X_2 and X_1 are also ICIS, then applying the Theorem 2.2 to X_2 and X_1,

$$\mu(X_2) + \mu(X_1) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle I_1^2(f|\Sigma(f)), J[I_1^2(f|\Sigma(f)), p \circ f]\rangle} \qquad (1)$$

Since X_1 is 0-dimensional,

$$\mu(X_1) = deg(p \circ f, I_1^2(f|\Sigma(f))) - 1 = 2m_0(f(D_1^2(f|\Sigma(f)))) - 1 \tag{2}$$

Then

$$\mu(D_1^2(f|\Sigma(f))) + 2m_0(f(D_1^2(f|\Sigma(f)))) - 1 = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle I_1^2(f|\Sigma(f)), J[I_1^2(f|\Sigma(f)), p \circ f]\rangle} \tag{3}$$

To end this proof we need to show that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle I_1^2(f|\Sigma(f)), J[I_1^2(f|\Sigma(f)), p \circ f]\rangle} = 2m_1(f(D_1^2(f|\Sigma(f)))) + 3\sharp A_{(1,2)} + 3\sharp A_3 + 6\sharp A_{(1,1,1)}.$$

This equality appears when we study the multiplicity $m_1(f(D_1^2(f|\Sigma(f))))$, we remember that this is the multiplicity where the 0-stable types appear.

Choose a s-parameters versal unfolding F of f,

$$F : D_1^2(F|\Sigma(F)) \subset \mathbb{C}^s \times \mathbb{C}^n \longrightarrow F(D_1^2(F|\Sigma(F))) \subset \mathbb{C}^s \times \mathbb{C}^3, \quad F(x,u) = (u, f_u(x)).$$

From the linearity of the generic projection p we have

$$\Sigma(((\pi_s, p) \circ F)|D_1^2(F|\Sigma(F))) = V(I_1^2(F|\Sigma(F)), J((\pi_s, p) \circ F, I_1^2(F|\Sigma(F)))) = V \subset \mathbb{C}^n \times \mathbb{C}^s$$

Since the multiplicity $m_1(f(D_1^2(f|\Sigma(f))))$ is controlled by the degree of the projection $\pi_s : \mathbb{C}^n \times \mathbb{C}^s \to \mathbb{C}^s$ restrict to V, or in other words, by the length of $e_J(f)$ of the maximal ideal m_s in $\mathcal{O}_s$. Then

$$e_J(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle I_1^2(f|\Sigma(f)), J[I_1^2(f|\Sigma(f)), p \circ f]\rangle}.$$

The possible components of V are the closure of the sets:

$$F^{-1}(P_1(F(D_1^2(F|\Sigma(F))), \pi_s)),\ F^{-1}(A_3),\ F^{-1}(A_{(1,2)}) \text{ and } F^{-1}(A_{(1,1,1)}).$$

Then we need to count the contribution of the degree of the projection (π_s, p) restrict to each one of these components. Choose a generic parameter u and neighborhoods $U_2 \subset \mathbb{C}^s \times \mathbb{C}^n, U_1 \subset \mathbb{C}^s$ in such a way that for any point in U_1 there are $e_J(f)$ pre-images in $V \cap U_2$ counting multiplicities.

Hence

$$e_J(f) = \sum_{x \in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{s+n,x}}{\langle m_s, I_1^2(F|\Sigma(F)), J((\pi_s, p) \circ f, I_1^2(F|\Sigma(F)))\rangle} =$$

$$\sum_{x \in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\langle I_1^2(f_u|\Sigma(f_u)), J(p \circ f_u, I_1^2(f_u|\Sigma(f_u)))\rangle}$$

where $S = {\pi_s}^{-1}(0) \cap V$. From the genericity of the parameter u, we consider that f_u is stable.

We remark that $F^{-1}(P_1(F(D_1^2(F|\Sigma(F))), \pi_s))$ contributes $2m_1(f(D_1^2(f|\Sigma(f))))$ with this degree, we need now to count the contribution of the 0-stable types.

To count this contribution, we use the fact that the stable germs in $\mathcal{O}(n,3)$, with $n > 3$ are the suspension of an A_k singularity which appears in $\mathcal{O}(3,3)$, and in this case, the double points set of this singularity in $\mathcal{O}(n,3)$ coincides with the double points set of the singularity in $\mathcal{O}(n,3)$, see [[8], pp.378] for more details.

Therefore we use the calculation done by V. H. J. Peres in [[15], pp.11] to get that the contribution of the type A_3 is 3, the contribution of the type $A_{(1,2)}$ is also 3 and of the type $A_{(1,1,1)}$ is 6.

Then, since

$$e_J(f) = \sum_{x \in S} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\langle I_1^2(f_u|\Sigma(f_u)), J(p \circ f_u, I_1^2(f_u|\Sigma(f_u)))\rangle} = deg(((\pi_s, p) \circ F)|V)$$

and $F|V$ is finite and bimeromorphic, we get

$$deg(((\pi_s, p) \circ F)|V) = deg((\pi_s, p)|P_1(F(D_1^2(F|\Sigma(F))), \pi_s) =$$

$$2m_1(f(D_1^2(f|\Sigma(f)))) + 3\sharp A_3 + 3\sharp A_{(1,2)} + 6\sharp A_{(1,1,1)}.$$

On the other side,

$$e_J(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle I_1^2(f|\Sigma(f)), J(p \circ f, I_1^2(f|\Sigma(f))\rangle}.$$

Then

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle I_1^2(f|\Sigma(f)), J(p \circ f, I_1^2(f|\Sigma(f))\rangle} =$$

$$2m_1(f(D_1^2(f|\Sigma(f)))) + 3\sharp A_3 + 3\sharp A_{(1,2)} + 6\sharp A_{(1,1,1)} \tag{4}$$

From these, we get the equality (IV)

$$2m_0(f(D_1^2(f|\Sigma(f)))) - 2m_1(f(D_1^2(f|\Sigma(f)))) + \mu(D_1^2(f|\Sigma(f))) = 3\sharp A_{(1,2)} + 3\sharp A_3 + 6\sharp A_{(1,1,1)} + 1$$

■

5. RELATIONS AMONG THE INVARIANTS OF THE STABLE TYPES IN $\Sigma(F)$

To obtain the relations among the invariants in $\Sigma(f)$ we use the result of Teissier, given in [18] pp.481, where it is shown how the absolute polar multiplicities of a d-dimensional hypersurface X with isolated singularity are related to the Milnor numbers $\mu^{(k)}(H) = \mu(X \cap H^k)$, with H^k being a generic hyperplane of dimension k, we have

$$m_k(X) = \mu^{(k+1)}(X) + \mu^{(k)}(X), 0 \leq k \leq d-1 \qquad (*)$$

We remark that this result is also valid for ICIS, as we see in [4], pag.210.

To apply these results we remember that the strata of $\Sigma(f)$ are the regular parts of $\Sigma(f)$, $\Sigma^{(n-2,1)}(f)$ and $D_1^2(f|\Sigma(f))$, $\Sigma(f)$ is a two dimensional ICIS and for corank one map germs, the set $\Sigma^{(n-2,1)}(f)$ is also an ICIS.

If we consider a map $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0)$, with $f^{-1}(0) = X^d$ and $p_1 : \mathbb{C}^n \to \mathbb{C}$ a linear generic projection, if $V(p_1, f)$ is an ICIS, from the Theorem 2.2 and the definition of polar multiplicities, we have

$$m_d(X^d) = \mu(X^d) \cap p_1^{-1}(0) + \mu(X^d) \qquad (**)$$

Then we show the following:

Theorem 5.1. *Let $f \in \mathcal{O}(n,3)$ with $n > 3$ be a finitely determined map germ, then:*

$$m_2(\Sigma(f)) - m_1(\Sigma(f)) + m_0(\Sigma(f) = \mu(\Sigma(f)) + 1. \qquad (I)$$

Morever, if f is of corank one,

$$m_1(\Sigma^{n-2,1}(f)) - m_0(\Sigma^{n-2,1}(f)) = \mu(\Sigma^{n-2,1}(f)) - 1, \qquad (II)$$

$$m_1(\Sigma(f)) = m_0(\Sigma^{n-2,1}(f)), \qquad (III)$$

$$m_1(D_1^2(f|\Sigma(f))) - m_0(D_1^2(f|\Sigma(f))) = \mu(D_1^2(f|\Sigma(f))) - 1. \qquad (IV)$$

Proof: To show the first equality we use the fact that $\Sigma(f)$ is an ICIS with dimension 2. From $(*)$

$$m_0(\Sigma(f)) = \mu^{(1)}(\Sigma(f)) + \mu^{(0)}(\Sigma(f)) \qquad (1)$$

$$m_1(\Sigma(f)) = \mu^{(2)}(\Sigma(f)) + \mu^{(1)}(\Sigma(f)) \qquad (2)$$

From (1) we obtain

$$\mu^{(1)}(\Sigma(f)) = m_0(\Sigma(f)) - 1 \qquad (3)$$

and from (2):

$$\mu^{(2)}(\Sigma(f)) = m_1(\Sigma(f)) - \mu^{(1)}(\Sigma(f)) \tag{4}$$

Moreover, from (**) we get

$$m_2(\Sigma(f)) = \mu(\Sigma(f) \cap p_2^{-1}(0)) + \mu(\Sigma(f)) \tag{5}$$

Then we obtain

$$m_2(\Sigma(f)) - \mu(\Sigma(f) \cap p_2^{-1}(0)) = \mu(\Sigma(f)), \tag{6}$$

But

$$\mu(\Sigma(f) \cap p_2^{-1}(0)) = \mu^{(2)}(\Sigma(f)) \tag{7}$$

From (7) and (6) we get

$$m_2(\Sigma(f)) - \mu^{(2)}(\Sigma(f)) = \mu(\Sigma(f)) \tag{8}$$

From (4) and (8), we obtain

$$m_2(\Sigma(f)) - m_1(\Sigma(f)) + \mu^{(1)}(\Sigma(f)) = \mu(\Sigma(f)) \tag{9}$$

From (3) and (9):

$$m_2(\Sigma(f)) - m_1(\Sigma(f)) + -m_0(\Sigma(f)) - 1 = \mu(\Sigma(f))$$

and the equality (I) follows.

If we suppose that f is of corank 1, $\Sigma^{n-2,1}(f)$ is an ICIS 1-dimensional, then (*) and (**) hold, or

$$m_0(\Sigma^{n-2,1}(f)) = \mu^{(1)}(\Sigma^{n-2,1}(f)) + \mu^{(0)}(\Sigma^{n-2,1}(f)) \tag{10}$$

and this is equivalent to

$$\mu^{(1)}(\Sigma^{n-2,1}(f)) = m_0(\Sigma^{n-2,1}(f)) - 1 \tag{11}$$

and

$$m_1(\Sigma^{n-2,1}(f)) = \mu(\Sigma^{n-2,1}(f) \cap p_1^{-1}(0)) + \mu(\Sigma^{n-2,1}(f)). \tag{12}$$

But

$$\mu(\Sigma^{n-2,1}(f) \cap p_1^{-1}(0)) = \mu^{(1)}(\Sigma^{n-2,1}(f)) \tag{13}$$

from (13) and (12) it follows that

$$m_1(\Sigma^{n-2,1}(f)) = \mu^{(1)}(\Sigma^{n-2,1}(f)) + \mu(\Sigma^{n-2,1}(f)). \tag{14}$$

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Then, from (11) and (14) we obtain

$$m_1(\Sigma^{n-2,1}(f)) - m_0(\Sigma^{n-2,1}(f)) = \mu(\Sigma^{n-2,1}(f)) - 1,$$

Therefore the equality (III) follows from the definition of the polar multiplicities and using the genericity of the projections, as we can see in the proof of 4.2.

The proof of the equality (IV) is analogous to the proof of the equality (II), since $D_1^2(f|\Sigma(f))$ is 1-dimensional. ∎

6. THE INVARIANTS IN THE STABLE TYPES OF $X(F_T)$

We consider now the set $X(f_t) = \overline{\left(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t)\right)}$.

Call $h_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ the function that defines $f^{-1}(\Delta(f_t))$ in $\mathbb{C}^n$, and denote by $X_o(f_t) : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ the function $X_0(f_t) := h_t(x, z)/J^2[f_t]$ which defines $X(f_t)$.

As we are considering $n > 3$ the set $X(f_t)$ is an hypersurface, possibly with non-isolated singularities, then we can not apply the Theorem 2.2 to the hypersurface $X(f_t)$ in $\{t\} \times \mathbb{C}^n$, since it is not with isolated singularity.

On the other side, we apply the results of Lê and Teissier given in [11], and of Gaffney and Gassler in [5] to prove the Whitney equisingularity of the regular set of $X(F) = \overline{(F^{-1}(\Delta(F)) - \Sigma(F))}$ along the parameter space $T = \mathbb{C} \times \{0\}$.

The main invariants used to obtain the Whitney equisingularity of the family $X(f_t)$ are the Lê numbers, which are the generalization of the Milnor number for handling non-isolated hypersurface singularities. See [13] for the definition of these numbers.

Proposition 6.1. *The pair $(X(F) - \Sigma(X_0(F)), T)$ is Whitney equisingular if, and only if, the Lê numbers $\lambda^i(X_0(f_t)/H^k)$ and $\lambda^i(X_0(f_t))$ are constants on T for all $i = 1, \dots, k-1$, $k = 1, ..., n-2$, where H^k is a generic k-dimensional linear subspace in $\mathbb{C}^n$.*

Proof: To proof of this result we follow the proof done by M. J. Saia and V. H. J. Peres in [16] for the case $n = p$.

From Gaffney and Gassler in [5], p. 726 we see that

$$\chi^{(k)} = m_k(X_0(f_t)) + \sum_{i=0}^{k} (-1)^{k-i} \lambda^i(X_0(f_t)), \tag{1}$$

here $\chi^{(k)}$ denotes the reduced Euler characteristic of the Milnor fibre of $X_0(f_t)$ restricted to a generic k-dimensional linear subspace in $\mathbb{C}^n$.

We denote by χ^* the sequence $\chi^* = \left(\chi^{(n)}, \cdots, \chi^{(2)}\right)$. From the theorem (5.3.1) p. 95 of [10], we see that $X(F) - \Sigma(X_0(F))$ is Whitney equisingular along the parameter space if, and only if, χ^* is constant.

If the Lê numbers of $X(f_t)$ and the Lê numbers of all generic planar sections of $X(f_t)$ are constant on T, we apply the equality (1) for $\chi^{(k)}$ and the following equalities given by Gaffney and Gassler in [5], p. 710:

$$\lambda^0(X_0(f_t)/H^j) = \lambda^{n-j}(X_0(f_t)) + m_{n-j}(X_0(f_t)),\ \text{for } j = 2, ..., n-1, \tag{2}$$

$$\lambda^{k-i}(X_0(f_t)/H^k) = \lambda^{n-i}(X_0(f_t)) \text{ for } i = 1, \ldots, k-1,$$

to obtain that χ^* is constant, hence the pair $(X(F) - \Sigma(X_0(F)), T)$ is Whitney equisingular.

We remark that here $m_k(X_0(f_t))$ denotes the *relative polar multiplicity.*

On the other side, if the pair $(X(F) - \Sigma(X_0(F)), T)$ is Whitney equisingular, then χ^* is constant. Moreover we obtain that all relative polar multiplicities are also constant, see [18] chapter V theorem 2.1. From the equality (1) we obtain that λ^j are constant for all j. From the equality (2) we obtain that $\lambda^j(X_0(f_t)/H^k)$ are constants for each generic k-plane H^k. ■

Remark 6. 1. From this Proposition and also from the theorem (5.3.1) of [10] we obtain that the total space $X(F)$ has a stratification by the stable types and the parameter space T, such that the condition W_f, therefore the Whitney equisingularity, holds for every pair of strata, except possibly over T.

However, we can use the Lê Numbers associated to the sets $X(f_t)$ to obtain the Whitney equisingularity of the full set $X(F)$ along the parameter space.

Let $F : (\mathbb{C} \times \mathbb{C}^n, (0,0)) \to (\mathbb{C} \times \mathbb{C}^3, (0,0))$ with $n > 3$ be a 1-parameter unfolding of a finitely determined map germ $f \in \mathcal{O}(n, 3)$.

THEOREM 6.1. *Suppose that the stratification by the stable types of F has only the parameter space $T = \mathbb{C} \times \{0\}$ in $\mathbb{C} \times \mathbb{C}^3$ as a locus of instability. Then the pair $(X(F), T)$ is Whitney equisingular if, and only if, the sequence $(\gamma_1(X_0(f_t)), \ldots, \gamma_{n-1}(X_0(f_t)), \chi^*)$ is independent of t.*

Proof: First we remark that if the closure of a stratum S of $V(X_0(F)) = X(F)$ is the image of a component of the exceptional divisors of the Blow up $B = Bl_{J(X_0(F))}\mathbb{C}^{n+1}$, from the fact that the condition A_F of Thom is generic and also from dimensional reazons, this component is the conormal space of $\bar{S}$. Then, applying the theorem 6.3 of [7], the Blow up of this component by the pullback of γ_T has a exceptional divisor which is equidimensional over T. From the theorem V.1.2 of [18]), the pair (S, T) satisfies the condition W_F, hence the Whitney equisingularity.

Therefore it is enough to show that each stable type is the image of a component of the exceptional divisor of the Blow up $B = Bl_{J(X_0(F))}\mathbb{C}^{n+1}$.

First we show this for the mono germs. Here it is sufficient to show that if $f_{\mathcal{Q}}$ represents a minimal stable type $\mathcal{Q}$ which appears with positive dimension in F, then the relative polar curve of $X(f_{\mathcal{Q}})$ at the origin is not empty, but this curve is empty if and only if the intersection $H_1 \cap \ldots \cap H_{n-1} \cap Bl_{J(h)}\mathbb{C}^n$ is empty if and only if the fiber over the origin is not a component.

Therefore we conclude that the origin is the image of a component of the exceptional divisor. Since $X(f_{\mathcal{Q}})$ is a cartesian product along the strata of $X(F)$ which are different from T, the stratum which represents the stable type $\mathcal{Q}$ is also the image of a component of the exceptional divisor.

We remember that if the multiplicity of the relative polar curve is not zero, then the polar curve is not empty. Massey showed in ([14], p. 365) that this multiplicity is the number of spheres in the homotopy type of the link of the singularity, independently of the dimension of the singular locus. Here we only need to show now that this number is greater that zero.

For this we apply the theorem 3.1 of Damon in [1], pp. 14, which tell us that for t and ϵ small enough, $(X_0(f_{\mathcal{Q}}))^{-1}(0) \cap B_\epsilon$ or $(X(f_{\mathcal{Q}}) \cap B_\epsilon)$ is independent of the stabilization $f_{\mathcal{Q}}$ and is homotopically equivalent to a bouquet of n real spheres, since $n > 3$, this is different from zero.

Therefore the multiplicity of the relative polar curve is not zero, and we finish the proof for mono germs.

Now we consider the stable types of the multi germs. Here we have that if two smooth sheets of $X(F)$ intersect, then this has codimension 0 in $\Sigma(X_0(F))$, so this stratum must be the image of a component of the exceptional divisor. Otherwise, we can assume that the set $X(f_t)$ is locally the union of two hypersurfaces X_1 and X_2 embedded in $\mathbb{C}^k \times \mathbb{C}^{n-k}$, with equations $g_1(z_1, \ldots, z_k) = 0$ and $g_2(z_{k+1}, \ldots, z_n) = 0$. There are two cases: one with $k = n-1$ and $g_2 = z_n$ and the other is with both $g_1,\ g_2$ defining singular hypersurfaces.

Suppose first that $h_2 = z_n$. Then we can assume that ϕ_1 parametrizes a branch of the polar curve of h_1 and $h_{1_{z_1}} \circ \phi_1$ is zero for $1 \leq i \leq n-2$. As h_1 is in the integral closure of $J(h_1)$ we obtain that $(h_1/h_{1_{z_{n-1}}}) \circ \phi_1$ is an analytic germ ψ^n. Call $\varphi = (\phi_1, \psi_n)$ then if $f = z_n h_1 f_{z_i} \circ \varphi = 0$ for $1 \leq i \leq n-2$ and $(f_{z_n} - f_{z_{n-1}}) \circ \varphi = (h_1 - z_n h_{1_{z_{n-1}}}) \circ \varphi = 0$. Hence φ parametrizes the branch of the polar curve of f.

Now suppose that $f = h_1 h_2$ and assume that ϕ_1 and ϕ_2 are parametrizations of the branches of the polar curves h_1 and h_2, respectively. Then $\varphi = (\phi_1, \phi_2)$ parametrizes a polar surface of f and we obtain $f_{z_1} \circ \varphi = 0$ for $1 \leq i \leq k$ and $k+1 \leq i \leq n$. Therefore, for an appropriate choice of A and B, $(Af_{z_k} + Bf_{z_n}) \circ \varphi$ defines a curve of singularities which branch is parametrized by ψ, then $\varphi \circ \psi$ parametrizes a branch of the polar curve of f and we are done. ■

From the equivalence between the sequences $(\gamma_1(X_0(f_t)), \ldots, \gamma_{n-1}(X_0(f_t)), \chi^*)$, $(\gamma_1(X_0(f_t)), \ldots, \gamma_{n-1}(X_0(f_t)), \lambda_2(X_0(f_t)), \ldots, \lambda_n(X_0(f_t)))$ and the equations 2, we obtain the following:

Corollary 6.1. *The pair par $(X(f_t), T)$ is Whitney equisingular if and only if the Lê numbers $\lambda^i(X_0(f_t))$, $1 \leq i \leq n$ and $\lambda^k(X_0(f_t)|H^{n-k})$, $1 \leq k \leq n-1$ are constant in the origin for any t.*

7. THE MAIN RESULTS

From the main Theorem of Gaffney, we need the constancy of $(6n+9)$ invariants to guarantee the Whitney equisingularity, for example, in the case $n=4$ we need 33 invariants.

To minimize this number we apply all results shown here to obtain first a result for the case of map germs with any corank. In this case we reduce the number of invariants from $6n+9$ to $2n+13$.

THEOREM 7.1. *Let $F:(\mathbb{C}\times\mathbb{C}^n,(0,0))\to(\mathbb{C}\times\mathbb{C}^3,(0,0))$ with $n>3$ be a 1-parameter unfolding of a finitely determined map germ $f\in\mathcal{O}(n,3)$. Suppose that the stratification by the stable types of F has only the parameter space $T=\mathbb{C}\times\{0\}$ in $\mathbb{C}\times\mathbb{C}^3$ as a locus of instability.*

Then the family is Whitney equisingular if and only if the following numbers are constant at the origin for all f_t:

$\mu(\Sigma(f_t))$, $m_1(\Sigma(f_t))$, $m_0(\Sigma^{(n-2,1)}(f_t))$, $m_1(\Sigma^{(n-2,1)}(f_t))$, $m_0(D_1^2(f_t|\Sigma(f_t)))$, $m_1(\Delta(f_t))$, $m_0(f_t(\Sigma^{(n-2,1)}(f_t)))$, $m_1(f(\Sigma^{(n-2,1)}(f_t)))$, $m_0(f_t(D_1^2(f_t|\Sigma(f_t))))$, $m_1(f_t(D_1^2(f_t|\Sigma(f_t)))), v\ m_1(D_1^2(f_t|\Sigma(f_t)))$, $\lambda^i(X_0(f_t))$, *for* $1\le i\le n$, $\lambda^k(X_0(f_t)|H^{n-k})$, *for* $1\le k\le n-1$.

In the case of corank one map germs, we minimize the number of invariants to obtain the following:

THEOREM 7.2. *Let $F(t,x)=(t,f_t(x))$ be an unfolding of a finitely determined map germ and of corank one $f\in\mathcal{O}(n,3), n>3$. Suppose that the stratification by the stable types of F has only the parameter space $T=\mathbb{C}\times\{0\}$ in $\mathbb{C}\times\mathbb{C}^3$ as a locus of instability.*

Then the family is Whitney equisingular if and only if the following numbers are constant at the origin for all f_t:

$\mu(\Sigma(f_t)), \mu(\Sigma^{(n-2,1)}(f_t)), m_1(\Sigma(f_t)), m_0(D_1^2(f_t|\Sigma(f_t))), m_1(D_1^2(f_t|\Sigma(f_t))), m_1(\Delta(f_t))$, $m_0(f_t(D_1^2(f_t|\Sigma(f_t))))$, $m_1(f_t(D_1^2(f_t|\Sigma(f_t))))$, $\lambda^i(X_0(f_t))$, *for* $1\le i\le n$, $\lambda^k(X_0(f_t)|H^{n-k})$, *for* $1\le k\le n-1$.

Here we reduce the number of invariants from $6n+9$ to $2n+7$.

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