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Valuations compatible with a projection

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Given an N-dimensional germ of analytic hypersurface $\mathcal{H}$, a finite projection $\pi : \mathcal{H} \longrightarrow \mathbb{C}^N$ and a valuation ν on the ring the ring of convergent series in N variables, we study the valuations on the ring $\mathcal{O}_{\mathcal{H}}$ that extend $\pi^*\nu$. All these valuations are described when ν is a monomial valuation whose weight vector is not orthogonal to any of the faces of the Newton Polyhedron of the discriminant of the projection π. This description is done in terms of the Puiseux parameterizations of $\mathcal{H}$ with exponents in a cone. October, 2004 ICMC-USP

1. INTRODUCTION

Let $\mathcal{H}$ be an irreducible germ of analytic hypersurface at the origin in $\mathbb{C}^{N+1}$, let $\pi : \mathcal{H} \longrightarrow (\mathbb{C}^N, 0)$ be a finite projection and denote by $\mathcal{R}$ the ring of convergent series in N variables. Given a valuation $\nu_\sim : \mathcal{R} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$, we want to describe all the valuations $\nu : \mathcal{O}_{\mathcal{H}} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ that *extend* $\nu_\sim$. That is, that make the diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathcal{H}} & & \stackrel{\nu}{\longrightarrow} & & \mathbb{R}_{\geq 0} \cup \{\infty\} \\ & \nwarrow^{\pi^*} & & \nearrow^{\nu_\sim} & \\ & & \mathcal{R} & & \end{array} \qquad (1)$$

commute.

In this note all these valuations are described when $\nu_\sim$ is a monomial valuation whose weight vector is not orthogonal to any of the faces of the Newton Polyhedron of the discriminant of the projection π. This description is done in terms of the Puiseux parameterizations of $\mathcal{H}$.

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In all what follows $\mathcal{H}$ will be a germ of analytic hypersurface embedded in $(\mathbb{C}^{N+1}, 0)$ defined by $F(x_1, \ldots, x_N, y) = 0$, where $F = y^d + a_{d-1}y^{d-1} \cdots + a_0$ is an irreducible polynomial of degree d in y and $a_i \in \mathcal{R}$. The projection

$$\begin{array}{cccc} \pi : & \mathcal{H} & \longrightarrow & \mathbb{C}^N \\ & (x_1, \ldots, x_N, y) & \mapsto & (x_1, \ldots, x_N) \end{array} \tag{2}$$

will be supposed to be finite. The discriminant of the projection π will be denoted by δ. That is

$$\delta(x_1, \ldots, x_N) = \text{Resultant}_y(F, \frac{\partial}{\partial y}F).$$

The ideal of $\mathcal{R}[y]$ generated by F will be denoted by $\mathcal{I}$. So that $\mathcal{O}_{\mathcal{H}} = \frac{\mathcal{R}[y]}{\mathcal{I}}$ and π^* is the natural inclusion $\mathcal{R} \hookrightarrow \frac{\mathcal{R}[y]}{\mathcal{I}}$.

2. MONOMIAL VALUATIONS

Given a Laurent series $\varphi = \sum_{\alpha \in \mathbb{Z}^N} a_\alpha x^\alpha$, the *set of exponents of* φ is the set

$$\mathcal{E}(\varphi) := \{\alpha \in \mathbb{Z}^N \mid a_\alpha \neq 0\}.$$

When $\mathcal{E}(\varphi)$ is finite, φ is a Laurent polynomial. When $\mathcal{E}(\varphi)$ is contained in the first orthant, φ is a series with non-negative exponents.

A subset of $\mathbb{R}^N$ of the form $\sigma = \{\lambda_1 v_1 + \cdots + \lambda_r v_r \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$ for some $v_1, \ldots, v_r \in \mathbb{Q}^N$ is called a (rational convex polyhedral) *cone*. A cone is said to be *strongly convex* when it contains no non-trivial linear subspace.

Let $\sigma \subset \mathbb{R}^N$ be a strongly convex cone. The set of *formal Laurent series with exponents in* σ

$$\mathbb{C}[[\sigma]] = \{\varphi \mid \mathcal{E}(\varphi) \subset \sigma\}$$

is a ring with the natural sum and product.

Definition 2.1. The *Newton polyhedron* of a series $\phi \in \mathcal{R}$ is the convex hull in $\mathbb{R}^N$ of the set $\mathcal{E}(\phi) + {\mathbb{R}_{\geq 0}}^N$ and is denoted by NPϕ. Let V be a vertex of NPϕ. The *cone of NPϕ associated to* V is the cone

$$\sigma_V = \{v \in \mathbb{R}^N \mid (V + \lambda v) \in \text{NP}\phi \quad \text{for some positive real number} \quad \lambda\}.$$

Remark 2. 1. The cone σ_V is always strongly convex and contains the positive orthant.

Now for $k \in \mathbb{N}$ consider the ring of series

$$\mathbb{C}[[\sigma]]_{\frac{1}{k}} := \left\{ \sum_{\alpha \in \left(\frac{1}{k}\mathbb{Z}\right)^N \cap \sigma} a_\alpha x^\alpha \mid a_\alpha \in \mathbb{C} \right\}.$$

When k divides k' there is a natural inclusion $\mathbb{C}[[\sigma]]_{\frac{1}{k}} \hookrightarrow \mathbb{C}[[\sigma]]_{\frac{1}{k'}}$. So, it makes sense to consider the *ring of formal Puiseux series with exponents in* σ

$$\mathbb{C}[[\sigma]]^{\wp} := \bigcup_{k \in \mathbb{N}} \mathbb{C}[[\sigma]]_{\frac{1}{k}},$$

We will work only with cones σ containing the first orthant, for such σ we have the following diagram of inclusions:

$$\begin{array}{ccccc} \mathcal{R} & \longrightarrow & \mathbb{C}[[\sigma]] & \longrightarrow & \mathrm{Fr}(\mathbb{C}[[\sigma]]) \\ & & \downarrow & & \downarrow \\ & & \mathbb{C}[[\sigma]]^{\wp} & \longrightarrow & \mathrm{Fr}(\mathbb{C}[[\sigma]]^{\wp}) \end{array}$$

where $\mathrm{Fr}(\mathcal{A})$ stands for the field of fractions of $\mathcal{A}$.

The *dual* of a cone σ is the cone $\sigma^\vee := \{v \in \mathbb{R}^N \mid u \cdot v \geq 0, \forall u \in \sigma\}$. A cone $\sigma \subset \mathbb{R}^N$ is strongly convex if and only if the interior of $\sigma^\vee$ (as a subset of $\mathbb{R}^N$) is not empty.

Definition 2.2. Let σ be a strongly convex cone. Given $w \in \sigma^\vee$ the map $\nu_w : \mathbb{C}[[\sigma]]^{\wp} \longrightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$\nu_w \phi = \min_{\alpha \in \mathcal{E}(\phi)} w \cdot \alpha \tag{3}$$

is a valuation called the *monomial* valuation with weight w.

By restriction, ν_w induces a valuation in all subrings of $\mathbb{C}[[\sigma]]^{\wp}$ that we will still denote by ν_w. The extension of ν_w from $\mathcal{R}$ to $\mathbb{C}[[\sigma]]^{\wp}$ is not unique. Anyhow the following holds:

Lemma 2.1. *There exists a unique way to extend ν_w from the ring of series with exponents in σ to the ring of Puiseux series with exponents in σ.*

Proof:

$$\mathbb{C}[[\sigma]]_{\frac{1}{k}} = \mathbb{C}[[\sigma]][x_1^{\frac{1}{k}}, \ldots, x_N^{\frac{1}{k}}] = \frac{\mathbb{C}[[\sigma]][t_1, \ldots, t_N]}{(\{t_i^k - x_i\}_{i=1,\ldots,N})}$$

The field extension

$$\mathrm{Fr}\left(\mathbb{C}[[\sigma]]\right) \hookrightarrow \mathrm{Fr}\left(\mathbb{C}[[\sigma]]\right)(x_1^{\frac{1}{k}}, \ldots, x_N^{\frac{1}{k}})$$

is a normal algebraic extension. Its Galois group is formed by the automorphisms

$$g_{(j_1,\ldots,j_N)} : x_i^{\frac{1}{k}} \mapsto \xi^{j_i} x_i^{\frac{1}{k}}; \qquad \xi^k = 1, \quad (j_1, \ldots, j_N) \in \mathbb{N}^N.$$

by a theorem of Ostrowski and Krull [4, F,Theorem 1] two extensions of a given valuation are conjugate. Since $\nu_w(g_{(j_1,\ldots,j_N)}(\phi)) = \nu_w \phi$ we have the result.
Q.E.D.

Remark 2. 2. Let $\mathbb{C}[\sigma]$ denote the ring of Laurent polynomials with exponents in σ. That is $\mathbb{C}[\sigma] = \{\varphi \in \mathbb{C}[[\sigma]] \mid \#\mathcal{E}(\varphi) < \infty\}$. Then $\mathbb{C}[[\sigma]]$ is the completion of $\mathbb{C}[\sigma]$ with respect to the valuation ν_w for any w in the interior of $\sigma^\vee$. The field of fractions of $\mathbb{C}[\sigma]$ is the same as the field of fractions of $\mathbb{C}[x_1, \ldots, x_N]$. Then, $\mathbb{C}[[\sigma]]$ is a ring of dimension N.

3. PARAMETRIZATIONS WITH EXPONENTS IN A CONE.

J. McDonald showed in [3] that given a polynomial P in $\mathbb{C}[x_1, \ldots, x_N][y]$, for any $w \in \mathbb{R}^N$ of rationally independent coordinates, there exists a strongly convex cone σ, with $w \in \sigma^\vee$ such that P has a root in $\mathbb{C}[[\sigma]]^\wp$. Moreover he gives an algorithm to compute such series. Then, P. González Pérez showed in [2] that σ may be chosen to be a cone of the Newton polyhedron of the discriminant of P with respect to y.

Definition 3.1. Let $\sigma \subset \mathbb{R}^N$ be a cone. For $\varrho \in (\mathbb{R}_{>0})^N$, the *$\sigma$-wedge* of polyradius ϱ is the set

$$\mathrm{W}(\sigma, \varrho) := \left\{ z \in (\mathbb{C}^*)^N;\ \tau(z)^u \leq \varrho^u,\ \forall\, u \in \sigma \cap \mathbb{Z}^N \right\}$$

where $\tau(z_1, \ldots, z_N) = (|z_1|, \ldots, |z_N|)$.

Denote by δ the discriminant of F with respect to y. For each vertex V of the Newton Polyhedron of δ let σ_V be the cone of NPδ associated to V (Definition 2.1). By [1, Proposition 5.1], there exists $\varrho_V \in (\mathbb{R}_{>0})^N$ such that the σ_V-wedge of polyradius ϱ_V does not intersect the zero locus of δ.

Definition 3.2. A connected component of $\pi^{-1}(\mathrm{W}(\sigma_V, \varrho_V)) \cap \mathcal{H}$ will be called a *σ_V-branch* of $\mathcal{H}$. Given a σ_V-branch C of $\mathcal{H}$ the degree of the covering $\pi : \mathrm{C} \longrightarrow \mathrm{W}(\sigma_V, \varrho_V)$ will be denoted by d_{C}.

Remark 3. 1. Let $\mathcal{B}_V$ be the set of σ_V-branches of $\mathcal{H}$, $d = \sum_{\mathrm{C} \in \mathcal{B}_V} d_{\mathrm{C}}$, d being the degree of F in y.

The following proposition is proved in [1]:

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PROPOSITION 3.1. *Let V be a vertex of NPδ and let σ_V be the cone of NPδ associated to V. Given a σ_V-branch* C *of $\mathcal{H}$, there exist a σ_V-wedge W and a series $\varphi_C \in \mathbb{C}[[\sigma_V]]$, convergent on W such that*

$$\begin{array}{cccc} \Phi_{\mathrm{C}}: & W & \longrightarrow & \mathbb{C}^{N+1} \\ & (x_1,\ldots,x_N) & \mapsto & ({x_1}^{d_{\mathrm{C}}},\ldots,{x_N}^{d_{\mathrm{C}}},\varphi_{\mathrm{C}}(x)) \end{array}$$

parameterizes C*. That is:*

$$\left\{({x_1}^{d_{\mathrm{C}}},\ldots,{x_N}^{d_{\mathrm{C}}},\varphi_{\mathrm{C}}(x)) \mid (x_1,\ldots,x_N) \in W\right\} = \mathrm{C}. \tag{4}$$

Remark 3. 2. Let ζ be a primitive d_{C}-root of unity, the set of functions with property (4) is

$$\left\{\varphi_{\mathrm{C}}(\zeta^{i_1}x_1,\ldots,\zeta^{i_N}x_N) \mid i_j \in \{1,\ldots,d_{\mathrm{C}}\}\right\} = \left\{\varphi_{\mathrm{C}}^{(1)},\ldots,\varphi_{\mathrm{C}}^{(d_{\mathrm{C}})}\right\}$$

and it has exactly d_{C} elements.

Remark 3. 3. The irreducibility of $\mathcal{H}$ implies that a polynomial $h \in \mathcal{R}$ is an element of $\mathcal{I}$ if and only if it vanishes on C. That is

$$h({x_1}^{d_{\mathrm{C}}},\ldots,{x_N}^{d_{\mathrm{C}}},\varphi_{\mathrm{C}}(x_1,\ldots,x_N)) = 0$$

which is equivalent to $h(x_1,\ldots,x_N,\varphi_{\mathrm{C}}({x_1}^{\frac{1}{d_{\mathrm{C}}}},\ldots,{x_N}^{\frac{1}{d_{\mathrm{C}}}})) = 0$.

4. σ-BRANCHES AND PRIMARY DECOMPOSITION.

Let V be a vertex of NPδ and let C be a σ_V-branch of $\mathcal{H}$. We will denote by $\mathcal{J}_{\mathrm{C}}$ the kernel of the morphism

$$\begin{array}{ccc} \mathbb{C}[[\sigma_V]][y] & \longrightarrow & \mathbb{C}[[\sigma_V]] \\ h(x_1,\ldots,x_N,y) & \mapsto & h({x_1}^{d_{\mathrm{C}}},\ldots,{x_N}^{d_{\mathrm{C}}},\varphi_{\mathrm{C}}(x)). \end{array}$$

where φ_{C} is as in proposition 3.1.

Since $\mathbb{C}[[\sigma_V]]$ is an integral domain, $\mathcal{J}_{\mathrm{C}}$ is a prime ideal of $\mathbb{C}[[\sigma_V]][y]$. By remark 3.3 we have $\mathcal{I} = \mathcal{J}_{\mathrm{C}} \cap \mathcal{R}[y] \quad$ for any $\quad \mathrm{C} \in \mathcal{B}_V$.

PROPOSITION 4.1. *Let $\mathcal{I}^{\sigma_V}$ be the extension of $\mathcal{I}$ to the ring $\mathbb{C}[[\sigma_V]][y]$ via the natural inclusion. Then*

$$\mathcal{I}^{\sigma_V} = \bigcap_{\mathrm{C} \in \mathcal{B}_V} \mathcal{J}_{\mathrm{C}}.$$

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Proof: Set $\phi_{\mathrm{C}}^{(i)} := \varphi_{\mathrm{C}}^{(i)}(x_1{}^{\frac{1}{d_{\mathrm{C}}}}, \ldots, x_N{}^{\frac{1}{d_{\mathrm{C}}}})$, $i = 1, \ldots d_{\mathrm{C}}$, where the $\varphi_{\mathrm{C}}^{(i)}$ are as in remark 3.2. Each $\varphi_{\mathrm{C}}^{(i)}$ is a root of F as a polynomial in y. Then $\prod_{i=1}^{d_{\mathrm{C}}}(y - \phi_{\mathrm{C}}^{(i)})$ divides F as an element of $\mathbb{C}[[\sigma_V]]^{\wp}[y]$. This, together with remark 3.1, implies

$$F = \prod_{\mathrm{C}\in\mathcal{B}_V} \prod_{i=1}^{d_{\mathrm{C}}}(y - \phi_{\mathrm{C}}^{(i)}). \tag{5}$$

Let $\mathcal{K}_{\mathrm{C}}^{(i)}$ be the kernel of the morphism

$$\begin{array}{ccc} \mathbb{C}[[\sigma_V]]^{\wp}[y] & \longrightarrow & \mathbb{C}[[\sigma_V]]^{\wp} \\ h(x,y) & \mapsto & h(x, \phi_{\mathrm{C}}^{(i)}(x)). \end{array}$$

By remark 3.3 we have

$$\mathcal{J}_{\mathrm{C}} = \mathcal{K}_{\mathrm{C}}^{(i)} \cap \mathbb{C}[[\sigma_V]][y] \quad \text{for any} \quad i \in \{1, \ldots, d_{\mathrm{C}}\}. \tag{6}$$

Let $\mathcal{I}^{\sigma_V{}^{\wp}}$ be the extension of $\mathcal{I}$ to the ring $\mathbb{C}[[\sigma_V]]^{\wp}[y]$ via the natural inclusion. Equation (5) implies $\mathcal{I}^{\sigma_V{}^{\wp}} = \bigcap_{\mathrm{C}\in\mathcal{B}_V} \bigcap_{i=1}^{d_{\mathrm{C}}} \mathcal{K}_{\mathrm{C}}^{(i)}$, and the conclusion follows from (6).
Q.E.D.

PROPOSITION 4.2. *The only prime ideals $\mathcal{P}$ of $\mathbb{C}[[\sigma_V]][y]$ with the property $\mathcal{P} \cap \mathcal{R}[y] = \mathcal{I}$ are of the form $\mathcal{J}_{\mathrm{C}}$ with $\mathrm{C} \in \mathcal{B}_V$.*

Proof: It follows from proposition 4.1 and remark 2.2.
Q.E.D.

5. THE THEOREM

Given $w \in \mathbb{R}_{>0}{}^N$, take a vertex V of the Newton polyhedron of δ such that $w \in \sigma^\vee$. Each σ_V-branch C of $\mathcal{H}$ induces a valuation

$$\begin{array}{rccc} \nu_{\mathrm{C}} : & \mathcal{O}_{\mathcal{H}} & \longrightarrow & \mathbb{R} \cup \infty \\ & h(x_1, \ldots, x_N, y) & \mapsto & \dfrac{1}{d_{\mathrm{C}}} \nu_w h(x_1{}^{d_{\mathrm{C}}}, \ldots, x_N{}^{d_{\mathrm{C}}}, \varphi_{\mathrm{C}}) \end{array}$$

that extends ν_w.

THEOREM 5.1. *Let $w \in \mathbb{R}_{>0}{}^N$ be a vector non-orthogonal to any of the faces of the Newton Polyhedron of δ, and let V be the only vertex of $NP\delta$ such that w belongs to the dual of σ_V. Then all the valuations that extend ν_w are the ones induced by the σ_V-branches of $\mathcal{H}$.*

We start by proving a lemma:

LEMMA 5.1. *Let w be a vector in the interior of ${\sigma_V}^\vee$, and let $\nu : \mathcal{O}_\mathcal{H} \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a valuation that extends ν_w. There exists a σ_V-branch C of $\mathcal{H}$ and a valuation $\bar{\nu}$: $\frac{\mathbb{C}[[\sigma_V]][y]}{\mathcal{J}_\mathrm{C}} \longrightarrow \mathbb{R} \cup \{\infty\}$ that makes the diagram*

$$\begin{array}{ccccc} \mathbb{C}[[\sigma_V]] & \hookrightarrow & \dfrac{\mathbb{C}[[\sigma_V]][y]}{\mathcal{J}_\mathrm{C}} & \hookleftarrow & \mathcal{O}_\mathcal{H} \\ & \searrow \nu_w & \downarrow \bar{\nu} & \nu \swarrow & \\ & & \mathbb{R} \cup \{\infty\} & & \end{array}$$

commutative.

Proof: An element $h \in \mathbb{C}[[\sigma]][y]$ is written as

$$h = \sum_{i=0}^{\deg h} \psi_i y^i, \quad \text{where} \qquad \psi_i = \sum_{\alpha \in \mathbb{Z}^N \cap \sigma} a_\alpha^{(i)} x^\alpha$$

for some $A \in \mathbb{Z}^N$. For each $i \in \{0, \ldots, \deg h\}$ and $j \in \mathbb{Z}$ set

$$\psi_i^{(j)} := \sum_{\substack{\alpha \in \mathbb{Z}^N \cap \sigma \\ j \leq w \cdot \alpha < j+1}} a_\alpha^{(i)} x^\alpha.$$

Since w is in the interior of $\sigma^\vee$, for all j, the set $\{\alpha \in \mathbb{Z}^N \cap \sigma \mid j \leq w \cdot \alpha < j+1\}$ is finite and then the $\psi_i^{(j)}$'s are Laurent polynomials. We have

$$\psi_i = \sum_{j=0}^{\infty} \psi_i^{(j)} \quad \text{for all} \quad i \in \{1, \ldots, \deg h\}.$$

Let $\mathcal{R}[y]_\mathcal{I}$ be the localization of $\mathcal{R}[y]$ with respect to $\mathcal{I}$. Since $\mathcal{I} \cap \mathcal{R} = \{0\}$, we have

$$\psi_i^{(j)} \in \mathcal{R}[y]_\mathcal{I}, \qquad \forall j \in \mathbb{Z} \,\text{and}\, \forall i \in \{1, \ldots, \deg h\}$$

Let $\hat{\nu} : \mathcal{R}_\mathcal{I} \longrightarrow \mathbb{R}$ be the extension of the morphism given by the composition

$$\mathcal{R}[y] \longrightarrow \mathcal{O}_\mathcal{H} \xrightarrow{\nu} \mathbb{R} \cup \infty.$$

Since ν extends ν_w and $\psi_i^{(j)} \in \mathrm{Fr}(\mathcal{R})$, we have $\hat{\nu}\left(\psi_i^{(j)}\right) = \nu_w\left(\psi_i^{(j)}\right) \geq j$. Then

$$\hat{\nu}\left(\sum_{i \in \{1, \ldots, \deg h\}} \psi_i^{(K)} y^i\right) \geq K + \min\{0, \nu(y) \deg h\}, \qquad \forall K. \tag{7}$$

Set

$$\tau_K := \hat{\nu}\left(\sum_{j=0}^{K} \sum_{i \in \{1,\dots,\deg h\}} \psi_i^{(j)} y^i\right).$$

Inequality (7) implies that either $\tau_K \geq K + \min\{0, \nu(y)\deg h\}$ for all K or there exists K such that $\tau_l = \tau_K$ for all $l > K$. Then it makes sense to define:

$$\tilde{\nu} h := \lim_{K \longrightarrow \infty} \tau_K.$$

By construction, $\tilde{\nu}(hh') = \tilde{\nu}(h) + \tilde{\nu}(h')$ and $\tilde{\nu}(h+h') \geq \min\{\tilde{\nu}(h), \tilde{\nu}(h')\}$, then $\tilde{\nu}$ induces a valuation $\bar{\nu} : \dfrac{\mathbb{C}[[\sigma]][y]}{\tilde{\nu}^{-1}(\infty)} \longrightarrow \mathbb{R} \cup \infty$.

The ideal $\tilde{\nu}^{-1}(\infty)$ is prime and $\tilde{\nu}^{-1}(\infty) \cap \mathcal{R}[y] = \mathcal{I}$. Then, by proposition 4.2 there exists $\mathrm{C} \in \mathcal{B}_V$ such that $\tilde{\nu}^{-1}(\infty) = \mathcal{J}_{\mathrm{C}}$.

Q.E.D.

Proof of theorem: Let C and $\bar{\nu} : \frac{\mathbb{C}[[\sigma]][y]}{\mathcal{J}_{\mathrm{C}}} \longrightarrow \mathbb{R} \cup \{\infty\}$ be as in the lemma. Let $\mathcal{K}_{\mathrm{C}}^{(1)}$ be as defined in the proof of proposition 4.1, and let $\bar{\bar{\nu}}$ be an extension of $\bar{\nu}$ to $\dfrac{\mathbb{C}[[\sigma_V]]^{\wp}[y]}{\mathcal{K}_{\mathrm{C}}^{(1)}}$.

Given $h \in \mathcal{R}[y]$,

$$h(x,y) = h(x, \phi_{\mathrm{C}}^{(1)}) + (y - \phi_{\mathrm{C}}^{(1)}) g(x,y), \quad \text{where} \quad g(x,y) \in \mathbb{C}[[\sigma]]^{\wp}.$$

So,

$$\nu(h) = \bar{\bar{\nu}}(h) = \bar{\bar{\nu}}(h(x, \phi_{\mathrm{C}}^{(1)})) \overset{\text{lemma 2.1}}{=} \nu_w h(x, \phi_{\mathrm{C}}^{(1)})) = \nu_{\mathrm{C}} h.$$

Q.E.D.

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