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On role of the equal-area condition in internal transition layer stationary solutions to a class of reaction-diffusion systems

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We present necessary conditions for the formation of internal transition layers in stationary solutions to some singularly perturbed reaction-diffusion systems. In particular we prove that the well-known equal-area condition which is always assumed in a typical set of sufficient conditions for existence of such solutions is actually a necessary hypothesis. May, 2004 ICMC-USP

1. INTRODUCTION

The prime concern in this paper is to present a necessary condition for the formation of internal transition layers for stationary solutions in N-dimensional domains of a singularly perturbed reaction-diffusion system which often appears in the literature.

The system which is the object of our analysis takes the general form

$$\begin{cases} u_t = \varepsilon\, div\,(h_1(x)\nabla u) + f(x,u,\mathbf{v}), \ \text{ in } I\!R^+ \times \Omega \\ \mathbf{v}_t = div\,(\mathbf{h_2}(x)\nabla \mathbf{v}) + \mathbf{g}(x,u,\mathbf{v}), \ \text{ in } I\!R^+ \times \Omega \\ \dfrac{\partial\, \mathbf{v}}{\partial\, \hat{n}} = 0 \text{ (or } \mathbf{v} = (0,0,...,0) \text{ on } I\!R^+ \times \partial\Omega, \end{cases}$$

where Ω is a smooth domain in $\mathbb{R}^N$ and the bold letters stand for vector-valued functions. However in order to put our work into perspective let us consider a simpler system of

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reaction-diffusion equations of activator-inhibitor type:

$$\begin{cases} u_t = \varepsilon\triangle u + f(u,v), & (t,x) \in I\!R^+ \times \Omega \\ v_t = \triangle v + g(u,v), & (t,x) \in I\!R^+ \times \Omega \\ \dfrac{\partial u}{\partial \hat{n}} = \dfrac{\partial v}{\partial \hat{n}} = 0, & (t,x) \in R^+ \times \partial\Omega, \end{cases} \tag{1.1}$$

where ε is a small positive parameter and Ω a smooth domain in $I\!R^N$, $N \geq 1$.

Part of the available literature on this problem is devoted to the study of (1.1) in the context of spatial pattern formation as it appears in many different fields such as mathematical biology, chemical reactions, morphogenesis, combustion, etc.. See [6], for instance, for a survey on this issue.

Roughly speaking we will say that a uniformly bounded family $\Phi^\epsilon = (u_\epsilon, v_\epsilon)$ of stationary (meaning that $u_t = v_t = 0$) solutions to (1.1) develops internal transition layer as $\epsilon \to 0$ if the component u_ϵ exhibits a sharp spatial transition between two different states. These solutions will be referred to as i.t.l.s. solutions, for short, and a rigorous definition will be provided.

Typically the setting in which the issue of spatial pattern formation in reaction-diffusion systems is considered involves the choice of a specific parameter region (the rates of diffusion and/or reaction) and the geometry of the zero-level set (nullcline) of the reaction terms f and g.

For one-dimensional domains, and using different techniques, existence (sometimes stability too) of i.t.l.s. solutions to (1.1) has been established in [4], [5], [14], [11], [6], [12], for instance. There is a vast literature on the subject but the references above best suit our purposes.

However, regardless of the particular technique used, whenever proving the existence of i.t.l.s. solutions to (1.1), the following hypotheses are tacitly assumed. The zero set of f,

$$Z = \{(u,v) \in I\!R^2 : f(u,v) = 0\}$$

has at least two different solutions $u = h_-(v)$ and $u = h_+(v)$, $h_-(v) < h_+(v)$, in a suitable domain which contains a real number $v = v^*$ satisfying

$$\int_{h_-(v^*)}^{h_+(v^*)} f(s, v^*)\, ds = 0.$$

Usually Z is supposed to take a sigmoidal form in the (u,v)-plane. This assumption is known as the equal-area condition (or rule) and it is always assumed as a sufficient condition for existence of i.t.l.s. solutions to (1.1). Sometimes it does not appear explicitly but in an equivalent form, as a Melnikov integral, for instance. See [10], for this matter.

It might seems at first sight that this hypothesis is not necessary for proving existence of such solutions. We give herein a rigorous mathematical proof that this is not the case. Rather it is a necessary condition.

Of course for each phenomenon the mathematical system models there is a physical mechanism underlying the equal-area condition whenever internal transition layer is created.

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An immediate conclusion of our results is that if the above equal-area condition on f does not hold then, as long as concentration phenomenon is concerned, we can only expect formation of spikes and/or boundary layer for stationary solutions of the system considered, just to mention the simplest geometric configurations that can occur (see example A.4 in Applications).

See [3], [2] and [9], for instance, for cases of a single scalar equation where the equal-area condition does not hold and spike and boundary layer solutions are obtained.

The present work is an extension to a class of systems of results obtained in [1] for a single scalar elliptic equation and at the same time an improvement of the approach used therein. In order to be more specific let us briefly describe one particular case of the main result in [1].

Let the constants α and β, $\alpha < \beta$, satisfy $f(x,\alpha) = f(x,\beta) = 0$, $\forall x \in \Omega \subset I\!R^N, N \geq 1$ and consider a smooth $(N-1)$-dimensional compact manifold without boundary Γ such that $\Gamma \subset \Omega$.

Suppose that the boundary value problem

$$\begin{cases} \varepsilon\, div\,(h_1(x)\nabla u) + f(x,u) = 0, \quad x \in \Omega \\ \dfrac{\partial u}{\partial \hat{n}} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has a family $\{u_\varepsilon\}$ of solutions which develops inner transition layer with interface Γ connecting the states α to β. Then necessarily

$$\int_\Gamma \{\int_\alpha^\beta f(x,s)ds\}\, x \cdot \hat{\eta}(x)\, dS = 0. \tag{1.3}$$

where $\hat{\eta}$ stands for the outward unit normal vector on Γ. In particular if f does not depend on x then

$$\int_\alpha^\beta f(s)ds\, = 0. \tag{1.4}$$

Herein we let α and β be functions of the space-variable x and in order to obtain any meaningful conclusion we generalize the Pohozaev procedure by working with a vector field $X(x)$ which satisfies some specific properties. If as in [1] we had worked with the space-position vector field, i.e. $X(x) = x$, then we would only have obtained our results at price of assuming that the interface be part of the boundary of a star-shaped set.

The following remark will help understanding the advantage of working with a vector-field $\widetilde{X}(x)$ which when restricted to the interface Γ it coincides with the normal vector-field $\hat{\eta}(x)$ on Γ, rather then working with the space-position vector field.

If in [1] we had allowed the interface Γ to intersect $\partial\Omega$ in a proper way (as we do herein) then still for the case α and β constants and f independent of x, (1.3) would become

$$\{\int_\alpha^\beta f(s)ds\} \int_\Gamma x \cdot \hat{\eta}(x)\, dS = 0.$$

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Then we would only recover the equal-area condition at the price of requiring the interface Γ to be a subset of the boundary of a star-shaped set, as the above equality shows. This is so because in [1] we used the vector-field $X(x) = x$. Resorting to the vector-field $\widetilde{X}$, described above and used in the present work, would allows us to recover the equal-area condition without the star-shape condition since then $x \cdot \hat{\eta}(x) = 1$ on Γ.

As an illustration we describe the corresponding version of the above equal-area condition for (1.1) which is obtained in the present work. Let $\Phi^{\epsilon} = (u_{\epsilon}, v_{\epsilon})$ be a family of i.t.l.s. solutions to (1.1) with interface $\Gamma \subset \overline{\Omega}$ in the sense that $\Phi^{\epsilon} \overset{\varepsilon \to 0}{\longrightarrow} (u_0, v_0)$, uniformly on compact sets of $\overline{\Omega} \backslash \Gamma$, where

$$u_0(x) = \alpha(x)\chi_{\Omega_\alpha}(x) + \beta(x)\chi_{\Omega_\beta}(x),\ x \in \Omega = \Omega_\alpha \cup \Gamma \cup \Omega_\beta$$

and χ_A stands for the characteristic function of the set A. Then we show that necessarily $f(\alpha(x),\, v_0(x)) = 0 = f(\beta(x),\, v_0(x))$, $\forall x \in \Omega \backslash \Gamma$ and there must exist constants $\bar{\alpha}, \bar{\beta}\, (\bar{\alpha} < \bar{\beta})$ and $\bar{v}$ such that

$$\int_{\bar{\alpha}}^{\bar{\beta}} f(s, \bar{v}) ds = 0, \tag{1.5}$$

where $\bar{\alpha} = \alpha(\bar{x})$ and $\bar{\beta} = \beta(\bar{x})$, for some $\bar{x} \in \Gamma$.

2. NECESSITY FOR FORMATION OF INTERNAL LAYERS

The main theorem is stated in a more general framework than those considered in the references supplied. Although existence of i.t.l.s. solutions to the full system considered below seems to be difficulty our results imply that whenever trying to do so the appropriate equal-area condition must be assumed.

Henceforth the following system will be considered:

$$\begin{cases} u_t = \varepsilon\, div\,(h_1(x)\nabla u) + f(x, u, \mathbf{v}), \ \ x \in \Omega \\ \mathbf{v}_t = div\,(\mathbf{h_2}(x)\nabla \mathbf{v}) + \mathbf{g}(x, u, \mathbf{v}), \ \ x \in \Omega \\ \dfrac{\partial\, \mathbf{v}}{\partial\, \hat{n}} = 0 \text{ (or } \mathbf{v} = (0, 0, ..., 0) \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω a smooth domain in $I\!R^N$, $N \geq 1$, $0 < \varepsilon \leq \varepsilon_0$ for some small ε_0; $\mathbf{g} = (g_1, g_2, ..., g_n)$ and f and g_i are functions in $C^1(\Omega \times I\!R \times I\!R^n)$, $\mathbf{h_2} = (h_{2,1}, h_{2,2}, ...h_{2,n})$ with $\mathbf{h_2}\nabla\mathbf{v} = (h_{2,1}\nabla v_1, ..., h_{2,n}\nabla v_n)$, $h_1, h_{2,i} \in C^{1,\nu}(\Omega)$, $0 < \nu < 1$, satisfying $0 < m < h_1, h_{2,i} < M$, for $i = 1, ...n$, and some constants m and M. Since our definition of internal transition layer will be local in space we do not need any boundary condition on u. On the other hand the boundary condition on $\mathbf{v}$ is used just once for technical reasons. This boundary condition could have been suppressed as well at the price of adding another (not restrictive) hypothesis in the definition of boundary layer. We will comment on this in the appropriate place.

We now state and justify our definition.

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DEFINITION 2.2.1. *Let $\mathcal{U}$ be an open connect set in Ω and let $\Gamma \subset \overline{\mathcal{U}}$ be an $(N-1)$-dimensional smooth (at least C^2) compact connected orientable manifold whose boundary $\partial\Gamma$ is such that $\partial\Gamma \cap \partial\Omega$ is a smooth $(N-2)$-dimensional submanifold of $\partial\Omega$.*

We will say that an ε-family of stationary solutions to (2.1)

$$\Phi^\varepsilon = \{(u_\varepsilon, \mathbf{v}_\varepsilon) \in [C^1(\overline{\mathcal{U}}) \cap C^2(\mathcal{U})]^{N+1},\ 0 < \varepsilon < \varepsilon_0\}$$

develops internal transition layer, as $\varepsilon \to 0$, in $\mathcal{U}$ with interface Γ if:

- *The family Φ^ε is bounded in $\overline{\Omega}$ uniformly for $0 < \varepsilon < \varepsilon_0$.*
- *$u_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} u_0$, uniformly on compact sets of $\overline{\mathcal{U}}\backslash\Gamma$, where u_0 is given by*

$$u_0(x) = \alpha(x)\chi_{\mathcal{U}_\alpha}(x) + \beta(x)\chi_{\mathcal{U}_\beta}(x) \tag{2.2}$$

for some functions α and β in $C^0(\mathcal{U})$, $\alpha(x) < \beta(x)$ for $x \in \Gamma$ and $\mathcal{U} = \mathcal{U}_\alpha \cup \Gamma \cup \mathcal{U}_\beta$, where $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ are disjoint open connect sets.

- *$\mathbf{v}_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \mathbf{v}_0$ uniformly in $\overline{\mathcal{U}}$.*

In this case we will refer to Φ^ε, as a family of i.t.l.s. solutions to (2.1) in $\mathcal{U}$ with interface Γ .

This definition is consistent with known existence results for the one-dimensional case, when $\mathbf{v}$ is a scalar function. Indeed consider (1.1) with $\Omega = (0\,,1) = I$ and $0 < \varepsilon < \varepsilon_0$. Then the existence of a family Φ^ε of i.t.l.s. solutions (as defined above) is proved, for instance, in [4]. See also [14] and [11] for related results. Actually Φ^ε is a $C^2_\varepsilon(I) \times C^1(\bar{I})$ bounded family where $C^p_\varepsilon(\bar{I})$ is the space of p-times continuous differentiable functions on $\bar{I}$ with the norm $\|u\|_{C^p_\varepsilon} = \sum_{j=0}^{p} |\varepsilon^j \frac{d^j}{dx^j} u(x)|$.

The above definition, which is local in space, suffices for our purposes and it allows for the existence of more than one transition-layer surface (interface) in Ω.

REMARK 2.2.1. *As in [1] we could equally well have considered the case in which the interface Γ does not intersect $\partial\Omega$. Actually this case is somehow easier and will be omitted.*

REMARK 2.2.2. *The question of how the interface Γ of an eventual family of i.t.l.s. solutions to (2.1) intersects the boundary of Ω is in general very difficult. It is known that in some simple scalar equations the intersection is orthogonal. However since this is not the issue herein only restriction on the smoothness of the intersection will be assumed.*

Next theorem states what is the main result of the present work.

THEOREM 2.2.1. *Let $\mathcal{U} \subset \Omega \subset I\!\!R^N$ be an smooth open bounded connect set and $\Phi^\varepsilon = \{(u_\varepsilon, \mathbf{v}_\varepsilon)\}_{0<\varepsilon<\varepsilon_0}$ a family of i.t.l.s. solutions to problem (2.1), in $\mathcal{U}$ with interface Γ. Then $f(x, u_0(x), \mathbf{v}_0(x)) = 0$ on $\mathcal{U}\backslash\Gamma$ and*

$$\int_\Gamma \left\{ \int_{\alpha(x)}^{\beta(x)} f(x, s, \mathbf{v}_0(x)) ds \right\} dS = 0, \tag{2.3}$$

where dS stands for the element of $(N-1)$-dimensional surface measure.

The following lemma will play an important role in the proof of the Theorem 2.1.

Lemma 2.2.1. *Under the conditions and notations of Theorem 2.2.1 we have*

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\partial\mathcal{U}} |\nabla u_\varepsilon(x)|^2 \, dS = 0.$$

Proof: To prove this fact it suffices to show that

a) $$\lim_{\varepsilon\to 0} \left|\varepsilon^{1/2}\, \nabla u_\varepsilon(x)\right| = 0\,, \text{ a.e. in } \partial\mathcal{U}\backslash\partial\Gamma,$$

and

b) $\exists\ M>0$ such that $\left|\varepsilon^{1/2}\, \nabla u_\varepsilon(x)\right| \le M$, a.e. in $x\in\partial\mathcal{U}\backslash\partial\Gamma$, uniformly for $0<\varepsilon<\varepsilon_0$.

Once this has been accomplished an application of Lebesgue Bounded Convergence Theorem will conclude the proof.

Due to smoothness of $\partial\Omega$ we can take without loss of generality $\mathcal{U}$ smooth and such that $\partial\mathcal{U}\cap\partial\Omega=\Gamma$. This will prevent us from taking on Schauder estimates on portions of $\partial\Omega$, which is a delicate and very technical matter, and at same time this type of neighborhood of Γ will suffice for our purposes.

A standard procedure will be used and therefore we only sketch the proof. It is based on a blow-up technique and Schauder estimates that have been used in the scalar case. Therefore only the points in which the proof differs from the scalar case will be stressed. See [1], for more details.

Firstly, for $\overline{x}\in\partial\mathcal{U}\backslash\partial\Gamma$ we define a C^2 local change of variables Σ, with $\Sigma(\overline{x})=0$, which straightens $\partial\mathcal{U}$ near $\overline{x}$ and then set

$$\tilde{u}_\varepsilon(y)=u_\varepsilon(\Sigma^{-1}(y)), \quad \text{for} \quad y\in\overline{B^+_\rho},$$

where $\overline{B^+_\rho}$ stands for the positive hemisphere of the ball of radius ρ and center at the origin. Let us consider $\{\varepsilon_k\}$ any positive sequence converging to 0. Now define scaled functions $\omega_k(z)$ and $\theta_k(z)$ by $\omega_k(z)=\tilde{u}_{\varepsilon_k}\left(\varepsilon_k^{1/2}z\right)$, $\theta_k(z)=\tilde{\mathbf{v}}_{\varepsilon_k}\left(\varepsilon_k^{1/2}z\right)$ for $z\in\overline{B^+_{\rho/\varepsilon_k^{1/2}}}$.

All the coefficients in the new differential equation for ω_k are C^ν bounded, uniformly in k. For a fixed ρ, we set

$$\rho_k \stackrel{\text{def}}{=} \left(\rho/\varepsilon_k^{1/2}\right) \stackrel{k\to\infty}{\longrightarrow} \infty\ .$$

Let R_m be a monotone increasing sequence of positive numbers such that $R_m\to+\infty$, as $m\to\infty$. For each m, there is k_m such that $2R_m<\rho_k$, for $k\ge k_m$.

Since $\{u_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0}$ and $\{\mathbf{v}_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0}$ are bounded in $\overline{\mathcal{U}}$, uniformly on ε, it follows that $\|\theta_k\|_{C\ \overline{B^+_{2R_m}}}$, $\|\omega_k\|_{C\ \overline{B^+_{2R_m}}} \le K_1$, for some constant K_1 which is independent of k.

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Thus by [7], Theorem 8.24, we conclude that θ_k and ω_k are locally C^ν bounded in $\overline{B^+_{2R_m}}$, uniformly in k.

Interior Schauder estimates in B_{R_m} (here the fact that $\partial\mathcal{U} \cap \partial\Omega = \Gamma$ comes into play) yield that ω_k is $C^{2,\nu}$ bounded in $\overline{B^+_{2R_m}}$, uniformly for $k \geq k_m$. Then by a diagonal process we can extract a subsequence, still labeled $\{\omega_k\}$, such that $\omega_k \to \omega_o$ in $C^2_{\text{loc}}\left(\mathbb{R}^N_+\right)$ where $\mathbb{R}^N_+ = \left\{z \in \mathbb{R}^N : z_N \geq 0\right\}$.

Consequently, for $B^+_1 = \{z \in \mathbb{R}^N : z_N \geq 0 \; and \; |z| \leq 1\}$ we have $|\,\omega_k - \omega_0\,|_{C^2(B^+_1)} \to 0$. But from the definition of ω_k we conclude that $\omega_0 \equiv \beta(\overline{x})$ or $\omega_0 \equiv \alpha(\overline{x})$, and so $\omega_0(z)$ is a constant function in B^+_1.

In particular $\lim_{k\to\infty} |\nabla\omega_k(0)| = 0$ and then if $\varepsilon_k \to 0,\ \varepsilon_k \in (0, \varepsilon_0)$,

$$\lim_{k\to\infty} \left|\varepsilon_k^{1/2}\, \nabla u_{\varepsilon_k}(\overline{x})\right| = 0 \ ,$$

for any $\overline{x} \in \partial\mathcal{U}\backslash\partial\Gamma$. Since $\partial\mathcal{U} \cap \partial\Gamma$ has zero $(N-1)$-dimension surface measure, a) follows. Finally standard Schauder estimates may be evoked to obtain b). Thus our claim is proved. ■

But note that if $\Phi_\varepsilon = (u_\varepsilon, v_\varepsilon)$ is a family of i.t.l.s. solutions on $\mathcal{U}$ then it still will be on any smooth open $\hat{\mathcal{U}} \subset \mathcal{U}$ containing Γ. Consequently making $u_t = 0$ in the first equation of (2.1), integrating and using the Divergence Theorem and Lemma 2.1 we conclude that for any $\hat{\mathcal{U}}$ such that $\Gamma \subset \hat{\mathcal{U}} \subset \mathcal{U}$

$$\lim_{\varepsilon\to 0} \int_{\hat{\mathcal{U}}} f(x, u_\varepsilon(x), \mathbf{v}_\varepsilon(x)) = 0,$$

But for $\varepsilon \to 0$, the family $(u_\varepsilon, \mathbf{v}_\varepsilon) \to (u_0, \mathbf{v}_0)$ uniformly in any compact set $K \subset \hat{\mathcal{U}}\backslash\Gamma$. So by boundness and regularity of f we conclude that for any open set $\hat{\mathcal{U}}$ such that $\hat{\mathcal{U}} \subset \mathcal{U}$,

$$\int_{\hat{\mathcal{U}}} f(x, u_0(x), \mathbf{v}_0(x)) = 0$$

We have thus proved the following

Lemma 2.2.2. *Under the conditions and notations of Theorem 2.2.1 we have that* $f(x, u_0(x), \mathbf{v}_0(x)) = 0$ *for any* $x \in \mathcal{U}\backslash\Gamma$.

Proof of Theorem 2.1: Let $\hat{\eta}$ stands for the normal vector field to Γ (which by hypothesis is C^2) and let us take a C^1 vector field $X : \overline{\mathcal{U}} \to \mathbb{R}^N$ so that when restrict to Γ it coincides with $\hat{\eta}$.

As in the Pohozaev procedure, making $u_t = 0$ in the first equation of (2.1), multiplying it by $X(x).\nabla u_\varepsilon$ and integrating over $\mathcal{U}$ we obtain

$$\int_{\mathcal{U}} \{\varepsilon \mathrm{div}(h_1(x)\nabla u_\varepsilon)(X(x)\cdot\nabla u_\varepsilon) + f(x,u_\varepsilon,\mathbf{v}_\varepsilon)\, X(x)\cdot\nabla u_\varepsilon\}\; dx = 0 \tag{2.4}$$

Working with the first term of this equality and using that

$$\mathrm{div}[X(x)\cdot\nabla u\; h_1\nabla u] = X(x)\cdot\nabla u\;\mathrm{div}(h_1\nabla u) + h_1\left[\sum_{i,k=1}^{N}\frac{\partial X_k}{\partial x_i}u_{x_k}u_{x_i} + X(x)\cdot\nabla(\frac{|\nabla u|^2}{2})\right]$$

along with the divergence theorem, it follows that

$$\begin{aligned}
&-\;\varepsilon\int_{\partial\mathcal{U}} h_1(x)\;\; X(x)\cdot\nabla u_\varepsilon\;\frac{\partial u_\varepsilon}{\partial\hat{n}}\;dS + \frac{\varepsilon}{2}\int_{\partial\mathcal{U}} h_1(x)|\nabla u_\varepsilon|^2\, X(x)\cdot\hat{n}dS\\
&-\;\frac{\varepsilon}{2}\int_{\mathcal{U}}|\nabla u_\varepsilon|^2 X(x)\cdot\nabla h_1\,dx - \frac{\varepsilon}{2}\int_{\mathcal{U}} h_1(x)|\nabla u_\varepsilon|^2\mathrm{div}X(x)\,dx\\
&+\;\varepsilon\int_{\mathcal{U}}\sum_{i,k=1}^{N} h_1\frac{\partial X_k}{\partial x_i}u_{x_k}u_{x_i} = \int_{\mathcal{U}} f(x,u_\varepsilon,v_\varepsilon)\;\, X(x)\cdot\nabla u_\varepsilon\;dx.
\end{aligned} \tag{2.5}$$

We claim that the left hand side of this equality goes to 0, as $\varepsilon\to 0$. In fact, this holds for the first and second terms by virtue of Lemma 2.1.

By utilizing energy estimates on $\mathcal{U}$ applied to the first equation of (2.1) (with $u_t = 0$) and Lemma 2.2 we obtain

$$\lim_{\varepsilon\to 0}\,\varepsilon\int_{\mathcal{U}}|\nabla u_\varepsilon(x)|^2\,dx \;=\; 0.$$

So the third, fourth and fifth terms of (2.5) approach zero, as $\varepsilon\to 0$, too. Hence

$$\lim_{\varepsilon\to 0}\int_{\mathcal{U}} f(x,u_\varepsilon,\mathbf{v}_\varepsilon)\,X(x)\cdot\nabla u_\varepsilon\,dx = \;0. \tag{2.6}$$

Note that if $F(x,u) = \int_\theta^u f(x,s,\mathbf{v})\,ds$

$$\mathrm{div}[\,X(x)F(x,u)] = f(x,u,\mathbf{v})\nabla u\cdot X(x) + \int_\theta^u X(x)\cdot\nabla_x f(x,s,\mathbf{v})ds + \mathrm{div}X(x)F(x,u) \tag{2.7}$$

where $\nabla_x f(x,s,\mathbf{v}(x)) = \partial_x f(x,s,\mathbf{v}(x)) + \sum\partial_{3,i}f(x,s,\mathbf{v}(x))\nabla v_i(x)$, $\partial_x f(x,s,t_1,t_2,..t_n)$ is the gradient of f with respect to x and $\partial_{3,i}(x,s,t_1,t_2,...t_n)$ is the partial derivative of f with respect to t_i, $i = 1,2,...n$. The Divergence Theorem yields

$$\int_{\mathcal{U}} f(x,u_\varepsilon,\mathbf{v}_\varepsilon)\,X(x)\cdot\nabla u_\varepsilon\;dx \;= \int_{\partial\mathcal{U}} X(x)\cdot\hat{n}\int_\theta^{u_\varepsilon} f(x,s,\mathbf{v}_\varepsilon)\,ds\,dS \tag{2.8}$$

$$-\int_{\mathcal{U}}\{\int_{\theta}^{u_\varepsilon} X(x)\cdot\nabla_x f(x,s,\mathbf{v}_\varepsilon)ds + \mathrm{div}X(x)\int_{\theta}^{u_\varepsilon} f(x,s,\mathbf{v}_\varepsilon)ds\}dx$$

Now each element of the right-hand term of (2.8) will be analyzed.

By hypothesis, $\{u_\varepsilon\}$ and $\{\mathbf{v}_\varepsilon\}$ converge uniformly on compact sets $K\subset\overline{\mathcal{U}}\backslash\Gamma$ and $\overline{\mathcal{U}}$, respectively. So by regularity of f we obtain for any $x\in\overline{\mathcal{U}}\backslash\Gamma$ that

$$\int_{\theta}^{u_\varepsilon(x)} f(x,s,\mathbf{v}_\varepsilon(x))\,ds \stackrel{\varepsilon\to 0}{\longrightarrow} \int_{\theta}^{u_0(x)} f(x,s,\mathbf{v}_0(x))\,ds,$$

with the same result when we take $\partial_x f$ or $\partial_{3,i}f$, $i=i,2,...n$, instead of f. Then applying Lebesgue Convergence Theorem we conclude that

$$\int_{\partial\mathcal{U}} X(x)\cdot\hat{n}\int_{\theta}^{u_\varepsilon} f(x,s,\mathbf{v}_\varepsilon)\,ds\,dS \stackrel{\varepsilon\to 0}{\longrightarrow} \int_{\partial\mathcal{U}} X(x)\cdot\hat{n}\int_{\theta}^{u_0} f(x,s,\mathbf{v}_0)\,ds\,dS \tag{2.9}$$

and

$$\int_{\mathcal{U}}\int_{\theta}^{u_\varepsilon} X(x)\,\partial_x f(x,s,\mathbf{v}_\varepsilon)ds\ dx \stackrel{\varepsilon\to 0}{\longrightarrow} \int_{\mathcal{U}}\int_{\theta}^{u_0} X(x)\,\partial_x f(x,s,\mathbf{v}_0)ds\ dx. \tag{2.10}$$

Recalling that $\mathcal{U}\backslash\Gamma=\mathcal{U}_\alpha\cup\mathcal{U}_\beta$, for $\sigma\in\{\alpha,\ \beta\}$ we obtain

$$\int_{\mathcal{U}_\sigma} \mathrm{div}X(x)\int_{\theta}^{u_\varepsilon} f(x,s,\mathbf{v}_\varepsilon)ds\ dx \stackrel{\varepsilon\to 0}{\longrightarrow} \int_{\mathcal{U}_\sigma} \mathrm{div}X(x)\int_{\theta}^{\sigma} f(x,s,\mathbf{v}_0)ds\ dx. \tag{2.11}$$

We also have that for $i=1,2,...n$

$$X(x)\int_{\theta}^{u_\varepsilon} \partial_{3,i}f(x,s,\mathbf{v}_\varepsilon(x))\,ds \to X(x)\int_{\theta}^{u_0} \partial_{3,i}f(x,s,\mathbf{v}_0(x))\,ds \tag{2.12}$$

strongly in $L^2(\mathcal{U})$. Then if $\nabla\mathbf{v}_\varepsilon$ to converge weakly in $L^2(\mathcal{U})$ we will have

$$\int_{\mathcal{U}}\int_{\theta}^{u_\varepsilon} X(x)\cdot\nabla_x f(x,s,\mathbf{v}_\varepsilon(x))ds\,dx \to \int_{\mathcal{U}}\int_{\theta}^{u_0} X(x)\cdot\nabla_x f(x,s,\mathbf{v}_0(x))ds\,dx. \tag{2.13}$$

In order to establish the weak convergence of $\{\mathbf{v}_\varepsilon\}$ in $H^1(\mathcal{U})$ note that energy estimates on the second equation of (2.1) (with $\mathbf{v}_t=0$) and boundness of $\mathbf{g}$, u_ε, and $\mathbf{v}_\varepsilon$ give us the boundness of $\nabla\mathbf{v}_\varepsilon$ in $L^2(\Omega)$, uniformly on ε.

Moreover $\mathbf{v}_\varepsilon\to\mathbf{v}_0$ uniformly in $\overline{\mathcal{U}}$. So $\mathbf{v}_\varepsilon$ is bounded in $H^1(\mathcal{U})$ and $\nabla\mathbf{v}_\varepsilon$ converges weakly to $\nabla\mathbf{v}_0$ in $L^2(\Omega)$.

Passing to the limit in (2.8), as $\varepsilon \to 0$, and using (2.6), (2.9) to (2.11) and (2.13) we obtain

$$\begin{aligned} 0 \ &= \int_{\partial \mathcal{U}} X(x) \cdot \hat{n} \int_{\theta}^{u_0(x)} f(x,s,\mathbf{v}_0(x)) ds\, dS \\ &- \int_{\mathcal{U}_\alpha} \{ \int_{\theta}^{\alpha(x)} X(x) \cdot \nabla_x f(x,s,\mathbf{v}_0(x)) \ + \ \mathrm{div} X(x) f(x,s,\mathbf{v}_0(x))\, ds \}\, dx \\ &- \int_{\mathcal{U}_\beta} \{ \int_{\theta}^{\beta(x)} X(x) \cdot \nabla_x f(x,s,\mathbf{v}_0(x)) \ + \ \mathrm{div} X(x) f(x,s,\mathbf{v}_0(x))\, ds \}\, dx\,. \end{aligned}$$

By (2.7) and Lemma 2.2 it holds that

$$0 = \int_{\partial\mathcal{U}} X(x) \cdot \hat{n} \ \ F(x, u_0(x))\ dS - \int_{\mathcal{U}_\alpha} \mathrm{div}\{X(x) F(x,\alpha(x))\} dx - \int_{\mathcal{U}_\beta} \mathrm{div}\{X(x) F(x,\beta(x))\} dx.$$

The Divergence Theorem implies

$$\int_\Gamma \{ \int_{\alpha(x)}^{\beta(x)} f(x,s,\mathbf{v}_0(x)) ds \, \} \, X(x) \cdot \hat{\eta} \ \ dS = 0. \tag{14}$$

Due to the way X was taken we obtain

$$\int_\Gamma \{ \int_{\alpha(x)}^{\beta(x)} f(x,s,\mathbf{v}_0(x)) ds \, \} \, \hat{\eta} \cdot \hat{\eta} \ dS = \int_\Gamma \{ \int_{\alpha(x)}^{\beta(x)} f(x,s,\mathbf{v}_0(x)) ds \, \} \ dS = 0,$$

thus proving (2.3). ■

REMARK 2.2.3. *It is worthwhile to note that boundary condition* $\mathbf{v}_\varepsilon = 0$ *or* $\frac{\partial \mathbf{v}_\varepsilon}{\partial \hat{\eta}} = 0$ *in* $I\!R^+ \times \partial\Omega$ *was need in order to obtaining (2.13). Without the boundary condition on* $\mathbf{v}_\varepsilon$*, the same conclusion could have been obtained had we required boundness of* $\{\mathbf{v}_\varepsilon\}$ *in* $C^1(\overline{\Omega})$*, uniformly in* ε*, in Definition 2.2.1.*

This additional hypothesis is not restrictive since the existence of a family $\Phi^\varepsilon = (u_\varepsilon, \mathbf{v}_\varepsilon)$ *which satisfies also this condition is proved, for instance, in [14], for the case* $\Omega = [0,1]$.

In the next results our goal is to recover the equal-area condition on f.

COROLLARY 2.2.1. *If* $\Omega = (0,1)$ *then the interface* Γ *is a point* $\overline{x} \in (0,1)$ *and condition (2.3) becomes*

$$\int_{\alpha(\overline{x})}^{\beta(\overline{x})} f(\overline{x}, s, \mathbf{v}_0(\overline{x}))\ ds = 0 \tag{15}$$

which is the known equal-area condition for f.

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Corollary 2.2.2. *If Φ^ε is a family of i.t.l.s. solutions to (2.1) as in Theorem 2.1, then there must exist $\overline{x} \in \Gamma$ such that $(\overline{x}, \alpha(\overline{x}), \mathbf{v}_0(\overline{x}))$ and $(\overline{x}, \beta(\overline{x}), \mathbf{v}_0(\overline{x}))$ are roots of f and*

$$\int_{\alpha(\overline{x})}^{\beta(\overline{x})} f(\overline{x}, s, \mathbf{v}_0(\overline{x}))\ ds = 0. \tag{16}$$

Proof: It follows from the continuity of α, β, $\mathbf{v}_0$ and f. ■

Next we provide sufficient conditions so that $\alpha(x)$, $\beta(x)$ and $\mathbf{v}_0(x)$ be constant functions, thus recovering the equal-area formula for f.

First of all we observe that if $\mathbf{g}$ is allowed to depend on ε, i.e., $\mathbf{g} = \mathbf{g}(\varepsilon, x, u, \mathbf{v})$ is a continuous function in any point $(\varepsilon, x, u, \mathbf{v})$ then the conclusions of Theorem 2.1 still remains true. In particular we have

Corollary 2.2.3. *With the notation of Theorem 2.1 let us take $f = f(u, \mathbf{v})$ and $\mathbf{g} = \mathbf{g}(\varepsilon, x, u, \mathbf{v})$. If $\{(u_\varepsilon, \mathbf{v}_\varepsilon)\}$ is a family of i.t.l.s. solutions in Ω with interface Γ and if $\mathbf{g}(\varepsilon, x, u_\varepsilon(x), \mathbf{v}_\varepsilon(x)) \overset{\varepsilon\to 0}{\longrightarrow} \mathbf{0}$ a.e. in Ω then $\mathbf{v}_0$ is a constant vector-valued function. Moreover if for any constant vector $\mathbf{c} = (c_1, c_2, ..., c_n)$ there holds that the set $\{s : f(s, \mathbf{c}) = 0\}$ is discrete, then α_0 and β_0 are also constant functions. In this case (2.3) simplifies to*

$$\int_{\alpha_0}^{\beta_0} f(s, \mathbf{v}_0) ds = 0 \tag{17}$$

Proof: Since $(u_\varepsilon, \mathbf{v}_\varepsilon)$ satisfies (2.1) (with time derivatives vanishing), by multiplying the second equation by $\mathbf{v}_\varepsilon$, integrating on Ω and passing to the limit as $\varepsilon \to 0$, we conclude that $\mathbf{v}_0$ is a constant vector-valued function.

By our hypotheses, $f(u_\varepsilon, \mathbf{v}_\varepsilon) \overset{\varepsilon\to 0}{\to} f(u_0, \mathbf{v}_0)$, uniformly in compact sets $K \subset \Omega\backslash\Gamma$. Thus Lemma 2.2 implies $f(u_0(x), \mathbf{v}_0) = 0$ and therefore $u_0(x) \in \{s; f(s, \mathbf{v}_0) = 0\}$ for any $x \in \Omega\backslash\Gamma$. But this is a discrete set and $u_\varepsilon \overset{\varepsilon\to 0}{\to} u_0$ uniformly in compact sets $K \subset \Omega\backslash\Gamma$. Therefore $u_0 = \alpha\, \chi_{\Omega_\alpha} + \beta\, \chi_{\Omega_\beta}$ is a constant function on each connect component of $\Omega\backslash\Gamma$, i.e., α and β are constant functions on Ω_α and Ω_β, respectively. But in particular $(u_\varepsilon, \mathbf{v}_\varepsilon)$ is a ε-family of i.t.l.s. solutions in a open set $\mathcal{U} \subset \Omega$ with $\mathcal{U}$ as in Theorem 2.1. Consequently (2.3) holds and by above considerations it simplifies to (2.17) . ■

Remark 2.2.4. *Under the hypothesis of Corollary 2.2.3 the nodal curve of f must intersect the set $\{s : f(s, \mathbf{v}_0) = 0\}$ at least three times in order that (2.17) holds.*

3. APPLICATIONS

We single out four examples from the extensive existing bibliography concerning existence of internal transition layers for such systems and conclude that the different forms of the equal-area assumed therein are in fact necessary conditions.

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A.1 Spatial dependent reactions terms.

The following model problem is considered, for instance, in [8]:

$$\begin{cases} u_t = \varepsilon^2 u_{xx} + (1-u^2)(u-a) - v - \dfrac{k}{\omega} sin(\omega x + b) \\ v_t = \dfrac{1}{\sigma} v_{xx} + (\delta u - v), \qquad x \in \ [0,1] \\ u_x = v_x = 0 \text{ for } x = 0 = 1. \end{cases} \tag{1}$$

where $a \in (-1,0)$, $\varepsilon > 0$ and small, $\sigma > 0$ and small, $\delta > 0$, $k > 0$, $\omega > 0$ and $b \in I\!R$.

In particular the existence of a family of stationary solution to (3.1) which develops internal transition layer, as $\varepsilon \to 0$, is proved therein.

Corollary 2.2.1 implies that a necessary condition for existence of such a family is that for some $x_0 \in [0,1]$ the following holds

$$\int_{h_-(v(x_0))}^{h_+(v(x_0))} [(1-\xi^2)(\xi - a) - v(x_0) - \frac{k}{\omega}\sin(\omega x_0 + b)]d\xi = 0.$$

But this is just hypothesis A.2 (p. 372) in [8], assumed therein as a sufficient condition.

Note that in the notation of Corollary 2.1 above equation reads

$$\tilde{v} = \frac{1}{\beta(x_0) - \alpha(x_0)} \int_{\alpha(x_0)}^{\beta(x_0)} [(1-\xi^2)(\xi - a)]\, d\xi$$

where $\tilde{v} = v(x_0) + \frac{k}{\omega}\sin(\omega x_0 + b)$.

A.2 A rescaled system regarding instability of patterns.

Let us consider the following reaction-diffusion system

$$\begin{cases} u_t = \varepsilon^2 \triangle u + f(u,v), \\ v_t = D\triangle v + g(u,v), \ \ (x,t) \in \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \hat{n}} = \dfrac{\partial v}{\partial \hat{n}} = 0, \ (x,t) \in \partial\Omega \times (0,\infty) \end{cases} \tag{2}$$

where u is the activator, v is the inhibitor, Ω is a smooth domain in $I\!R^N$, $D > 0$ and ε a small positive parameter. The nullcline of f is sigmoidal and consists of three smooth curves $u = h_-(v)$, $u = h_0(v)$ and $u = h_+(v)$ defined on the intervals I_-, I_0 and I_+, respectively. Also if $\min I_- = \underline{v}$ and $\max I_+ = \overline{v}$ then the inequality $h_-(v) < h_0(v) < h_+(v)$ holds for $I^* = (\underline{v}, \overline{v})$ and $h_+(v)$ (resp., $h_-(v)$) coincides with $h_0(v)$ at only one point $v = \overline{v}$ (resp., $v = \underline{v}$), respectively.

In [13], (2) is assumed to satisfy a set of hypotheses, which we denote by $\mathcal{H}$, among which we mention the following one concerning f:

- $J(v) = \int_{h_-(v)}^{h_+(v)} f(\xi, v)d\xi$ has one isolated zero at $v = v^* \in I^*$.

Under the hypotheses $\mathcal{H}$, they suppose the existence of a family of i.t.l.s. solutions $(u_\varepsilon, v_\varepsilon)$ to (2), whose interface S_ε is smooth up to $\varepsilon = 0$. Under this hypothesis it is proved that this family of i.t.l.s. solutions becomes unstable for ε small.

By "smooth up to $\varepsilon = 0$" it is meant that there exists an $(N-1)$-dimensional smooth compact connected manifold S_0 without boundary in $I\!R^N$ such that $S_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} S_0$.

In order to capture the morphology of the patterns which should be very intricate in the limit and based on the balance of the bulk force and the mean curvature effect they formally derive that the rate of shrinking of this patterns is of order $\varepsilon^{1/3}$. See [13], p. 1103. Then by a suitable scaling, the resulting rescaled equations capture the morphology of the magnified patterns. After performing the change of variable $X = \frac{x-x^*}{\varepsilon^{1/3}}$ for a suitable $x^* \in R^N$, the rescaled system becomes

$$\begin{cases} \tilde{u}_t = \tilde{\varepsilon}^2 \triangle \tilde{u} + f(\tilde{u}, \tilde{v}), \\ \nu \tilde{\varepsilon} \tilde{v}_t = D \triangle \tilde{v} + \tilde{\varepsilon} g(\tilde{u}, \tilde{v}), \ \ (X,t) \in \Omega_\varepsilon \times (0,\infty) \\ \dfrac{\partial \tilde{u}}{\partial \hat{n}} = \dfrac{\partial \tilde{v}}{\partial \hat{n}} = 0, \ (X,t) \in \partial\Omega_\varepsilon \times (0,\infty) \end{cases} \tag{3}$$

where $\tilde{\varepsilon} = \varepsilon^{2/3}$ and the rescaled domain Ω_ε satisfies $\Omega_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \widetilde{\Omega}$ with $|\widetilde{\Omega}| < \infty$. This convergence may be any one as long as the hypotheses of Theorem 2.1 on the tubular neighborhood $\mathcal{U}$ of Γ are satisfied.

It turns out however that the system for the stationary solutions to (3) is just (2.1) when $h_1 = 1$, $h_2 = D$ and $\mathbf{g} = \varepsilon g(u,v)$, namely,

$$\begin{cases} \tilde{\varepsilon}^2 \triangle \tilde{u} + f(\tilde{u}, \tilde{v}) = 0, \ \ X \in \Omega_\varepsilon \\ D \triangle \tilde{v} + \tilde{\varepsilon} g(\tilde{u}, \tilde{v}) = 0, \ \ X \in \Omega_\varepsilon \\ \dfrac{\partial \tilde{u}}{\partial \hat{n}} = \dfrac{\partial \tilde{v}}{\partial \hat{n}} = 0, \ X \in \partial\Omega_\varepsilon \end{cases} \tag{4}$$

Thus Corollary 2.2.3 applies to any family $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ of solutions to (3.4) which develops internal transition layer with interface S_0. In our notation $v_0 = v^*$, $\alpha_0 = h_-(v^*)$ and $\beta_0 = h_+(v^*)$. We conclude that $\tilde{v}_\varepsilon \to v_0$ where $J(v_0) = 0$ and $v_0 \equiv$constant. The point to be stressed here is that the equal-area condition $J(v_0) = 0$ assumed in [13] is actually a necessary condition.

A.3 The FitzHugh-Nagumo system .

Let us now consider the well-known FitzHugh-Nagumo system (F-N system, for short) which by its turn is a simplified version of the Hodgkin-Huxley equations, which models electrical impulses travelling in the axon of the squid:

$$\begin{cases} u_t = \varepsilon \triangle u + h(u) + v, \ \ (t,x) \in I\!R^+ \times \Omega \\ v_t = d \triangle v + \delta u - \gamma v, \ \ (t,x) \in I\!R^+ \times \Omega \\ \dfrac{\partial u}{\partial \hat{n}} = \dfrac{\partial v}{\partial \hat{n}} = 0, \quad (t,x) \in R^+ \times \partial\Omega. \end{cases} \tag{5}$$

Here ε, δ and γ are positive constants whereas d is a nonnegative one. For the case $\Omega = [0,1]$, existence of i.t.l.s. solutions to F-N system above has long been related to the existence of fronts and pulses for F-N system in the positive real line. Therefore according to (1.5) if one expects to obtain a family (u_ϵ, v_ϵ) of i.t.l.s. solutions to F-N system above then h must satisfy the following hypothesis: there are constants α, β $(\beta > \alpha)$ and $\bar{v}$ such that $h(\alpha) = h(\beta) = \bar{v}$ and

$$\frac{1}{(\beta - \alpha)} \int_\alpha^\beta h(s)ds = \bar{v}.$$

Hence if this condition is violated one can only expect existence of pulses for the F-N system in the real line.

A.4 A model from morphogenesis.

As another application let us consider the system introduced in [15] in the context of morphogenesis and which inspired many related works:

$$\begin{cases} u_t = d_1 \triangle u - u + (u^p/v^q), \ \ (t,x) \in I\!R^+ \times \Omega \\ \tau v_t = d_2 \triangle v - v + (u^r/v^s), \ \ (t,x) \in I\!R^+ \times \Omega \\ \dfrac{\partial u}{\partial \hat{n}} = \dfrac{\partial v}{\partial \hat{n}} = 0, \quad (t,x) \in R^+ \times \partial\Omega, \end{cases} \tag{6}$$

where d_1, d_2, p, q, r, τ are positive constant, $s \geq 0$ and

$$0 < \frac{p-1}{q} < \frac{r}{s+1}.$$

Our results give a rigorous proof to the heuristic fact that solutions to (3.6) do not develop internal transition layers as $d_1 \longrightarrow 0$. This will follow from Corollary 2.2 along with the fact that for each fixed v, say $\bar{v}$, the line $(u, \bar{v})$ intersects the graph of the function $v = u^{p-1/q}$ at most twice thus making it impossible for (2.16) to hold. Therefore, as $d_1 \to 0$, among the simplest geometric configuration possible, stationary solutions to (3.16) can only develop formation of spikes and/or boundary layer.

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