CADERNOS DE MATEMÁTICA **01**, 151–171 May (2004)
ARTIGO NÚMERO SMA#202

# Strictly positive definite kernels on subsets of the complex plane

Valdir A. Menegatto

*Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil*
E-mail: menegatt@icmc.usp.br

Claudemir P. Oliveira

*Instituto de Ciências Exatas, Departamento de Matemática e Computação, Universidade Federal de Itajubá, Caixa Postal 50, 37500-903 Itajubá, MG Brazil*
E-mail: oliveira@unifei.edu.br

Ana P. Peron*

*Departamento de Matemática, Centro de Ciêmcias Exatas, Universidade Estadual de Maringá, Avenida Colombo 5790, 87020-900 Maringá, PR Brazil*
E-mail: apperon@uem.br

In this paper we seek for inner product dependent strictly positive definite kernels on subsets of $\mathbb{C}$. We present separated necessary and sufficient conditions in order that a positive definite kernel on $\mathbb{C}$ be strictly positive definite. One emphasis is on strictly positive definite kernels on the unit circle. Since positive definite kernels on the circle were already characterized in [1], the study in this case reduces to the determination of what kind of positive definite kernels are indeed strictly positive definite. For other subsets, we begin with a quite general positive definite kernel on the whole $\mathbb{C}$ and find conditions in order that it is strictly positive definite on the subset. For some classes of subsets, the results are final.

**Keywords and phrases.** positive definite, strictly positive definite, circle, Vandermonde matrices. May, 2004 ICMC-USP

## 1. INTRODUCTION

Recently, positive definite kernels on spheres and similar manifolds have been employed in a variety of problems by many researchers (see [2,4,14] and references therein). Large families of positive definite kernels have been found to be very useful in the construction of the so-called spherical radial-basis function interpolants. In particular, if $S$ is either the

* First and third authors Partially Supported by PPROCAD-CAPES, grant #0092/01-0



Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

unit circle $\Omega_2$ in $\mathbb{C}$ or the unit circle $S^1$ in $\mathbb{R}^2$, these interpolants are functions of the form

$$z \in S \longrightarrow \sum_{\mu=1}^{n} c_\mu f(z \cdot z_\mu), \tag{1.1}$$

in which $z_1, z_2, \ldots, z_n$ are distinct points of $S$, $c_1, c_2, \ldots, c_n$ are complex numbers, $\cdot$ is the inner product of the space where $S$ lives, and $f$ is an appropriate real or complex function. If the kernel $(z, w) \in S \times S \mapsto f(z \cdot w)$ is positive definite on $S$, i.e.,

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} f(z_\mu \cdot z_\nu) \geq 0, \tag{1.2}$$

whenever $n$ is a positive integer, $\{z_1, z_2, \ldots, z_n\} \subset S$, and $\{c_1, c_2, \ldots, c_n\} \subset \mathbb{C}$, the resulting interpolation matrix $(f(z_\mu \cdot z_\nu))$ is already nonnegative definite. Its invertibility, i.e., its positive definiteness, will depend on $f$ and on the interpolation points.

The main issue in this paper is to search for a characterization of the continuous strictly positive definite kernels on $S$. A positive definite kernel $(z, w) \in S \times S \mapsto f(z \cdot w)$ is *strictly positive definite of order* $n$ ($n \geq 1$) if the quadratic form (1.2) is positive whenever the $z_\mu$ are different from each other and at least one of the $c_\mu$ is nonzero. A strictly positive definite of all orders is termed *strictly positive definite* on $S$. In particular, a strictly positive definite kernel produces a positive definite interpolation matrix for all choices of the interpolation points. The above definitions can be naturally extended to other subsets of $\mathbb{C}$ as the reader can easily verify himself.

According to Theorem 3.1 in [11], a continuous kernel $(z, w) \in \Omega_2 \times \Omega_2 \mapsto f(z \cdot w)$ is positive definite if and only if $f$ is representable in the form

$$f(z) = \sum_{k \in \mathbb{Z}} a_k(f) z^k, \quad |z| = 1, \quad a_k(f) \geq 0, \quad \sum_{k \in \mathbb{Z}} a_k(f) < \infty. \tag{1.3}$$

If $(z, w) \in S^1 \times S^1 \mapsto g(z \cdot w)$ is positive definite, the function $g$ must satisfy (see [6,16,17])

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} g(\cos(\theta_\mu - \theta_\nu)) \geq 0, \tag{1.4}$$

whenever $n$ is a positive integer, the $c_\mu$ are complex numbers and $\{\theta_1, \theta_2, \ldots, \theta_n\} \subset [0, 2\pi)$. In particular, $(z, w) \in \Omega_2 \times \Omega_2 \mapsto g(\mathrm{Re}\,(z \cdot w))$ is positive definite and we can write

$$g(\mathrm{Re}\, z) = \sum_{k \in \mathbb{Z}} b_k(g) z^k, \quad |z| = 1, \quad b_k(g) \geq 0, \quad \sum_{k \in \mathbb{Z}} b_k(g) < \infty. \tag{1.5}$$

In polar coordinates, this reads

$$g(\cos\theta) = \sum_{k \in \mathbb{Z}} b_k(g) e^{ik\theta}, \quad \theta \in \mathbb{R}, \quad b_k(g) \geq 0, \quad \sum_{k \in \mathbb{Z}} b_k(g) < \infty. \tag{1.6}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Since the above series defines an even function of $\theta$, $b_k(g) = b_{-k}(g)$, $k \in \mathbb{Z}$. Thus, we infer that $g$ is a real function having the form

$$g(t) = \sum_{k\in\mathbb{Z}_+} a_k(g)T_k(t), \quad -1 \leq t \leq 1, \quad a_k(g) \geq 0, \quad \sum_{k\in\mathbb{Z}_+} a_k(g) < \infty, \tag{1.7}$$

in which $T$ is the Tchebyshev polynomial (of first kind) of degree $k$. This characterization matches that of Schoenberg in [17]. To the best of our knowledge, there is no complete characterization of positive definite kernels on other relevant subsets of $\mathbb{C}$ in the sense defined here.

For a positive definite kernel $(z, w) \in \Omega_2 \times \Omega_2 \mapsto f(z\cdot w)$ and $n$ distinct points $z_1, z_2 \ldots, z_n$ on $\Omega_2$, it holds

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} f(z_\mu \cdot z_\nu) = \sum_{k\in K(f)} a_k(f) \sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu}\,(z_\mu \cdot z_\nu)^k, \tag{1.8}$$

in which $K(f) := \{k : a_k(f) > 0\}$. Due to the nonnegative definiteness of the matrices $((z_\mu \cdot z_\nu)^k)$, $k \in K(f)$, the matrix $(f(z_\mu \cdot z_\nu))$ is positive definite if and only if

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu}(z_\mu \cdot z_\nu)^k > 0 \tag{1.9}$$

for some $k \in K(f)$. Thus, the strict positive definiteness of order $n$ of the kernel depends upon the set $K(f)$ only and not on the actual values of the coefficients $a_k(f)$. This remark opens space for the following additional definition: a subset $K$ of $\mathbb{Z}$ *induces strict positive definiteness* (SPD) of order $n$ on $\Omega_2$, if every positive definite kernel $(z, w) \in \Omega_2 \times \Omega_2 \mapsto f(z \cdot w)$ for which $K(f) = K$, is strictly positive definite of order $n$ on $\Omega_2$. The above remarks and definitions apply to $S^1$ as well (see [16] for details).

SPD on $\Omega_2$ has not been investigated yet while many results on SPD on $S^1$ can be found in [6, 8, 16, 18]. To this day, no easy to check characterization of SPD on $S^1$ was found, but SPD of order 2 (see [10]). A complete characterization for strict positive definiteness on spheres of $\mathbb{R}^m$, $m \geq 3$ and $\mathbb{C}^q$, $q \geq 3$, were given in [2] and [12] respectively. Characterizations for strict positive definiteness on the unit sphere of a real or complex Hilbert space can be found in [9] and [12].

In this paper, we study strict positive definiteness on $\Omega_2$ and more generally on subsets of $\mathbb{C}$. For subsets of $\Omega_2$, we begin with a kernel $f$ as in (1.3) and find conditions on $K(f)$ so that it is strict positive definite on the subset. Since the positive definite kernels on other relevant subsets of $\mathbb{C}$ are not characterized yet, the procedure we adopt in the other cases is slightly different. If the kernel $(z, w) \in \mathbb{C} \times \mathbb{C} \mapsto f(z \cdot w)$ is positive definite then Corollary 4.7 in [1] reveals that $f$ must have the form

$$f(z) = \sum_{k,l\in\mathbb{Z}_+} a_{k,l}(f) z^k \overline{z}^l, \quad a_{k,l}(f) \geq 0, \tag{1.10}$$

where the series converges for all $z \in \mathbb{C}$. Thus, given a function $f$ as above and a subset $B$ of $\mathbb{C}$, we will seek for conditions in order that $(z, w) \in B \times B \mapsto f(z \cdot w)$ be strictly positive definite. The strictly positive definite kernels having this form comprehend a large family of kernels.

This paper is organized as follows. In Section 2, we compare SPD on $\Omega_2$ with SPD on $S^1$. In Section 3, we present some separated necessary and sufficient conditions for the induction of SPD on $\Omega_2$. In Section 4, we present conditions on a function $f$ as in (1.10) that ensure the strict positive definiteness of the kernel $(z, w) \in B \times B \mapsto f(z \cdot w)$. For two different classes of sets, we present conditions which turn out be necessary and sufficient.

## 2. Strict positive definiteness on $\Omega_2$ and on $S^1$

SPD on the circle was introduced and first studied in [6,18], considering the circle as a subset of $\mathbb{R}^2$. Later, SPD was considered in [8] and [16], where separated necessary and sufficient conditions for the induction of SPD on $S^1$ were given. An easy to check and complete characterization was never found. Our aim in this section is to explore the relationship between the concepts of SPD on $\Omega_2$ and on $S^1$. We refer the reader to [6,8,18,10] for acquaintance with SPD on $S^1$. The results in this section will confirm that to identify sets that induce SPD on $\Omega_2$ is more complicated than to identify sets that induce SPD on $S^1$. It will be shown here that SPD on $S^1$ corresponds to SPD on $\Omega_2$ of sets that are symmetric with respect to 0.

An alternative formulation for SPD on $\Omega_2$ is presented in Lemma 2.1 below. Hereafter, the inner product of $\mathbb{C}$ will be denoted trough its standard notation.

**Lemma 2.1** *Let $K$ be a subset of $\mathbb{Z}$ and $n$ a positive integer. The following assertions are equivalent:*
$(i)$ *$K$ induces SPD of order $n$ on $\Omega_2$;*
$(ii)$ *There exists no nonzero function of the form*

$$z \longmapsto \sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu z}, \quad c_1, c_2, \ldots, c_n \in \mathbb{C}, \tag{2.1}$$

*where $\theta_1, \theta_2, \ldots, \theta_n$ are distinct points in $[0, 2\pi)$, that vanishes on $K$.*
**Proof.** Writing $z_\mu = \exp(i\theta_\mu)$, $\theta_\mu \in [0, 2\pi)$, equation (1.8) becomes

$$\sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} f(z_\mu \overline{z_\nu}) = \sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} \sum_{k \in K} a_k(f) e^{i(\theta_\mu - \theta_\nu)k} = \sum_{k \in K} a_k(f) \left| \sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu k} \right|^2. \tag{2.2}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

The rest follows. ■

A similar result for SPD on $S^1$ can be found in [16]. It uses subsets of $\mathbb{Z}_+$ and requires real coefficients in $(ii)$. Theorem 2.2 and Corollary 2.3 below allows us to consider SPD on $S^1$ as a particular case of SPD on $\Omega_2$.

**Theorem 2.2** *Let $K$ be a subset of $\mathbb{Z}_+$. The following are equivalent:*
*(i) The set $\{k : |k| \in K\}$ induces SPD of order $n$ on $\Omega_2$;*
*(ii) $K$ induces SPD of order $n$ on $S^1$.*
**Proof.** Write $K^{'} := \{k : |k| \in K\}$. Let $\theta_1, \theta_2, \ldots, \theta_n$ be distinct points in $[0, 2\pi)$. If $c_1, c_2, \ldots, c_n$ are real numbers and

$$\sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu k} = 0, \quad k \in K, \tag{2.3}$$

then

$$\sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu k} = 0, \quad k \in K'. \tag{2.4}$$

Hence, if $K'$ induces SPD of order $n$ on $\Omega_2$, Lemma 2.1 forces all the $c_\mu$ to be zero. Thus, $(i)$ implies $(ii)$. Conversely, if $d_1, d_2, \ldots, d_n$ are complex numbers and

$$\sum_{\mu=1}^{n} d_\mu e^{i\theta_\mu k} = 0, \quad k \in K', \tag{2.5}$$

then, the symmetric nature of $K'$ allows us to conclude that

$$\sum_{\mu=1}^{n} \overline{d_\mu}\, e^{i\theta_\mu k} = 0, \quad k \in K'. \tag{2.6}$$

Thus,

$$\sum_{\mu=1}^{n} (\operatorname{Re} d_\mu)\, e^{i\theta_\mu k} = \sum_{\mu=1}^{n} (\operatorname{Im} d_\mu)\, e^{i\theta_\mu k} = 0, \quad k \in K'. \tag{2.7}$$

In particular,

$$\sum_{\mu=1}^{n} (\operatorname{Re} d_\mu)\, e^{i\theta_\mu k} = \sum_{\mu=1}^{n} (\operatorname{Im} d_\mu)\, e^{i\theta_\mu k} = 0, \quad k \in K. \tag{2.8}$$

If $K$ induces SPD of order $n$ on $S^1$, the above relations yield

$$\operatorname{Re} d_\mu = \operatorname{Im} d_\mu = 0, \quad \mu = 1, 2, \ldots, n, \tag{2.9}$$

whence all the $d_\mu$ are zero. By Lemma 2.1, $K'$ induces SPD of order $n$ on $\Omega_2$. ■

**Corollary 2.3** *If a subset $K$ of $\mathbb{Z}$ induces SPD of order $n$ on $\Omega_2$ then so does the set $\{|k| : k \in K\}$ on $S^1$. The converse is true whenever $K$ is symmetric in the sense that $K = \{-k : k \in K\}$.*

To see that the converse of Corollary 2.3 is not true in general consider the set $K = \{-1, 0, 1, 2\}$. Being of cardinality 4, it does not induce SPD of order 5 on $\Omega_2$ (see Corollary 3.5 in Section 3 ahead). However, $\{|k| : k \in K\} = \{0, 1, 2\}$ induces SPD of order 5 on $S^1$ by Theorem 1 in [15].

## 3. Strict positive definiteness on $\Omega_2$

In this section, we study SPD on $\Omega_2$ and its subsets. Our results include separated necessary and sufficient conditions for the induction of SPD on $\Omega_2$. Some of them come from ideas present in the study of SPD on $S^1$ while others are genuinely of complex nature. Unfortunately, all the conditions do not complete characterize SPD on $\Omega_2$.

Our first important result establishes the invariance of SPD on $\Omega_2$ by integer translations. The reader is advised that a similar invariance property does not hold for SPD on $S^1$ because translation invariance does not preserve symmetry (see Corollary 2.3).

**Theorem 3.1** *Let $K$ be a subset of $\mathbb{Z}$, $n$ a positive integer and $l$ an integer. These properties are equivalent:*
*$(i)$ $K$ induces SPD of order $n$ on $\Omega_2$;*
*$(ii)$ $K + l := \{k + l : k \in K\}$ induces SPD of order $n$ on $\Omega_2$.*
**Proof.** Let $\theta_1, \theta_2, \ldots, \theta_n$ be distinct points in $[0, 2\pi)$ and $c_1, c_2, \ldots, c_n$ complex numbers. If $(i)$ holds and

$$\sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu(k+l)} = 0, \quad k \in K, \tag{3.1}$$

then Lemma 2.1 implies that $c_\mu \exp(i\theta_\mu l) = 0$, $\mu = 1, 2, \ldots, n$. It is now clear that $c_1 = c_2 = \cdots = c_n = 0$. Thus, $K + l$ induces SPD of order $n$ on $\Omega_2$ by Lemma 2.1. If $(ii)$ holds and

$$\sum_{\mu=1}^{n} c_\mu e^{i\theta_\mu k} = 0, \quad k \in K, \tag{3.2}$$

then

$$\sum_{\mu=1}^{n} c_\mu e^{-i\theta_\mu l} e^{i\theta_\mu(k+l)} = 0, \quad k \in K. \tag{3.3}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Applying Lemma 2.1 once again, we conclude that $c_\mu \exp(-i\theta_\mu l) = 0$, $\mu = 1, 2, \ldots, n$, that is, $c_1 = c_2 = \cdots = c_n = 0$. ■

Likewise, a reflection with respect to 0 does not alter the SPD of the set.

**Theorem 3.2** *A subset $K$ of $\mathbb{Z}$ induces SPD on $\Omega_2$ if and only if $-K := \{-k : k \in K\}$ does.*

Next, we identify sets that induce SPD of order 2 on $\Omega_2$. The result indicates how complicated can be to exhibit an easy to check condition characterizing SPD of order $n \geq 3$ on $\Omega_2$ and $S^1$.

**Theorem 3.3** *A subset $K$ of $\mathbb{Z}$ induces SPD of order 2 on $\Omega_2$ if and only if the difference set $\{k - l : k, l \in K\}$ possesses a relatively prime subset.*

**Proof.** The proof is an adaptation of arguments already used in [7,10,13]. Invoking the definition of SPD of order 2 it is readily seen that $K$ induces SPD of order 2 on $\Omega_2$ if and only if

$$f(1)^2 - f(z)f(\overline{z}) > 0, \quad z \in \Omega_2 \setminus \{1\}, \tag{3.4}$$

whenever $(z, w) \in \Omega_2 \times \Omega_2 \mapsto f(z\,\overline{w})$ is a positive definite kernel with $K(f) = K$. But, this is equivalent to the condition

$$\sum_{\substack{k,l \in K \\ k \neq l}} a_k(f) a_l(f) \left(1 - z^{k-l}\right) > 0, \quad z \in \Omega_2 \setminus \{1\}, \tag{3.5}$$

that is, to the requirement that the system

$$z^m = 1, \quad m \in \{k - l : k, l \in K\} \setminus \{0\}, \tag{3.6}$$

have no solution in $\Omega_2 \setminus \{1\}$. This last assertion is equivalent to $\{k - l : k, l \in K\}$ having a relatively prime subset. ■

Theorem 3.4 below contains a necessary condition for the induction of SPD on $\Omega_2$. The proof is included for completeness once similar arguments were already used in the proof of Theorem 3.1 in [13]. An alternative proof can be elaborated via arguments used in the proof of Lemma 12 in [6].

**Theorem 3.4** *Let $K$ be a subset of $\mathbb{Z}$ and $n$ a positive integer. In order that $K$ induce SPD of order $n$ on $\Omega_2$ it is necessary that it have a nonempty intersection with the arithmetic progressions*

$$n\mathbb{Z} + j := \{nl + j : \ l \in \mathbb{Z}\}, \quad j = 0, 1, \ldots, n - 1. \tag{3.7}$$

**Proof.** We assume that $K$ induces SPD of order $n$ on $\Omega_2$ and that $K \cap (n\mathbb{Z} + j) = \emptyset$, for some $j$ in the set $\{0, 1, \ldots, n - 1\}$ and reach a contradiction. The complex numbers

$c_\mu = \exp(-i2\pi\mu j/n)$, $\mu = 1, 2, \ldots, n$, are certainly not zero. Since $k - j \not\equiv 0 \bmod n$, $k \in K$, we have that

$$\sum_{\mu=1}^{n} c_\mu e^{i2\pi\mu k/n} = \sum_{\mu=1}^{n} e^{i2\pi\mu(k-j)/n} = e^{i2\pi(k-j)/n} \frac{e^{i2\pi(k-j)} - 1}{e^{i2\pi(k-j)/n} - 1} = 0, \quad k \in K, \tag{3.8}$$

in contradiction with Lemma 2.1. ■

Below, we use Theorem 3.4 to further illustrate the translation invariance phenomenon explained in Theorem 3.1. We show that $\{0, 1\}$ is the only set of cardinality 2 that induces SPD of order 3 on $S^1$. Due to Theorem 3.4, the sets $\{-2, 0, 2\}$, $\{-3, 0, 3\}$, and $\{-2, -1, 1, 2\}$ do not induce SPD of order 3 on $\Omega_2$. Thus, $\{0, 2\}$, $\{0, 3\}$ and $\{1, 2\}$ do not induce SPD of order 3 on $S^1$. To see that a set $K$ of the form $\{\alpha, \beta\}$, $\alpha + \beta \geq 4$, does not induce SPD of order 3 on $S^1$, consider $f(t) := T_\alpha(t) + T_\beta(t)$ and the points

$$x_j = (\cos(j-1)\theta, \sin(j-1)\theta), \quad j = 1, 2, 3, \quad \theta = 2\pi/(\alpha + \beta). \tag{3.9}$$

on $S^1$. These points are obviously different from each other and

$$(f(x_\mu \cdot x_\nu)) = \begin{pmatrix} f(1) & f(\cos\theta) & f(\cos 2\theta) \\ f(\cos\theta) & f(1) & f(\cos\theta) \\ f(\cos 2\theta) & f(\cos\theta) & f(1) \end{pmatrix}. \tag{3.10}$$

The determinant of the above matrix is

$$2(2 - f(\cos 2\theta)) \left[2 + f(\cos 2\theta) - f(\cos\theta)^2\right]. \tag{3.11}$$

Since $\cos j\alpha\theta = \cos j\beta\theta$, $j = 1, 2$, the expression inside brackets vanishes. Therefore, the matrix is singular.

**Corollary 3.5** *Let $K$ and $n$ be as in Theorem 3.4. If $K$ induces SPD of order $n$ on $\Omega_2$ then its cardinality is at least $n$. In particular, a set that induces SPD on $\Omega_2$ is infinite.*

Theorem 3.4 takes the following form when we are dealing with plain SPD on $\Omega_2$.

**Theorem 3.6** *Let $K$ be a subset of $\mathbb{Z}$. In order that $K$ induce SPD on $\Omega_2$ it is necessary that it have an infinite intersection with each one of the arithmetic progressions*

$$n\mathbb{Z} + j, \quad j = 0, 1, \ldots, n-1, \quad n \in \mathbb{Z}_+ \setminus \{0\}. \tag{3.12}$$

**Proof.** Suppose that $K$ induces SPD on $\Omega_2$. Due to Theorem 3.4, $K$ has a nonempty intersection with every set listed in (3.12). Next, we assume that $K \cap (n\mathbb{Z} + j)$ is finite for some $n \geq 1$ and some $j < n$ and reach a contradiction. Let $k_1, k_2, \ldots, k_l$ be the elements of $K \cap (n\mathbb{Z} + j)$ and define $m := \max\{|k_\mu| : \mu = 1, 2, \ldots, l\}$. If $m > 0$, pick an integer $k = 4nmp + 2nm + j$ in $K \cap (4nm\mathbb{Z} + 2nm + j)$. In particular, $k$ belongs to $K \cap (n\mathbb{Z} + j)$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

If $p \geq 0$, then $k \geq 2nm + j > m$. If $p < 0$, then

$$k \leq -4nm + 2nm + j < -2mn + n \leq -2mn + mn = -mn \leq -m. \tag{3.13}$$

In both cases, we have a contradiction to the definition of $m$. If $m = 0$, we pick an integer $k = 4n^2p + 2n^2 + j$ in $4n^2\mathbb{Z} + 2n^2 + j$. If $p \geq 0$, then $k \geq 2n^2 + j > 0 = m$. If $p < 0$, then $k \leq -4n^2 + 2n^2 + j = -2n^2 + j < -2n^2 + n < 0 = m$. Once again, we have a contradiction to the definition of $m$. ■

A more intricate proof of Theorem 3.6 can be elaborated on by mimicking the proof of Corollary 5.2 in [16]. If the reader is asking himself about how far the condition given in Theorem 3.6 is from being sufficient for the induction of SPD on $\Omega_2$, our guess is that either it is not sharp enough or an additional necessary condition is needed. Our next theorems may help the reader to form his own conclusion on that. The first one, is another result that can be adapted from Lemma 12 in [6].

**Theorem 3.7** *Let $l$ and $m$ be positive integers with $l \geq m$. Let $K$ be a subset of $\mathbb{Z}$ such that $K \cap (l\mathbb{Z} + j) \neq \emptyset$, $j = 0, 1, \ldots, m-1$. If $\theta_1, \theta_2, \ldots, \theta_m$ are pairwise distinct module $2\pi$, $r + \{l\theta_\mu : \mu = 1, 2, \ldots, m\} \subset 2\pi\mathbb{Z}$ for some real $r$, and $L(z) := \sum_{\mu=1}^m c_\mu \exp(i\theta_\mu z)$ annihilates $K$ then $L$ is identically zero.*

**Proof.** Use the hypothesis on $K$ to select integers $l\alpha_j + j$ in $K \cap (l\mathbb{Z} + j)$, $j = 1, 2, \ldots, m$. If $L$ annihilates $K$ then

$$\sum_{\mu=1}^{m} c_\mu e^{i\theta_\mu(l\alpha_j + j)} = 0, \quad j = 1, 2, \ldots, m. \tag{3.14}$$

The hypothesis on the $\theta_\mu$ allows us to reduce (3.14) to

$$e^{-ir\alpha_j} \sum_{\mu=1}^{m} c_\mu e^{i\theta_\mu j} = 0, \quad j = 1, 2, \ldots, m. \tag{3.15}$$

Since the matrix of this system is Vandermonde-like associated to the distinct points $\exp(i\theta_\mu)$, $\mu = 1, 2, \ldots, m$, all the scalars $c_\mu$ must be zero. ■

In the next results, we use the following equivalence relation on $[0, 2\pi)$:

$$\theta \sim \phi \text{ if and only if } \frac{\theta - \phi}{2\pi} \in \mathbb{Q}. \tag{3.16}$$

The equivalence class of an element $\theta$ of $[0, 2\pi)$ is written as $[[\theta]]$.

**Theorem 3.8** *Let $K$ be a subset of $\mathbb{Z}$ containing an arithmetic progression of length at least $m$ and let $\theta_1, \theta_2, \ldots, \theta_m$ be distinct points in $[0, 2\pi)$. Assume that the classes $[[\theta_\mu]]$, $\mu = 1, 2, \ldots, m$, are pairwise distinct. If the functional $L(z) := \sum_{\mu=1}^m c_\mu \exp(i\theta_\mu z)$ annihilates $K$ then $L$ is identically zero.*

**Proof.** Let $p+q\{0,1,\ldots,l-1\}$ be the arithmetic progression of length $l \geq m$ contained in $K$. The classes $[[\theta_\mu]]$, $\mu = 1,2,\ldots,m$, being pairwise distinct, we have that $\{q(\theta_\mu-\theta_\nu) : \mu \neq \nu\} \not\subset 2\pi\mathbb{Z}$, whence $e^{i\theta_\mu q} \neq e^{i\theta_\nu q}$, $\mu \neq \nu$. The condition $L(p+qj) = 0$, $j = 0,1,\ldots,m-1$, produces the system

$$\sum_{\mu=1}^{m} c_\mu e^{ip\theta_\mu} e^{iqj\theta_\mu} = \sum_{\mu=1}^{m} c_\mu e^{ip\theta_\mu} \left(e^{iq\theta_\mu}\right)^j = 0, \quad j = 0,1,\ldots,m-1. \tag{3.17}$$

Once again, the matrix of the system being Vandermonde-like associated to distinct points, we conclude that all the $c_\mu$ must be zero. ■

Theorem 3.9 presents a sole sufficient condition for the induction of SPD on $\Omega_2$.

**Theorem 3.9** *Let $K$ be a subset of $\mathbb{Z}$ satisfying the following condition: for every positive integer $n$, the sets $K \cap (n\mathbb{Z}+j)$, $j = 0,1,\ldots,n-1$, contain arithmetic progressions of length $n$. Then $K$ induces SPD on $\Omega_2$.*

**Proof.** Let $\theta_1,\theta_2,\ldots,\theta_m$ be distinct points in $[0,2\pi)$ and suppose that the functional $L(z) := \sum_{\mu=1}^{m} c_\mu \exp(i\theta_\mu z)$ annihilates $K$. We show that $L = 0$ analyzing two distinct cases. First assume that all the $\theta_\mu$ belong to a same equivalence class. Without loss of generality we can assume that $\theta_1 < \theta_2 < \cdots < \theta_m$ and that $\theta_\mu \in [[\theta_1]]$, $\mu = 1,2,\ldots,m$. Then, we can write

$$\theta_\mu = \theta_1 + 2\pi q_\mu, \quad q_\mu \in \mathbb{Q} \cap [0,1), \quad \mu = 1,2,\ldots,m, \tag{3.18}$$

with the $q_\mu$ pairwise distinct. The functional $L$ takes the form

$$L(z) = e^{i\theta_1 z}\left(c_1 + \sum_{\mu=2}^{m} c_\mu e^{i2\pi q_\mu z}\right) := e^{i\theta_1 z} L_1(z). \tag{3.19}$$

Obviously, $L(z) = 0$ if and only if $L_1(z) = 0$. Since the $q_\mu$ are rational, there is a positive integer $l \geq m$ such that $\{lq_\mu : \mu = 1,2,\ldots,m\} \subset \mathbb{Z}$. Since $K \cap (l\mathbb{Z}+j) \neq \emptyset$, $j = 0,1,\ldots,l-1$, Theorem 3.7 is applicable. Thus, $L_1 = L = 0$.
Next, we assume that the $\theta_\mu$'s generate at least two different equivalence classes. After re-ordering the $\theta_\mu$'s if necessary, we can write

$$\{\theta_1,\theta_2,\ldots,\theta_m\} = \cup_{s=1}^{l} [[\theta_s]]', \quad 1 < l \leq m, \tag{3.20}$$

in which $[[\theta_s]]' \subset [[\theta_s]]$, $s = 1,2,\ldots l$. Also, we can write

$$[[\theta_s]]' = \{\theta_s + 2\pi q_{s,\nu} : \nu = 1,2,\ldots,\nu_s\}, \quad q_{s,\nu} \in \mathbb{Q} \cap [0,1), \quad s = 1,2,\ldots,l, \tag{3.21}$$

in which $\nu_s$ is the cardinality of $[[\theta_s]]'$. We observe that, for each $s \in \{1,2,\ldots,l\}$, the rational numbers $q_{s,\nu}$, $\nu = 1,2,\ldots,\nu_s$ are pairwise distinct. Renaming the $c_\mu$ according to

(3.21), the functional $L$ takes the form

$$L(z) = \sum_{s=1}^{l} \left( \sum_{\nu=1}^{\nu_s} c_{\nu,s} e^{i(\theta_s + 2\pi q_{s,\nu})z} \right) = \sum_{s=1}^{l} \left( \sum_{\nu=1}^{\nu_s} c_{\nu,s} e^{i2\pi q_{s,\nu} z} \right) e^{i\theta_s z}, \tag{3.22}$$

To proceed, let us choose an integer $\eta \geq 1 + \max\{l, \nu_1, \nu_2, \ldots, \nu_l\}$ such that $\eta q_{s,\nu} \in \mathbb{Z}$, $\nu = 1, 2, \ldots, \nu_s$, $s = 1, 2, \ldots, l$. Next, fix $j \in \{0, 1, \ldots, \eta - 1\}$. Due to our hypothesis on $K$, the set $K \cap (\eta\mathbb{Z} + j)$ contains an arithmetic progression of length $\eta$. The inner sums in (3.22) remain fixed when $z$ runs over $\eta\mathbb{Z} + j$. In particular, the same is true when $z$ runs over this arithmetic progression. Denoting the arithmetic progression by $p + q\{0, 1, \ldots, \eta - 1\}$, the condition $L(p + q\mu) = 0$, $\mu = 0, 1, \ldots, \eta - 1$, reduces (3.22) to

$$\sum_{s=1}^{l} \left( \sum_{\nu=1}^{\nu_s} c_{\nu,s} e^{i2\pi q_{s,\nu} j} \right) e^{i\theta_s p}\, e^{i\theta_s q\mu} = 0, \quad \mu = 0, 1, \ldots, \eta - 1. \tag{3.23}$$

Since the $\theta_\mu$'s present in (3.22) are from different equivalence classes, the same argument used in the proof of Theorem 3.8 holds, that is, $e^{i\theta_\mu q} \neq e^{i\theta_\nu q}$, $\mu \neq \nu$. In addition, we conclude that

$$\sum_{\nu=1}^{\nu_s} c_{\nu,s} e^{i2\pi q_{s,\nu} j} = 0, \quad s = 1, 2, \ldots, l. \tag{3.24}$$

Since $j$ was arbitrary in $\{0, 1, \ldots, \eta - 1\}$, (3.24) holds for all $j$'s in this set. For a fixed $s$, (3.21) implies that $q_{s,\mu} - q_{s,\nu} \notin \mathbb{Z}$, $\mu \neq \nu$. Hence, each system

$$\sum_{\nu=1}^{\nu_s} c_{\nu,s} e^{i2\pi q_{s,\nu} j} = 0, \quad j = 0, 1, \ldots, \nu_s \tag{3.25}$$

is Vandermonde-like associated to distinct points. Thus, all the $c_{\nu,s}$ vanish and $L = 0$. ■

**Example 3.10** *If $F \subset \mathbb{Z}$ is finite then $\mathbb{Z} \setminus F$ induces SPD on $\Omega_2$.*

The preliminary version of Theorem 3.9 we had proved carried the following pair of hypotheses on $K$:
($i$) $K \cap (n\mathbb{Z} + j) \neq \emptyset$, $j = 0, 1, \ldots, n-1$, $n = 1, 2, \ldots$;
($ii$) $K$ contains arbitrarily long arithmetic progressions.

The proof had a flaw which was unnoticed for a long period of time. We do not know whether these two hypotheses are sufficient to guarantee the SPD of $K$ on $\Omega_2$. We observe that the above conditions are certainly independent of each other. Indeed, $2\mathbb{Z}$ is an example of set satisfying Condition ($ii$) but not ($i$). As for an example in the other way direction one may proceed like this: let $A_1, A_2, \ldots$ be an enumeration of the family of sets

$$\{n\mathbb{Z} + j : n = 1, 2, \ldots; j = 0, 1, \ldots, n-1\}. \tag{3.26}$$

Inductively, pick $k_\mu \in A_\mu$ so that $k_\mu > 2k_{\mu-1}$ for every $\mu$. Then the set $K = \{k_1, k_2, \ldots\}$ satisfies Condition $(ii)$ and contains no arithmetic progression of length 3.

Next, we address an apparently different sufficient condition for the induction of SPD of on $\Omega_2$. An argument similar to the one used in the proof of Theorem 3.11 was used in [8].

**Theorem 3.11** *Let $K$ be a subset of $\mathbb{Z}$. If $K$ contains arbitrarily long arithmetic progressions $p_j + q_j\{0, 1, \ldots, j\}$ and the set $\{q : q \text{ is prime and divides some } q_j\}$ is infinite then $K$ induces SPD on $\Omega_2$.*

**Proof.** Let $\theta_1, \theta_2, \ldots, \theta_m$ be distinct points in $[0, 2\pi)$ and consider the set $\Lambda := \{(\theta_\mu - \theta_\nu)/2\pi : \mu \neq \nu\}$. For each element in this set choose a prime number $p$ according to the following rule: if the element is rational, let $p$ be a divisor of its denominator. Otherwise, let $p$ be any prime. This choice produces a finite set of primes associated to $\Lambda$. Hence, due to our hypothesis, we can choose an index $j$ so that $q_j$ is not divisible by any of these prime numbers. We can assume, in addition, that $q_j \geq m$. If $L(z) := \sum_{\mu=1}^m c_\mu \exp(i\theta_\mu z)$, in which the $\theta_\mu$ are distinct in $[0, 2\pi)$, it is easily seen that the system $L(p_j + q_j\mu) = 0$, $\mu = 0, 1, \ldots, j$, contains a subsystem whose matrix is Vandermonde-like associated to the distinct points $\exp(iq_j\theta_\mu)$, $\mu = 1, 2, \ldots, m$. Thus, $L = 0$. ■

## 4. Strict positive definiteness on subsets of $\mathbb{C}$

In this section we study strictly positive definite kernels on other subsets of $\mathbb{C}$. Here, we lack of a characterization of the positive definite kernels on the subset. Even in the case in which the subset is the whole $\mathbb{C}$, a characterization is not available yet. Given a subset $B$ of $\mathbb{C}$, the idea is them to consider kernels of the form

$$f(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{k,l}(f) z^k \overline{z}^l, \quad z \in \mathbb{C}, \tag{4.1}$$

in which all coefficients $a_{k,l}(f)$ are nonnegative and the series is convergent for all $z$ in the image of the kernel $(z, w) \in B \times B \mapsto z\,\overline{w}$ and to search for conditions in order that the kernel $(z, w) \in B \times B \mapsto f(z\,\overline{w})$ be strictly positive definite. A similar problem in the real context was proposed and solved by Pinkus in [15].

The case $B = \Omega_2$ will be not considered here because the analysis reduces to that of Section 2. Indeed, given $f$ as in (4.1) with a convergent series when $z \in \Omega_2$, it is very easy to see that $(z, w) \in \Omega_2 \times \Omega_2 \mapsto f(z\,\overline{w})$ is strictly positive definite if and only if $\{k - l : a_{k,l}(f) > 0\}$ induces SPD on $\Omega_2$.

We observe that some of the results we present in this section can be put in a more general context. Indeed, it suffices to consider absolutely convergent series of the form

$$f(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{k,l}(f) \varphi_{k,l}(z), \quad z \in \mathbb{C}. \tag{4.2}$$

Here, $\{\varphi_{k,l}\}$ is a complex polynomial of degree $k$ in the variable $z \in \mathbb{C}$ and of degree $l$ in $\overline{z}$, fulfilling one or both conditions below:

$(i)$ $\varphi_{k,l}$ is even when $k+l$ is even and it is odd otherwise;
$(ii)$ For each pair $(k,l)$, the kernel $(z,w) \in B \times B \mapsto \varphi_{k,l}(z\,\overline{w})$ is well-defined and positive definite.

We begin with the following elementary, but important, lemma.

**Lemma 4.1** *Let $B \subset \mathbb{C}$ and $f$ a function as in* (4.1). *If $(z,w) \in B \times B \mapsto f(z\,\overline{w})$ is strictly positive definite and $0 \in B$ then $a_{0,0}(f) > 0$.*
**Proof.** If $a_{0,0}(f) = 0$, then any set of the form $\{z_1 = 0, z_2, \ldots, z_n\}$ will produce a matrix $(f(z_\mu \overline{z_\nu}))$ having an empty row. Thus, the kernel $(z,w) \in B \times B \mapsto f(z\,\overline{w})$ cannot be positive definite. ∎

We call a subset $B$ of $\mathbb{C}$ *symmetric* whenever it is symmetric with respect to the origin of $\mathbb{C}$. For a subset $K$ of $\mathbb{Z}_+ \times \mathbb{Z}_+$ we write

$$K_e := \{(k,l) \in K : k+l \in 2\mathbb{Z}_+\} \tag{4.3}$$

and

$$K_o := \{(k,l) \in K : k+l \in 1 + 2\mathbb{Z}_+\}. \tag{4.4}$$

In particular, for a function as in (4.1), we write $K_o(f) := K(f)_o$ e $K_e(f) := K(f)_e$, in which $K(f) := \{(k,l) : a_{k,l}(f) > 0\}$.

The proof of our next result requires two elementary results on special matrices. We quote them below without proof but refer the reader to [5, p. 407] and [5, p. 458]. The first one is this: If $G$ is the Gram matrix of a subset $\{z_1, z_2, \ldots, z_k\}$ of $\mathbb{C}^q$ then the rank of $G$ is the maximum number of independent vectors in the set. The second one is this: If $A_1$ and $A_2$ are nonnegative definite matrices of same size then the Schur product $A_1 \circ A_2$ of $A_1$ and $A_2$ has rank at most (rank $A_1$)(rank $A_2$). The symbol $|K|$ will stand for the cardinality of the set $K$.

**Lemma 4.2** *Let $f$ be as in* (4.1). *If $K(f)$ is finite and $B$ is a subset of $\mathbb{C}$ with more than $|K(f)|$ elements then $(z,w) \in B \times B \mapsto f(z\,\overline{w})$ is not strictly positive definite.*

**Proof.** Let $z_1, z_2, \ldots, z_n$ be distinct points in $B$ where $n > |K(f)|$. The Gram matrix $(z_\mu\,\overline{z_\nu})$ has rank at most 1. Hence, the same is true of the Schur product with $\mu\nu$-entry $(z_\mu\,\overline{z_\nu})^k (z_\nu\,\overline{z_\mu})^l$. The lemma follows. ∎

**Lemma 4.3** *Let $f$ be as in* (4.1). *If $B \subset \mathbb{C}$ is symmetric, $B \neq \{0\}$, and either $K_e(f)$ or $K_o(f)$ is empty then $(z,w) \in B \times B \mapsto f(z\,\overline{w})$ is not strictly positive definite.*
**Proof.** We only prove the lemma in the case in which $K_o(f)$ is empty, being the other case similar. Take $2n$ distinct points $z_1, z_2, \ldots, z_{2n}$ in $B$ such that $z_{n+\mu} = -z_\mu$, $\mu = 1, 2, \ldots, n$. Write $A$ to denote the $2n \times 2n$ matrix with $\mu\nu$ entry $f(z_\mu\,\overline{z_\nu})$ and for each $\mu \in \{1, 2, \ldots, n\}$, let $v(\mu, \mu+n)$ be the vector in $\mathbb{C}^{2n}$ having 1 as its $\mu^{\text{th}}$ component, $-1$ as its $(\mu+n)^{\text{th}}$ component, and 0 elsewhere. Since $K(f) = K_e(f)$, it follows that the $\eta^{\text{th}}$ component of

the vector $Av(\mu,\mu+n)$ is precisely

$$f(z_\eta\,\overline{z_\mu}) - f(z_\eta\,\overline{z_{\mu+n}}) = \sum_{(k,l)\in K_e(f)} a_{k,l}(f)\left[(z_\eta\,\overline{z_\mu})^k(z_\mu\overline{z_\eta})^l - (-1)^{k+l}(z_\eta\,\overline{z_\mu})^k(z_\mu\overline{z_\eta})^l\right] = 0. \tag{4.5}$$

Thus, $A$ is singular because its rank is at most $n$. ■

**Theorem 4.4** *Let $f$ be as in* (4.1)*. Assume that $B\subset\mathbb{C}$ is infinite and symmetric. If $(z,w)\in B\times B\mapsto f(z\,\overline{w})$ is strictly positive definite then both sets $K_e(f)$ and $K_o(f)$ are infinite.*

**Proof.** If $K_o(f)$ is finite, pick $n>1+|K_o(f)|$ and take $2n$ distinct points $z_1,z_2,\ldots,z_{2n}$ in $B$ such that $z_{\mu+n}=-z_\mu$, $\mu=1,2,\ldots,n$. Define

$$A_o := \sum_{(k,l)\in K_o(f)} a_{k,l}(f)(z_\mu\,\overline{z_\nu})^k(z_\nu\overline{z_\mu})^l \tag{4.6}$$

and

$$A_e := \sum_{(k,l)\in K_e(f)} a_{k,l}(f)(z_\mu\,\overline{z_\nu})^k(z_\nu\overline{z_\mu})^l \tag{4.7}$$

so that

$$A := f(z_\mu\,\overline{z_\nu}) = A_o + A_e. \tag{4.8}$$

The proof of Lemma 4.2 implies that the rank of $A_o$ is less than $n-1$. Lemma 4.3 reveals that the rank of $A_e$ is at most $n$. Thus, $A$ has rank at most $2n-1$. If $K_e(f)$ is finite, a similar argument can be used to reach the same conclusion. ■

**Lemma 4.5** *Let $\{z_1,z_2,\ldots,z_n\}$ be a subset of $\mathbb{C}\setminus\{0\}$ and $K$ a subset of $\mathbb{Z}_+\times\mathbb{Z}_+$. Assume that the following condition holds: if $|z_\mu|=|z_\nu|$, $\mu\neq\nu$ then $z_\mu=-z_\nu$. If both $K_e$ and $K_o$ are infinite then the system*

$$\sum_{\mu=1}^{n} c_\mu z_\mu^k\overline{z_\mu}^l = 0, \quad (k,l)\in K, \tag{4.9}$$

*has a unique solution $(c_1,c_2,\ldots,c_n)\in\mathbb{C}^n$.*

**Proof.** Multiply Equation (4.9) by $z_m^{-k}(\overline{z_m})^{-l}$, in which $z_m$ has maximum modulus among all the $z_\mu$. The resulting equation has the form

$$\sum_{|w_\mu|=1} c_\mu w_\mu^k\overline{w_\mu}^l + \sum_{|w_\mu|<1} c_\mu w_\mu^k\overline{w_\mu}^l = 0, \quad (k,l)\in K \tag{4.10}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

in which $w_\mu := z_\mu/z_m$, $\mu = 1, 2, \ldots, n$. Since $\lim_{p\to\infty} w_\mu^p = 0$ when $|\omega_\mu| < 1$, given $\epsilon > 0$, there is an integer $N(\mu) > 0$ such that

$$|c_\mu w_\mu^k \overline{w_\mu}^l| < \frac{\epsilon}{n}, \quad (k,l) \in K, \quad k+l \geq N(\mu). \tag{4.11}$$

Hence, there is an $N > 0$ such that

$$\left| \sum_{|w_\mu|<1} c_\mu w_\mu^k \overline{w_\mu}^l \right| < \epsilon, \quad (k,l) \in K, \quad k+l \geq N. \tag{4.12}$$

Looking at (4.10), we conclude that

$$\left| \sum_{|w_\mu|=1} c_\mu w_\mu^k \overline{w_\mu}^l \right| < \epsilon, \quad (k,l) \in K, \quad k+l \geq N. \tag{4.13}$$

Due to our hypothesis on the $z_\mu$, the sum in (4.13) reduces to either

$$|c_m| < \epsilon, \quad (k,l) \in K, \quad k+l \geq N, \tag{4.14}$$

or

$$|c_m + c_\mu(-1)^{k+l}| < \epsilon, \quad (k,l) \in K, \quad k+l \geq N, \tag{4.15}$$

for some $\mu$. Due to our hypothesis on $K$, we conclude that $c_m = 0$ in the first case. As for the other, $c_m + c_\mu = c_m - c_\mu = 0$, that is, $c_m = c_\mu = 0$. Now, we apply the same procedure to the sum resulting from (4.10) when we eliminate the $c_\mu$ which are zero. After finitely many steps, we conclude that all the $c_\mu$ are zero. ■

This is our first characterization of SPD on a subset of $\mathbb{C}$.

**Theorem 4.6** *Let $f$ be as in* (4.1). *Let $B$ be an infinite and symmetric subset of $\mathbb{C} \setminus \{0\}$ and satisfying the following condition: if $z, w \in B$, $z \neq w$, and $|z| = |w|$ then $z = -w$. Then, $(z,w) \in B \times B \mapsto f(z\,\overline{w})$ is strictly positive definite if and only if $K_e(f)$ and $K_o(f)$ are infinite.*

**Proof.** Necessity was proved in Theorem 4.4. As for the sufficiency, let $z_1, z_2, \ldots, z_n$ be distinct points in $B$ and consider the associated quadratic form

$$Q := \sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} f(z_\mu\, \overline{z_\nu}). \tag{4.16}$$

We show that if $K_e(f)$ and $K_o(f)$ are infinite, the condition $Q = 0$ implies $c_\mu = 0$, $\mu = 1, 2, \ldots, n$. To do that, we first observe that

$$Q = \sum_{(k,l)\in K(f)} a_{k,l}(f) \left| \sum_{\mu=1}^{n} c_\mu z_\mu^k \overline{z_\mu}^l \right|^2. \tag{4.17}$$

Thus, if $Q = 0$ then

$$\sum_{\mu=1}^{n} c_\mu z_\mu^k \overline{z_\mu}^l = 0, \quad (k,l) \in K(f). \tag{4.18}$$

Under the hypothesis of the theorem, Lemma 4.5 is applicable. It follows that all the $c_\mu$ are zero. ∎

Recalling Lemma 4.1, we have the following version of the previous theorem.

**Theorem 4.7** *Let $f$ be as in* (4.1). *Let $B$ be an infinite and symmetric subset of $\mathbb{C}$ satisfying the following conditions: $\{0\} \subset B$ and if $z, w \in B \setminus \{0\}$, $z \neq w$, and $|z| = |w|$ then $z = -w$. Then, $(z,w) \in B \times B \mapsto f(z\,\overline{w})$ is strictly positive definite if and only if $a_{0,0}(f) > 0$ and both, $K_e(f)$ and $K_o(f)$, are infinite.*

**Proof.** Necessity follows from Lemma 4.1 and Theorem 4.4. As for the sufficiency, it suffices to prove that the matrix $(f(z_\mu \overline{z_\nu}))$ is positive definite whenever $n \geq 1$, $z_1, z_2, \ldots, z_n$ are distinct points in $B$, and 0 is among the $z_\mu$. Once again we consider the associated quadratic form

$$Q := \sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu} f(z_\mu\, \overline{z_\nu}). \tag{4.19}$$

If $n = 1$, the condition $a_{0,0}(f) > 0$ implies that $Q = c_1 \overline{c_1} f(0) > 0$, when $c_1 \neq 0$. If $n \neq 1$, we can set $z_1 = 0$. We assume that at least one $c_\mu$ is nonzero and prove that $Q > 0$. If $c_1 = 0$, then

$$Q = \sum_{\mu,\nu=2}^{n} c_\mu \overline{c_\nu} f(z_\mu\, \overline{z_\nu}) > 0, \tag{4.20}$$

due to the previous theorem. If $c_1 \neq 0$, write

$$\begin{aligned} Q &= \sum_{\mu,\nu=1}^{n} c_\mu \overline{c_\nu}\, [f(z_\mu\, \overline{z_\nu}) - f(0)] + f(0) \left| \sum_{\mu=1}^{n} c_\mu \right|^2 \\ &= \sum_{\mu,\nu=2}^{n} c_\mu \overline{c_\nu}\, [f(z_\mu\, \overline{z_\nu}) - f(0)] + f(0) \left| \sum_{\mu=1}^{n} c_\mu \right|^2 . \end{aligned}$$

If $c_\mu = 0$, $\mu \neq 1$, the second summand above is positive. Otherwise, the first one is positive due to the previous theorem. In any case, $Q > 0$. ∎

¿From now on, we will try to establish a similar theorem for sets $B$ intersecting circles in more than 2 points. We begin with a generalization of Lemma 4.5. We use the following equivalence relation on a finite subset $B$ of $\mathbb{C}$:

$$z \simeq w \ \text{ if and only if } \ |z| = |w|. \tag{4.21}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

The equivalence class of an element $z$ of $B$ is denoted by $\lceil z \rfloor$. The symbol $\arg z$ will represent the sole argument of the complex number $z$ in $[0, 2\pi)$.

**Lemma 4.8** *Let $B$ be a finite subset of $\mathbb{C} \setminus \{0\}$ and $K$ a subset of $\mathbb{Z}_+ \times \mathbb{Z}_+$. Assume that the following condition holds: if $w \in B$ and $\lceil w \rfloor$ has at least two elements then there exists a real number $r$ such that $\arg z \in r + 2\pi\mathbb{Q}$, $z \in \lceil w \rfloor$. If $\{k - l : (k, l) \in K\} \cap (n\mathbb{Z} + j) \neq \emptyset$, $j = 0, 1, \ldots, n-1$, $n = 1, 2, \ldots$ then the system*

$$\sum_{z \in B} c_z z^k \overline{z}^l = 0, \quad (k, l) \in K, \tag{4.22}$$

*has a unique solution $(c_z)_{z \in B}$.*

**Proof.** We adapt the proof of Lemma 4.5 to the situation under consideration. We recall that the proof of Theorem 3.6 reveals that the condition $\{k - l : (k, l) \in K\} \cap (n\mathbb{Z} + j) \neq \emptyset$, $j = 0, 1, \ldots, n-1$, $n = 1, 2, \ldots$ translates itself into the following one: every set $\{k - l : (k, l) \in K\} \cap (n\mathbb{Z} + j)$ is infinite. Let $w$ be an element of $B$ having maximum modulus. Given $\epsilon > 0$, there is an integer $N = N(\epsilon)$ such that

$$\left| \sum_{z \in \lceil w \rfloor} c_z e^{i(\arg z - \arg w)(k-l)} \right| = \left| \sum_{z \in \lceil w \rfloor} c_z \frac{z^k}{w^k} \frac{\overline{z}^l}{\overline{w}^l} \right| < \epsilon, \quad (k, l) \in K, \quad k + l \geq N. \tag{4.23}$$

If $\lceil w \rfloor = \{w\}$, the arguments at the end of the proof of Lemma 4.5 can be entirely reproduced. Indeed, if $K$ satisfies $\{k - l : (k, l) \in K\} \cap (n\mathbb{Z} + j) \neq \emptyset$, $j = 0, 1, \ldots, n-1$, $n = 1, 2, \ldots$ then $K_e$ and $K_o$ are infinite. Hence, we conclude that $c_w = 0$. If $\lceil w \rfloor$ has cardinality $m > 1$, the hypothesis on $K$ allows us to select a pair $(k_0, l_0)$ in $K$ so that $\{(k_0 - l_0)(\arg z - \arg w) : z \in \lceil w \rfloor\} \subset 2\pi\mathbb{Z}$. For $j \in \{0, 1, \ldots, m-1\}$ fixed, the set $\{k - l : (k, l) \in K\} \cap ((k_0 - l_0)\mathbb{Z} + j)$ contains infinitely many elements $(k, l)$ such that $k + l \geq N$. On the other hand, for such an element, say $(k_0 - l_0)\alpha + j$, we have that

$$\sum_{z \in \lceil w \rfloor} c_z e^{i(\arg z - \arg w)((k_0 - l_0)\alpha + j)} = \sum_{z \in \lceil w \rfloor} c_z e^{i(\arg z - \arg w)j}. \tag{4.24}$$

Thus, since $\epsilon$ is arbitrary, we conclude that

$$\sum_{z \in \lceil w \rfloor} c_z e^{i(\arg z - \arg w)j} = 0, \quad j = 0, 1, \ldots, m-1. \tag{4.25}$$

The matrix of this system is a Vandermonde corresponding to $m$ distinct points. Therefore, $c_z = 0$, $z \in \lceil w \rfloor$. We now repeat this procedure to the sum resulting from (4.22) when we eliminate these $c_z$'s which are 0. After finitely many steps, we conclude that all the $c_z$'s are zero. ■

**Theorem 4.9** *Let $f$ be as in (4.1) and $B$ a subset of $\mathbb{C} \setminus \{0\}$. Assume that the following conditions hold:*

*(i) If* $w \in B$ *and* $\lceil w \rfloor$ *has at least two elements then there exists a real number* $r$ *such that* $\arg z \in r + 2\pi\mathbb{Q}$, $z \in \lceil w \rfloor$;
*(ii) Given* $n \geq 1$, *there exists an equivalence class* $\lceil w \rfloor$ *containing* $n$ *equally spaced points. Then,* $(z, w) \in B \times B \mapsto f(z\,\overline{w})$ *is strictly positive definite if and only if*

$$\{k - l : (k, l) \in K(f)\} \cap (n\mathbb{Z} + j) \neq \emptyset, \quad j = 0, 1, \ldots, n-1, \quad n = 1, 2, \ldots. \tag{4.26}$$

**Proof.** Sufficiency follows from the previous lemma. Necessity follows from arguments similar to those used in the proof of Theorem 3.4. ■

**Example** $B = \{se^{i2\pi q} : s \in (0, \infty),\, q \in \mathbb{Q} \cap [0, 1]\}$ *satisfies the conditions required in the previous theorem.*

**Corollary 4.10** *Let* $f$ *be as in* (4.1) *and* $B$ *be a subset of* $\mathbb{C}$. *Assume that the following conditions hold:*
*(i) If* $w \in B$ *and* $\lceil w \rfloor$ *has at least two elements then there exists a real number* $r$ *such that* $\arg z \in r + 2\pi\mathbb{Q}$, $z \in \lceil w \rfloor$;
*(ii) Given* $n \geq 1$, *there exists an equivalence class* $\lceil w \rfloor$ *containing* $n$ *equally spaced points. Then,* $(z, w) \in B \times B \mapsto f(z\,\overline{w})$ *is strictly positive definite if and only if* $a_{0,0}(f) > 0$ *and* $\{k - l : (k, l) \in K(f)\} \cap (n\mathbb{Z} + j) \neq \emptyset$, $j = 0, 1, \ldots, n-1$, $n = 1, 2, \ldots$.

We close the paper with a sufficient condition for strict positive definiteness on $\mathbb{C}$.

**Theorem 4.11** *Let* $f$ *be as in* (4.1). *If* $a_{0,0}(f) > 0$ *and for every* $n$, *the sets* $\{k - l : (k, l) \in K(f)\} \cap (n\mathbb{Z} + j)$, $j = 0, 1, \ldots, n-1$ *contain an arithmetic progression of length* $n$, *then* $(z, w) \in \mathbb{C} \times \mathbb{C} \mapsto f(z\,\overline{w})$ *is strictly positive definite.*
**Proof.** Let $B$ be a finite subset of $\mathbb{C} \setminus \{0\}$. Assuming the conditions in the statement of the theorem, we will show that the system

$$\sum_{z \in B} c_z z^k \overline{z}^l = 0, \quad (k, l) \in K(f) \tag{4.27}$$

has just one solution. The procedure is the same used in other proofs in the paper. Let $w$ be an element of $B$ having maximum modulus. We break our analysis into two cases. If $\lceil w \rfloor = \{w\}$, the procedure used in the proof of Lemma 4.5 can be repeated here. Indeed, the condition on $K(f)$ in the statement of the theorem implies that $K_e(f)$ and $K_o(f)$ are infinite. Thus, $c_w = 0$. If $\lceil w \rfloor$ has at least two elements, we repeat the procedure used in the proofs of Lemmas 4.5 and 4.8. Given $\epsilon > 0$, there is an integer $N = N(\epsilon)$ such that

$$\left| \sum_{z \in \lceil w \rfloor} c_z \frac{z^k}{w^k} \frac{\overline{z}^l}{\overline{w}^l} \right| < \epsilon, \quad (k, l) \in K(f), \quad k + l \geq N. \tag{4.28}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Writing $z = |w|e^{i\theta_z}$, $z \in \lceil w \rfloor$, inequality (4.28) reduces itself to

$$\left| \sum_{z\in\lceil w\rfloor} c_z e^{i(\theta_z-\theta_w)(k-l)} \right| < \epsilon, \quad (k,l) \in K(f), \quad k+l \geq N. \tag{4.29}$$

We now proceed analyzing two sub-cases:
If $\theta_z \in [[\theta_w]]$, $z \in \lceil w \rfloor$, we can write

$$\theta_z - \theta_w = 2\pi q_z, \quad q_z \in \mathbb{Q} \cap (-1,1), \quad z \in \lceil w \rfloor \tag{4.30}$$

and (4.29) becomes

$$\left| \sum_{z\in\lceil w\rfloor} c_z e^{i2\pi q_z(k-l)} \right| < \epsilon, \quad (k,l) \in K(f), \quad k+l \geq N. \tag{4.31}$$

Next, we choose an integer $m \geq |\lceil w \rfloor| + 1$ such that $mq_z \in \mathbb{Z}$, $z \in \lceil w \rfloor$. Recalling our assumption on $K(f)$, we can refine our choice of $m$ so that the following additional feature holds: for $j \in \{0,1,\ldots,m-1\}$, the set $\{k-l : (k,l) \in K(f)\} \cap (m\mathbb{Z}+j)$ contains an element $k-l = m\alpha_j + j$ such that $k+l \geq N$. Returning to (4.31), we obtain

$$\left| \sum_{z\in\lceil w\rfloor} c_z e^{i2\pi q_z j} \right| < \epsilon, \quad j = 0,1,\ldots m-1. \tag{4.32}$$

Since $\epsilon$ was arbitrary and the whole procedure above was independent of $\epsilon$, we conclude that

$$\sum_{z\in\lceil w\rfloor} c_z e^{i2\pi q_z j} = 0, \quad j = 0,1,\ldots,|\lceil w \rfloor|. \tag{4.33}$$

Obviously, $q_z - q_\zeta \in (-2,2)$, $z,\zeta \in \lceil w \rfloor$. If $q_z - q_\zeta = 0$ then $z = \zeta$ and if $|q_z - q_\zeta| = 1$, then $\theta_z - \theta_\zeta = \pm 2\pi$. It follows that $q_z - q_\zeta \notin \mathbb{Z}$, $z,\zeta \in \lceil w \rfloor$, $z \neq \zeta$. In particular, the system (4.33) has just one solution, that is, $c_z = 0$, $z \in \lceil w \rfloor$.
If not all $\theta_z$ belong to the class $[[\theta_w]]$, we can write

$$\{\theta_z - \theta_w : z \in \lceil w \rfloor\} = \left( \cup_{s=1}^{L} [[\theta_s]]' \right) - \theta_w, \quad 1 < L \leq |\lceil w \rfloor|, \tag{4.34}$$

in which $[[\theta_s]]' \subset [[\theta_s]]$ and $\{[[\theta_s]] : s = 1,2,\ldots,L\}$ are the distinct classes generated by arguments of elements in $\lceil w \rfloor$. In addition, we write

$$[[\theta_s]]' = \{\theta_s + 2\pi q_{s,\nu} : \nu = 1,2,\ldots,\nu_s\}, \quad q_{s,\nu} \in \mathbb{Q} \cap [0,1), \quad s = 1,2,\ldots,L, \tag{4.35}$$

in which $\nu_s$ is of cardinality of $[[\theta_s]]'$. We observe that, for each $s \in \{1,2,\ldots,L\}$, the rational numbers $q_{s,\nu}$, $\nu = 1,2,\ldots,\nu_s$ are pairwise distinct. Expression (4.29) takes the form

$$\left|\sum_{s=1}^{L}\left(\sum_{\nu=1}^{\nu_s} c_{\nu,s}e^{i2\pi q_{s,\nu}(k-l)}\right)e^{i(\theta_s-\theta_w)(k-l)}\right| < \epsilon, \quad (k,l)\in K, \quad k+l\geq N. \tag{4.36}$$

To proceed, we choose an integer $\eta \geq 1+\max\{L,\nu_1,\nu_2,\ldots,\nu_L\}$ and such that $\eta q_{s,\nu} \in \mathbb{Z}$, $\nu = 1,2,\ldots,\nu_s$, $s = 1,2,\ldots,L$. In view of our hypothesis on $K(f)$, we can assume that $\eta$ is large enough so that the following conditions are fulfilled: the sets $\{k-l : (k,l)\in K\}\cap(\eta\mathbb{Z}+j)$, $j = 0,1,\ldots,\eta-1$, contain arithmetic progressions of length at least $\max\{L,\nu_1,\nu_2,\ldots,\nu_L\}$ and every element in these progressions come from pairs $(k,l)$ for which $k+l\geq N$. Let us denote these progressions by $p_j+q_j\{0,1,\ldots,\eta-1\}$. Returning to (4.36),

$$\left|\sum_{s=1}^{L}\left(\sum_{\nu=1}^{\nu_s} c_{\nu,s}e^{i2\pi q_{s,\nu}j}\right)e^{i(\theta_s-\theta_w)(p_j+q_j\mu)}\right| < \epsilon, \quad \mu,j = 0,1,\ldots\eta-1. \tag{4.37}$$

Since $\epsilon$ was arbitrary and the procedure does not depend on it, we conclude that

$$\sum_{s=1}^{L}\left(\sum_{\nu=1}^{\nu_s} c_{\nu,s}e^{i2\pi q_{s,\nu}j}\right)e^{i(\theta_s-\theta_w)p_j}e^{i(\theta_s-\theta_w)q_j\mu} = 0, \quad \mu,j = 0,1,\ldots\eta-1. \tag{4.38}$$

For every $j\in\{0,1,\ldots,\eta-1\}$, $(\theta_s-\theta_w)q_j \neq (\theta_r-\theta_w)q_j$ when $r\neq s$. Thus,

$$\left(\sum_{\nu=1}^{\nu_s} c_{\nu,s}e^{i2\pi q_{s,\nu}j}\right)e^{i(\theta_s-\theta_w)p_j} = 0, \quad s = 1,2\ldots,L, \quad j = 0,1,\ldots,\eta-1, \tag{4.39}$$

that is,

$$\sum_{\nu=1}^{\nu_s} c_{\nu,s}e^{i2\pi q_{s,\nu}j} = 0, \quad s = 1,2\ldots,L, \quad j = 0,1,\ldots,\eta-1. \tag{4.40}$$

For a fixed $s$, $q_{s,\nu}-q_{s,\mu}\notin\mathbb{Z}$, $\mu\neq\nu$. Thus, $c_{\nu,s}=0$, $\nu = 1,2,\ldots,\nu_s$, $s = 1,2,\ldots,L$, and, consequently, $c_z = 0$, $z\in\lceil w\rfloor$. We now repeat the argument to the next $z\in B$ of maximum modulus, after eliminating the elements of $\lceil w\rfloor$ from the sum in (4.26). After finitely many steps, we conclude that all the $c_z$ are zero. ■

## REFERENCES


1. Berg, C.; J. P. R. Christensen and P. Ressel, Harmonic analysis on semigroups. Theory of positive definite and related functions. Graduate Texts in Mathematics, 100. Springer-Verlag, New York, (1984).
2. Chen, Debao; V. A. Menegatto and Xingping Sun, A necessary and sufficient condition for strictly positive definite functions on spheres. *Proc. Amer. Math. Soc.* **131**(9) 2733-2740 (2003).
3. Cheney, E. W., Approximation using positive definite functions. Approximation theory VIII, Vol. 1 (College Station, TX, 1995), 145-168, Ser. Approx. Decompos., 6, World Sci. Publishing, River Edge, NJ (1995).


Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

4. Fasshauer, G. E. and L. L. Schumaker, Scattered data fitting on the sphere. Mathematical methods for curves and surfaces, II (Lillehammer, 1997), 117-166, Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN (1998).
5. Horn, R. A. and C. R. Johnson, Matrix analysis. Cambridge University Press, Cambri-dge-New York, (1985).
6. Light, W. A. and E. W. Cheney, Interpolation by periodic radial basis functions, *J. Math. Anal. Appl.*, **168** 111-130 (1992).
7. Menegatto, V. A., Interpolation on the complex Hilbert sphere using positive definite and conditionally negative definite kernels. *Acta Math. Hungar.* **75**(3) 215-225 (1997).
8. Menegatto, V. A., Strictly positive definite kernels on the circle, *Rocky Mountain J. Math.*, **25**(3) 1149-1163 (1995).
9. Menegatto, V. A., Strictly positive definite kernels on the Hilbert sphere. *Appl. Anal.* **55**(1-2) 91-101 (1994).
10. Menegatto, V. A., Strict positive definiteness on spheres. *Analysis* **19**(3) 217-233(1999).
11. Menegatto, V. A. and A. P. Peron, Positive definite kernels on complex spheres. *J. Math. Anal. Appl.* **254**(1) 219-232 (2001).
12. Menegatto, V. A. and A. P. Peron, Strict positive definiteness on spheres via disk polynomials. *Int. J. Math. Math. Sci.* **31**(12) 715-724 (2002).
13. Menegatto, V. A. and A. P. Peron, A complex approach to strict positive definiteness on spheres. *Integral Transform. Spec. Funct.* **11**(4) 377-396 (2001).
14. Narcowich, F. J., Recent developments in approximation via positive definite functions. Approximation theory IX, Vol. 2 (Nashville, TN, 1998), 221-242, Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN (1998).
15. Pinkus, A., Strictly positive definite functions on a real inner product space, *Adv. Comp. Math.* **20** 263-271 (2004).
16. Ron, A. and Xingping Sun, Strictly positive definite functions on spheres in Euclidean spaces, *Math. Comp.*, **65**(216) 1513-1530 (1996).
17. Schoenberg, I. J., Positive definite functions on spheres. *Duke Math. J.* **9** 96-108 (1942).
18. Sun, Xingping and V. A. Menegatto, Strictly positive definite functions on the complex Hilbert sphere. *Adv. Comput. Math.* **11** 105-119 (1999).
19. Xu, Yuan and E. W. Cheney, Strictly positive definite functions on spheres. *Proc. Amer. Math. Soc.* **116**(4) 977-981 (1992).


Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC