CADERNOS DE MATEMÁTICA **05**, 121–133 May (2004)
ARTIGO NÚMERO SMA#200

Control element approximation for parabolic differential equations with final observation

Alexei Prilepko

Department of Mathematical Analysis, Moscow State University, Vorobjevy Gory, Moscow 119899, Russia

Sergey Piskarev

Scientific Research Computer Center, Moscow State University, Vorobjevy Gory, Moscow 119899, Russia
E-mail: piskarev@srcc.msu.su

S.-Y. Shaw

Department of mathematics, National Central University, Chung-Li 32054, Taiwan
E-mail: shaw@math.ncu.edu.tw

This paper devoted to approximation of control element for abstract parabolic equation. The presentation is given on general approximation scheme, which includes finite element methods, finite difference schemes, projection methods.
May, 2004 ICMC-USP

Mathematics Subject Classification by Zentralblatt MATH: 65J20, 65J22, 34G10, 47D06.
Keywords and phrases: Abstract differential equations, C_0-semigroups, Banach spaces, discretization methods, difference schemes, discrete semigroups, inverse problems, Fredholm equations.

1. INTRODUCTION

Let $B(E)$ denote the Banach algebra of all linear bounded operators on a complex Banach space E. The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$. We denote by $\sigma(B)$ the spectrum of the operator B, by $\rho(B)$ the resolvent set of B, by $\mathcal{N}(B)$ the null space of B and by $\mathcal{R}(B)$ the range of B.

Let A be a generator of analytic C_0-semigroup $\exp(\cdot A)$, defined on a Banach space E. Consider in a Banach space E the problem

$$u'(t) = Au(t) + \mathscr{F}(t)f, \tag{1}$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

with

$$u(0) = u^0 \in D(A),\, u(T) = u^T \in D(A), \tag{2}$$

where $\mathscr{F}(\cdot) \in C^1([0,T])$ is given function and $f \in E$ is an unknown element. Follow [4],[18] the mild solution of such problem (1)-(2) is given by the formula

$$u(t) = \exp(tA)u^0 + \int_0^t \exp((t-s)A)\mathscr{F}(s) f ds,\, t \in [0,T], \tag{3}$$

with such an element $f \in E$ that conditions (2) are satisfied. Since, we will assumed that A generates analytic C_0-semigroup, then in this case $u(t) \in D(A)$ for any function $\mathscr{F}(\cdot) \in C^1([0,T])$, see [4], [17].

Applying A in (3) one gets

$$Au(t) = \exp(tA)Au^0 - \mathscr{F}(t)f + \exp(tA)\mathscr{F}(0)f + \int_0^t \exp((t-s)A)\mathscr{F}(s)'_s f ds.$$

If one makes the choice of the function $\mathscr{F}(\cdot)$ such that $\mathscr{F}(T) = 1$ and use condition $u(T) = u^T$, then we get the Fredholm second order equation in the form

$$If - K(T)f := If - \exp(TA)\mathscr{F}(0)f - \int_0^T \exp\Big((T-s)A\Big)\mathscr{F}(s)'_s ds f = G(T), \tag{4}$$

where $G(T) = \exp(TA)Au^0 - Au^T$. Such problem is very important in many physical applications, see [12].

So the main purpose of our paper is the construction of algorithm for approximation of an element f, which solve the problem $(1)-(2)$ or by other words we want to solve equation (4). We give the algorithm on general approximation scheme, which includes finite element and finite difference methods and projection methods.

The technique we developed here could be applied for more general situation, when operator A generates integrated semigroup or C-semigroup, see [5] - [7]. Moreover, in the proofs concerning compactness of integral from non-semigroup families it is convenient to use results from [19]. We show this in forthcoming papers.

2. EXISTENCE OF CONTROL ELEMENT *F*

The main question is a solvability of the problem (1)-(2). It is clear that in the case of compact operator $K(T)$ the operator $I - K(T)$ is Fredholm operator of index 0. Most of the results on the existence of solution of problem (1)-(2) are concerned to compactness or positivity property of resolvent of operator A. So the existence of bounded inverse operator $(I-K(T))^{-1}$ follows practically from condition $\mathcal{N}(I-K(T)) = \{0\}$ and compact convergence of resolvent, see Theorem 3.2 and Step 4 of the proof of Theorem 5.1. There are

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

some theorems proved, say, in [11], [12], [13], which guarantee that condition $\mathcal{N}(I-K(T)) = \{0\}$ holds.

Namely, let us list some results which could be applied here. Consider in a Banach space E the problem of finding an element f from relations:

$$u'(t) = Au(t) + \mathscr{F}(t)f,\ t \in [0,T], \tag{1}$$

$$u(0) = u^0,\ \int_0^T u(t)d\mu(t) = u^T,\ u^0, u^T \in D(A). \tag{2}$$

THEOREM 2.1. *[13] Let $\mu(\cdot)$ be nondecreasing on $[0,T]$ and continuous on the right at $t = 0$. Let $\mathscr{F}(t) \geq 0$ for $0 \leq t \leq T$ and $\int_0^T \mathscr{F}(t)d\mu(t) \neq 0$ and $\|\exp(tA)\| \leq Me^{-\beta t}$ with $\beta > 0, t \geq 0$. Suppose that anyone of the following conditions is fulfilled as well*

a) $\mathscr{F}(t)'_t \geq 0$ for $0 \leq t \leq T$,

b) the function $\mu(\cdot)$ is convex upward on $[0,T]$.

Then the Problem $(1)-(2)$ is well-posed.

THEOREM 2.2. *[13] Let $\mu(\cdot)$ be nondecreasing on $[0,T]$ and continuous on the right at $t = 0$. Let $\mathscr{F}(t) \geq 0$ for $0 \leq t \leq T$ and the operator $0 \leq (\int_0^T \mathscr{F}(t)d\mu(t))^{-1} \in B(E)$. Let the C_0-semigroup is positive. Assume also that $\sigma(A) \subset \{\lambda \in \mathbb{C} : Re\lambda < 0\}$. Then the kernel of problem $(1)-(2)$ is trivial and thus for any $u^0, u^T \in D(A)$ this problem have at most one solution.*

Approximation of positive C_0-semigroups on general approximation scheme is described in [10]. One can find there analogy of Theorem (ABC) for positive C_0-semigroups and also stability condition for the scheme like (5).

THEOREM 2.3. *[13] Let the conditions of Theorem 2.2 be fulfilled. Suppose also that the C_0-semigroup $\exp(tA)$ is compact. Then the problem $(1)-(2)$ is well-posed.*

3. GENERAL APPROXIMATION SCHEME

The general approximation scheme, due to [15], [16] can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n, p_n \in B(E, E_n), n \in \mathbb{N} = \{1, 2, \cdots\}$, with the property:

$$\|p_n x\|_{E_n} \to \|x\|_E \text{ as } n \to \infty \text{ for any } x \in E.$$

DEFINITION 3.1. The sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be $\mathcal{P}$-convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$ and we write this $x_n \to x$.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

DEFINITION 3.2. The sequence of bounded linear operators $B_n \in B(E_n), n \in \mathbb{N}$, is said to be $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, such that $x_n \to x$ one has $B_n x_n \to Bx$. We write then $B_n \to B$.

For general examples of notions of $\mathcal{P}-$convergence see for instance [14].

Remark 3. 1. If we put $E_n = E$ and $p_n = I$ for each $n \in \mathbb{N}$, where I is the identity operator on E, then Definition 3.1 leads to the traditional pointwise convergent bounded linear operators which we denote by $B_n \to B$.

DEFINITION 3.3. A sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be $\mathcal{P}$-*compact* if for any $\mathbb{N}' \subseteq \mathbb{N}$ there exist $\mathbb{N}'' \subseteq \mathbb{N}'$ and $x \in E$ such that $x_n \to x$, as $n \to \infty$ in $\mathbb{N}''$.

DEFINITION 3.4. A sequence of operators $\{B_n\}$, $B_n \in B(E_n)$, $n \in \mathbb{N}$, *converges compactly* to an operator $B \in B(E)$ if $B_n \to B$ and the following compactness condition holds:

$$\|x_n\|_{E_n} = O(1) \Longrightarrow \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

DEFINITION 3.5. A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in \mathbb{N}$, is said to be stably convergent to an operator $B \in B(E)$ iff $B_n \to B$ and $\|B_n^{-1}\|_{B(E_n)} = O(1), n \to \infty$. We will write this as: $B_n \to B$ stably.

DEFINITION 3.6. A sequence of operators $\{B_n\}, B_n \in B(E_n)$, is called regularly convergent to the operator $B \in B(E)$ iff $B_n \to B$ and the following implication holds:

$$\|x_n\|_{E_n} = O(1) \;\&\; \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact} \Longrightarrow \{x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

We write this as: $B_n \to B$ regularly.

THEOREM 3.1. *[16] Let $C_n, Q_n \in B(E_n)$, $C, Q \in B(E)$ and $\mathcal{R}(Q) = E$. Assume also that $C_n \to C$ compactly and $Q_n \to Q$ stably. Then $Q_n + C_n \to Q + C$ converge regularly.*

THEOREM 3.2. *[16] For $B_n \in B(E_n)$ and $B \in B(E)$ the following conditions are equivalent*:

(i) *$B_n \to B$ regularly, B_n are Fredholm operators of index* 0 *and* $\mathcal{N}(B) = \{0\}$;
(ii) *$B_n \to B$ stably and $\mathcal{R}(B) = E$*;
(iii) *$B_n \to B$ stably and regularly*;

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

(iv) *if one of conditions* (i)–(iii) *holds, then there exist* $B_n^{-1} \in B(E_n), B^{-1} \in B(E)$, *and* $B_n^{-1} \to B^{-1}$ *regularly and stably.*

In the case of unbounded operators, and we know in general infinitesimal generators are unbounded, we consider the notion of *compatibility*.

Definition 3.7. The sequence of closed linear operators $\{A_n\}, A_n \in \mathcal{C}(E_n), n \in I\!N$, are said to be compatible with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}, x_n \in D(A_n) \subseteq E_n, n \in I\!N$, such that $x_n \to x$ and $A_n x_n \to Ax$. We write (A_n, A) are compatible.

Note, that (A_n, A) are compatible if resolvents converge $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$. Usually in practice Banach spaces E_n are finite dimensional, although, in general, say for the case of a closed operator A, we have $\dim E_n \to \infty$ and $\|A_n\|_{B(E_n)} \to \infty$ as $n \to \infty$.

4. DISCRETIZATION OF SEMIGROUPS

Let us consider the well-posed Cauchy problem in the Banach space E with operator $A \in \mathcal{C}(E)$

$$\begin{aligned} &u'(t) = Au(t),\ t \in [0, \infty), \\ &u(0) = u^0 \in E, \end{aligned} \tag{1}$$

where operator A generates C_0-semigroup $\exp(\cdot A)$. It is well-known that the C_0-semigroup gives the solution of (1) by the formula $u(t) = \exp(tA)u^0$ for $t \geq 0$. The theory of well-posed problems and numerical analysis of these problems have been developed extensively, see for instance [2], [3], [8]. Let us consider on the general discretization scheme the semidiscrete approximation of the problem (1) in the Banach spaces E_n :

$$\begin{aligned} &u_n'(t) = A_n u_n(t),\ t \in [0, \infty), \\ &u_n(0) = u_n^0 \in E_n, \end{aligned} . \tag{2}$$

with the operators $A_n \in \mathcal{C}(E_n)$, such that they generate C_0-semigroups, which are consistent with the operator $A \in \mathcal{C}(E)$ and $u_n^0 \to u^0$.

4.1. The simplest discretization schemes

We have the following version of Trotter-Kato's Theorem on general approximation scheme:

Theorem 4.1. [14] (*TheoremABC*) *Assume that* $A \in \mathcal{C}(E), A_n \in \mathcal{C}(E_n)$ *and they generate* C_0*-semigroups. The following conditions* (A) *and* (B) *are equivalent to condition* (C).

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \cap_n\ \rho(A_n)$ *such that the resolvents converge* $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$;

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

(B) Stability. There are some constants $M \geq 1$ and ω, which are not depending on n and such that $\| \exp(tA_n)\| \leq M \exp(\omega t)$ for $t \geq 0$ and any $n \in N$;
(C) Convergence. For any finite $T > 0$ one has $\max_{t\in[0,T]} \| \exp(tA_n)u_n^0 - p_n \exp(tA)u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \to u^0$ for any $u_n^0 \in E_n, u^0 \in E$.

Remark 4. 1. The condition (A) in the contents of these Theorems is equivalent to compatibility of operators (A_n, A).

THEOREM 4.2. *[8] Let operators A and A_n generate analytic C_0-semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1).*
(A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ such that the resolvents converge $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$;
(B_1) Stability. There are some constants $M_2 \geq 1$ and ω_2 independent of n such that for any $Re\lambda > \omega_2$

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_2}{|\lambda - \omega_2|} \text{ for all } n \in I\!N;$$

(C_1) Convergence. For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$ we have

$$\max_{\eta\in\Sigma(\theta,\mu)} \| \exp(\eta A_n)u_n^0 - p_n \exp(\eta A)u^0\| \to 0$$

as $n \to \infty$ whenever $u_n^0 \to u^0$. Here $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$, and $\Sigma(\theta) = \{z \in \mathbb{C} : |arg\, z| \leq \theta\}$.

We can assume that conditions (A) and (B) for the corresponding semigroup case are satisfied without any restriction of generality if any discretization processes in time are considered. We denote by $T_n(\cdot)$ a family of discrete semigroups as in [3], i.e. $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = [\frac{t}{\tau_n}]$, as $\tau_n \to 0, n \to \infty$. The generator of discrete semigroup is defined by $\breve{A}_n = \frac{1}{\tau_n}(T_n(\tau_n) - I_n) \in B(E_n)$ and so $T_n(t) = (I_n + \tau_n\breve{A}_n)^{k_n}$, where $t = k_n\tau_n$.

THEOREM 4.3. *[14] $(Theorem\, ABC - discr)$ The following conditions (A) and (B') are equivalent to condition (C').*
(A) Consistency. There exists $\lambda \in \rho(A) \cap \cap_n \rho(\breve{A}_n)$ such that the resolvents converge $(\lambda I_n - \breve{A}_n)^{-1} \to (\lambda I - A)^{-1}$;
(B') Stability. There are some constants $M_1 \geq 1$ and $\omega_1 \in I\!R$ such that

$$\|T_n(t)\| \leq M_1 \exp(\omega_1 t) \text{ for } t \in \overline{I\!R}_+ = [0, \infty), n \in I\!N;$$

(C') Convergence. For any finite $T > 0$ one has $\max_{t\in\ [0,T]} \|T_n(t)u_n^0 - p_n \exp(tA)u^0\| \to 0$ as $n \to \infty$, whenever $u_n^0 \to u^0$ for any $u^0 \in E, u_n^0 \in E_n$.

THEOREM 4.4. *[14] Assume that $A \in \mathcal{C}(E), A_n \in \mathcal{C}(E_n)$ and they generate C_0-semigroup. Assume also that conditions (A) and (B) of Theorem 4.1 hold. Then the implicit difference*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

scheme

$$\frac{\overline{U}_n(t+\tau_n)-\overline{U}_n(t)}{\tau_n}=A_n\overline{U}_n(t+\tau),\ \overline{U}_n(0)=u_n^0, \tag{3}$$

is stable, i.e. $\|(I_n-\tau_nA_n)^{-k_n}\|\le M_1e^{\omega_1 t}, t=k_n\tau_n\in\overline{I\!R}_+$, *and gives an approximation to the solution of the problem* (1), *i.e.* $\overline{U}_n(t)\equiv(I_n-\tau_nA_n)^{-k_n}u_n^0\to\exp(tA)u_n^0$ $\mathcal{P}$*-converges uniformly with respect to* $t=k_n\tau_n\in[0,T]$ *as* $u_n^0\to u^0, n\to\infty, k_n\to\infty, \tau_n\to 0$.

THEOREM 4.5. [*8*] *Assume that conditions* (*A*) *and* (B_1) *of Theorem* 4.2 *hold and condition*

$$\tau_n\|A_n\|\le 1/(M+2),\ n\in I\!N \tag{4}$$

is fulfilled. Then the difference scheme

$$\frac{U_n(t+\tau_n)-U_n(t)}{\tau_n}=A_nU_n(t),\ U_n(0)=u_n^0, \tag{5}$$

is stable and gives an approximation to the solution of the problem (1), *i.e.* $U_n(t)\equiv(I_n+\tau_nA_n)^{k_n}u_n^0\to u(t)$ *discretely* $\mathcal{P}$*-converge uniformly with respect to* $t=k_n\tau_n\in[0,T]$ *as* $u_n^0\to u^0, n\to\infty, k_n\to\infty, \tau_n\to 0$.

Let us introduce the following equivalent conditions:
(B_1') Stability. There are constants M',ω' such that

$$\|\exp(tA_n)\|\le M'e^{\omega' t},\ \|A_n\exp(tA_n)\|\le\frac{M'}{t}e^{\omega' t},\ t\in I\!R_+.$$

(B_1'') Stability. There are constants M'',ω'' and $\tau^*>0$ such that

$$\|(I-\tau_nA_n)^{-k}\|\le M''e^{\omega'' k\tau_n},\ \|k\tau_nA_n(I-\tau_nA_n)^{-k}\|\le M''e^{\omega'' k\tau_n},\ 0<\tau_n<\tau^*, n,k\in I\!N.$$

THEOREM 4.6. *The conditions* (*A*) *and* (B_1') *are equivalent to the condition* (C_1).

Remark 4. 2. Conditions $(B_1),(B_1')$ and (B_1'') are equivalent.

5. APPROXIMATION OF CONTROL ELEMENT *F*

Let A_n be a generator of analytic and compact C_0-semigroup $\exp(\cdot A_n)$. Consider in a Banach space E_n the problem

$$u_n'(t)=A_nu_n(t)+\mathscr{F}_n(t)f_n, \tag{1}$$

with

$$u_n(0) = u_n^0 \in D(A_n),\ u_n(T) = u_n^T \in D(A_n). \tag{2}$$

The solution of problem (1)-(2) is given by the formula

$$u_n(t) = \exp(tA_n)u_n^0 + \int_0^t \exp((t-s)A_n)\mathscr{F}_n(s)f ds,\ t \in [0,T],$$

and corresponding second order Fredholm equation can be written in the form:

$$I_n f_n - K_n(T)f_n := I_n f_n - \exp(TA_n)\mathscr{F}_n(0)f_n - \int_0^T \exp((T-s)A_n)\mathscr{F}_n(s)'_s f_n ds = G_n(T), \tag{3}$$

where $G_n(T) = \exp(TA_n)A_n u_n^0 - A_n u_n^T$. Before we formulate our main results just recall that condition $\mathcal{N}(I - K(T)) = \{0\}$ could be obtained from Theorems in section 2.

Theorem 5.1. *Assume that functions $\mathscr{F}_n(\cdot), \mathscr{F}(\cdot) \in C^1([0,T])$ and they converge $\mathscr{F}_n(t)' \to \mathscr{F}(t)'$ uniformly in $t \in [0,T]$. Let conditions $(A), (B_1')$ be satisfied and $u_n^0 \to u^0, u_n^T \to u^T$ in a such way that $G_n(T) \to G(T)$. Assume also that $\mathcal{N}(I - K(T)) = \{0\}$, operator $(\lambda I - A)^{-1}$ is compact and $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly. Then solutions of the problems (3) exist and converge to the solution of the problem (4), i.e. $f_n \to f$.*

Proof. The proof falls naturally into 4 steps.

Step 1. First, let us show that the compact convergence of resolvents $R(\lambda; A_n) \to R(\lambda; A)$ is equivalent to the compact convergence of C_0-semigroups $\exp(tA_n) \to \exp(tA)$ for any $t > 0$. Let $\|x_n\| = O(1)$. Then from the estimate $\|A_n \exp(tA_n)\| \le \frac{M}{t} e^{\omega t}$, which is exactly condition (B_1'), we obtain the boundedness in n of the sequence $\{(A_n - \lambda I_n)\exp(tA_n)x_n\}$ for any fixed $t > 0$. Because of the compact convergence of resolvents, we obtain the compactness of the sequence $\{\exp(tA_n)x_n\}$.

The necessity will be proved if for the measure of noncompactness $\mu(\cdot)$ (for the definition, see [16]), we establish that $\mu\big(\{(\lambda I_n - A_n)^{-1}x_n\}\big) = 0$ for any $\|x_n\| = O(1)$. We have

$$\mu\big(\{(\lambda I_n - A_n)^{-1}x_n\}\big) = \mu\Big(\{\int_0^\infty e^{-\lambda t}\exp(tA_n)x_n\}\Big) \le \mu\Big(\{\int_0^q e^{-\lambda t}\exp(tA_n)x_n dt\}\Big) +$$

$$\mu\Big(\{\int_Q^\infty e^{-\lambda t}\exp(tA_n)x_n dt\}\Big) + \mu\Big(\{\exp(\epsilon A_n)\int_q^Q e^{-\lambda t}\exp\big((t-\epsilon)A_n\big)x_n dt\}\Big).$$

Two first terms can be made less than ϵ by the choice of q, Q. The last term is equal to zero because of the compact convergence $\exp(\epsilon A_n) \to \exp(\epsilon A)$ for any $0 < \epsilon < q$.

Step 2. Consider the operators $K(T)$ and $K_n(T)$ defined by (4) and (3) on the spaces E and E_n. The operator $K(T)$ defined by (4) is compact in E. Indeed, we obtain that the operator

$$K_\epsilon(t) = \exp(\epsilon A) \int_0^{t-\epsilon} \exp\big((t-s-\epsilon)A\big)\mathscr{F}(s)'_s ds$$

is a product of compact and bounded operators. Moreover $\|K_\epsilon(t) - K(t)\| \leq C\epsilon$ for any $t \in (0,T]$, where

$$K(t) = \int_0^t \exp\big((t-s)A\big)\mathscr{F}(s)'_s ds$$

and $0 < \epsilon < t$. Then it follows that the operator $K(t) : E \to E$ is compact for the same $t > 0$. For $t = 0$, the operator $K(0)$ is also compact.

Step 3. It is easy to see that $K_n(T) \to K(T)$. To show that $K_n(T) \to K(T)$ compactly, we assume that $\|f_n\|_{E_n} = O(1)$. Now $\{K_n(T)f_n\}$ is $\mathcal{P}$-compact because of representation

$$K_{\epsilon,n}(t) = \exp(\epsilon A_n) \int_0^{t-\epsilon} \exp\big((t-s-\epsilon)A_n\big)\mathscr{F}_n(s)'_s ds$$

and one can easy verify the vanishing of the noncompactness measure $\mu(\{K_n(t)f_n\}) = 0$ for all $t \in [0,T]$, taking into an account that $\|K_{\epsilon,n}(T) - K_n(T)\| \leq C\epsilon$.

Step 4. Now $I_n \to I$ stably and $K_n(T) \to K(T)$ compactly. Hence it follows from Theorem 3.1 that $I_n - K_n(T) \to I - K(T)$ regularly. Moreover, the nullspace $\mathcal{N}\big(I - K(T)\big) = \{0\}$ and the operators $I_n - K_n(T)$ are Fredholm of index zero. Then it follows from Theorem 3.2 that $I_n - K_n(T) \to I - K(T)$ stably, i.e. $\big(I_n - K_n(T)\big)^{-1} \to \big(I - K(T)\big)^{-1}$. Since $G_n(T) \to G(T)$, one gets $f_n = (I_n - K_n(T))^{-1}G_n(T) \to (I - K(T))^{-1}G(T) = f$. The Theorem is proved.

One can find that solution of the problem (3) with $u^0 = 0$ according to Theorem **??**, since $r(K_n(T)) < 1$, could be organized as follows:

$$f_{n,j+1} = K_n(T)f_{n,j} - A_n u_n^T,\ j = 0, 1, \cdots, \tag{4}$$

with initial condition $f_{n,0} = 0$. The value $K_n(T)f_{n,j}$ is nothing else as a solution of Cauchy problem

$$v_n'(t) = A_n v_n(t) + \mathscr{F}_n(t)'_t f_{n,j},\ v_n(0) = \mathscr{F}_n(0)f_{n,j}, \tag{5}$$

at the point T, i.e.

$$K_n(T)f_{n,j} = v_n(T; f_{n,j}) = \exp(TA_n)\mathscr{F}_n(0)f_{n,j} + \int_0^T \exp((T-s)A_n)\mathscr{F}_n(s)'_s f_{n,j} ds. \tag{6}$$

So (4) could be written in the form, starting from $f_{n,0} = 0$,

$$f_{n,j+1} = v_n(T; f_{n,j}) - A_n u_n^T,\ j = 0, 1, \cdots. \tag{7}$$

Moreover, $f_{n,j} \to f_n$ as $j \to \infty$ since $r(K_n(T)) < 1$.

There are different ways how one can calculate $v(T; f_{n,j})$. One can use directly Theorems 4.4, 4.5 or maybe some hire order difference schemes for approximation of $\exp(TA_n)$, say as in [2, 9], and then apply some quadrature formula for approximation the term $\int_0^T \exp((T-s)A_n)\mathscr{F}_n(s)'_s f_{n,j} ds$.

In this paper we consider just the simplest way which comes from Theorem 4.4. In case of Theorem 4.5 we have to assume stability condition, but the other considerations are the same. So following the scheme (3) we consider approximation of the equation (1) by

$$\frac{\overline{U}_n(t+\tau_n) - \overline{U}_n(t)}{\tau_n} = A_n \overline{U}_n(t+\tau) + \mathscr{F}_n(t+\tau_n) f_n, \, \overline{U}_n(0) = u_n^0, \quad t = k\tau_n. \qquad (8)$$

The solution of the scheme (8) can be written in the form

$$\overline{U}_n(t) = (I_n - \tau_n A_n)^{-k} u_n^0 + \Sigma_{l=0}^{k-1} (I_n - \tau_n A_n)^{-k+l} \mathscr{F}_n(l\tau_n + \tau_n) f_n \tau_n, \quad t = k\tau_n. \quad (9)$$

To construct approximation of operator $K_n(T)$ in (6) we just consider the simplest formula $(T = k_n \tau_n)$

$$\tilde{K}_n(T) := (I_n - \tau_n A_n)^{-k_n} \mathscr{F}_n(0) + \Sigma_{l=0}^{k_n-1} (I_n - \tau_n A_n)^{-k_n+l} \frac{\mathscr{F}_n(l\tau_n + \tau_n) - \mathscr{F}_n(l\tau_n)}{\tau_n} \tau_n. \qquad (10)$$

Let us remark that we have some interesting relation between (10) and (9). Namely, let us put $u_n^0 = 0$ in (9) and apply operator A_n. We have integrating by parts $(t = k\tau_n)$

$$A_n \overline{U}_n(t) = \Sigma_{l=0}^{k-1} \tau_n A_n (I_n - \tau_n A_n)^{-k+l} \mathscr{F}_n(l\tau_n + \tau_n) f_n =$$

$$= \Sigma_{l=0}^{k-1} (I_n - \tau_n A_n)^{-k+l} \mathscr{F}_n(l\tau_n + \tau_n) f_n - \Sigma_{j=0}^{k-1} (I_n - \tau_n A_n)^{-k+j} \mathscr{F}_n(j\tau_n) f_n +$$

$$+ (I_n - \tau_n A_n)^{-k} \mathscr{F}_n(0) f_n - (I_n - \tau_n A_n)^{-0} \mathscr{F}_n(k\tau_n) =$$

$$(I_n - \tau_n A_n)^{-k} \mathscr{F}_n(0) f_n + \Sigma_{l=0}^{k-1} (I_n - \tau_n A_n)^{-k+l} \frac{\mathscr{F}_n(l\tau_n + \tau_n) - \mathscr{F}_n(l\tau_n)}{\tau_n} f_n \tau_n - \mathscr{F}_n(t) f_n. \qquad (11)$$

THEOREM 5.2. *Let C_0-semigroup $\exp(\cdot A_n)$ be positive and compact. Assume also that $\mathscr{F}_n(t) \geq 0, \mathscr{F}_n(t)'_t \geq 0$ for any $t \in [0, T]$. Define operator $\tilde{K}_n(T)$ as in (10). Then $r(\tilde{K}_n(T)) < 1$.*

Proof. The operator $\tilde{K}_n(T)$ is positive and compact , so by Krein -Rutman Theorem there are $\lambda_0 \geq 0$ and $0 \preceq f_n^0 \neq 0$ such that $\tilde{K}_n(T) f_n^0 = \lambda_0 f_n^0$, and, moreover, $r(\tilde{K}_n(T)) = \lambda_0$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Assume now in contradiction that $\lambda_0 \geq 1$. Let us consider the operator in (11) with $f_n = f_n^0$. One gets that $A_n\overline{U}_n(T)$ is positive for positive f_n^0. So if we apply $(-A_n)^{-1}$, then because of positiveness of C_0-semigroup $\exp(\cdot A_n)$ the resolvent $(-A_n)^{-1}$ is also positive and $(-A_n)^{-1}A_n\overline{U}_n(T) \succeq 0$, which means that $0 \succeq \overline{U}_n(T)$. From the other hand from (9) it follows that $\overline{U}_n(T) \succeq 0$ for $f_n^0 \succeq 0$ and $u_n^0 = 0$. This means that $\overline{U}_n(T) = 0$, which can happen just in case of $f_n^0 = 0$. But this contradicts to $f_n^0 \neq 0$. The Theorem is proved.

¿From Theorem 5.2 it follows that one can organize the process

$$\tilde{f}_{n,j+1} = \tilde{K}_n(T)\tilde{f}_{n,j} - A_n u_n^T,\ j = 0, 1, \ldots, \tag{12}$$

which converge $\tilde{f}_{n,j} \to \tilde{f}_n$ as $j \to \infty$, where $\tilde{f}_n$ is a solution of problem

$$\tilde{f}_n = \tilde{K}_n(T)\tilde{f}_n - A_n u_n^T. \tag{13}$$

Theorem 5.3. *Let C_0-semigroups $\exp(\cdot A_n)$ be positive and analytic. Assume also that functions $\mathscr{F}_n(\cdot), \mathscr{F}(\cdot) \in C^1([0,T])$ and they converge $\mathscr{F}_n(t)' \to \mathscr{F}(t)'$ uniformly in $t \in [0,T]$. Let conditions $(A), (B_1')$ be satisfied and $u_n^0 \to u^0, u_n^T \to u^T$ in a such way that $G_n(T) \to G(T)$. Assume also that $\mathcal{N}(I - K(T)) = \{0\}$, operator $(\lambda I - A)^{-1}$ is compact and $(\lambda I_n - A_n)^{-1} \to (\lambda I - A)^{-1}$ compactly and $\mathscr{F}_n(\cdot) \in C^3([0,T])$ and $|\mathscr{F}_n(t)'''_{ttt}| \leq$ constant, $t \in [0,T]$. Then solutions of the problems (13) exist and converge to the solution of the problem (4), i.e. $\tilde{f}_n \to f$ as $n \to \infty$.*

Proof. If $\tilde{K}_n(T) \to K(T)$ compactly, then the statement of the Theorem follows the same way as in the Step 4 of Theorem 5.1.

So we are going to show that $\tilde{K}_n(T) \to K(T)$ compactly. To do this it is enough to prove that $\|\tilde{K}_n(T) - K_n(T)\| \to 0$ as $n \to \infty$, since the statement $K_n(T) \to K(T)$ compactly is already proved in Theorem 5.1. One can write

$$K_n(T) - \tilde{K}_n(T) = \exp(TA_n)\mathscr{F}_n(0) - (I_n - \tau_n A_n)^{-k_n}\mathscr{F}_n(0) +$$

$$+ \int_0^T \exp((T-s)A_n)\mathscr{F}_n(s)'_s ds - \Sigma_{l=0}^{k_n-1}(I_n - \tau_n A_n)^{-k_n+l}\,\frac{\mathscr{F}_n(l\tau_n + \tau_n) - \mathscr{F}_n(l\tau_n)}{\tau_n}\,\tau_n,$$

where $k_n\tau_n = T$. In [1], p. 765, it is proved under condition (B_1) that $\|\exp(tA_n) - (I_n - \tau_n A_n)^{-k_n}\| \leq C\frac{\tau_n}{t}e^{\omega t}$ as $k_n \to \infty$ and $\tau_n k_n = t$. Let us consider now the difference

$$\Sigma_{l=0}^{k_n-1}\int_{l\tau_n}^{(l+1)\tau_n} \exp((T-s)A_n)\mathscr{F}_n(s)'_s ds - \Sigma_{l=0}^{k_n-1}(I_n - \tau_n A_n)^{-k_n+l}\,\frac{\mathscr{F}_n(l\tau_n + \tau_n) - \mathscr{F}_n(l\tau_n)}{\tau_n}\,\tau_n.$$

To finish with the demonstration we have to use

$$\pm\,\Sigma_{l=0}^{k_n-1}\int_{l\tau_n}^{(l+1)\tau_n} \exp((T-s)A_n)\,\frac{\mathscr{F}_n(l\tau_n + \tau_n) - \mathscr{F}_n(l\tau_n)}{\tau_n}\,ds$$

terms. Indeed, it is easy to show that difference

$$\Sigma_{l=0}^{k_n-1}\int_{l\tau_n}^{(l+1)\tau_n}\exp((T-s)A_n)\mathscr{F}_n(s)'_s ds-\Sigma_{l=0}^{k_n-1}\int_{l\tau_n}^{(l+1)\tau_n}\exp((T-s)A_n)\,\frac{\mathscr{F}_n(l\tau_n+\tau_n)-\mathscr{F}_n(l\tau_n)}{\tau_n}\,ds$$

converge to zero as $k_n \to \infty$ and $T = k_n\tau_n$, since

$$\int_{l\tau_n}^{(l+1)\tau_n}\exp((T-s)A_n)\Big(\mathscr{F}_n(s)'_s-\frac{\mathscr{F}_n(l\tau_n+\tau_n)-\mathscr{F}_n(l\tau_n)}{\tau_n}\Big)ds$$

is estimated by $C\int_{l\tau_n}^{(l+1)\tau_n}|\mathscr{F}_n(s)'_s-\frac{\mathscr{F}_n(l\tau_n+\tau_n)-\mathscr{F}_n(l\tau_n)}{\tau_n}|ds = O(\tau_n^2)$. The second term from $\pm$ construction could be estimated as

$$\|\Sigma_{l=0}^{k_n-1}\int_{l\tau_n}^{(l+1)\tau_n}\Big(\exp((T-s)A_n)-(I_n-\tau_nA_n)^{-k_n+l}\Big)ds\,\frac{\mathscr{F}_n(l\tau_n+\tau_n)-\mathscr{F}_n(l\tau_n)}{\tau_n}\|\le$$

$$\le C\,\Sigma_{l=0}^{k_n-2}\int_{l\tau_n}^{(l+1)\tau_n}\|\exp((T-s)A_n)-\exp((T-l\tau_n)A_n)\|ds\,+\,C\tau_n+$$

$$C\,\Sigma_{l=0}^{k_n-1}\|\exp((T-l\tau_n)A_n)-(I_n-\tau_nA_n)^{-k_n+l}\|\,\tau_n\le C\Big(\Sigma_{l=0}^{k_n-2}\tau_n/(k_n-l-1)+\Sigma_{l=0}^{k_n-1}\tau_n/(k_n-l)\Big).$$

We took into an account here also that $\|\exp((T-s)A_n)-\exp((T-l\tau_n)A_n)\|\le C\,\frac{\tau_n}{k_n\tau_n-l\tau_n-\tau_n}$ for any $s\in[l\tau_n,(l+1)\tau_n]$, $0\le l\le k_n-2$. The Theorem is proved.

ACKNOWLEDGMENTS

The part of this article was done during second author's visits to National Central University of Taiwan, Taiwan, and to University of Sao Paulo, Sao Carlos, Brazil. The supports of grant from Russian Foundation for Basic Research (01-01-00398), grant NSC 91-2115-M-008-003 from the National Science Council of Taiwan and grant 02/10699-7 from Brazil are gratefully acknowledged.

REFERENCES

1. H. Fujita and A. Mizutani. On the finite element method for parabolic equations, I; approximation of holomorphic semi-groups. *J. Math. Soc. Japan*, 28(4):749–771, 1976.
2. Guidetti D., Karasozen B. and Piskarev S. Approximation of abstract differential equations. To appear in Journal of Mathematical Science. 2004.
3. T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
4. S. G. Krein. *Linear differential equations in Banach space*. American Mathematical Society, Providence, R.I., 1971. Translated from the Russian by J. M. Danskin, Translations of Mathematical Monographs, Vol. 29.

5. C.-C. Kuo. On α-times integrated C-semigroups and the abstract Cauchy problem, Studia Math. 142 (2000), 201-217.

6. Y.-C. Li and S.-Y. Shaw. On generators of integrated C-semigroups and C-cosine functions, Semigroup Forum 47 (1993), 29-35.

7. Y.-C. Li and S.-Y. Shaw. N-times integrated C-semigroups and the abstract Cauchy problem, Taiwanese J. Math. 1 (1997), 75-102.

8. S. Piskarev. Approximation of holomorphic semigroups. *Tartu Riikl. Ül. Toimetised*, (492):3–14, 1979.

9. S. I. Piskarev, "Error estimates in the approximation of semigroups of operators by Padé fractions," *Izv. Vuzov, Mat.*, **4**, 33–38 (1979).

10. S. Piskarev. Approximation of positive C_0-semigroups of operators. *Differentsial' nye Uravneniya*, 27(7):1245–1250, 1287, 1991.

11. A. I. Prilepko and A. B. Kostin. An estimate for the spectral radius of an operator and the solvability of inverse problems for evolution equations. *Rossiĭskaya Akademiya Nauk. Matematicheskie Zametki.* 53, 1: 89–94,1993.

12. A.I. Prilepko, D. G. Orlovsky, I.A. Vasin. Methods for solving inverse problems in mathematical physics. Pure and Applied Mathematics, Marcel Dekker. 231. New York, NY: Marcel Dekker. 2000.

13. A. I. Prilepko and I. V. Tikhonov. Reconstruction of the inhomogeneous term in an abstract evolution equation. Izv. Ross. Akad. Nauk Ser. Mat. **58** (1994), no. 2, 167–188

14. T. Ushijima. Approximation theory for semi-groups of linear operators and its application to approximation of wave equations. *Japan. J. Math. (N.S.)*, 1(1):185–224, 1975/76.

15. G. Vainikko. *Funktionalanalysis der Diskretisierungsmethoden.* B. G. Teubner Verlag, Leipzig, 1976. Mit Englischen und Russischen Zusammenfassungen, Teubner-Texte zur Mathematik.

16. G. Vainikko. Approximative methods for nonlinear equations (two approaches to the convergence problem). *Nonlinear Anal.*, 2:647–687, 1978.

17. V. V. Vasilev, S. G. Krein, and S. Piskarev. Operator semigroups, cosine operator functions, and linear differential equations. In *Mathematical analysis, Vol. 28 (Russian)*, Itogi Nauki i Tekhniki, pages 87–202, 204. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1990. Translated in J. Soviet Math. **54** (1991), no. 4, 1042–1129.

18. Vasiliev V.V., Piskarev S.I. Differential Equations in Banach Spaces. Theory of Cosine Operator-Functions. To appear in Journal of Mathematical Science. 2004.

19. J. Voigt. On the convex compactness property for the strong operator topology. *Note Mat.*, **12**, 259–269 (1992).

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC