CADERNOS DE MATEMÁTICA **05**, 1–15 May (2004)
ARTIGO NÚMERO SMA#185

Orbit Structure of certain $\mathbb{R}^2$-actions on the solid torus

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We describe in this paper the orbit structure of C^2-actions of $\mathbb{R}^2$ on the solid torus $S^1 \times D^2$ having $S^1 \times \{0\}$ and $S^1 \times \partial D^2$ as the only compact orbits and $S^1 \times \{0\}$ as their singular set. May, 2004 ICMC-USP

Key Words: foliation, action of $\mathbb{R}^2$, solid torus, circle orbit.

1. INTRODUCTION

A foliation on a compact orientable 3-dimensional manifold N can be obtained by considering the orbits of an action φ of $\mathbb{R}^2$ on N. The geometric description and characterization of locally free C^2-actions, that is, when all the orbits are of codimension 1 in N, are given in [4], [10] and [1]. To our knowledge, the case when φ is not locally free has no been dealt with in the literature. The aim of this paper is to initiate this study.

We consider here a family of C^2-actions φ of $\mathbb{R}^2$ on the solid torus $N = S^1 \times D^2$ having $\mathcal{O}_0 = S^1 \times \{0\}$ and $\mathcal{O} = S^1 \times \partial D^2$, as the only compact orbits and with singular set (orbits with codimension greater than 1) $\text{Sing}(\varphi) = S^1 \times \{0\}$. We denote the set of all the above actions by $\mathcal{A}$, $\mathcal{O}_p$ the *φ-orbit* of p, G_p the *isotropy group* of p, and G_0 and G the isotropy groups of $\mathcal{O}_0$ and $\mathcal{O}$, respectively.

The possible φ-orbits are homeomorphics to a point, a line, a circle, a cylinder, a plane or a torus. In order to describe the asymptotic behavior of orbits, the topological concept

* Partially supported by FAPESP of Brazil Grant 98/13400-5.
† Partially supported by FAPESP of Brazil Grant 99/11311-8 and 02/09425-0.

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of *limit set* of $\mathcal{O}_p$, denoted by $\lim \mathcal{O}_p$, is essential. We show in Proposition 2.2 that $\mathcal{O}_0 \subset \lim \mathcal{O}_p$ for each $p \in N \setminus \mathcal{O}$. Consequently, the set C of points $p \in N$ such that $\mathcal{O}_p$ is a $S^1 \times \mathbb{R}$-orbit, is a disjoint union of $C_0 = \{ p \in C \,;\, G_p \subset G_0^0 \}$ and $C_1 = \{ p \in C \,;\, G_p \cap G_0^0 = \{0\} \}$, where G_0^0 is the connected component of G_0 that contains the origin.

We give in this paper a complete geometric description of the orbit structure of $\varphi \in \mathcal{A}$ in the complement of S_Γ (the solid k-tube of φ at $\mathcal{O}_0$ associated to k-tube Γ, see Definition 3.3). More precisely,

Theorem A. *If $\varphi \in \mathcal{A}$, then there exist an integer $k \geq 0$ and a φ-invariant neighborhood U of $\mathcal{O}$, homeomorphic to $T^2 \times (0,1]$, such that Γ, the frontier of U, is a k-tube of φ at $\mathcal{O}_0$ and all the orbits inside $U \setminus \mathcal{O}$ have the same topological type. Furthermore, one of the following cases occurs for each $p \in U \setminus \mathcal{O}$:*

(1) *$\mathcal{O}_p$ is a $S^1 \times \mathbb{R}$-orbit in C_0, $C_1 = \emptyset$ and $\lim \mathcal{O}_p = \mathcal{O}_0 \cup \mathcal{O}$. In particular, $U = N \setminus \mathcal{O}_0$.*

(2) *$\mathcal{O}_p$ is a $S^1 \times \mathbb{R}$-orbit in C_1, $C_0 = \emptyset$ and $\lim \mathcal{O}_p = \Gamma \cup \mathcal{O}$.*

(3) *$\mathcal{O}_p$ is a $\mathbb{R}^2$-orbit dense in $U \cup \Gamma$, and $\lim \mathcal{O}_p = \Gamma \cup U$.*

In case (1) of Theorem A, $k = 0$ and $\Gamma = \mathcal{O}_0 = S_\Gamma$. When $k > 0$, the orbit structure of φ in the interior of S_Γ appears to be very complicated, although in this case the above theorem states that $C_0 = \emptyset$.

We prove in Proposition 3.1 the existence of a closed embedded 2-disk Σ in N with $\partial\Sigma \subset \mathcal{O}$ which is in general position with respect to $\mathscr{F}_\varphi$, the foliation defined by φ. Furthermore, the induced foliation on Σ is given by a vector field Z_φ.

Theorem A′. *If $\varphi \in \mathcal{A}$, then there exist an integer $k \geq 0$ and a Z_φ-invariant neighborhood V of $\partial\Sigma$ in Σ homeomorphic to $S^1 \times (0,1]$ such that Γ, the frontier of V in Σ, is a k-petal of Z_φ at $\mathcal{O}_0 \cap \Sigma$ and all orbits inside $V \setminus \partial\Sigma$ have the same topological type. Furthermore, one of the following cases occurs for each $p \in V \setminus \partial\Sigma$:*

(1) *$\mathcal{O}_p(Z_\varphi)$ is periodic and $V = \Sigma \setminus (\mathcal{O}_0 \cap \Sigma)$.*

(2) *$\mathcal{O}_p(Z_\varphi)$ is homeomorphic to $\mathbb{R}$ and $\alpha(p) \cup \omega(p) = \partial\Sigma \cup \Gamma$.*

The concept of k-petal is analogous to the k-tube (see Definition 3.2). In case (1) of Theorem A′, $k = 0$. Theorems A and A′ are related as follows. Theorem A (1) is satisfied if and only if and Theorem A′ (1) is true, and (2) or (3) in Theorem A is true if and only if (2) in Theorem A′ is true. We shall prove Theorems A and A′ simultaneously.

The paper is organized as follows. We study in the next section the action $\varphi \in \mathcal{A}$ in neighborhoods of compact orbits and obtain topological and asymptotic properties of orbits in C_0 and C_1. In Section 3 we introduce the concept of general position, show the existence of an embedded 2-disk Σ in N which is in general position with respect to the foliation defined by φ, and give the proofs of Theorems A and A'.

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The results of our investigation can be used to study 3-manifolds that admit a Heegaard splitting of genus, since these are obtained by gluing two copies of $S^1 \times D^2$ by an diffeomorphism of $\partial(S^1 \times D^2)$.

ACKNOWLEDGMENTS

The authors would like to thank J. L. Arraut for suggesting the problem and for many stimulating conversations.

2. PROPERTIES OF ACTIONS IN $\mathcal{A}$

We start by studying the action $\varphi \in \mathcal{A}$ in neighborhoods of compact orbits. We then establish some properties that will be used in the proof of the main theorems.

We take $D^2 = \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 \leq 1\}$. For each $w \in \mathbb{R}^2 \setminus \{0\}$, φ induces a C^2-flow $(\varphi_w^t)_{t\in\mathbb{R}}$ given by $\varphi_w^t(p) = \varphi(tw, p)$ and its corresponding C^1- vector field X_w is defined by $X_w(p) = D_1\varphi(0,p) \cdot w$. If $\{w_1, w_2\}$ is a basis of $\mathbb{R}^2$, then $\{X_{w_1}, X_{w_2}\}$, called a set of *infinitesimal generators* of φ, determines completely the action φ. Moreover, $[X_{w_1}, X_{w_2}] = 0$.

2.1. The orbit structure in neighborhoods of $\mathcal{O}_0$ and $\mathcal{O}$

Let S be a smooth compact surface and denote by $\mathfrak{X}^r(S)$, $r \geq 1$, the set of C^r vector fields on S. Let $X \in \mathfrak{X}^r(S)$ with a finite number of singularities, all contained in the interior of S when $\partial S \neq \emptyset$. Suppose that $f \in \mathrm{Diff}^r(S)$ preserves the orbits of X. Let M be the manifold obtained from $\mathbb{R} \times S$ by identifying (z,p) with $(z-1, f(p))$. The suspension of f defines a C^r foliation $\mathscr{F}(X,f)$ of M, which is the image of the foliation of $\mathbb{R} \times S$ whose leaves are $\mathbb{R} \times \mathcal{O}_p(X)$ by the quotient map. The foliation $\mathscr{F}(X,f)$ is called the *suspension of X by f*.

In $N = S^1 \times D^2$ we consider coordinates (θ, x) where $\theta \in S^1$ and $x = (x_1, x_2) \in D^2$. In the rest of the paper $p_0 = (\theta_0, x_0) \in \mathcal{O}$, $q_0 = (\theta_0, 0) \in \mathcal{O}_0$ and $S_1 = \partial D_{\theta_0}$, where $D_{\theta_0} = \{\theta_0\} \times D^2$.

Proposition 2.1. *If $\varphi \in \mathcal{A}$, then there exist neighborhoods W_0 of $\mathcal{O}_0$, W_1 of $\mathcal{O}$ and, for $i = 0, 1$, a C^2 diffeomorphism $f_i : A_i \to U_i$ and $Y_i \in \mathfrak{X}^1(A_i)$, where A_0, U_0 and A_1, U_1 are neighborhoods in D_{θ_0} of q_0 and p_0, respectively, such that f_i preserves the orbits of Y_i and $\mathscr{F}_\varphi|_{W_i}$ is topologically equivalent to the suspension of Y_i by f_i.*

The proof of Proposition 2.1 follows from Lemma 2.1 and 2.3 below, using the "charts adapted to the compact orbits".

Let $\varphi \in \mathcal{A}$ and H a 1-dimensional subspace of $\mathbb{R}^2$ such that $\mathbb{R}^2 = H \oplus G_0^0$. Let $\{w_1, w_2\}$ be a basis of $\mathbb{R}^2$ such that w_1 and w_2 generate the subgroups $G_0 \cap H$ and G_0^0, respectively. Note that $\{X_i = X_{w_i}; i = 1, 2\}$ is a set of infinitesimal generators of φ such that $\mathcal{O}_0$ is a periodic orbit of X_1 of period one, and $X_2(q) = 0$ for every $q \in \mathcal{O}_0$. We say that $\{X_1, X_2\}$ is a set of *infinitesimal generators adapted to φ at $\mathcal{O}_0$*.

Let $I^3(\varepsilon) = \{(\theta, x) \in N; |\theta - \theta_0| < \varepsilon \text{ and } |x| < \varepsilon\}$ and $h : V \to I^3(\varepsilon)$ be a chart of N at q_0 such that if $(\theta, x_1, x_2) \in I^3(\varepsilon)$, then the vector fields X_i in this chart can be written as

$$\begin{aligned} X_1(\theta, x_1, x_2) &= \frac{\partial}{\partial \theta}, \\ X_2(\theta, x_1, x_2) &= a(x_1, x_2)\frac{\partial}{\partial \theta} + b(x_1, x_2)\frac{\partial}{\partial x_1} + c(x_1, x_2)\frac{\partial}{\partial x_2}. \end{aligned} \tag{1}$$

The above chart is called *adapted to* $\mathcal{O}_0$ *at* q_0. The vector field

$$\widehat{Y}_2(x_1, x_2) = b(x_1, x_2)\frac{\partial}{\partial x_1} + c(x_1, x_2)\frac{\partial}{\partial x_2} \tag{2}$$

defined on $A_0(\varepsilon) = \{(\theta_0, x) \in N; |x| < \varepsilon\}$ has only q_0 as singularity.

Remark 2. 1. Note that $\{X_1, \widehat{Y}_2\}$ defines a local $\mathbb{R}^2$-action $\widehat{\varphi}$ on $I^3(\varepsilon)$ and $\mathcal{O}_{(\theta,x)}(\widehat{\varphi}) = \mathcal{O}_{(\theta,x)}(h \circ \varphi \circ h^{-1})$ for each $(\theta, x) \in I^3(\varepsilon)$.

Let U_0 be a neighborhood of q_0 in D_{θ_0} such that the Poincaré diffeomorphism of X_1 at q_0, $f_0 : A_0(\varepsilon) \to U_0$, is well defined. Note that f_0 is of class C^2. For $\varepsilon > 0$ sufficiently small, let $\tau : A_0(\varepsilon) \to [0, 1 + \varepsilon)$ be the time of the first return map. Let W_0 denote the interior of $\cup_{q \in \mathrm{cl}(A_0(\varepsilon))}\{X_1^t(q); 0 \le t \le \tau(q)\}$. We have the following result.

LEMMA 2.1. *There exist* $\varepsilon > 0$ *and a neighborhood* W_0 *of* $\mathcal{O}_0$ *such that* f_0 *preserves the orbits of* $\widehat{Y}_2$ *and* $\mathscr{F}_\varphi|_{W_0}$ *is topologically equivalent to the suspension of* $\widehat{Y}_2$ *by* f_0.

Suppose that $\{w_1, w_2\}$ is a basis of $\mathbb{R}^2$ such that w_1 and w_2 generate the isotropy group G of $\mathcal{O}$. Write $X_i = X_{w_i}$; $i = 1, 2$. Note that if $q \in \mathcal{O}$, then for $i \in \{1, 2\}$ the orbit of X_i by q is periodic of period one. Without loss of generality, we can assume that for each $\theta \in S^1$, $\{\theta\} \times \partial D^2$ is an orbit of X_1 and for each $x \in \partial D^2$, $S^1 \times \{x\}$ is an orbit of X_2. We shall say that $\{X_1, X_2\}$ is a set of *infinitesimal generators adapted to* φ *at* $\mathcal{O}$. The holonomy of $\mathcal{O}$ as a leaf of $\mathscr{F}_\varphi$ yields information on the orbit structure of φ in a neighborhood of $\mathcal{O}$.

We consider in N the coordinates system (θ, x) where $\theta \in S^1$, and $x \in D^2$ is given in polar coordinates (ρ, r). Let S_i, $i = 1, 2$, be the circle orbit of X_i through p_0, that is, $S_1 = \{\theta_0\} \times \partial D^2$ and $S_2 = S^1 \times \{x_0\}$. For $\varepsilon \in (0, 1)$ let $A_1(\varepsilon) = \{(\theta_0, \rho, r) \in N; r > 1 - \varepsilon\}$ and $A_2(\varepsilon) = \{(\theta, \rho_0, r) \in N; r > 1 - \varepsilon\}$, where $x_0 = (\rho_0, 1)$. For simplicity of notation we write (ρ, r) and (θ, r) instead of (θ_0, ρ, r) and (θ, ρ_0, r), respectively. Since S_1 (S_2) is transverse to the orbits of X_2 (X_1), there exists $\varepsilon > 0$ such that $A_1(\varepsilon)$ and $A_2(\varepsilon)$ are transverse to the orbits of X_2 and X_1, respectively. Consequently, $A_1(\varepsilon)$, $A_2(\varepsilon)$ and $J_\varepsilon = A_1(\varepsilon) \cap A_2(\varepsilon)$ are transverse to the orbits of φ. Let $\delta > 0$, $I(\delta) = (-\delta, \delta)$ and for $i = 1, 2$ we consider the C^2-maps $h_i : A_i(\varepsilon) \times I(\delta) \to N$ defined by $h_1(q, t) = X_2^t(q)$ and $h_2(q, t) = X_1^t(q)$. There exists $\delta > 0$ such that $h_i|_{A_i(\varepsilon) \times I(\delta)}$ is a diffeomorphism onto its image V_i. Moreover, in the coordinates (h_1^{-1}, V_1) the infinitesimal generators of φ are of

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the form

$$\begin{aligned} X_1(\rho,r,t) &= a_1(\rho,r)\frac{\partial}{\partial t} + b_1(\rho,r)\frac{\partial}{\partial \rho} + c_1(\rho,r)\frac{\partial}{\partial r} \\ X_2(\rho,r,t) &= \frac{\partial}{\partial t}\,, \end{aligned} \tag{3}$$

and in the coordinates (h_2^{-1}, V_2) the infinitesimal generators of φ are of the form

$$\begin{aligned} X_1(\theta,r,t) &= \frac{\partial}{\partial t} \\ X_2(\theta,r,t) &= a_2(\theta,r)\frac{\partial}{\partial t} + b_2(\theta,r)\frac{\partial}{\partial \theta} + c_2(\theta,r)\frac{\partial}{\partial r}. \end{aligned} \tag{4}$$

The maps h_i are called a *cylindrical coordinate system adapted to* $\mathcal{O}$ *at* S_i , $i=1,2$. The vector fields

$$\widehat{X}_1 = b_1(\rho,r)\frac{\partial}{\partial \rho} + c_1(\rho,r)\frac{\partial}{\partial r} \text{ and } \widehat{X}_2 = b_2(\theta,r)\frac{\partial}{\partial \theta} + c_2(\theta,r)\frac{\partial}{\partial r} \tag{5}$$

define a local flow on $A_1(\varepsilon)$ and $A_2(\varepsilon)$, respectively. Furthermore, $S_i \subset A_i(\varepsilon)$ is an orbit or $\widehat{X}_i$, $i=1,2$.

Note that $p_0 \in J_\varepsilon$ and the map $\alpha_i : [0,1] \to A_i(\varepsilon)$ given by $\alpha_i(\tau) = \widehat{X}_i^\tau(p_0)$ is a parametrization of S_i , $i=1,2$. Let $P_i : (J_\varepsilon, p_0) \to (J_\varepsilon, p_0)$ be the Poincaré map of α_i and

$$\mathrm{Hol} : \pi_1(\mathcal{O}, p_0) \cong \mathbb{Z}^2 \to \mathrm{Diff}^2(J_\varepsilon, p_0) \tag{6}$$

the holonomy of $\mathcal{O}$ as a leaf of the foliation $\mathscr{F}_\varphi$. Then $P_i = \mathrm{Hol}([\alpha_i])$.

Remark 2. 2. Note that $\{\widehat{X}_1\,, X_2\}$ and $\{X_1\,, \widehat{X}_2\}$ define two local $\mathbb{R}^2$-actions $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ on $A_1(\varepsilon)\times I(\delta)$ and $A_2(\varepsilon)\times I(\delta)$, respectively, where $\widehat{X}_1$ and $\widehat{X}_2$ are given in Equation 5. Moreover,

$$\mathcal{O}_{(\rho,r,t)}(\widehat{\varphi}_1) = \mathcal{O}_{(\rho,r,t)}(h_1\circ\varphi\circ h_1^{-1}) \text{ and } \mathcal{O}_{(\theta,r,t)}(\widehat{\varphi}_2) = \mathcal{O}_{(\theta,r,t)}(h_2\circ\varphi\circ h_2^{-1}).$$

The following result is a particular case of Lemma 2.4 in [2]. The condition of C^2 differentiability is necessary.

Lemma 2.2. *There exists* $\varepsilon \in (0,1)$ *such that for each* $i \in \{1,2\}$ *one of the cases holds:*

(a) $P_i|_{J_\varepsilon} = \mathrm{id}$; *that is, every* $\widehat{X}_i$*-orbit near* S_i *is periodic.*

(b) *Either* $P_i|_{J_\varepsilon}$ *or* $(P_i|_{J_\varepsilon})^{-1}$ *is a topological contraction, i.e. every* $\widehat{X}_i$*-orbit near* S_i *spirals towards* S_i *.*

Furthermore, if P_1 *(resp.* P_2*) satisfies (a), then* P_2 *(resp.* P_1*) satisfies (b).*

The following lemma completes the proof of Proposition 2.1.

LEMMA 2.3. *There exist $\varepsilon > 0$, a neighborhood W_1 of $\mathcal{O}$, a neighborhood U_1 in D_{θ_0} of S_1 and a C^2-diffeomorphism $f_1 : A_1(\varepsilon) \to U_1$ that preserves the orbits of $\widehat{X}_1$ such that $\mathscr{F}_\varphi|_{W_1}$ is topologically equivalent to the suspension of $\widehat{X}_1$ by f_1.*

Proof. Since $X_2^1(S_1) = S_1$, there exist $\varepsilon > 0$ such that $X_2^1(A_1(\varepsilon)) \subset V_1$. Consequently, the time of first return map $\tau : A_1(\varepsilon) \to [0, 1+\delta)$ is a C^2 map. If $U_1 = \cup_{q\in A_1(\varepsilon)} X_2^{\tau(q)}(q)$, then the C^2 map $f_1 : A_1(\varepsilon) \to U_1$, defined by $f_1(q) = X_2^{\tau(q)}(q)$ is a diffeomorphism that preserves the orbits of $\widehat{X}_1$. Let W_1 be the interior of $\cup_{q\in \mathrm{cl}(A_1(\varepsilon))}\{X_2^t(q); 0 \le t \le \tau(q)\}$. By taking a smaller ε if necessary, f_1 and $\widehat{X}_1$ induces a local diffeomorphism of J_ε, which coincides with P_2. Therefore, the holonomy of $\mathcal{O}$ as a leaf of the foliation obtained by suspension of $\widehat{X}_1$ by f_1 is the same holonomy of $\mathcal{O}$ as a leaf of $\mathscr{F}_\varphi$, which is given by Lemma 2.2. Thus, $\mathscr{F}_\varphi|_{W_1}$ is topologically equivalent to the suspension of $\widehat{X}_1$ by f_1. ▌

Lemma 2.2 yields a natural decomposition of the family $\mathcal{A}$ into a disjoint union:

$$\mathcal{A} = \mathcal{A}_{S^1} \cup \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p,$$

where $\mathcal{A}_{S^1} = \{\varphi \in \mathcal{A}; P_1 = \mathrm{id}\}$, $\mathcal{A}_{\mathbb{R}}^c = \{\varphi \in \mathcal{A}; P_2 = \mathrm{id}\}$ and $\mathcal{A}_{\mathbb{R}}^p = \{\varphi \in \mathcal{A}; P_1, P_2 \neq \mathrm{id}\}$.

LEMMA 2.4. *Let $\varphi \in \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p$. Then there exists an orientation preserving C^2 diffeomorphism $f : S^1 \to S^1$ such that for each $m \in \mathbb{N}$ either $\mathrm{Fix}(f^m) = \emptyset$ or $\mathrm{Fix}(f^m) = S^1$.*

Proof. As $\varphi \in \mathcal{A}_{\mathbb{R}}^c \cup \mathcal{A}_{\mathbb{R}}^p$, there exists a C^2 embedding $F : S^1 \to A_1(\varepsilon)$ such that $S = F(S^1)$ is transverse to $\widehat{X}_1$. Then the diffeomorphism f_1, as in Lemma 2.3, induces a C^2 diffeomorphism $f : S^1 \to S^1$ defined by $f(q) = F^{-1}(S \cap \mathcal{O}_{f_1\circ F(q)}(\widehat{X}_1))$. If $q \in S^1$ is a fixed point of f^m, then $f_1^m(F(q)) \in \mathcal{O}_{F(q)}(\widehat{X}_1)$ with $F(q) \neq p_0$. Without loss of generality we can assume that $F(q) \in J_\varepsilon$. Hence $P_2^m(F(q)) = F(q)$, and it follows from Lemma 2.2 that $P_2^m = \mathrm{id}$, that is, $f^m = \mathrm{id}$. Therefore $f : S^1 \to S^1$ preserves orientation (otherwise it would have exactly two fixed points, see [6, Exercise 11.2.4]). ▌

LEMMA 2.5. *If $\varphi \in \mathcal{A}$, there exists a φ-invariant neighborhood U of $\mathcal{O}$ such that all φ-orbits inside $U \setminus \mathcal{O}$ have the same topological type and one of the following possibilities occurs for each $p \in U \setminus \mathcal{O}$:*

(1) *$\mathcal{O}_p$ is a $S^1 \times \mathbb{R}$-orbit, $\mathrm{cl}(\mathcal{O}_p) \setminus \mathcal{O}_p$ has two connected components, with $\mathcal{O}$ being one of them, and $\varphi \in \mathcal{A}_{S^1} \cup \mathcal{A}_{\mathbb{R}}^c$.*
(2) *$\mathcal{O}_p$ is a $\mathbb{R}^2$-orbit which is dense in U, and $\varphi \in \mathcal{A}_{\mathbb{R}}^p$.*

Proof. A classical result in foliation theory states that the leaves structure of a foliation in the neighborhood of a compact leaf is determined by the holonomy of this leaf, see [3]. The holonomy of the orbit $\mathcal{O}$, given in Lemma 2.2, guarantees that the φ-orbits by points in U_0, the saturated of $J_\varepsilon \setminus \{p_0\}$ by φ, either are all homeomorphics to $S^1 \times \mathbb{R}$, or all homeomorphics to $\mathbb{R}^2$. We take $U = U_0 \cup \mathcal{O}$. If every orbit inside U_0 is homeomorphic to $S^1 \times \mathbb{R}$, then part (1) of the Lemma follows from Lemma 2.3. Assume now that every orbit inside U_0 is homeomorphic to $\mathbb{R}^2$. In this case $\varphi \in \mathcal{A}^p_{\mathbb{R}}$. Let $f \in \mathrm{Diff}^2_+(S^1)$ be the diffeomorphism given in Lemma 2.4 and $\tau(f)$ its rotation number. We claim that $\tau(f) \in \mathbb{R}\backslash\mathbb{Q}$. Otherwise, if $\tau(f) \in \mathbb{Q}$, then f has at least a periodic point. It follows from Lemma 2.4 that $f^m = \mathrm{id}$ for some $m \in \mathbb{N}$. This implies that $P_2^m = \mathrm{id}$, contradicting the fact that P_2 satisfies (b) of Lemma 2.2. Therefore $\tau(f) \in \mathbb{R}\backslash\mathbb{Q}$. Since f is of class C^2, by Denjoy's Theorem [6, Theorem 12.1.1] f is topologically conjugate to the rotation in S^1 given by $R_{\tau(f)}(\theta) = \theta + \tau(f)$. Therefore, the set $\{f^n(q)\}_{n\in\mathbb{Z}_+}$ is dense in S^1 for each $q \in S^1$. Consequently, every orbit inside U_0 is dense in U_0. ▮

2.2. Asymptotic properties

The *limit set* of $\mathcal{O}_p$ is a φ-invariant compact set given by $\lim \mathcal{O}_p = \cap_{i=1}^{\infty} \mathrm{cl}(\mathcal{O}_p \setminus K_i)$, where K_i is a compact subset of $\mathcal{O}_p$, $K_i \subset K_{i+1}$, $\mathcal{O}_p = \cup_{i=1}^{\infty} K_i$ and cl denotes the closure of a set. It is not difficult to show that $\mathrm{cl}(\mathcal{O}_p) = \mathcal{O}_p \cup \lim \mathcal{O}_p$. The notions of minimal and exceptional minimal sets that we use here are the standard ones.

Proposition 2.2. *If $\varphi \in \mathcal{A}$, then:*

(i) $\mathcal{O}$ and $\mathcal{O}_0$ are the only minimal sets of φ;
(ii) $\mathcal{O}_0 \subset \lim \mathcal{O}_p$ for each $p \in N \setminus \mathcal{O}$. Consequently $G_p \subset G_0$.

Proof. (i). Suppose that μ is a minimal set of φ such that $\mu \neq \mathcal{O}$ and $\mathcal{O}_0$. Then μ is also a minimal set of the action φ' of $\mathbb{R}^2$ on $N' = N \setminus (\mathcal{O} \cup \mathcal{O}_0)$ given by $\varphi' = \varphi|_{\mathbb{R}^2 \times N'}$. Consequently, as φ' has no exceptional minimal sets (see [11, Theorem 8]), either $\mu = \mathcal{O}_p$ for some $p \in N'$, or $\mu = N'$. If $\mu = \mathcal{O}_p$, then $\mathcal{O}_p$ is a compact orbit of φ, contradicting the fact that $\varphi \in \mathcal{A}$. If $\mu = N'$, then $\mathrm{cl}(\mu)$, the closure of μ in N, contains $\mathcal{O}$ and $\mathcal{O}_0$. This contradicts the fact of μ be a minimal set of φ. This completes the proof of (i).

(ii). Suppose that (ii) is not true, i.e. there exists $p \in N \setminus \mathcal{O}$ such that $\mathcal{O}_0 \not\subset \lim \mathcal{O}_p$. Since $\mathrm{cl}(\mathcal{O}_p) = \mathcal{O}_p \cup \lim \mathcal{O}_p$, we have $\mathcal{O}_0 \not\subset \mathrm{cl}(\mathcal{O}_p)$. Therefore, as the φ-invariant compact set $\mathrm{cl}(\mathcal{O}_p)$ contains a minimal set μ which by (i) coincides with $\mathcal{O}$, we have $\mathcal{O} \subset \mathrm{cl}(\mathcal{O}_p)$. If $\mathcal{O}_p$ is a $S^1 \times \mathbb{R}$-orbit, then by item (1) of Lemma 2.5, there exits a connected component Δ of $\mathrm{cl}(\mathcal{O}_p) \setminus \mathcal{O}_p$ such that $\mathcal{O} \cap \Delta = \emptyset$. Let $\mu' \subset \Delta$ be a minimal set of φ. By (i), $\mu' = \mathcal{O}_0$, which contradicts the fact that $\Delta \cap \mathcal{O}_0 = \emptyset$. This contradiction shows, in this case, that $\mathcal{O}_0 \subset \mathrm{cl}(\mathcal{O}_p)$. Finally, assume that $\mathcal{O}_p$ is a $\mathbb{R}^2$-orbit. Then the neighborhood U of $\mathcal{O}$, as in Lemma 2.5(2) satisfies $\mathrm{cl}(\mathcal{O}_p) \cap U = U$, and $\mathrm{Front}(U)$ satisfies $\mathrm{Front}(U) \cap \mathcal{O}_0 = \emptyset = \mathrm{Front}(U) \cap \mathcal{O}$. So, $\mathrm{Front}(U)$ contains a minimal set which is neither $\mathcal{O}_0$ nor $\mathcal{O}$, contradicting (i). This concludes the proof of (ii). ▮

Remark 2. 3. For each $\varphi \in \mathcal{A}$, let C be the set of points $p \in N$ such that $\mathcal{O}_p$ is a $S^1 \times \mathbb{R}$-orbit. Since $G_p \subset G_0$, it follows that C is a disjoint union of the sets C_0 and C_1, with:

$$C_0 = \{\, p \in C\, ;\, G_p \subset G_0^0 \,\} \ \text{ and } \ C_1 = \{\, p \in C\, ;\, G_p \cap G_0^0 = \{0\}\}.$$

Let $\varphi \in \mathcal{A}$, $q \in C_0$ and $\{w_1, w_2\}$ a base of $\mathbb{R}^2$ such that the vector fields $\{X_1 = X_{w_1}, X_2 = X_{w_2}\}$ are infinitesimal generators adapted to φ at $\mathcal{O}_0$ with w_2 a generate of G_q.

LEMMA 2.6. *Given $p \in \mathcal{O}_0$ and a neighborhood V_p of p, there exists a neighborhood $U_p \subset V_p$ of p such that $\mathcal{O}_{q'}(X_2) \subset V_p$ for all $q' \in \mathcal{O}_q \cap U_p$.*

Proof. Since the orbits of X_2 by points of $\mathcal{O}_q$ are periodic of period 1, it is sufficient to show that there exists a neighborhood U_p of p such that $X_2^t(U_p) \subset V_p$, for all $t \in [0,1]$. Since $X_2^t(p) = p$, $t \in [0,1]$, then there exist a neighborhood $U_{p,t}$ of p and an open interval $I_t \subset [0,1]$ that contains t such that $X_1^s(U_{p,t}) \subset V_p$, for all $s \in I_t$. There exists a subfamily $\{I_{t_i}; i = 1, \ldots, k\}$ of $\{I_t; t \in [0,1]\}$ that covers $[0,1]$. The statement now follows by taking $U_p = \cap_{i=1}^k U_{p,t_i}$. ▌

PROPOSITION 2.3. *If $\varphi \in \mathcal{A}$, then $\lim \mathcal{O}_q = \mathcal{O}_0 \cup \mathcal{O}$ for every $q \in C_0$.*

Proof. Let $q \in C_0$. Since $G_q \subset G_0^0$, we take as $\{X_1, X_2\}$ infinitesimal generators adapted to φ at $\mathcal{O}_0$ such that w_2 generates G_q, and (h, V_{q_0}) a chart adapted to $\mathcal{O}_0$ at q_0. By Lemma 2.6, there exists a neighborhood U_{q_0} of q_0 such that $\mathcal{O}_{q'}(X_2) \subset V_{q_0}$ for every $q' \in \mathcal{O}_q \cap U_{q_0}$. We first show that $\lim \mathcal{O}_q$ (which in this case is equal to $\mathrm{cl}(\mathcal{O}_q) \setminus \mathcal{O}_q$) has two connected components Δ_0 and Δ with $\mathcal{O}_0 \subset \Delta_0$.

By Lemma 2.1, $\mathscr{F}|_\varphi$ is the suspension, in a neighborhood of $\mathcal{O}_0$, of $\widehat{Y}_2$ by the Poincaré diffeomorphism $f_0 : A_0(\varepsilon) \to U_0$ of X_1 at q_0.

Let $\varepsilon > 0$ such that $A_0(\varepsilon), U_0 \subset U_{q_0}$. Since $\mathcal{O}_0 \subset \mathrm{cl}(\mathcal{O}_q)$, there exists $q' \in \mathcal{O}_q \cap A_0(\varepsilon)$ such that $\mathcal{O}_{q'}(X_2)$ and $\mathcal{O}_{f_0(q')}(X_2) \subset V_{q_0}$. Consequently, the $\widehat{Y}_2$-orbits γ_1 and γ_2 by q' and $f_0(q')$, respectively, are periodic, contained in $A_0(\varepsilon)$ and satisfy $f_0(\gamma_1) = \gamma_2$. Furthermore, $f_0(\gamma_1) \neq \gamma_2$, otherwise $\mathcal{O}_{q'}$ would be a compact φ-orbit. We can assume that γ_2 is contained in the interior of γ_1 in $A_0(\varepsilon)$. Let $A \subset A_0(\varepsilon)$ be the ring limited by γ_1 and γ_2 and $B \subset \mathcal{O}_q$ the closed cylinder whose boundary is $\gamma_1 \cup \gamma_2$. Then $T = A \cup B$ is an embedded topological torus containing $\mathcal{O}_0$ in its interior and such that $N \setminus T$ has two connected components N_0 and N_1, with $\mathcal{O}_0 \subset N_0$, see Figure 1.

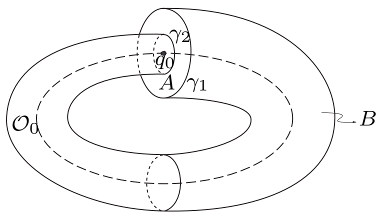

FIG. 1.

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If $\mathcal{O}^+ = \{X_1^t(\gamma_1); t > 0\}$ and $\mathcal{O}^- = \{X_1^t(\gamma_1); t < 0\}$, then $\mathcal{O}_q = \mathcal{O}^- \cup \gamma_1 \cup \mathcal{O}^+$. Since X_1 is transverse to $A_0(\varepsilon) \supset A$, then $\mathcal{O}^+ \subset N_0$ and $\mathcal{O}^- \subset N_1$. Consequently, $\mathrm{cl}(\mathcal{O}_q) \setminus \mathcal{O}_q$ has two connected components, Δ_0 and Δ. Since $\mathcal{O}_0 \subset \mathrm{cl}(\mathcal{O}_q)$, we can assume that $\mathcal{O}_0 \subset \Delta_0$.

Finally, we show that $\Delta = \mathcal{O}$ and $\Delta_0 = \mathcal{O}_0$. We have $\mathcal{O} \subset \Delta$, otherwise φ has a minimal set $\mu \subset \Delta$, with $\mu \neq \mathcal{O}_0$ and $\mathcal{O}$, contradicting Proposition 2.2(i). It follows then from Lemma 2.5(1) that $\Delta = \mathcal{O}$. We consider a sequence $\{\gamma_k\}_{k\in\mathbb{N}}$, where $\gamma_k = f_0(\gamma_{k-1})$ is a closed orbit of $\widehat{Y}_2$. If $\mathrm{int}(\gamma_k)$ denotes the interior of γ_k in $A_0(\varepsilon)$, then $\gamma_k \subset \mathrm{int}(\gamma_{k-1})$. We claim that $\cap_{k\in\mathbb{N}}\mathrm{int}(\gamma_k) = \{q_0\}$, otherwise $\cap_{k\in\mathbb{N}}\mathrm{int}(\gamma_k)$ is an open 2-disk in $A_0(\varepsilon)$ whose boundary γ is a closed orbit of $\widehat{Y}_2$ such that $f_0(\gamma) = \gamma$, and thus the φ-orbit that contains γ would to be homeomorphic to T^2. Therefore, $\cap_{k\in\mathbb{N}}\mathrm{int}(\gamma_k) = \{q_0\}$. Since φ is given by a suspension, in a neighborhood of $\mathcal{O}_0$, it follows that $\Delta_0 = \mathcal{O}_0$, which concludes the proof. ▌

Remark 2. 4. Proposition 2.3 is not necessarily true when $q \in C_1$. For example, the closure of a $S^1 \times \mathbb{R}$-orbit in a k-tube of φ at $\mathcal{O}_0$ (see Definition 3.3) does not contain $\mathcal{O}$.

2.3. C_0 versus C_1

We need the following result from [3] (Corollary 2, p. 176).

LEMMA 2.7. *Let $\varphi : G \times M \to M$ be a locally free action of a simply connected Lie group G on M, a C^∞ manifold with $dim(M) = dim(G) + 1 \geq 3$. If $\mathcal{O}$ is an orbit of φ and $i : \mathcal{O} \to M$ is the canonic immersion, then $i_* : \pi_1(\mathcal{O}) \to \pi_1(M)$ is injective, i.e. if γ is a closed curve in $\mathcal{O}$ homotopic to a constant in M, then γ is homotopic to a constant in $\mathcal{O}$.*

THEOREM 2.1. *Suppose that $\mathcal{O}_q$ is a $S^1 \times \mathbb{R}$-orbit of $\varphi \in \mathcal{A}$ and let $\gamma \subset \mathcal{O}_q$ be a simple closed curve which is not homotopic to a constant in $\mathcal{O}_q$. Then,*

(a) $\mathcal{O}_q \subset C_0$ if and only if γ is homotopic to a constant in N.
(b) if $\varphi \in \mathcal{A}_{S^1}$, then $C = C_0 = N \setminus (\mathcal{O}_0 \cup \mathcal{O})$.
(c) $\mathcal{O}_q \subset C_0$ if and only if $\varphi \in \mathcal{A}_{S^1}$.
(d) if $\mathcal{O}_q \subset C_0$, then γ bounds a closed 2-disk $D \subset N$ such that $\mathcal{O}_0 \cap D \neq \emptyset$.
(e) $\mathcal{O}_q \subset C_1$ if and only if γ is not homotopic to a constant in N.
(f) $\mathcal{O}_q \subset C_1$, if and only if $\varphi \in \mathcal{A}^c_{\mathbb{R}} \cup \mathcal{A}^p_{\mathbb{R}}$.

Proof. (a). Let $\{X_1, X_2\}$ be infinitesimal generators adapted to φ at $\mathcal{O}_0$. If $\mathcal{O}_q \subset C_0$, then we take w_2 as the generator of G_q. Let $p \in \mathcal{O}_0$ and V_p a neighborhood of p. By Proposition 2.2 and Lemma 2.6, there exist a neighborhood U_p of p and a point $q' \in U_p \cap \mathcal{O}_q$ such that $\mathcal{O}_{q'}(X_2) \subset V_p$. We can take V_p sufficiently small such that $\mathcal{O}_{q'}(X_2)$ bounds a closed 2-disk inside V_p, i.e. $\mathcal{O}_{q'}(X_2)$ is homotopic to a constant in

N. Since γ is simple, then $\mathcal{O}_{q'}(X_2)$ and γ (or $-\gamma$) are homotopic in $\mathcal{O}_q$, and therefore γ is homotopic to a constant in N. Conversely, suppose that γ is null homotopic in N and $\mathcal{O}_q \subset C_1$. Since $\mathbb{R}^2 = H \oplus G_0^0$, we can assume that H is generated by G_q. Let $f_0 : A_0(\varepsilon) \to U_0$ be the Poincaré diffeomorphism of X_1 at $q_0 \in \mathcal{O}_0$. Let $V(\varepsilon) = S^1 \times D_\varepsilon$ where $D_\varepsilon = \{(x_1, x_2) \in D^2; x_1^2 + x_2^2 < \varepsilon\}$ and $\pi : V(\varepsilon) \to S^1$ given by $\pi(\theta, x) = \theta$. There exists $\varepsilon > 0$ such that $\pi^{-1}(\theta)$ is transverse to X_1, for each $\theta \in S^1$. Since $\mathcal{O}_0 \subset \mathrm{cl}(\mathcal{O}_q)$, there exist $p \in \mathcal{O}_q \cap A_0(\varepsilon)$ and $n \in \mathbb{Z}$ such that $\mathcal{O}_p(X_1) \subset V(\varepsilon)$ is periodic of period n, i.e. $f_0^n(p) = p$. Then $\pi^{-1}(\theta_0) \cap \mathcal{O}_p(X_1)$ has exactly n elements, and consequently every $\pi^{-1}(\theta)$, $\theta \in S^1$, contains exactly n points of $\mathcal{O}_p(X_1)$. In particular, $\pi(\mathcal{O}_p(X_1)) = \mathcal{O}_0$, and thus $\mathcal{O}_p(X_1)$ is not homotopic to a constant in N. This is not possible since $\mathcal{O}_p(X_1)$ is homotopic to γ (or $-\gamma$) in $\mathcal{O}_q$, which by hypothesis is homotopic to a constant in N. This contradiction show that $\mathcal{O}_q \subset C_0$ and completes the proof of (a).

(b). If $\varphi \in \mathcal{A}_{S^1}$, by Lemma 2.5 and item (a), there exists a φ-invariant neighborhood U of $\mathcal{O}$ such that $U \setminus \mathcal{O} \subset C_0$. It follows from Proposition 2.3 that $U = N \setminus \mathcal{O}_0$.

(c). We assume that $q \in C_0$ and $\varphi \notin \mathcal{A}_{S^1}$. By Proposition 2.3, $\mathcal{O} \subset \mathrm{cl}(\mathcal{O}_q)$, and consequently it follows from Lemma 2.5 that $\varphi \in \mathcal{A}_{\mathbb{R}}^c$. Since $P_2 = \mathrm{id}$, there exists a simple closed curve $\gamma \subset \mathcal{O}_q$ which is neither homotopic to a constant in $\mathcal{O}_q$, nor homotopic to a constant in N, contradicting item (a). This shows that $\varphi \in \mathcal{A}_{S^1}$. Conversely, if $\varphi \in \mathcal{A}_{S^1}$, it follows from (b), that $\mathcal{O}_q \subset C_0$.

(d). Let $q \in C_0$ and φ' be the action of $\mathbb{R}^2$ on $N' = N \setminus (\mathcal{O}_0 \cup \mathcal{O})$ defined by $\varphi'(g, p) = \varphi(g, p)$. By item (a), γ bounds a closed 2-disk $D \subset N$. If $\mathcal{O}_0 \cap D = \emptyset$, by Lemma 2.7, γ is homotopic to a constant in $\mathcal{O}_q$, since $\mathcal{O}_q$ is also a φ'-orbit.

(e)-(f). Since $C = C_0 \cup C_1$ with $C_0 \cap C_1 = \emptyset$, it follows that (e) (resp. (f)) is equivalent to (a) (resp. (c)). ∎

3. PROOF OF THE MAIN RESULTS

Theorems A and A′ are proved simultaneously. We chose Σ a 2-disk embedded in N, in general position with respect to $\mathscr{F}_\varphi$.

DEFINITION 3.1. Let $\varphi \in \mathcal{A}$ and $\mathscr{F}_0$ be the restriction of $\mathscr{F}_\varphi$ to $N \setminus \mathcal{O}_0$. An embedding $g : D^2 \to N$ with $g(0) \in \mathcal{O}_0$ is said to be in *general position* with respect to $\mathscr{F}_\varphi$ if g is transverse to $\mathscr{F}_\varphi$ at $g(0)$ and, for every distinguished map f of $\mathscr{F}_0$, the map $(f \circ g)|_{D^2 \setminus \{0\}}$ is locally of Morse type. The submanifold $g(D^2)$ is said to be in *general position* with respect to $\mathscr{F}_\varphi$.

Remark 3. 1. If $g : D^2 \to N$ with $g(0) \in \mathcal{O}_0$ in general position with respect to $\mathscr{F}_\varphi$, then g induces a foliation $\mathscr{F}^*$ in $\Sigma = g(D^2)$ whose leaves are the connected components of the intersection of the leaves of $\mathscr{F}_\varphi$ with $g(D^2)$. Furthermore, $\mathscr{F}^*$ has a finite number of singularities that are centers or saddles, except maybe for $g(0)$. The singularities of $\mathscr{F}^*$ are the points where Σ is tangent to a leaf of $\mathscr{F}_\varphi$.

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PROPOSITION 3.1. *If $\varphi \in \mathcal{A}$, then there exists Σ a closed 2-disk embedded in N in general position with respect to $\mathscr{F}_\varphi$, such that $\partial\Sigma \subset \mathcal{O}$ and the foliation in Σ induced by $\mathscr{F}_\varphi$ is given by a vector field $Z_\varphi \in \mathfrak{X}^2(\Sigma)$.*

Proof. Let $j : D^2 \to N$ be the inclusion such that $j(D^2) = D_{\theta_0} = \{\theta_0\} \times D^2$, for some $\theta_0 \in S^1$. Given $\varepsilon > 0$ and an integer $r \geq 2$, with an adaptation of Haefliger's techniques [5], we obtain a C^∞-embedding $g : D^2 \to N$ in general position with respect to $\mathscr{F}_\varphi$, that is ε-close to j in the C^r-topology, and that coincides with j in neighborhoods of ∂D^2 and 0.

The foliation $\mathscr{F}^*$, in a neighborhood of $q_0 = (\theta_0, 0)$, is given by the vector field $\widehat{Y}_2$, as in the Equation 2. Thus, by Remark 3.1, $\mathscr{F}^*$ is C^2 locally orientable, and consequently by [3, Proposition 2, pg. 123], $\mathscr{F}^*$ is C^2 orientable. Set $\Sigma = g(D^2)$. It follows that the foliation $\mathscr{F}^*$ is given by a vector field $Z_\varphi \in \mathfrak{X}^2(\Sigma)$. ▮

Remark 3. 2. There exists $\varepsilon > 0$ such that $A_0(\varepsilon)$ and $A_1(\varepsilon)$ are contained in Σ. We can take Z_φ such that $Z_\varphi|_{A_0(\varepsilon)} = \widehat{Y}_2$ and $Z_\varphi|_{A_1(\varepsilon)} = \widehat{X}_1$, where $\widehat{Y}_2$ and $\widehat{X}_1$ are the vector fields given in Equations (2) and (5), respectively. Furthermore, by a small isotopy of Σ in a neighborhood of each singularity different of q_0, we may assume that no two singularities of Z_φ in $\Sigma \setminus \{q_0\}$ are on the same leaf of $\mathscr{F}_\varphi$. Thus we can assume that there is no connection between two different saddles of Z_φ in $\Sigma \setminus \{q_0\}$.

As a consequence of Theorem 2.1(a), we have:

COROLLARY 3.1. *If $Z_\varphi \in \mathfrak{X}^2(\Sigma)$ is a vector field induced by $\varphi \in \mathcal{A}$ and γ is a closed orbit of Z_φ whose interior in Σ contains q_0, then the φ-orbit that contains γ is a $S^1 \times \mathbb{R}$-orbit that is contained in C_0.*

A *limit cycle* Γ of Z_φ is a non-empty limit set $\alpha(\gamma)$ or $\omega(\gamma)$, of some orbit γ of Z_φ such that $\gamma \cap \Gamma = \emptyset$, and which does not consist of a singular point. By Poincaré-Bendixson Theorem, if $q_0 \notin \Gamma$, then Γ is either a periodic orbit or a graph of Z_φ. In the last case, by taking Z_φ as in Remark 3.2, Γ is the union of a saddle p with one or two self-connections of p. Since Γ is connected, if $\Gamma \neq \partial\Sigma$, then $\Sigma \setminus \Gamma$ has at least two connected components, a neighborhood of $\partial\Sigma$ being one of them.

PROPOSITION 3.2. *Let $\varphi \in \mathcal{A}^c_{\mathbb{R}} \cup \mathcal{A}^p_{\mathbb{R}}$ and $\Gamma \neq S_1$ be a limit cycle of Z_φ. Then $q_0 \in \Gamma$ and $\Gamma \setminus Sing(Z_\varphi)$ has a finite number of connected components, each one of them contend q_0 in its closure.*

Proof. We start by showing that $q_0 \in \Gamma$. Let $\mathcal{O}_\Gamma$ be the φ-orbit containing Γ, and $q \in \Gamma$ a regular point of Z_φ. Since Γ is a limit cycle, the holonomy transformation of $\mathcal{O}_\Gamma$ at q associated to Γ is not trivial. Consequently, Γ is not homotopic to a constant in $\mathcal{O}_\Gamma$. If $q_0 \notin \Gamma$, we claim that $q_0 \in R(\Gamma)$, where $R(\Gamma)$ is the union of the connected components of $\Sigma \setminus \Gamma$ that do not contain $\partial\Sigma$. Indeed, if $q_0 \notin R(\Gamma)$, then Γ is homotopic to a constant in $\Sigma \setminus \{q_0\}$, and hence, homotopic to a constant in $N \setminus (\mathcal{O}_0 \cup \mathcal{O})$. Consequently, by Lemma 2.7, Γ is null homotopic in $\mathcal{O}_\Gamma$. But this contradicts the fact that Γ is a limit cycle of Z_φ. Therefore, $q_0 \in R(\Gamma)$. By taking Z_φ as in Remark 3.2, Γ is either a simple closed curve or the union of two simple closed curves Γ_1 and Γ_2 with $\Gamma_1 \cap \Gamma_2 = \{q\} = \mathrm{Sing}(Z_\varphi) \cap \Gamma$. Since $q_0 \in R(\Gamma)$, it follows that either Γ or Γ_i, for some $i = 1, 2$, is not homotopic to a constant in $\mathcal{O}_\Gamma$. However these are homotopic to a constants in N. Thus, $\mathcal{O}_\Gamma$ is a $S^1 \times \mathbb{R}$-orbit, and by Theorem 2.1(a), $\mathcal{O}_\Gamma \subset C_0$. Hence, by item (c) of Theorem 2.1, $\varphi \in \mathcal{A}_{S^1}$. This contradiction proves that $q_0 \in \Gamma$.

Finally, by Poincaré-Bendixon Theorem, $\Gamma \setminus \mathrm{Sing}(Z_\varphi)$ has a finite number of connected components. Let $s \subset \Gamma$ be a separatrix of $p \in \Gamma \cap \mathrm{Sing}(Z_\varphi)$. We show that $\mathrm{cl}(s) = s \cup \{p, q_0\}$. If this is not the case, since Z_φ has not connection between two different saddles in $\Sigma \setminus \{q_0\}$, then $\gamma = s \cup \{p\}$ is a closed simple curve such that $q_0 \notin \mathrm{int}(\gamma)$. Then, γ is homotopic to a constant in $N \setminus (\mathcal{O}_0 \cup \mathcal{O})$, and by Lemma 2.7, γ is homotopic to a constant in $\mathcal{O}_\Gamma$. Again, this contradicts the fact that $\Gamma \supset \gamma$ is a cycle limit and completes the proof. ∎

Remark 3. 3. Note that if $Z_\varphi \in \mathfrak{X}^2(\Sigma)$ has no limit cycle, then q_0 is the only singularity of Z_φ.

Let $\{w_1, w_2\}$ be a base of $\mathbb{R}^2$ such that the vector fields $\{X_1 = X_{w_1}, X_2 = X_{w_2}\}$ are infinitesimal generators adapted to φ at $\mathcal{O}$.

Lemma 3.1. *If $\varphi \in \mathcal{A}_{S^1} \cup \mathcal{A}^c_{\mathbb{R}}$, then there exist a neighborhood V of $\mathcal{O}$ and a C^2 function $u_i : V \to \mathbb{R}^2$ such that $u_i(p_0) = w_i$ and for each $q \in V$, $u_i(q)$ generates G_q.*

Proof. Let $I^2(\varepsilon) = \{(x_1, x_2) \in \mathbb{R}^2; |x_1|, |x_2| < \varepsilon\}$ and $h : V_{p_0} \to I^2(\varepsilon) \times (-\varepsilon, 0]$ be a chart at p_0 with $h(p_0) = 0$, such that, if (x_1, x_2, x_3) are the coordinates of $I^2(\varepsilon) \times (-\varepsilon, 0]$. Then $h_* X_k = \partial/\partial x_k$ for $k = 1, 2$. Let $D_k = D_k(\varepsilon) = \{(x_1, x_2, x_3) \in I^2(\varepsilon) \times (-\varepsilon, 0]; x_k = 0\}$ and $\Sigma_k = \Sigma_k(\varepsilon) = h^{-1}(D_k)$ for $k = 1, 2$. The function $\tau_k : V_{p_0} \to (-\varepsilon, \varepsilon)$ given by $\tau_k(q) = -x_k(q)$, where $h(q) = (x_1(q), x_2(q), x_3(q))$, is such that $X_k^{\tau_k(q)}(q) \in \Sigma_k$, for $k = 1, 2$. Since $X_k^1(p_0) = p_0$, $k = 1, 2$, it follows that there exists $0 < \delta < \varepsilon$ such that $X_k^1(\Sigma_k(\delta)) \subset V_{p_0}$, $k = 1, 2$. Let $\Sigma_{p_0} = \Sigma_{p_0}(\delta) = \Sigma_1(\delta) \cap \Sigma_2(\delta)$. Σ_{p_0} is a transverse section to $\mathcal{O}$ at p_0. Consider the functions $u_i : \Sigma_p \to \mathbb{R}^2$ given by

$$
\begin{aligned}
u_1(q) &= (1 + \tau_1(X_1^1(q)))w_1 + \tau_2(X_1^1(q))w_2, \text{ if } i = 1,\\
u_2(q) &= \tau_1(X_2^1(q))w_1 + (1 + \tau_2(X_2^1(q)))w_2, \text{ if } i = 2.
\end{aligned}
\tag{7}
$$

It can be shown that every orbit of $X_{u_i(q)}$ inside $\mathcal{O}_q$, $q \in \Sigma_{p_0}$, is periodic of period one and $u_i(p_0) = w_i$. We can extend the functions u_i to the open set $V = \cup_{q \in \Sigma_{p_0}} (\mathcal{O}_q \cap V_{p_0})$ by defining $u_i(q) = u_i(\Sigma_{p_0} \cap \mathcal{O}_q)$. ∎

Proposition 3.3. *If $\varphi \in \mathcal{A}_{S^1} \cup \mathcal{A}^c_{\mathbb{R}}$, then there exists a φ-invariant neighborhood U of $\mathcal{O}$ such that for each $q \in U \setminus \mathcal{O}$, $\mathcal{O}_q$ is a $S^1 \times \mathbb{R}$-orbit. Moreover, one of the following occurs:*

(1) *if* $\varphi \in \mathcal{A}_{S^1}$, *then* $q \in C_0$ *and* G_q *is generated by* w_1;
(2) *if* $\varphi \in \mathcal{A}^c_{\mathbb{R}}$, *then* $q \in C_1$ *and* G_q *is generated by* w_2.

Proof. We take U as in Lemma 2.5, and assume it contains the neighborhood V given in Lemma 3.1. Since $\mathcal{O} \subset \mathrm{cl}(\mathcal{O}_q)$, $q \in U$, then $G_q \subset G$. Consequently, (1) and (2) follow by Theorem 2.1 and Lemma 3.1. ▮

Definition 3.2. Let Z be a vector field on $\mathbb{R}^2$, p a singularity of Z, $k \geq 0$ an integer and $\Gamma = \cup_{i=0}^k \gamma_i$, where $\gamma_0 = p$ and γ_i, $i = 1, \dots, k$, is a regular orbit of Z. Γ is said to be a *k-petal* of Z at p if $\mathrm{cl}(\gamma_i) \setminus \gamma_i = \{p\}$ and $\mathrm{cl}(\gamma_i)$ is the frontier of a open 2-disk D_i such that $D_i \cap D_j = \emptyset$ for $j = 1, \dots, k$ with $j \neq i$.

Definition 3.3. Let $\varphi \in \mathcal{A}$, $k \geq 0$ an integer and $\Gamma = \cup_{i=0}^k \mathcal{O}_i$, where $\mathcal{O}_i$, $i = 1, \dots, k$, is a $S^1 \times \mathbb{R}$-orbit. Γ is said to be a *k-tube* of φ at $\mathcal{O}_0$ if $\mathrm{cl}(\mathcal{O}_i) \setminus \mathcal{O}_i = \mathcal{O}_0$ and $\mathrm{cl}(\mathcal{O}_i)$ is the frontier of a open solid torus T_i such that $T_i \cap T_j = \emptyset$ for $j = 1, \dots, k$ with $j \neq i$. The set $S_\Gamma = \cup_{i=1}^k \mathrm{cl}(T_i)$ is said to be a *solid k-tube* of φ at $\mathcal{O}_0$ associated to Γ.

Remark 3. 4. Every $S^1 \times \mathbb{R}$-orbit in a k-tube is contained in C_1.

3.1. Proof of Theorems A and A′

We consider the following two cases separately.

(*i*) *The case when* $\varphi \in \mathcal{A}_{S^1}$. We shall show that there exists a closed 2-disk Σ embedding in N transverse to $\mathscr{F}_\varphi$ such that the induced vector field Z_φ satisfies Theorem A′(1). Theorem A (1) follows then by Corollary 3.1 and Proposition 2.3.

By Theorem 2.1, $\mathcal{O}_q \subset C_0$ for each $q \in N \setminus (\mathcal{O}_0 \cup \mathcal{O})$. Let U be the φ-invariant neighborhood of $\mathcal{O}$, as in Proposition 3.3, and $A = \cup_{q \in J} \mathcal{O}_q(X_1)$, where J is a segment in J_ε with end points p_0 and p_1. It is not hard to show that A is homeomorphic to $S^1 \times [0,1]$ and is transverse to X_2, and consequently is transverse to $\mathscr{F}_\varphi$. Given a neighborhood V_0 of q_0, by Lemma 2.6, there exists a neighborhood U_0 of q_0 such that $\mathcal{O}_q(X_1) \subset V_0$, for each $q \in U_0 \cap \mathcal{O}_{p_1}$. Since $\mathcal{O}_0 \subset \mathrm{cl}(\mathcal{O}_{p_1})$, there exists $t \in \mathbb{R}$ such that $X_2^t(p_1) \in U_0$ and, consequently, $\mathcal{O}_{X_2^t(p_1)}(X_1) = X_2^t(\mathcal{O}_{p_1}(X_1)) \subset V_0$. The ring $A_2 = X_2^t(A)$ is the union of closed orbits of X_1, transverse to $\mathscr{F}_\varphi$, with $\partial A_2 = \mathcal{O}_{X_2^t(p_0)}(X_1) \cup \mathcal{O}_{X_2^t(p_1)}(X_1)$. We consider V_0 as the domain of a chart adapted to φ at q_0. Then there exists a closed 2-disk $D \subset V_0$ transverse to $\mathscr{F}_\varphi$ such that $\partial D = \mathcal{O}_{X_2^t(p_1)}(X_1)$. Let $X \in \mathfrak{X}^2(D)$ be the vector field whose orbits are the connected components of the intersection of D with the φ-orbits. We claim that every orbit of X inside $D \setminus \mathcal{O}_0$ is periodic. Otherwise, there exists $q \in D$ such that $\omega_X(q) = \gamma$ (or $\alpha_X(q) = \gamma$), where $\gamma \subset D$ is a closed orbit of X and $q \notin \gamma$. Then $\mathcal{O}_\gamma$, the φ-orbit containing γ, is contained in $\mathrm{cl}(\mathcal{O}_q)$ and is different from $\mathcal{O}_0$ and $\mathcal{O}$. But this contradicts Proposition 2.3. Therefore all the orbits of X in

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$D \setminus \mathcal{O}_0$ are periodics. Let $\widetilde{D} = A_2 \cup D$, $f : D^2 \to \widetilde{D} \subset N$ a homeomorphism and D_0 a neighborhood of $f^{-1}(\partial D)$ in D^2. We take a C^∞ embedding $g : D^2 \to N$, arbitrarily close to f, in the C^0-topology, such that $g|_{(D^2 \setminus D_0)} = f|_{(D^2 \setminus D_0)}$ and transverse to $\mathscr{F}_\varphi$. Then, the foliation $\mathscr{F}^*$, in $\Sigma = g(D^2)$ induced by $\mathscr{F}_\varphi$, is given by a vector field $Z_\varphi \in \mathfrak{X}^2(\Sigma)$ whose orbits in $\Sigma \setminus (\mathcal{O}_0 \cap \Sigma)$ are all periodics.

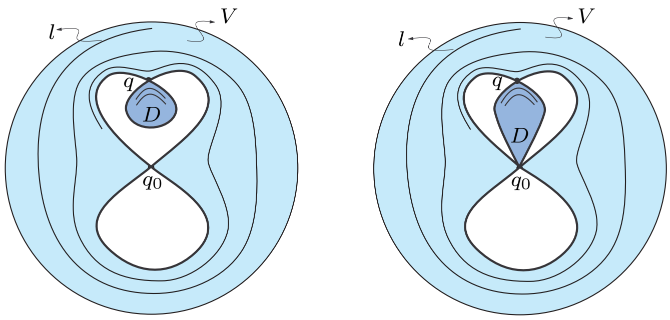

FIG. 2. The φ-orbit that contain l intersects the region D.

(ii) *The case when* $\varphi \in \mathcal{A}^c_{\mathbb{R}} \cup \mathcal{A}^p_{\mathbb{R}}$. In this case we take Z_φ as in Remark 3.2. Let V be the saturated of $A_1(\varepsilon)$ by Z_φ. Then V is homeomorphic to $S^1 \times (0,1]$ and every orbit of Z_φ inside $V \setminus S_1$ is homeomorphic to $\mathbb{R}$. Hence, the frontier Γ of V in Σ, either coincides with $\{q_0\}$ or is a limit cycle of Z_φ.

We suppose that Γ is a limit cycle of Z_φ. By Lemma 2.3, there exist a neighborhood W_1 of $\mathcal{O}$, a neighborhood U_1 of S_1 in D_{θ_0} and a C^2 diffeomorphism $f_1 : A_1(\varepsilon) \to U_1$ that preserves orbits of $\widehat{X}_1$ such that $\mathscr{F}_\varphi|_{W_1}$ is the suspension of $\widehat{X}_1 = Z_\varphi|_{A_1(\varepsilon)}$ by f_1. Then, as V is invariant by Z_φ, we obtain an extension $F_1 : V \to V$ of f_1, that is given by $F_1(q) = Z^t_\varphi(f_1(Z^{-t}_\varphi(q)))$, where $t \in \mathbb{R}$ is such that $Z^{-t}_\varphi(q) \in S$ with $S \subset A_1(\varepsilon)$ a circle transverse to Z_φ. Note that, by definition, F_1 preserves the orbits of Z_φ, and thus $\mathscr{F}_\varphi|_U$ is the suspension of $Z_\varphi|_V$ by F_1, where U is as in Lemma 2.5. Therefore, $V = U \cap \Sigma$. We claim that Γ is a k-petal, i.e. $\Gamma \cap \mathrm{Sing}(Z_\varphi) = \{q_0\}$. Suppose that there exists $q \in \Gamma \cap \mathrm{Sing}(Z_\varphi)$ with $q \neq q_0$. Since Γ is a limit cycle, q is necessarily a saddle. Hence, as Σ is in general position with respect to $\mathscr{F}_\varphi$, every φ-orbit by points in V has points in $\Sigma \setminus \mathrm{cl}(V)$, see Figure 2. But this contradicts the fact that $V = U \cap \Sigma$. This contradiction shows that Γ is a k-petal of Z_φ at q_0, and consequently $\mathrm{Front}(U)$ is a k-tube of φ at $\mathcal{O}_0$. Theorem A (2) and (3) then follow from Theorem 2.1.

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