

Continuation and asymptotics of solutions to semilinear parabolic equations with critical nonlinearities

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In this paper we discuss continuation properties and asymptotic behavior of ε -regular solutions to abstract semilinear parabolic problems in case when the nonlinear term satisfies critical conditions. A necessary and sufficient condition for global in time existence of ε -regular solutions is given. We also formulate sufficient conditions to construct a piecewise smooth extension for ε -regular solutions (continuation beyond maximal time of existence for ε -regular solutions). Applications to strongly damped wave equations and to higher order semilinear parabolic equations are finally discussed. In particular global solvability and the existence of a global attractor for $u_{tt} + \eta(-\Delta_D)^{\frac{1}{2}}u_t + (-\Delta_D)u = f(u)$ in $H_0^1(\Omega) \times L^2(\Omega)$ is achieved in case when a nonlinear term f satisfies a critical growth condition and a dissipativeness condition. Similar result is obtained for a $2m$ -th order semilinear parabolic initial boundary value problem in a Hilbert space $H_{2,\{B_j\}}^m(\Omega)$. May, 2004 ICMC-USP

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1. INTRODUCTION

Let X be a Banach space and $A : D(A) \subset X \rightarrow X$ be a sectorial operator. Denote by X^α , $\alpha \geq 0$ be the fractional power spaces associated to A . Consider semilinear parabolic

equations of the form

$$\begin{aligned} \dot{u}(t) + Au(t) &= F(u(t)), \quad t > 0 \\ u(0) &= u_0 \in X^1 \end{aligned} \quad (1.1)$$

where $F : X^1 \rightarrow X^\alpha$, $\alpha \geq 0$, is a locally Lipschitz continuous map. A mild solution to (1.1) is a function $u(\cdot, u_0) \in C([0, \tau_{u_0}), X^1)$ which satisfies the variation of constants formula

$$u(t, u_0) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}F(u(s, u_0))ds. \quad (1.2)$$

We say that (1.1) is locally well posed in X^1 if for any $u_0 \in X^1$ there is a unique mild solution $t \mapsto u(t, u_0)$ to (1.1) defined on a maximal interval of existence $[0, \tau_{u_0})$ which depends continuously on the initial data u_0 .

If $\alpha > 0$ then (1.1) is locally well posed (see [HE, CD]). If in addition F takes bounded subsets of X^1 in bounded subsets of X^α then either $\tau_0 = +\infty$ or $\lim_{t \rightarrow \tau_{u_0}^-} \|u(t, u_0)\|_{X^1} = +\infty$ (see [HE, CD]).

If $\alpha = 0$ the local well posedness is a more delicate matter. Usually it is not true that the problem is locally well posed if F is assumed to be a Lipschitz continuous map from \mathcal{X}^1 into X (e.g. if $F(u) = -2Au$). In this case, following [AC], we define ε -regular solutions as the functions from $C([0, \tau], X^1) \cap C((0, \tau], \mathcal{X}^{1+\varepsilon})$ which satisfy (1.2) and we say that (1.1) is locally well posed if for each $u_0 \in X^1$ there is a unique ε -regular solution to (1.1) defined on a maximal interval of existence which depends continuously on the initial data u_0 . Also, according to [AC], we say that F is an ε -regular map relative to the pair (X^1, X) if there are constants $C > 0$, $\rho > 1$, $\varepsilon \in (0, \frac{1}{\rho})$, and $1 > \gamma \geq \rho\varepsilon$ such that

$$\|F(v) - F(w)\|_{X^\gamma} \leq C\|v - w\|_{X^{1+\varepsilon}} \left(1 + \|v\|_{X^{1+\varepsilon}}^{\rho-1} + \|w\|_{X^{1+\varepsilon}}^{\rho-1}\right) \quad \text{for } v, w \in X^{1+\varepsilon}. \quad (1.3)$$

The following result, taken from [AC], ensures local well posedness in the case $\alpha = 0$.

Proposition 1.1.1. *If (1.3) holds and $v_0 \in X^1$, then there are $r > 0$ and $\tau > 0$ such that for any $u_0 \in B_{X^1}(v_0, r) = \{u_0 \in X^1; \|u_0 - v_0\|_{X^1} \leq r\}$ there exists a unique ε -regular solution $u \in C([0, \tau], \mathcal{X}^1) \cap C((0, \tau], X^{1+\varepsilon})$ to (1.1). In addition, if $\rho\varepsilon < \gamma$ then, r can be chosen arbitrarily large.*

Whenever $\gamma(\varepsilon) > \rho\varepsilon$, the radius r in Proposition 1.1.1 can be chosen arbitrarily large so that the time of existence of ε -regular solution becomes uniform on bounded subsets of X^1 . We remark that the latter property may be sometimes preserved also when $\gamma(\varepsilon) = \rho\varepsilon$ (see Section 2). If this is the case, the following result is applicable.

Corollary 1.1.1. *Suppose (1.3) hold, $u_0 \in X^1$ and let $u(\cdot, u_0)$ be a unique ε -regular solution to (1.1) defined on its maximal interval of existence $[0, \tau_{u_0})$. Suppose also that the time of existence of ε -regular solution to (1.1) is uniform on arbitrarily large balls.*

Under these assumptions, if $\tau_{u_0} < \infty$ then $\limsup_{t \rightarrow \tau_{u_0}^-} \|u(t, u_0)\|_{X^1} = \infty$.

As for the continuation it is proved in [AC, Proposition 1] that for any ε -regular solution $u(\cdot, u_0)$ defined on the maximal interval of existence $[0, \tau_{u_0})$ we have

$$\limsup_{t \rightarrow \tau_{u_0}^-} \|u(t, u_0)\|_{X^{1+\varepsilon}} = \infty$$

whenever $\tau_{u_0} < \infty$. We remark that such a continuation result originates from a continuation result in the subcritical case. Indeed, if $u(\cdot, u_0)$ is an ε -regular solution to (1.1), then for each $s \in (0, \tau_{u_0})$ function $u(\cdot + s, u_0)$ is a mild solution through $u(0) = u(s, u_0) \in X^{1+\varepsilon}$ of a ‘subcritical’ problem

$$\dot{u} + A_\gamma u = F(u), \quad t > 0, \quad (1.4)$$

with A_γ being X_γ realization of \mathcal{A} and F satisfying (1.3) where $1 + \varepsilon > \gamma > \varepsilon$. In fact, if $u(\cdot, u_0)$ is an ε -regular solution defined on the interval $[0, \tau)$ with $\limsup_{t \rightarrow \tau} \|u(t, u_0)\|_{X^{1+\varepsilon}} < \infty$ and if $t_n \uparrow \tau$, then from Proposition 1.1.1 there is a positive number τ_1 such that any solution starting in $u(t_n, u_0)$ exists up to time τ_1 . Since, for large n , $t_n + \tau_1 > \tau$ we have the result.

Remark 1.1.1. *The above considerations are stating that continuation is related to locally uniform time of existence and convergence of the solution in an appropriate norm or uniform time of existence in bounded sets and boundedness of solutions in an appropriate norm.*

In the case when $F : X^1 \rightarrow X^\alpha$ and $\alpha > 0$, a procedure commonly used to prove that a solution to (1.1) is globally defined is to show that the X^1 norm of the solution remains bounded in the maximal interval of existence.

In the case $\alpha = 0$, to obtain global solutions we need to ensure that $X^{1+\varepsilon}$ remains bounded in the maximal interval of existence. This becomes a very cumbersome task since computations involving the $X^{1+\varepsilon}$ norm are, in general, quite difficult (not to say impossible). It becomes important, in the case $\alpha = 0$, to obtain any information that allow us to conclude global existence of solutions simply by the knowledge of bounds for the X^1 norm of the solution. Another important consideration is to give conditions under which solutions (in a sense to be specified) may be continued beyond the maximal time of existence when it is finite and, consequently, the $X^{1+\varepsilon}$ norm blows up.

We remark that in the study of the asymptotic behavior of solutions to semilinear parabolic problems it is of fundamental importance to be able to establish that solutions exist globally. Questions like the existence of global attractors can only be addressed after we have established the global existence of solutions. With this in mind, only in some very special cases, one can obtain the existence of global attractors for parabolic problems with critically growing nonlinearities. In [CC 2] one can find examples of semilinear strongly damped wave equations, with critically growing nonlinearities, for which one cannot show the global existence of solutions and consequently the existence of global attractors some of these are considered here.

Following [WA] we give a necessary and sufficient condition for the global in time continuation of ε -regular solutions. We also develop the idea of a piecewise smooth ε -regular

solution, which is a right-hand continuation of a non globally defined ε -regular solution $u(\cdot, u_0)$ onto an interval containing the maximal time of existence τ_{u_0} for the ε -regular solution. This will improve the existing results concerning solvability of abstract semilinear parabolic problems in case of critically growing nonlinearities. Final examples will show applicability of the above concepts in concrete problems; like an initial boundary value problem for a $2m$ -th order semilinear parabolic equation in $H_{2, \{B_j\}}^m(\Omega)$ with a nonlinear term growing like $|u|^{\frac{N+2m}{N-2m}}$ ($N > 2m$).

In applications much of our attention goes to a strongly damped wave equation $u_{tt} + \eta(-\Delta_D)^{\frac{1}{2}}u_t + (-\Delta_D)u = f(u)$ with f satisfying growth condition $\lim_{|s| \rightarrow \infty} \frac{|f'(s)|}{|s|^{\frac{N-2}{4}}} = 0$ ($N \geq 3$) and the dissipativeness condition $\limsup_{|s| \rightarrow \infty} \frac{f(s)}{s} \leq 0$. We show that in an energy space $H_0^1(\Omega) \times L^2(\Omega)$ this equation generates a dissipative semigroup of global solutions possessing a compact global attractor, which extends the results obtained in cited references.

Concerning the construction of a right-hand continuation of a non globally defined ε -regular solution to (1.1) we show that, although $\|u(t, u_0)\|_{X^{1+\varepsilon}}$ norm may 'blow up' as $t \rightarrow \tau_{u_0}^-$, such a continuation is possible based on the following concept of *piecewise ε -regular solutions*.

Definition 1.1.1. *Function $U : [0, \tau) \rightarrow X^1$ is called a piecewise ε -regular solution to (1.1) on $[0, \tau)$ iff $U(0) = u_0$ and, for each $T \in (0, \tau)$, the following conditions hold:*

(β_1) $U \in BWC([0, T], X^1)$ (the Banach space of all bounded and weakly continuous maps from $[0, T]$ to X^1),

(β_2) $U \in C([0, T], Z)$ for certain Banach space Z satisfying embedding property $X^1 \hookrightarrow Z$,

(β_3) there is a number $N_T \in \mathbb{N}$ and times $\tau_i \in [0, T]$, $i = 0, 1, \dots, N_T$ satisfying $0 = \tau_0 < \tau_1 < \dots < \tau_{N_T} < T =: \tau_{N_T+1}$ such that, for certain $\gamma > \varepsilon$,

$$U \in C([\tau_{i-1}, \tau_i), X^1) \cap C((\tau_{i-1}, \tau_i), X^{1+\gamma}) \cap C^1((\tau_{i-1}, \tau_i), X^{1+\zeta}), \quad i = 1, \dots, N_T+1, \quad 0 \leq \zeta < \gamma$$

and

$$\limsup_{t \rightarrow \tau_i^-} \|U(t)\|_{X^{1+\varepsilon}} = \infty \quad \text{whenever } i = 1, \dots, N_T,$$

(β_4) $\dot{U}(t) + A_0U(t) = F(U(t))$ for each $t \in (0, T) \setminus \{\tau_i; i = 1, \dots, N_T\}$.

If $U : [0, \infty) \rightarrow X^1$ is a piecewise ε -regular solution to (1.1) on $[0, \tau)$ for each $\tau > 0$, then U is called a *global piecewise ε -regular solution* to (1.1).

If $\tau < \infty$, $U : [0, \tau) \rightarrow X^1$ is a piecewise X^1 -solution to (1.1) on $[0, \tau)$ and τ is a limit of a strictly increasing sequence $\{\tau_i, i \in \mathbb{N}\}$ of times for which $\limsup_{t \rightarrow \tau_i^-} \|U(t)\|_{X^{1+\varepsilon}} = \infty$, then τ is called an *accumulation time of singular times*.

It will be assumed (and next verified in concrete examples) that local solutions resulting from Proposition 1.1.1 satisfy the condition

$$\|u(t, u_0)\|_{X^1} \leq g(u_0), \quad t \in [0, \tau_{u_0}). \quad (1.5)$$

From the point of view of further applications the bound in (1.5) will be called *energy estimate*.

Note that with (1.5), (1.3), and interpolation inequality

$$\|\phi\|_{X^{1+\varepsilon}} \leq \bar{c} \|\phi\|_{X^{1+\varepsilon\rho}}^{\frac{1}{\rho}} \|\phi\|_{X^1}^{1-\frac{1}{\rho}} \leq \tilde{c} \|\phi\|_{X^{1+\gamma}}^{\frac{1}{\rho}} \|\phi\|_{X^1}^{1-\frac{1}{\rho}}$$

we arrive at the subordination condition of the form

$$\|F(u(t, u_0))\|_{X^\gamma} \leq \tilde{g}(u_0) (1 + \|u(t, u_0)\|_{X^{1+\gamma}}), \quad (1.6)$$

where $\tilde{g}(u_0) = 2C (1 + \tilde{c}^\rho g^{\rho-1}(u_0)) + \|F(0)\|_{X^\gamma}$.

Mention should be made that in (1.6) the ‘distance on a scale’ between the spaces X^γ and $X^{1+\gamma}$ is equal to one. Furthermore, in this estimate we can hardly replace $\|u(t, u_0)\|_{X^{1+\gamma}}$ with any ‘weaker’ norm $\|u(t, u_0)\|_{X^{1+\gamma-\delta}}$. Therefore, the nonlinearity in (1.1) is *critical* not only from the point of view of local solvability of (1.1) in X^1 but also from the point of view of possibility of global in time continuation of a mild solution to (1.4) corresponding to an initial data from $X^{1+\varepsilon}$ (see [WA]).

As noticed in [WA], even if (1.5) holds the solution may cease to exist by oscillating when $t \rightarrow \tau_{u_0}^-$. To prevent such a behaviour we consider the following condition, which proves to be useful in further applications:

$$\begin{aligned} &\text{if } \tau_{u_0} < \infty, \text{ then there is a Banach space } Z \text{ for which } X^1 \hookrightarrow Z \\ &\text{and a map } [0, \tau_{u_0}) \ni t \rightarrow u(t, u_0) \in Z \text{ is uniformly continuous.} \end{aligned} \quad (1.7)$$

Section 2 is devoted to abstract results concerning solvability of parabolic equations in Banach spaces. In Theorem 2.2.1 a local well posedness result concerning almost critical nonlinearities is proved. In particular, for such maps, the time of existence of ε -regular solutions is shown to be uniform on bounded sets. In Theorem 2.2.2 we give a necessary and sufficient condition for the global in time existence of an ε -regular solution. We also show that a piecewise extension of an ε -regular solution can be constructed whenever the latter ceases to exist in a finite time.

In Sections 3 and 4, we discuss two examples of (1.1) with F satisfying certain critical condition relatively to a pair of spaces (X^1, X^0) . These are a strongly damped wave equation and a $2m$ -th order semilinear parabolic equation. Besides continuation of solutions the existence of a global attractor is also shown provided growth condition and dissipativeness condition are suitably formulated, which complements the results previously obtained in [CC 2, Theorem 5] and [CD, Theorem 5.3.1].

2. ABSTRACT RESULTS

Assume that X is a Banach space, A is a sectorial operator in X and X^α denote the fractional power spaces associated to A .

Definition 2.2.1. *We say that F is an almost critical ε -regular map relative to the pair (X^1, X) if there are constants $c > 0$, $\rho > 1$, $\varepsilon \in (0, \frac{1}{\rho})$ and, for any $\eta > 0$, there is $C_\eta > 0$*

such that

$$\begin{aligned} \|F(v) - F(w)\|_{X^{\rho\varepsilon}} &\leq c\|v - w\|_{X^{1+\varepsilon}} \left(C_\eta + \eta\|v\|_{X^{1+\varepsilon}}^{\rho-1} + \eta\|w\|_{X^{1+\varepsilon}}^{\rho-1} \right) \quad \text{for } v, w \in X^{1+\varepsilon}, \\ \|F(v)\|_{X^{\rho\varepsilon}} &\leq c\|v\|_{X^{1+\varepsilon}} \left(C_\eta + \eta\|v\|_{X^{1+\varepsilon}}^{\rho-1} \right) \quad \text{for } v \in X^{1+\varepsilon}. \end{aligned} \tag{2.1}$$

THEOREM 2.2.1. *Consider (1.1) and assume that F is an almost critical ε -regular map. Let $u_0 \in X^1$ and $B_{X^1}(v_0, r)$ be a ball in X^1 with radius $r > 0$ centered at v_0 .*

Then, given $r > 0$ there is a $\tau_0 > 0$ such that for each $u_0 \in B_{X^1}(v_0)$ there exists a unique ε -regular solution $u(\cdot, u_0)$ to (1.1). In addition,

- (i) $t^\varsigma \|u(t, u_0)\|_{X^{1+\varsigma}} \rightarrow 0$ as $t \rightarrow 0^+$, $0 < \varsigma < \rho\varepsilon$,
- (ii) $t^\varsigma \|u(t, u_1) - u(t, u_2)\|_{X^{1+\varsigma}} \leq C' \|u_1 - u_2\|_{X^1}$ for $t \in [0, \tau_0]$, $0 \leq \varsigma \leq \varsigma_0 < \rho\varepsilon$, $u_1, u_2 \in B_{X^1}(v_0, r)$,
- (iii) $u(t, u_0) \in C((0, \tau_0], X^{1+\rho\varepsilon}) \cap C^1((0, \tau_0], X^{1+\varsigma})$ for $0 \leq \varsigma < \rho\varepsilon$; in particular the solution, $u(t, u_0)$ satisfies both relations in (1.1).

In addition, if the solution $u(t, u_0)$ is bounded in X^1 in its maximal interval of existence, then it must exist for all $t \geq 0$.

2.1. Proof of Theorem 2.2.1

Most of the proof of Theorem 2.2.1 is the same as in [AC]. We need to ensure that we can obtain a uniform time of existence in bounded sets and then obtain the continuation result stated as the last part of the theorem. The later follows from Corollary 1.1.1.

Before we prove Theorem 3.3.2 we will need some technical lemmas.

LEMMA 2.2.1. *The operators $t^\alpha e^{-At} : X^1 \rightarrow X^{1+\alpha}$, $t > 0$, are bounded linear operators satisfying $\|t^\alpha e^{-At}\|_{L(X^1, X^{1+\alpha})} \leq M$, with M independent of t . Moreover, given a compact subset J of X^1 , we have*

$$\lim_{t \rightarrow 0^+} \sup_{x \in J} \|t^\alpha e^{-At} x\|_{X^{1+\alpha}} = 0.$$

Proof. See [AC] for a proof. ■

Let $\mathbf{B}(\cdot, \cdot) : (0, \infty) \times (0, \infty) \rightarrow (0, \infty)$, denote the beta function and define

$$\mathbf{B}_\varepsilon^\theta = \max_{0 \leq \xi \leq \theta} \{\mathbf{B}(\rho\varepsilon - \xi, 1 - \rho\varepsilon), \mathbf{B}(\rho\varepsilon - \xi, 1 - \rho\varepsilon)\}.$$

LEMMA 2.2.2. *Assume that F is an almost critical ε -regular map. If $u \in C((0, \tau], X^{1+\varepsilon})$ then, for all $0 \leq \theta < \rho\varepsilon$,*

$$t^\theta \left\| \int_0^t e^{-A(t-s)} F(u(s)) ds \right\|_{X^{1+\theta}} \leq cM \mathbf{B}_\varepsilon^\theta (t^{\rho\varepsilon} + [\lambda(t)]^\rho), \quad 0 < t \leq \tau,$$

where $\lambda(t) := \sup_{s \in (0, t]} \{s^\varepsilon \|u(s)\|_{X^{1+\varepsilon}}\}$.

Proof. Since F is an *almost critical* ε -regular map we have

$$\begin{aligned} t^\theta \left\| \int_0^t e^{-A(t-s)} F(u(s)) ds \right\|_{X^{1+\theta}} &\leq Mt^\theta \int_0^t (t-s)^{-1+\rho\varepsilon-\theta} \|F(u(s))\|_{X^\gamma} ds \\ &\leq cMt^\theta \int_0^t (t-s)^{-1+\rho\varepsilon-\theta} (C_\eta + \eta \|u(s)\|_{X^{1+\varepsilon}}^\rho) ds \\ &\leq cC_\eta Mt^\theta \int_0^t (t-s)^{-1+\rho\varepsilon-\theta} ds + c\eta Mt^\theta \int_0^t (t-s)^{-1+\rho\varepsilon-\theta} s^{-\rho\varepsilon} [s^\varepsilon \|u(s)\|_{X^{1+\varepsilon}}]^\rho ds \\ &= cM\mathbf{B}_\varepsilon^\theta [C_\eta t^{\rho\varepsilon} + \eta[\lambda(t)]^\rho], \end{aligned}$$

from which the lemma follows. ■ ■

LEMMA 2.2.3. *Let F be as before and $u, \bar{u} \in C((0, \tau], X^{1+\varepsilon})$ be such that $t^\varepsilon \|u(t)\|_{X^{1+\varepsilon}} \leq \mu$ and $t^\varepsilon \|\bar{u}(t)\|_{X^{1+\varepsilon}} \leq \mu$, for some $\mu > 0$. Then, for all $0 \leq \theta < \rho\varepsilon$, we have*

$$t^\theta \left\| \int_0^t e^{-A(t-s)} [F(u(s)) - F(\bar{u}(s))] ds \right\|_{X^{1+\theta}} \leq \Gamma_\theta(t) \sup_{s \in [0, \tau]} s^\varepsilon \|u(s) - \bar{u}(s)\|_{X^{1+\varepsilon}}$$

where

$$\Gamma_\theta(t) = cM\mathbf{B}_\varepsilon^\theta [C_\eta t^{\rho\varepsilon-\varepsilon} + 2\eta\mu^{\rho-1}].$$

Proof. As in the preceding lemma the proof follows from the fact that F is an *almost critical* ε -regular map; namely

$$\begin{aligned} t^\theta \left\| \int_0^t e^{-A(t-s)} [F(u(s)) - F(\bar{u}(s))] ds \right\|_{X^{1+\theta}} &\leq cC_\eta Mt^\theta \int_0^t (t-s)^{-1+\rho\varepsilon-\theta} s^{-\varepsilon} s^\varepsilon \|u(s) - \bar{u}(s)\|_{X^{1+\varepsilon}} ds \\ &+ c\eta Mt^\theta \int_0^t (t-s)^{-1+\rho\varepsilon-\theta} s^{-\rho\varepsilon} \left[(s^\varepsilon \|u(s)\|_{X^{1+\varepsilon}})^{\rho-1} + (s^\varepsilon \|\bar{u}(s)\|_{X^{1+\varepsilon}})^{\rho-1} \right] s^\varepsilon \|u(s) - \bar{u}(s)\|_{X^{1+\varepsilon}} ds \\ &= \Gamma_\theta(t) \sup_{t \in [0, \tau]} \{s^\varepsilon \|u(s) - \bar{u}(s)\|_{X^{1+\varepsilon}}\}. \end{aligned}$$

■ ■

Proof of Theorem 2.2.1: For the proof we refer the reader to [AC]. We only point out the changes needed to obtain the uniform time of existence in bounded subsets of X^1 . Given $r > 0$ let $\mu = 4Mr$. Choose $\eta > 0$ such that

$$c\eta M\mathbf{B}_\varepsilon^\varepsilon \mu^{\rho-1} = \frac{1}{8},$$

then

$$r = \frac{\mu}{4M} = \frac{1}{4M(8c\eta MB_\varepsilon^\varepsilon)^{\frac{1}{\rho-1}}} \quad (2.2)$$

Also, for v_0 fixed, choose $\tau_0 \in (0, 1]$ such that $\nu(t) = \max\{t^{\rho\varepsilon}, t^{(\rho-1)\varepsilon}\} < \delta$ for $t \in (0, \tau_0]$ and

$$\begin{aligned} \|t^\varepsilon e^{-At} v_0\|_{X^{1+\varepsilon}} &\leq \frac{\mu}{2}, \quad 0 \leq t \leq \tau_0, \\ cC_\eta M \delta \mathbf{B}_\varepsilon^\varepsilon &= \min\left\{\frac{\mu}{8}, \frac{1}{4}\right\}. \end{aligned} \quad (2.3)$$

These choices imply $\Gamma_\varepsilon(t) \leq \frac{1}{2}$ for $t \in (0, 1)$.

As in [AC], we search for solutions in

$$K(\tau_0) = \left\{ u \in C((0, \tau_0], X^{1+\varepsilon}) : \sup_{t \in (0, \tau_0]} t^\varepsilon \|u(t)\|_{X^{1+\varepsilon}} \leq \mu \right\}$$

endowed with the topology given by the metric

$$d(u, v)_{K(\tau_0)} = \sup_{t \in (0, \tau_0]} t^\varepsilon \|u(t) - v(t)\|_{X^{1+\varepsilon}}.$$

Assume that $u_0 \in X^1$ with $\|u_0 - v_0\|_{X^1} < r$ and on $K(\tau_0)$ define the map

$$T(u)(t) = e^{-At} u_0 + \int_0^t e^{-A(t-s)} F(u(s)) ds.$$

Let us prove that (uniformly for $u_0 \in B_{X^1}(v_0, r)$) T is well defined strict contraction from $K(\tau_0)$ into itself.

Let us first prove that T is a well-defined map and that $T(K(\tau_0)) \subset K(\tau_0)$. Firstly, exactly as in [AC], we obtain that

$$\text{if } u \in K(\tau_0), \text{ then } T(u) \in C((0, \tau_0], X^{1+\theta}) \text{ for all } \theta \in [0, \rho\varepsilon). \quad (2.4)$$

To show that $t^\varepsilon \|T(u)(t)\|_{X^{1+\varepsilon}} \leq \mu$, for all $t \in (0, \tau_0]$ we proceed as follows

$$\begin{aligned} t^\varepsilon \|T(u)(t)\|_{X^{1+\varepsilon}} &\leq \|t^\varepsilon e^{-At} u_0\|_{X^{1+\varepsilon}} + cMt^\varepsilon \int_0^t (t-s)^{-1+(\rho-1)\varepsilon} (C_\eta + \eta \|u(s)\|_{X^{1+\varepsilon}}^\rho) ds \\ &\leq \|t^\varepsilon e^{-At} u_0\|_{X^{1+\varepsilon}} + cC_\eta M \delta \mathbf{B}_\varepsilon^\varepsilon + c\eta M \mathbf{B}_\varepsilon^\varepsilon \mu^\rho \\ &\leq Mr + \|t^\varepsilon e^{-At} v_0\|_{X^{1+\varepsilon}} + cC_\eta \delta M \mathbf{B}_\varepsilon^\varepsilon + c\eta M \mathbf{B}_\varepsilon^\varepsilon \mu^\rho \leq \mu, \end{aligned}$$

which implies that T takes $K(\tau_0)$ into itself.

The next step is to prove that the map T is a contraction from $K(\tau_0)$ into itself, which follows from Lemma 2.2.3 by taking $\theta = \varepsilon$. Thus T is a strict contraction in $K(\tau_0)$ and

$$\|T(u) - T(v)\|_{K(\tau_0)} \leq \frac{1}{2} \|u - v\|_{K(\tau_0)}.$$

By the Banach contraction principle we have that T has a unique fixed point in $K(\tau_0)$. We will denote this fixed point by $U(t, u_0)$; it is defined for $\|u_0 - v_0\|_{X^1} < r$, $0 \leq t \leq \tau_0$. Note that, from (2.4), $U(\cdot, u_0) \in C((0, \tau_0], X^{1+\theta})$ for all $0 \leq \theta < \rho\varepsilon$. The remaining existence proof and the uniqueness follows exactly as in [AC]. The statement concerning global existence follows from Corollary 1.1.1. ■

An ε -regular map F may not possess the properties of Definition 2.2.1 so that Corollary 1.1.1 may not apply. Thus in the next theorem our concern is to prove a result concerning continuation of solutions to (1.1) in case of nonlinearities satisfying (1.3).

THEOREM 2.2.2. *Suppose (1.3) hold, $u_0 \in X^1$ and let $u(\cdot, u_0)$ be a unique ε -regular solution to (1.1) defined on its maximal interval of existence $[0, \tau_{u_0}) \subset [0, \infty)$.*

Then $\tau_{u_0} = \infty$ if and only if

$$\begin{aligned} & \text{for each } T \in (0, \infty) \text{ such that } [0, T) \subset [0, \tau_{u_0}) \text{ map} \\ & [0, T) \ni t \rightarrow u(t, u_0) \in X^1 \text{ is uniformly continuous.} \end{aligned} \quad (2.5)$$

In case when (2.5) fails but X^1 is a reflexive Banach space and both (1.5) and (1.7) are satisfied, there exist $\tau \in (\tau_{u_0}, \infty]$ and a unique extension $U : [0, \tau) \rightarrow X^1$ of $u(\cdot, u_0)$ such that U is a piecewise ε -regular solution to (1.1) on $[0, \tau)$ and either $\tau = \infty$ or τ is an accumulation time of singular times.

Proof. Note first that by smoothness of ε -regular solution reported in Proposition 1.1.1 we obtain immediately necessity of (2.5).

Suppose now that (2.5) holds and $\tau_{u_0} < \infty$ so that

$$[0, \tau_{u_0}) \ni t \rightarrow u(t, u_0) \in X^1 \text{ is uniformly continuous.}$$

If $t \rightarrow \tau_{u_0}^-$, then $u(t, u_0)$ possesses a (left-hand) limit w in a strong topology of X^1 . In particular if $t_n \rightarrow \tau_{u_0}^-$, then $w_n = u(t_n, u_0) \rightarrow w$ in X^1 and $\tau_{w_n} = \tau_{u_0} - t_n \rightarrow 0$, where $[0, \tau_{w_n})$ is a maximal interval of existence of $u(\cdot, w_n)$. It is thus clear that the ‘life’ times of the corresponding ε -regular solutions $u(\cdot, w_n)$ shrink to zero.

Observe however that, for any $r > 0$, almost all elements of $\{w_n\}$ lie in a ball $B_{X^1}(w, r)$ so that by Proposition 1.1.1 almost all of ε -regular solutions $u(\cdot, w_n)$ must exist up to certain positive time, which leads to a contradiction.

Now, if (2.5) fails, we know from the already proved equivalence that $\tau_{u_0} < \infty$, i.e.

$$\limsup_{t \rightarrow \tau_{u_0}^-} \|u(t, u_0)\|_{X^{1+\varepsilon}} = \infty$$

(see [AC, Proposition 1]). With (1.5) and (1.7) we thus need to construct an extension of an ε -regular solution $u(\cdot, u_0)$ to a function $U : [0, \tau) \rightarrow X^1$ with $\tau > \tau_{u_0}$ such that $(\beta_1) - (\beta_4)$ hold for each $T \in (0, \tau)$ and τ is an accumulation time of singular times unless τ is infinite.

Thanks to (1.7) there exist Z with $X^1 \subset Z$ and $u_1 \in Z$ such that

$$\lim_{t \rightarrow \tau_{u_0}^-} \|u(t, u_0) - u_1\|_Z = 0.$$

Since X^1 is reflexive, the energy inequality ensures that

if $t_n \rightarrow \tau_{u_0}^-$, then there is a subsequence $t_{n'} \rightarrow \tau_{u_0}^-$ and certain $\tilde{u}_1 \in X^1$ such that $u(t_{n'}, u_0) \rightharpoonup \tilde{u}_1$ weakly in X^1 .

Because of the embedding $X^1 \subset Z$ we then have

$$u(t_{n'}, u_0) \rightharpoonup \tilde{u}_1 \text{ weakly in } Z,$$

which necessitates that $u_1 = \tilde{u}_1$. As a consequence u_1 belongs to X^1 and $u(t, u_0)$ can be extended in a unique way to a function $U_0 \in C([0, \tau_{u_0}], Z) \cap BWC([0, \tau_{u_0}], X^1)$.

By Proposition 1.1.1 to the initial condition u_1 there will correspond a unique ε -regular solution $u(\cdot, u_1)$. Thus, in case when $\tau_{u_1} < \infty$, the above considerations allow us to find $u_2 \in X^1$ being a left-hand limit of $u(\cdot, u_1)$ in a weak topology of X^1 and in a strong topology of Z and thus extend $u(\cdot, u_1)$ in a unique way to $U_1 \in C([0, \tau_{u_0}], Z) \cap BWC([0, \tau_{u_1}], X^1)$.

If there is $k \in \mathbb{N}$ such that after $k + 1$ steps we will have $\tau_{u_k} = \infty$ then function $U : [0, \infty) \rightarrow X^1$, where

$$U(t) = \begin{cases} U_0(t), & t \in [0, \tau_{u_0}], \\ U_j \left(t - \sum_{l=0}^{j-1} \tau_{u_l} \right), & t \in \left(\sum_{l=0}^{j-1} \tau_{u_l}, \sum_{l=0}^j \tau_{u_l} \right], \quad j = 1, \dots, k, \end{cases} \quad (2.6)$$

satisfies requirements $(\beta_1) - (\beta_4)$.

Otherwise, continuing the described procedure we will obtain a sequence of maps

$$U_j \in C([0, \tau_{u_j}], Z) \cap BWC([0, \tau_{u_j}], X^1), \quad j = 0, 1, \dots,$$

so that condition (2.6) with $k \in \mathbb{N}$ will define a piecewise ε -regular solution to (1.1) on $[0, \tau)$, with $\tau := \sum_{j=0}^{\infty} \tau_{u_j}$. It is now evident that if $\tau < \infty$, then τ is accumulation time of singular times $\tau_j := \sum_{l=0}^j \tau_{u_l}$, $j \in \mathbb{N}$.

Theorem 2.2.2 is thus proved. \blacksquare

Remark 2.2.1. Note that although formulation of the condition (2.5) is in a vein of [WA], the global existence result of Theorem 2.2.2 is obtained in this paper as a straightforward consequence of the properties of ε -regular solutions.

3. STRONGLY DAMPED WAVE EQUATION IN AN ENERGY SPACE

Consider the problem

$$\begin{cases} u_{tt} + \eta(-\Delta)^{\frac{1}{2}} u_t + (-\Delta)u = f(u), & t > 0, \quad x \in \Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = v_0(x), & x \in \Omega, \\ u(t, x) = 0, & t \geq 0, \quad x \in \partial\Omega, \end{cases} \quad (3.1)$$

where Ω is a bounded smooth domain in \mathbb{R}^N , $N \geq 3$, and $A = (-\Delta)$ with Dirichlet boundary conditions. It is well known that A is a positive, self-adjoint operator with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and therefore $-A$ generates a compact analytic semigroup on $X = L^2(\Omega)$. Denote by X^α the fractional power spaces associated to the operator A ; that is, $X^\alpha = D(A^\alpha)$ endowed with the graph norm.

The problem (3.1) will be viewed in a product space $Y = X \times X^{-\frac{1}{2}}$, where $X^{-\frac{1}{2}} = (X^{\frac{1}{2}})'$, in a form of an ordinary differential equation:

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{F}(u), \quad t > 0, \quad \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \quad (3.2)$$

with \mathcal{A} and \mathcal{F} denoted in a matrix form by

$$\mathcal{A} = \begin{bmatrix} 0 & -I \\ A & \eta A^{\frac{1}{2}} \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} 0 \\ F \end{bmatrix}. \quad (3.3)$$

Here F is the Nemitskii map associated to f and $\mathcal{A} : D(\mathcal{A}) \subset Y^0 \rightarrow Y^0$ with

$$D(\mathcal{A}) = Y^1 = X^{\frac{1}{2}} \times X. \quad (3.4)$$

is a sectorial positive operator possessing compact resolvent and bounded imaginary powers (see [CC 1, Proposition 5]).

The main result in this paper concerning global solvability and the existence of a global attractor for (3.1) is

THEOREM 3.3.1. *Let $N \geq 3$. If $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies the growth condition*

$$\lim_{|s| \rightarrow \infty} \frac{|f'(s)|}{|s|^{\frac{4}{N-2}}} = 0 \quad (3.5)$$

and the dissipativeness condition

$$\limsup_{|s| \rightarrow \infty} \frac{f(s)}{s} \leq 0, \quad (3.6)$$

then (3.1) is globally well posed in $H_0^1(\Omega) \times L^2(\Omega)$ and possesses a compact global attractor \mathbf{A} which consists of the elements of $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

3.1. Local well posedness for (3.1)

LEMMA 3.3.1. *Let $\rho > 1$. If $f \in C^1(\mathbb{R}, \mathbb{R})$ fulfils*

$$\lim_{|s| \rightarrow \infty} \frac{|f'(s)|}{|s|^{1-\rho}} = 0, \quad (3.7)$$

then given $\eta > 0$ there is a positive constant C_η such that

$$|f(s_1) - f(s_2)| \leq |s_1 - s_2| (C_\eta + \eta|s_1|^{\rho-1} + \eta|s_2|^{\rho-1}) \quad (3.8)$$

and

$$|f(s)| \leq C_\eta + \eta|s|^\rho. \quad (3.9)$$

Proof. Condition (3.8) follows by a mean value theorem. Then (3.9) is obtained via Young inequality. \blacksquare

LEMMA 3.3.2. *Let $N \geq 3$, $\rho = \frac{N+2}{N-2}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy (3.8). Then, for each $\varepsilon \in [0, \frac{1}{2\rho}]$ and $\eta > 0$ there are a $c > 0$ and $C_\eta > 0$ such that*

$$\|f(w_1) - f(w_2)\|_{L^{\frac{2N}{N+2-2\gamma(\varepsilon)}}(\Omega)} \leq c \|w_1 - w_2\|_{H^{1+\varepsilon}(\Omega)} \left(C_\eta + \eta \|w_1\|_{H^{1+\varepsilon}(\Omega)}^{\rho-1} + \eta \|w_2\|_{H^{1+\varepsilon}(\Omega)}^{\rho-1} \right)$$

for $w_1, w_2 \in H^{1+\varepsilon}(\Omega)$.

Proof. It follows from (3.8) and from Hölder inequality that:

$$\begin{aligned} \|f(w_1) - f(w_2)\|_{L^{\frac{2N}{N+2-2\rho\varepsilon}}(\Omega)} &\leq \left[\int_{\Omega} [|w_1 - w_2| (C_\eta + \eta|w_1|^{\rho-1} + \eta|w_2|^{\rho-1})]^{\frac{2N}{N+2-2\rho\varepsilon}} \right]^{\frac{N+2-2\rho\varepsilon}{2N}} \\ &\leq \left[\int_{\Omega} |w_1 - w_2|^{\frac{2N}{N-2-2\varepsilon}} \right]^{\frac{N-2-2\varepsilon}{2N}} \left[\int_{\Omega} (C_\eta + \eta|w_1|^{\rho-1} + \eta|w_2|^{\rho-1})^{\frac{2N}{4+2(\varepsilon-\rho\varepsilon)}} \right]^{\frac{4+2(\varepsilon-\rho\varepsilon)}{2N}} \\ &\leq c \|w_1 - w_2\|_{L^{\frac{2N}{N-2-2\varepsilon}}(\Omega)} \left(\tilde{C}_\eta + \eta \|w_1\|_{L^{\frac{2N(\rho-1)}{4+2(\varepsilon-\rho\varepsilon)}(\Omega)}}^{\rho-1} + \eta \|w_2\|_{L^{\frac{2N(\rho-1)}{4+2(\varepsilon-\rho\varepsilon)}(\Omega)}}^{\rho-1} \right). \end{aligned}$$

Note that the following Sobolev embeddings hold

$$L^{\frac{2N}{N-2-2\varepsilon}}(\Omega) \supset H^{1+\varepsilon}(\Omega), \quad L^{\frac{2N(\rho-1)}{4+2(\varepsilon-\rho\varepsilon)}}(\Omega) \supset H^{1+\varepsilon}(\Omega)$$

since $\rho = \frac{N+2}{N-2}$. The proof is complete. \blacksquare

Let us recall that $Y^{1+\alpha} = X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}$, $\alpha \in [0, 1]$, and

$$X^{\frac{\alpha}{2}} \times L^q(\Omega) \subset Y^\alpha, \quad \frac{2N}{N+2-2\alpha} \leq q, \quad \alpha \in [0, 1]$$

(see [CC 1, Theorem 2, Corollary 1]). We thus get

Corollary 3.3.1. *Let $N \geq 3$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy (3.8) with $\rho = \frac{N+2}{N-2}$. Let \mathcal{F} be the map defined by*

$$\mathcal{F}(u) = \begin{bmatrix} 0 \\ F(u) \end{bmatrix} \quad (3.10)$$

where $F(u)$ is the Nemitskiĭ map associated to f .

Then, \mathcal{F} is an almost critical ε -regular map relatively to (Y^1, Y) for each $\varepsilon \in [0, \frac{1}{2\rho}]$. That is,

$$\begin{aligned} & \|\mathcal{F}(u) - \mathcal{F}(u')\|_{Y^{\rho\varepsilon}} \\ & \leq c \left\| \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} u' \\ v' \end{bmatrix} \right\|_{Y^{1+\varepsilon}} \left(C_\eta + \eta \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_{Y^{1+\varepsilon}}^{\rho-1} + \eta \left\| \begin{bmatrix} u' \\ v' \end{bmatrix} \right\|_{Y^{1+\varepsilon}}^{\rho-1} \right), \quad \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u' \\ v' \end{bmatrix} \in Y^{1+\varepsilon}, \end{aligned} \quad (3.11)$$

whenever $\varepsilon \in [0, \frac{1}{2\rho}]$.

We are now ready to formulate the following local well posedness result.

THEOREM 3.3.2. *Consider problem (3.2) corresponding to (3.1) with $f : \mathbb{R} \rightarrow \mathbb{R}$ as in Corollary 3.3.1. Let $\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix} \in Y^1$ and $B_{Y^1}(\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix}, r)$ be a ball in Y^1 with radius $r > 0$ centered at $\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix}$.*

Then, given $r > 0$ there is a $\tau_0 > 0$ such that for each $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B_{Y^1}(\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix}, r)$ there exists a unique ε -regular solution $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot, u_0, v_0)$ to (3.2). In addition,

- (i) $t^\varsigma \left\| \begin{bmatrix} u \\ v \end{bmatrix}(t, u_0, v_0) \right\|_{Y^{1+\varsigma}} \rightarrow 0$ as $t \rightarrow 0^+$, $0 < \varsigma < \frac{1}{2}$,
- (ii) $t^\varsigma \left\| \begin{bmatrix} u \\ v \end{bmatrix}(t, u_1, v_1) - \begin{bmatrix} u \\ v \end{bmatrix}(t, u_2, v_2) \right\|_{Y^{1+\varsigma}} \leq C' \left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_{Y^1}$ whenever $t \in [0, \tau_0]$, $0 \leq \varsigma \leq \varsigma_0 < \frac{1}{2}$, $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \in B_{Y^1}(\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix}, r)$,
- (iii) $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot, u_0, v_0) \in C((0, \tau_0], Y^{1+\frac{1}{2}}) \cap C^1((0, \tau_0], Y^{1+\varsigma})$ for $0 \leq \varsigma < \frac{1}{2}$; in particular the solution, $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot, u_0, v_0)$ satisfies both relations in (3.2).

In addition, if the solution $\begin{bmatrix} u \\ v \end{bmatrix}(t, u_0, v_0)$ is bounded in Y^1 in its maximal interval of existence, then it must exist for all $t \geq 0$.

The proof of Theorem 3.3.2 follows immediately from Theorem 2.2.1 and from Corollary 3.3.1.

3.2. Proof of Theorem 3.3.1

We first note that by (3.6) the $X^{\frac{1}{2}} \times X$ norm of solutions remains bounded uniformly in time. Indeed, multiplying the first equation entering (3.1) by $v = \dot{u}$ in $L^2(\Omega)$ we get

$$\frac{d}{dt} \mathcal{L} \left(\begin{bmatrix} u \\ v \end{bmatrix} \right) = -\eta \|A^{\frac{1}{2}} v\|_{L^2(\Omega)}^2 \leq 0. \quad (3.12)$$

where

$$\mathcal{L} \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \frac{1}{2} \|w_2\|_{L^2(\Omega)}^2 + \frac{1}{2} \|A^{\frac{1}{2}} w_1\|_{L^2(\Omega)}^2 - \int_{\Omega} \int_0^{w_1} f(s) ds dx, \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in Y^1, \quad (3.13)$$

so that by (3.6)

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix} (t, u_0, v_0) \right\|_{Y^1} \leq \text{const.} \left(\left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right\|_{Y^1} \right). \quad (3.14)$$

Considerations of Theorem 3.3.2 guarantee that (3.1) is globally well posed in $H_0^1(\Omega) \times L^2(\Omega)$. Thus, by (3.14) and compactness of the resolvent of \mathcal{A} , problem (3.1) generates a C^0 semigroup of global solutions $\{S(t)\}$ which is compact¹ and have bounded orbits of bounded sets. Dissipativeness condition (3.6) ensures next that the set of equilibria is bounded in $H_0^1(\Omega) \times L^2(\Omega)$ (see [CC 2, p. 297]). Combining this with the properties of the Lyapunov function \mathcal{L} defined in (3.13), we conclude that $\{S(t)\}$ is dissipative and thus possesses a global attractor \mathbf{A} (see [HA]).

As follows from [CC 2, Theorem 4], the global solutions $\begin{bmatrix} u \\ v \end{bmatrix} \in C([0, \infty), H_0^1(\Omega) \cap L^2(\Omega))$ obtained in this section fulfil the condition

$$\begin{bmatrix} u \\ v \end{bmatrix} \in C((0, \infty), H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)) \cap C^1((0, \infty), H_0^1(\Omega) \cap L^2(\Omega)).$$

This ensures the inclusion $\mathbf{A} \subset H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and thus the proof is complete. ■

3.3. Piecewise ε -regular solutions to (3.1)

Suppose now that instead of (3.7) we only know that

$$f \in C^1(\mathbb{R}, \mathbb{R}) \quad \text{and} \quad |f'(s)| \leq c \left(1 + |s|^{\frac{4}{N-2}} \right), \quad s \in \mathbb{R}. \quad (3.15)$$

Connecting (3.12), (3.14) and the relation

$$A^{-\frac{1}{2}} \dot{v} + \eta v + A^{\frac{1}{2}} u = A^{-\frac{1}{2}} f(u),$$

we obtain

$$u \in W^{1,1}((0, \tau_{u_0, v_0}), L^2(\Omega))$$

¹In case of almost critical nonlinearities compactness of a nonlinear semigroup in case when the resolvent of A is compact follows from the compactness of embeddings among fractional power spaces and property (ii) of Theorem 2.2.1.

and

$$\dot{u} \in W^{1,1}((0, \tau_{u_0, v_0}), H^{-1}(\Omega)).$$

The latter ensures that

$$[0, \tau_{u_0, v_0}) \ni t \longrightarrow \begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix} \in L^2(\Omega) \times H^{-1}(\Omega) \text{ is uniformly continuous,}$$

i.e. condition (1.7) holds with $Z = L^2(\Omega) \times H^{-1}(\Omega)$.

We thus conclude that

Corollary 3.3.2. *Under assumptions (3.15), (3.6) any non global ε -regular solution to (3.1) can be extended to a piecewise ε -regular solution.*

It is an open question whether a piecewise ε -regular solution resulting from Corollary 3.3.2 exists globally in time.

4. HIGHER ORDER PARABOLIC EQUATION IN A HILBERT SCALE

Let a triple $(A, \{B_j\}, \Omega)$, where Ω is a bounded subdomain of \mathbb{R}^N forms a regular elliptic boundary value problem as in [CD, pp. 29-30], so that A with the domain $W_{\{B_j\}}^{2m,p}(\Omega)$ is sectorial in $L^p(\Omega)$, $p \in (1, \infty)$. Suppose additionally that A in $X = L^2(\Omega)$ with the domain $X^1 = H_{2, \{B_j\}}^{2m}$ is selfadjoint and positive definite. For example $A = (-\Delta)^m$ with Dirichlet boundary conditions $\{B_j = \frac{\partial^j}{\partial \nu^j}, j = 0, \dots, m-1\}$ although any particular choice does not influence in an essential way further considerations of this section.

Denote by $[(X^\alpha, A_\alpha); \alpha \in \mathbb{R}]$ the fractional power scale generated by (X, A) , which is a Hilbert scale equivalent to the interpolation-extrapolation scale generated by $(L^2(\Omega), A)$ and $[\cdot, \cdot]_\alpha$ (see [AM, Theorem V.1.5.15]). In particular

$$X^\alpha \hookrightarrow H_{2, \{B_j\}}^{2m\alpha} \text{ for } \alpha \in [0, 1] \quad (4.1)$$

and $X^\alpha \doteq H_{2, \{B_j\}}^{2m\alpha}$ unless $2m\alpha - \frac{1}{2}$ coincides with the order of some of boundary operators B_j , $j = 0, \dots, m-1$.

Consider further an initial-boundary value problem (see [WA])

$$\begin{cases} u_t + Au = f(u), & t > 0, \ x \in \Omega \subset \mathbb{R}^N, \ N > 2m > 2, \\ B_0 u = \dots = B_{m-1} u = 0, \\ u(0, x) = u_0 \in H_{2, \{B_j\}}^m(\Omega) \end{cases} \quad (4.2)$$

and focus on the case of critical nonlinearity

$$f \in C^1(\mathbb{R}, \mathbb{R}), \ |f'(s)| \leq c \left(|s|^{\frac{N+2m}{N-2m}-1} + 1 \right), \ s \in \mathbb{R}. \quad (4.3)$$

A technical lemma below displays the ‘critical’ feature of the exponent in (4.3) and is crucial both for the discussion of ε -regular solutions and piecewise ε -regular solutions to (4.2).

LEMMA 4.4.1. *Let $N > 2m$, $\frac{2m}{N-2m} < \rho \leq \frac{N+2m}{N-2m} =: \rho^*$ and*

$$|f(s) - f(t)| \leq c|s - t|(1 + |s|^{\rho-1} + |t|^{\rho-1}), \quad s, t \in \mathbb{R}. \quad (4.4)$$

Then, for each $\varepsilon \in [0, \frac{1}{2\rho}]$ there is certain $\gamma(\varepsilon) \in [\rho\varepsilon, \frac{1}{2}]$ such that

$$\begin{aligned} & \|f(v(\cdot)) - f(w(\cdot))\|_{X^{-\frac{1}{2}+\gamma(\varepsilon)}} \\ & \leq \tilde{c}\|v - w\|_{X^{\frac{1}{2}+\varepsilon}} \left(1 + \|v\|_{X^{\frac{1}{2}+\varepsilon}}^{\rho-1} + \|w\|_{X^{\frac{1}{2}+\varepsilon}}^{\rho-1} \right), \quad v, w \in X^{\frac{1}{2}+\varepsilon}. \end{aligned} \quad (4.5)$$

Furthermore, if $\rho = \rho^$ then $\gamma(\varepsilon) = \rho^*\varepsilon$.*

Proof. We first note that by (4.1)

$$X^{\frac{1}{2}+\varepsilon} \hookrightarrow L^p(\Omega), \quad 1 \leq p \leq \frac{2N}{N-2m-4m\varepsilon}, \quad \frac{1}{2\rho} \geq \varepsilon \geq 0, \quad (4.6)$$

$$X^{\frac{1}{2}-\gamma(\varepsilon)} \hookrightarrow L^p(\Omega), \quad 1 \leq p \leq \frac{2N}{N-2m+4m\gamma(\varepsilon)}, \quad \frac{1}{2} \geq \gamma(\varepsilon) \geq 0,$$

and hence

$$L^q(\Omega) \hookrightarrow X^{-\frac{1}{2}+\gamma(\varepsilon)}, \quad q \geq \frac{2N}{N+2m-4m\gamma(\varepsilon)}. \quad (4.7)$$

From (4.7), (4.4) and Hölder inequality we then obtain

$$\begin{aligned} & \|f(v(\cdot)) - f(w(\cdot))\|_{X^{-\frac{1}{2}+\gamma(\varepsilon)}} \leq c' \|f(v(\cdot)) - f(w(\cdot))\|_{L^{\frac{2N}{N+2m-4m\gamma(\varepsilon)}}(\Omega)} \\ & \leq c' \|c|v - w|(1 + |v|^{\rho-1} + |w|^{\rho-1})\|_{L^{\frac{2N}{N+2m-4m\gamma(\varepsilon)}}(\Omega)} \\ & \leq c' c \|v - w\|_{L^{\frac{2N}{N-2m-4m\varepsilon}}(\Omega)} \|1 + |v|^{\rho-1} + |w|^{\rho-1}\|_{L^{\frac{2N}{4m+4m(\varepsilon-\gamma(\varepsilon))}}(\Omega)} \\ & \leq c'' \|v - w\|_{L^{\frac{2N}{N-2m-4m\varepsilon}}(\Omega)} \left(1 + \|v\|_{L^{\frac{2N(\rho-1)}{4m+4m(\varepsilon-\gamma(\varepsilon))}}(\Omega)}^{\rho-1} + \|w\|_{L^{\frac{2N(\rho-1)}{4m+4m(\varepsilon-\gamma(\varepsilon))}}(\Omega)}^{\rho-1} \right), \end{aligned} \quad (4.8)$$

where $c'' = cc' \left(1 + |\Omega|^{\frac{4m+4m(\varepsilon-\gamma(\varepsilon))}{2N(\rho-1)}} \right)$.

As a consequence of (4.6) to pass from the right hand side of (4.8) to (4.5) we need to guarantee that

$$\frac{2N(\rho-1)}{4m+4m(\varepsilon-\gamma(\varepsilon))} \leq \frac{2N}{N-2m-4m\varepsilon},$$

i.e.

$$\rho \leq \frac{N+2m-4m\gamma(\varepsilon)}{N-2m-4m\varepsilon}.$$

However, if $\gamma(\varepsilon) \geq \rho\varepsilon$ then

$$\rho \leq \frac{N + 2m - 4m\rho\varepsilon}{N - 2m - 4m\varepsilon}. \quad (4.9)$$

Evidently (4.9) holds if and only if $\rho \leq \frac{N+2m}{N-2m} = \rho^*$. Also, $\rho = \rho^*$ cannot be attained for $\gamma(\varepsilon) > \rho^*\varepsilon$. The proof is thus complete. \blacksquare

We will consider next

$$(\mathcal{X}, \mathcal{A}) = (X^{-\frac{1}{2}}, A_{-\frac{1}{2}})$$

and a fractional power scale $[(\mathcal{X}^\alpha, \mathcal{A}_\alpha); \alpha \in \mathbb{R}]$ (equivalently interpolation-extrapolation scale) generated by $(\mathcal{X}, \mathcal{A})$. We also recall that

$$(\mathcal{X}^\alpha, \mathcal{A}_\alpha) = (X^{\alpha-\frac{1}{2}}, A_{\alpha-\frac{1}{2}}) \text{ for } \alpha \in \mathbb{R},$$

except for equivalent norms.

From Lemma 4.4.1 we infer in particular that in a critical case $\rho = \rho^*$ function f defines a substitution operator F being an ε -regular map relative to the pair $(\mathcal{X}^1, \mathcal{X}) = (H_{2,\{B_j\}}^m(\Omega), [H_{2,\{B_j\}}^m(\Omega)]')$ such that a counterpart of (1.3) holds. Namely,

$$\|F(v) - F(w)\|_{\mathcal{X}^{\rho^*}} \leq C\|v - w\|_{\mathcal{X}^{1+\varepsilon}} \left(1 + \|v\|_{\mathcal{X}^{1+\varepsilon}}^{\rho^*-1} + \|w\|_{\mathcal{X}^{1+\varepsilon}}^{\rho^*-1}\right) \quad (4.10)$$

for each $v, w \in \mathcal{X}^{1+\varepsilon}$ and $\varepsilon \in \left[0, \frac{1}{2\rho^*}\right]$.

This shows that (4.2) can be written abstractly in a Banach space $\mathcal{X} = (H_{2,\{B_j\}}^m(\Omega))'$ as

$$\begin{aligned} \dot{u}(t) + \mathcal{A}u(t) &= F(u(t)), \quad t > 0 \\ u(0) &= u_0 \in \mathcal{X}^1, \end{aligned} \quad (4.11)$$

with $\mathcal{A} = A_{-\frac{1}{2}}$ and F being an ε -regular map with respect to $(\mathcal{X}^1, \mathcal{X}) = (H_{2,\{B_j\}}^m(\Omega), [H_{2,\{B_j\}}^m(\Omega)]')$.

Note that in this example nonlinearity is in fact critical with respect to $(\mathcal{X}^1, \mathcal{X})$. Note also that $\varepsilon = \frac{1}{2\rho^*}$ is allowed in (4.10), which implies that ε -regular solution $u(\cdot, u_0)$ to the problem (4.11) belongs to $C((0, \tau_{u_0}), \mathcal{X}^{1+\frac{1}{2}}) \cap C^1((0, \tau_{u_0}), \mathcal{X}^{1+\zeta})$, $\zeta \in [0, \frac{1}{2}]$, so that

$$u(\cdot, u_0) \in C([0, \tau_{u_0}), H_{2,\{B_j\}}^m(\Omega)) \cap C((0, \tau_{u_0}), H_{2,\{B_j\}}^{2m}(\Omega)) \cap C^1((0, \tau_{u_0}), H_{2,\{B_j\}}^m(\Omega)). \quad (4.12)$$

Multiplying both sides of the first equation entering (4.2) by \dot{u} in $L^2(\Omega)$ we obtain

$$\int_0^t \|\dot{u}(s, u_0)\|_{L^2(\Omega)}^2 ds + \frac{1}{2}\|u(t, u_0)\|_{X^{\frac{1}{2}}}^2 - \frac{1}{2}\|u_0\|_{X^{\frac{1}{2}}}^2 \leq \Lambda(u(t, u_0)) - \Lambda(u_0), \quad t \in (0, \tau_{u_0}). \quad (4.13)$$

where

$$\Lambda(u(t, u_0)) := \int_{\Omega} \int_0^{u(t, u_0)} f(s) ds.$$

Let $c_{1/2} > 0$ be such that $\|\phi\|_{L^2(\Omega)} \leq c_{1/2}\|\phi\|_{X^{\frac{1}{2}}}$ and suppose further that

$$\int_0^r f(s)ds \leq ar^2 + b, \quad r \in \mathbb{R} \quad \text{with certain } a \in \left(0, \frac{1}{4c_{1/2}^2}\right) \quad \text{and } b \in \mathbb{R}. \quad (4.14)$$

From (4.13) we then have

$$\int_0^t \|\dot{u}(s, u_0)\|_{L^2(\Omega)}^2 ds + \frac{1}{4}\|u(t, u_0)\|_{X^{\frac{1}{2}}}^2 - \frac{1}{2}\|u_0\|_{X^{\frac{1}{2}}}^2 \leq -\Lambda(u_0) + b|\Omega|, \quad t \in (0, \tau_{u_0}). \quad (4.15)$$

and thus

$$\|u(t, u_0)\|_{X^{\frac{1}{2}}}^2 \leq 2\|u_0\|_{X^{\frac{1}{2}}}^2 - 4\Lambda(u_0) + 4b|\Omega|, \quad t \in (0, \tau_{u_0}) \quad (4.16)$$

which is a counterpart of (1.5).

Furthermore, (4.15) implies that

$$u(\cdot, u_0) \in W^{1,1}((0, \tau_{u_0}), L^2(\Omega)) \quad (4.17)$$

and hence

$$[0, \tau_{u_0}] \ni t \rightarrow u(t, u_0) \in L^2(\Omega) \text{ is uniformly continuous,}$$

as required in (1.7).

Based on Theorem 2.2.2 we conclude that

Corollary 4.4.1. *Under the assumptions (4.3), (4.14), whenever $\tau_{u_0} < \infty$ there exists $\tau \in (\tau_{u_0}, \infty]$ and a unique extension $U : [0, \tau) \rightarrow H_{2, \{B_j\}}^m(\Omega)$ of ε -regular solution $u(\cdot, u_0)$ to the problem (4.11) corresponding to (4.2). Furthermore, τ is an accumulation time of singular times unless $\tau = \infty$.*

Suppose further that $b = 0$ in (4.14) and let us restrict our attention to initial data satisfying

$$u_0 \in H_{2, \{B_j\}}^m(\Omega), \quad C_{NG}C \left[2\|u_0\|_{H_{2, \{B_j\}}^m(\Omega)}^2 - 4\Lambda(u_0) \right]^{\frac{\rho^* - 1}{2}} =: C(u_0) < 1 \quad (4.18)$$

where constants C_{NG} and C are as in the estimates below:

$$\begin{aligned} \|\phi\|_{L^{2\rho^*}(\Omega)}^{\rho^*} &\leq \text{const.} \|\phi\|_{H_{2, \{B_j\}}^{2m}(\Omega)} \|\phi\|_{H_{2, \{B_j\}}^{m\rho^* - 1}(\Omega)} \leq C_{NG} \|\phi\|_{X^1} \|\phi\|_{X^{\frac{1}{2}}}^{\rho^* - 1}, \\ |f(s)| &\leq C(1 + |s|^{\rho^*}), \quad s \in \mathbb{R}. \end{aligned} \quad (4.19)$$

From (4.2) and (4.19) we then get

$$\begin{aligned} \|u(t, u_0)\|_{X^1} &\leq \|\dot{u}(t, u_0)\|_{L^2(\Omega)} + \|C(1 + |u(t, u_0)|^{\rho^*})\|_{L^2(\Omega)} \\ &\leq \|\dot{u}(t, u_0)\|_{L^2(\Omega)} + C|\Omega|^{\frac{1}{2}} + CC_{NG}\|u(t, u_0)\|_{X^{\frac{1}{2}}}^{\rho^* - 1} \|u(t, u_0)\|_{X^1}, \quad t \in (0, \tau_{u_0}). \end{aligned} \quad (4.20)$$

Connecting (4.20), (4.16), (4.18) we next obtain

$$\|u(t, u_0)\|_{X^1} \leq \frac{1}{1 - C(u_0)} \|\dot{u}(t, u_0)\|_{L^2(\Omega)} + \frac{C|\Omega|^{\frac{1}{2}}}{1 - C(u_0)}, \quad t \in (0, \tau_{u_0}). \quad (4.21)$$

Note that as a consequence of (4.21), (4.17) for a triple of Hilbert spaces $X^1 \hookrightarrow X^{\frac{1}{2}} \hookrightarrow X^{-1}$ we have

$$u(\cdot, u_0) \in L^2((0, \tau_{u_0}), X^1), \quad \dot{u}(\cdot, u_0) \in L^2((0, \tau_{u_0}), L^2(\Omega)) \hookrightarrow L^2((0, \tau_{u_0}), X^{-1}).$$

Thus, by [TE, Chapter II, Lemma 3.2],

$$[0, \tau_{u_0}) \ni t \rightarrow u(t, u_0) \in X^{\frac{1}{2}} \text{ is uniformly continuous.} \quad (4.22)$$

Since $X^1 = X^{\frac{1}{2}}$ the above relation is a counterpart of (2.5).

Under the assumptions (4.3) and (4.14) with $b = 0$ we thus proved that

Corollary 4.4.2. *If $u_0 \in H_{2, \{B_j\}}^m(\Omega)$ satisfies (4.18) then τ_{u_0} cannot be finite, i.e. ε -regular solution $u(\cdot, u_0)$ to (4.2) exists globally in time.*

Remark 4.4.1. *In [WA] existence of a global smooth solutions have been obtained under the assumptions similar to (4.3), (4.14) but with an additional requirements that*

$$\begin{aligned} sf(s) &\geq -c \left(|s|^{\frac{N+2m}{N-2m}+1} + 1 \right), \\ sf(s) &\leq c \left(|s|^{\frac{N+2m}{N-2m}+1-\varepsilon} + 1 \right), \quad |s| \geq 1. \end{aligned} \quad (4.23)$$

Condition (4.23) was used in [WA] to get uniform continuity in the energy space of the smooth solution corresponding to arbitrarily chosen initial data $u_0 \in W_{\{B_j\}}^{2m,p}(\Omega)$, $p > N + 1$. Since within the above approach we cover the case when $u_0 \in W_{2, \{B_j\}}^m(\Omega)$ and we do not assume a ‘structure’ condition (4.23), the above considerations complements the results of [WA].

In case when instead of (4.3) we assume that

$$f \in C^1(\mathbb{R}, \mathbb{R}), \quad \lim_{|s| \rightarrow \infty} \frac{|f'(s)|}{|s|^{\frac{4m}{N-2m}}} = 0, \quad (4.24)$$

then the Nemitskiĭ map associated to f will be almost critical ε -regular map relative to the pair of spaces $(H_{2, \{B_j\}}^m(\Omega), [H_{2, \{B_j\}}^m(\Omega)]')$ so that Theorem 2.2.1 will apply. If dissipativeness condition $\limsup_{|s| \rightarrow \infty} \frac{f(s)}{s} \leq 0$ is assumed, then (4.14) holds for any $a > 0$

with certain $b = b(a)$ so that uniform in time $H_{2,\{B_j\}}^m(\Omega)$ estimate of ε -regular solutions to (4.2) will be derived similarly as in formulas (4.13)-(4.16). Additionally, stationary solutions to (4.2) will remain bounded in $H_{2,\{B_j\}}^m(\Omega)$ based on the estimate

$$(Au, u)_{L^2(\Omega)} = \|u\|_{X^{\frac{1}{2}}}^2 = (f(u), u)_{L^2(\Omega)} \leq \delta \|u\|_{L^2(\Omega)}^2 + C_\delta$$

and compactness of the resolvent of A will translate into compactness of ω -limit sets of bounded sets. Since we have also shown regularity condition (4.12) the above considerations proves that

THEOREM 4.4.1. *If $N \geq 2m$ and (4.24), (3.6) hold then (4.2) is globally well posed in $H_{2,\{B_j\}}^m(\Omega)$ and possesses a compact global attractor which is a subset of $H_{2,\{B_j\}}^{2m}(\Omega)$.*

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