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C^ℓ-G-triviality of map germs and Newton polyhedra, $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$.

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We provide a sufficient condition for the C^ℓ-G-triviality of deformations of map germs $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ of type $f_t(x) = f(x) + th(x)$ which satisfy a non-degeneracy condition, where $0 \leq \ell < \infty$ and G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$. This condition is given in terms of the Newton filtration of the map germ h. May, 2004 ICMC-USP

1. INTRODUCTION

The determinacy of families of map germs which are trivial with respect to some equivalence relation is a fundamental subject in singularity theory. Many works are devoted to the characterization of the topological triviality with respect to various equivalence relations, see [3] or [4], for example. In the case of the C^ℓ-triviality for $0 < \ell < \infty$, there are few works in this subject, in [2] there are given estimates for the $C^\ell - \mathcal{R}$-triviality for families of germs of functions and in [6] there are given estimates for the $C^\ell - G$-triviality of families of map germs which are semi-weighted homogeneous. In this paper we provide new estimates on the C^ℓ-G-triviality $0 \leq \ell < \infty$, G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, of map germs which satisfy a non-degeneracy condition with respect to some fixed Newton filtration. We generalize the results given by Ruas and Saia in [6] on weighted homogeneous map germs satisfying a convenient Lojasiewicz condition. The results give an explicit order such that the C^ℓ geometrical structure of a non-degenerate polynomial map germ is preserved after

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higher order perturbations. Our method consists of constructing controled vector fields based on control functions determined by a convenient Newton polyhedron.

2. THE NEWTON FILTRATION

We denote the ring of real analytic germs by $\mathcal{E}_n$. The key tool to provide estimates for the C^ℓ-triviality of map germs which are not semi-weighted homogeneous is the determinacy of a convenient Newton filtration for $\mathcal{E}_n$ based in a Newton polyhedron. To construct a Newton polyhedron we fix an $n \times m$ matrix $A = (a_i^j)$, with $i = 1, ..., n,\ j = 1, ..., m$, $a^j = (a_1^j, \dots, a_n^j) \in \mathbb{Q}_+^n$ and $m \geq n$, such that the first n colluns of A are $(0, ..., 0, a_j^j, 0, ..., 0)$ with $a_j^j > 0$, for all $j = 1, ..., n$.

For example, when $n = 2$, let $A = \begin{pmatrix} a_1^1 & 0 & a_1^3 & \cdots & a_1^m \\ 0 & a_2^2 & a_2^3 & \cdots & a_2^m \end{pmatrix}$.

The support of A is denoted by $Supp(A) = \{a^j,\ j = 1, ..., m\}$, the Newton polyhedron $\Gamma_+(A)$ is the convex hull in $\mathbb{R}^n$ of the set $Supp(A) + \mathbb{R}_+^n$ and the Newton diagram of A, denoted $\Gamma(A)$, is the union of the compact faces of $\Gamma_+(A)$.

For a fixed Newton polyhedron $\Gamma_+(A)$ and for each $w = (w_1, \dots, w_n) \in \mathbb{R}_+^n$ we define:

Definition 2.1.

(a) $\ell(w) = \min\{\langle w, k\rangle : k \in \Gamma_+(g)\}$, $\langle w, k\rangle = \sum_{i=1}^n w_i k_i$.
(b) $\Delta(w) = \{k \in \Gamma_+(g) :\ \langle w, k\rangle = \ell(w)\}$.
(c) Two vectors $a, a' \in \mathbb{R}_+^{n*}$ are equivalent if $\Delta(a) = \Delta(a')$.

A vector w is called a primitive integer if it is the vector with minimum length in $\mathrm{C}(w) \cap (\mathbb{Z}_+^n - \{0\})$, where $\mathrm{C}(w)$ is the half ray emanating from 0 passing through w.

Since each $(n-1)$-dimensional face Δ of $\Gamma(g)$ is associated to a primitive integer $w \in \mathbb{R}_+^{n*}$ such that $\Delta = \Delta(w)$, we fix $v^k = (v_1^k, ..., v_n^k),\ k = 1, ..., r$, the set of primitive integers associated to the $(n-1)$-dimensional faces Δ of $\Gamma_+(A)$ and denote by M the number $M = m.m.c.\{\ell(v^k)\}$.

Definition 2.2. For any monomial $x^\alpha = x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ in $\mathcal{E}_n$, let

$$\varphi(\alpha) = \min_{k=1}^{r} \left\{ \frac{M}{\ell(v^k)} \langle \alpha, v^k\rangle \right\}.$$

For an analytic real germ $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ call

$$fil(f) := \inf\{\varphi(\alpha)\ /\ \alpha \in Supp(f)\}.$$

Associated to a Newton polyhedron $\Gamma_+(A)$ we define a control function, which is fundamental to describe the integrable controlled vector fields that guarantee the topological triviality.

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For any vector a^j of the matrix A and $p \in \mathbb{R}_+$, we call $pa^j = (pa_1^j, pa_2^j, \dots, pa_n^j)$.

We choose the number p as large as needed to have $2pa_i^j$ integer and define the smooth function $\rho(x)$, called *control* of $\Gamma_+(A)$:

$$\rho(x) = \left(\sum_{j=1}^{m} x^{2pa^j} \right)^{\frac{1}{2p}} = \left(\sum_{j=1}^{m} x_1^{2pa_1^j} x_2^{2pa_2^j} \cdots x_n^{2pa_n^j} \right)^{\frac{1}{2p}}. \tag{1}$$

DEFINITION 2.3. An analytic function germ $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ with Taylor series $f(x) = \sum_{\nu} c_\nu x^\nu$, where $\nu = (\nu_1, ..., \nu_n)$ is called A-homogeneous of degree d if $f(x) = \sum_{\nu \in \Gamma(\rho^d)} c_\nu x^\nu$.

LEMMA 2.1 ([9], p. 524). *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be a polynomial function germ with supp$(f) \in \Gamma_+(\rho^d)$. Then there exists a constant $c_1 > 0$ and a neighbourhood V of the origin such that $\|f(x)\| \le c_1 \rho(x)^d$ for all $x \in V$.*

In the sequel we shall consider the decomposition of an analytic function germ $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ in its A-homogenous parts

$$f(x) = H_d(x) + \cdots + H_\ell(x) + \cdots \quad , \ell \ge d$$

where each function germ H_i is A-homogenous of degree i.

For any germ g with Taylor series $g = \sum a_\alpha x^\alpha$, if Δ is a subset of a Newton polyhedron $\Gamma_+(A)$, we denote by g_Δ the germ $g_\Delta = \sum_{\alpha \in \Delta} a_\alpha x^\alpha$

DEFINITION 2.4. [[9], p. 525] We say that 0 is an A-isolated point of f if for each compact face γ of $\Gamma(\rho^d)$, the equation $f_\Delta(x) = 0$ does not have solution in $(\mathbb{R} - \{0\})^n$. In this case the function f is said to be A-isolated.

LEMMA 2.2 ([9], p. 525). *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be an analytic function germ written as $f(x) = H_d(x) + \cdots H_\ell(x) + \cdots$. If f is A-isolated, then there exists a real $c > 0$ such that $c\rho(x)^d \le \|f(x)\|$ for all x in a neighborhood of the origin.*

The next lemmas form the key tool to guarantee the class of differentiability of the controlled vector fields.

LEMMA 2.3. *For any analytic function germ $h : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ with h in the interior of $\Gamma_+(\rho^d)$:*

$$\lim_{x \to 0} \frac{h(x)}{\rho(x)^d} = 0.$$

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Proof: It is sufficient to prove that for all $a = (a_1, ..., a_n)$ in the interior of $\Gamma_+(\rho^d)$, $\lim_{x\to 0} x^a/\rho(x)^d = 0$, where $x^a = x_1^{a_1} \dots x_n^{a_n}$.

Suppose that this does not occur, hence there exists a constant $c > 0$ and a neighborhood V of the origin such that $\|x^a/\rho(x)^d\| \geq c$, for all x in V. Therefore the origin 0 is in the closure of the set $X := \{x \in \mathbb{R}^n \ / \ \|x\| \geq c\rho(x)^d\}$.

Since X is semi-analytic we apply the Curve Selection Lemma to conclude that there exists an analytic curve $\lambda : (0, \epsilon] \to X$, with $\lambda(0) = 0$ such that

$$\lambda_1(t) \sim t^{\alpha_1}, ..., \lambda_n(t) \sim t^{\alpha_n}.$$

Therefore, as $\rho(\lambda(t))^d \leq \frac{1}{c}\|\lambda(t)^a\|$ we obtain $\inf_j\{\langle da^j, \alpha\rangle\} \geq \langle a, \alpha\rangle$, but

$$\Delta(\alpha) := \{b \in \Gamma_+(\rho^d) \ / \ \langle b, \alpha\rangle = \ell(\alpha)\}$$

is a face of $\Gamma_+(\rho^d)$ with $\ell(\alpha) := \min\{\langle c, \alpha\rangle \ / \ c \in \Gamma_+(\rho^d)\}$.

¿From the fact that da^j, is one of the vertices of $\Delta(\alpha)$, we have $\langle da^j, \alpha\rangle = \ell(\alpha)$, however, $\langle a, \alpha\rangle \leq \langle da^j, \alpha\rangle = \ell(\alpha)$, hence $a \in \Gamma(\rho^d)$ and we obtain a contradiction to the hypothesis.$\|$

Lemma 2.4. *Let $h : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ in $\mathcal{E}_n$ such that $fil(h) \geq fil(\rho^d) + \ell R + 1$. Then $\frac{h(x)}{\rho(x)^d}$ is differentiable of class C^ℓ.*

Proof: The proof is done by induction on ℓ.

For $\ell = 1$, we obtain $fil(h(x)) \geq fil(\rho(x)^d) + R + 1$.

Since $f(x) = \frac{h(x)}{\rho(x)^d}$ we have

$$\nabla f(x) = \frac{1}{\rho(x)^{2d}}(\rho^d.\nabla h - h.\nabla\rho^d)$$

and also that

$$\begin{aligned} fil(\rho^d.\nabla h - h.\nabla\rho^d) &\geq fil(h) + fil(\rho^d) - R \\ &\geq 2fil(\rho^d) + 1 \\ &= fil(\rho^{2d}) + 1. \end{aligned}$$

Applying the Lemma 2.3 we obtain $\lim\limits_{x\to 0} \frac{1}{\rho^{2d}}(\rho^d.\nabla h - h.\nabla\rho^d) = 0$, and ∇f is continuous. Therefore f is of class C^1.

Suppose now that any function $f(x) = \frac{h_1(x)}{\rho(x)^d}$ satisfying

$$fil(h_1) \geq fil(\rho^d) + (\ell - 1)R + 1,$$

is of class $C^{\ell-1}$. Let $f(x) = \frac{h_1(x)}{\rho(x)^d}$ with $fil(h_1) \geq fil(\rho^d) + \ell R + 1$. Hence $\nabla f(x) = \frac{H(x)}{\rho(x)^d}$, with $fil(H(x)) \geq fil(\rho^d) + (\ell - 1)R + 1$, is of class $C^{\ell-1}$ therefore f is of class C^ℓ. $\|$

Ruas and Saia in [6] determined conditions for the C^ℓ-G- triviality,$\ell \geq 0$, $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, of weighted homogeneous map germs with isolated singularity, in terms of the weights and degrees. Here we generalize these results for the class of map germs that are A-homogenous for some fixed matrix A.

Definition 2.5. An analytic map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$; $f = (f_1, \dots, f_p)$ is a A-homogenous of degree $d = (d_1, \dots, d_p)$ if each f_i is A-homogenous of degree d_i.

The main idea is to choose, for each group $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, a convenient A-isolated function germ which is equivalent to a control function $\rho(x)$ associated to $\Gamma_+(A)$. First we shall do it for the group $\mathcal{R}$.

3. THE GROUP $\mathcal{R}$

For each polynomial map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ we denote M_I, for $I = \{i_1, ..., i_p\}$, a minor of order p of the Jacobian matrix df. Call $s_I := fil(M_I)$ and denote $\alpha := m.m.c.\{s_I\}$. The we define $N_{\mathcal{R}} f := \sum_I M_I^{2\alpha_I}$ where $\alpha_I = \alpha / s_I$.

Write $N_{\mathcal{R}} f = H_D + \cdots + H_L$, with $L > D$, and suppose that $N_{\mathcal{R}} f$ is A-isolated. From the Lemmas (2.1) e (2.2) we see that

$$N_{\mathcal{R}} f \leq \|H_D\| + \cdots + \|H_L\| \leq c_D \rho^D + \cdots + c_L \rho^L \leq (c_D + \cdots c_L)\rho^D \quad \text{and} \quad N_{\mathcal{R}} f \geq c\rho^D.$$

Hence there exist constants c_1 and $c_2 > 0$ such that $c_1 \rho^D \leq N_{\mathcal{R}} f \leq c_2 \rho^D$.

Consider now a deformation $f_t = f + t\theta$ of f and define $N_{\mathcal{R}} f_t := \sum_I M_{t_I}^{2\alpha_I}$.

Lemma 3.1. *Suppose that $N_{\mathcal{R}} f = \sum_I M_I^2 = H_D + \cdots + H_L$ is A-isolated for some Newton polyhedron $\Gamma_+(A)$. If $f_t = f + t\theta$ is a deformation of f with $fil(\theta_i) > fil(f_i)$, then there exist constants c_1 and $c_2 > 0$ and a neighborhood V of 0 such that for all $x \in V$*

$$c_1 \rho^D \leq N_{\mathcal{R}} f_t \leq c_2 \rho^D.$$

Proof: Since $N_{\mathcal{R}} f_t = N_{\mathcal{R}} f + t\Theta$ where Θ satisfies $fil(\Theta) > fil(N_{\mathcal{R}} f)$, we can write

$$N_{\mathcal{R}} f \leq N_{\mathcal{R}} f_t + \|\Theta\|, \text{ for all } 0 \leq t \leq 1.$$

From the Lemma 2.2 there exist a constant $c_1 > 0$ such that

$$c_1 \rho^D \leq N_{\mathcal{R}} f \leq N_{\mathcal{R}} f_t + \|\Theta\|.$$

Since $fil(\Theta) > fil(N_{\mathcal{R}} f)$, we obtain $\lim_{x \to 0} \Theta / \rho^D = 0$, therefore $c_1 \rho^D \leq N_{\mathcal{R}} f_t$.

On the other hand, $N_{\mathcal{R}} f_t \leq N_{\mathcal{R}} f + \|\Theta\| \leq c_2 \rho^D + \|\Theta\| \leq (c_2 + c_3)\rho^D$. ||

In order to show the main result of this section we fix a Newton polyhedron $\Gamma_+(A)$ with associated primitive integers v^j, and denote

$$R = \max_j \max_i \left\{ \frac{M}{\ell(v^j)} v_i^j \right\}, \text{ and } r = \min_j \min_i \left\{ \frac{M}{\ell(v^j)} v_i^j \right\}.$$

Let $f_t(x) = f(x) + t\theta(x)$, with $\theta = (\theta_1, ..., \theta_p)$, be a deformation of a polynomial map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$.

We prove now the following:

Proposition 3.1. *Suppose that $N_{\mathcal{R}} f := \sum_i M_I^{2\alpha_I}$ is A-isolated. Then*

(a) *If $fil(\theta_i) \geq fil(f_i) + \ell R - r + 1$, then for $t \in [0, 1]$, f_t is C^ℓ-$\mathcal{R}$-trivial for all $\ell \geq 1$;*

(b) *If $fil(\theta_i) \geq fil(f_i)$, then f_t is C^0-$\mathcal{R}$-trivial for all $t \in [0, 1]$.*

Proof: We follow the proof given by Ruas and Saia in [6] for the weighted homogeneous case.

(a) For each p minor M_{t_I} of df_t, we construct the vector field W_I defined by the cofactors of M_{t_I} :

$$W_I = \sum_{i=1}^n w_i \frac{\partial}{\partial x_i}, \text{ with } \begin{cases} w_i = 0, \text{ if } i \notin I \\ w_{i_m} = \sum_{j=1}^p N_{ji_m} (\frac{\partial f_t}{\partial t})_j, \text{ if } i_m \in I \end{cases}$$

Where N_{ji_m} denotes the $(p-1) \times (p-1)$ minor cofactor of element $\dfrac{\partial f_j}{\partial x_{i_m}}$ in the matrix df. Hence

$$\frac{\partial f_t}{\partial t} M_{t_I} = df(W_I).$$

Consider the vector field $W_{\mathcal{R}} := \sum_I M_I^{2\alpha_I - 1} W_I$, hence $N_{\mathcal{R}} f_t \cdot \frac{\partial f_t}{\partial t} = df_t(W_{\mathcal{R}})$ and

$$\begin{aligned} fil(W_{\mathcal{R}}) &= \min\{fil(M_I^{2\alpha_I - 1}) + fil(W_I)\} \\ &\geq \min\{2\alpha - fil(M_I) + fil(N_{ji_m}) + fil(\theta_j)\} \\ &\geq \min\{2\alpha - fil(M_I) + fil(M_I) - fil(\frac{\partial f_j}{\partial x_{i_m}}) + fil(\theta_j)\} \\ &\geq \min\{2\alpha - (fil(f_j) - r) + fil(\theta_j)\} \\ &\geq 2\alpha + \ell R + 1. \end{aligned}$$

Now consider the vector field $V : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}, 0$: $V(x) = \dfrac{W_{\mathcal{R}}}{N_{\mathcal{R}} f_t}$ which is of class C^ℓ from Lemmas (2.3) and (2.4).

The C^ℓ-triviality, for small values of t, follows from the equation $\frac{\partial f_t}{\partial t}(x, t) = (df_t)_x(V(x, t))$, and a similar argument shows that the result follows for all $t_0 \in [0, 1]$.

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(b) Since

$$\begin{aligned} fil(W_{\mathcal{R}}) &= fil(\sum_I M_I^{2\alpha_I-1}W_I) \\ &\geq \min\{2\alpha - fil(M_I) + fil(M_I) - fil(\frac{\partial f_j}{\partial x_{i_m}}) + fil(\theta_j)\} \\ &\geq \min\{2\alpha - (fil(f_j) - r) + fil(f_j) + R - r\} \\ &= 2\alpha + r = fil(\rho^D) + r \end{aligned}$$

and $fil(W_{\mathcal{R}}) \geq fil(\rho^D) + r = fil(\rho^D\|x\|)$, we see that $\frac{W_{\mathcal{R}}}{\rho^D\|x\|}$ is limited. Therefore, from the inqualities

$$\left\|\frac{W_{\mathcal{R}}}{N_{\mathcal{R}}f_t}\right\| \leq c\left\|\frac{W_{\mathcal{R}}}{\rho^D}\right\| \leq c'\,\|x\|$$

we obtain tha the vector field $\dfrac{W_{\mathcal{R}}}{N_{\mathcal{R}}f_t}$ is integrable. ‖

3.1. Examples

Example 3.1. Here we show that these estimates can not be improved.

Let $f_t(x,y) = (x^2+y^2)^2 + tx^ay^b$, we consider in this case $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which gives the usual filtration by the degree, then $fil(f_0) = 4$ and $R = r = 1$. From the proposition 3.1 we see that the family is C^ℓ-trivial if $fil(x^ay^b) = a + b \geq fil(f_0) + \ell = 4 + \ell$.

If we fix $x^ay^b = x^{p+1}$, then $fil(x^{5+p}) = 5 + p$, and f_t is C^{p+1}-trivial.

However, Kuiper in [8] showed that the family $f_t(x,y) = (x^2+y^2)^2 + tx^{p+1}$ is not C^{p+2}-trivial, for all $p \geq 0$.

Example 3.2. Let $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$; $f(x,y) = (xy, x^{2b+2} - y^{2b} + x^{2r}y^{2s})$, with

$$r + s = b, \quad r + 2s = b + 1 \quad \text{and} \quad r > s.$$

Fixing $br + (b+1)s < (b+1)b$, consider the matrix $A = \begin{pmatrix} \frac{1}{b} & 0 & \frac{r}{(b+1)b} \\ 0 & \frac{1}{b+1} & \frac{s}{(b+1)b} \end{pmatrix}$.

The corresponding Newton polyhedron $\Gamma_+(\rho^{2b(b+1)})$ has two faces with vertices $\{(2b+2,0),(2r,2s),(0,2b)\}$ and associate primitive integers $v_1 = (1,1)$ and $v_2 = (1,2)$.

Since $m.c.m\{\ell(v_1),\ell(v_2)\} = 2b(b+1)$, we obtain $fil(xy) = 2b+2$, $fil(y^{2b}) = 2b(b+1)$, $fil(x^{2r}y^{2s}) = 2b(b+1)$ and $fil(x^{2b+2}) = 2b(b+1)$, then $R = 2b$ and $r = b$.

The 2×2 minor of df is $M = -2((b+1)x^{2b+2} + by^{2b} + (r-s)x^{2r}y^{2s})$, which is A-homogenous of degree $2b(b+1)$

To apply the Proposition 3.1 to obtain the C^ℓ-triviality of a family $f_t = f + t(\theta_1, \theta_2)$ we need $fil(\theta_1) \geq b + 2b\ell + 3$ and $fil(\theta_2) \geq 2b^2 + b + 2b\ell + 1$.

For example, if we consider $(\theta_1, \theta_2) = (x^5y^9, y^{4(b+1)})$, $fil(\theta_1) = 14b + 14$ and $fil(\theta_2) = 4b^2 + 8b + 4$, then if $b \geq 3$, the number $\ell = 6$ satisfies the above inequalities and the family

$$f_t(x, y) = f(x, y) + t(\theta_1, \theta_2) = (xy + tx^5y^9, x^{2b+2} - y^{2b} + x^{2r}y^{2s} + ty^{4(b+1)})$$

is C^6-$\mathcal{R}$-trivial.

Example 3.3. Let $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$, given by $f(x, y) = y^7 + x^4y + x^9$.
Then $df = (4x^3y + 9x^8, 7y^6 + x^4)$, with minors $M_1 = 4x^3y + 9x^8$ and $M_2 = 7y^6 + x^4$.

Consider $A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 6 & 1 \end{pmatrix}$, with associate control $\rho(x, y) = (y^{12} + x^6y^2 + x^8)^{\frac{1}{2}}$

To check that $N^*_{\mathcal{R}}f = M_1^2 + M_2^2 = (4x^3y + 9x^8)^2 + (7y^6 + x^4)^2$ is A-isolated, call Δ_1 and Δ_2 the 1-dimensional compact faces of $\Gamma_+(\rho^{36})$, then $N^*_{\mathcal{R}}f|_{\Delta_1} = (4x^3y)^2 + (7y^6)^2$, $N^*_{\mathcal{R}}f|_{\Delta_2} = (4x^3y)^2 + (x^4)^2$ and

$$\{(x, y) \in \mathbb{R}^2 \ / \ N^*_{\mathcal{R}}f|_{\Delta_1}\} = 0 \subset \{(x, y) \in \mathbb{R}^2 \ / \ xy = 0\}.$$

and

$$\{(x, y) \in \mathbb{R}^2 \ / \ N^*_{\mathcal{R}}f|_{\Delta_2}\} = 0\} \subset \{(x, y) \in \mathbb{R}^2 \ / \ xy = 0\}.$$

As for any monomial x^ay^b, $\varphi(x^ay^b) = \min\{2(5a + 3b), 9(a + b)\}$, we obtain $fil(f) = \min\{\varphi(y^7), \varphi(x^4y), \varphi(x^9)\} = 42$, with $R = 10$ and $r = 6$.

Consider the family $f_t(x, y) = y^7 + x^4y + x^9 + tx^5y^7$, since $fil(x^5y^7) = 92$ it is enough to have $\ell = 5$ to apply the Proposition (3.1) and conclude that f_t is C^5-$\mathcal{R}$-trivial.

Example 3.4. The modified Briançon-Speder example
Let $f : \mathbb{K}^3, 0 \to \mathbb{K}, 0$, $f(x, y, z) = x^{15} + xy^7 + z^5$. For the case $\mathbb{K} = \mathbb{C}$, the family $F(x, y, z, t) = x^{15} + xy^7 + z^5 + ty^6z$ is topologically trivial, since it has constant Milnor number for all t. Briançon e Speder showed in [1] that the variety $F^{-1}(0)$ in $\mathbb{C}^4$ is not Whitney equisingular on the parametre space $0 \times \mathbb{C}$ at 0.
A full description of all equisingular deformations of f is given in [7]. The variety $F^{-1}(0)$, defined by $F(x, y, z, t) = f(x, y, z) + tx^ay^bz^c$ is Whitney equisingular if, and only if, $x^ay^bz^c \in \Gamma_+(\{(15, 0, 0), (0, 8, 0), (0, 0, 5), (1, 7, 0)\})$.

Here we consider the family $F : \mathbb{R}^4, 0 \to \mathbb{R}, 0$ with $F(x, y, z, t) = f(x, y, z) + tx^ay^bz^c$.

Ruas and Saia in [6] showed that if $a + 2b + 3c \geq 18$ the family is Whitney equisingular. This result was improved by Fernandes and Ruas in [5], they showed the Whitney equisingularity for the family if $a + 2b + 3c \geq 17$.

Since $df = (15x^{14} + y^7, 7xy^6, 5z^4)$, consider $A = \begin{pmatrix} 14 & 0 & 0 & 1 \\ 0 & 7 & 0 & 6 \\ 0 & 0 & 4 & 0 \end{pmatrix}$.

Therefore $N^*_{\mathcal{R}} f = M_1^2 + M_2^2 + M_3^2 = (15x^{14} + y^7)^2 + (7xy^6)^2 + (5z^4)^2$ is A-isolated.

As for any monomial $x^a y^b z^c$, $\varphi(a, b, c) = \min\{\langle (a, b, c), (6, 13, 21)\rangle, 3\langle (a, b, c), (4, 4, 7)\rangle\} = \min\{6a + 13b + 21c, 12a + 12b + 21c\}$.

Hence $fil(f) = 90$ since $\varphi(x^{15}) = 90$, $\varphi(xy^7) = 96$ and $\varphi(z^5) = 105$.

As $R = 21$ and $r = 6$, for the C^1-triviality of the family f_t we need $fil(x^a y^b z^c) \geq 105$.

Since the C^1-triviality implies the Whitney equisingularity, we obtain that $F^{-1}(0)$ is Whitney-equisingular along the parameter space if $\min\{6a + 13b + 21c, 12a + 12b + 21c\} \geq 105$.

4. THE GROUP $\mathcal{C}$

For each anlytic map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ we denote $N_{\mathcal{C}} f := \sum_{i=1}^p (f_i)^{2\beta_i}$ where

$$\beta_i := \frac{mmc\{fil(f_j), j = 1, ..., p\}}{fil(f_i)}.$$

If $N_{\mathcal{C}} f = H_D + \cdots + H_L$ is A-isolated for some matrix A, apply the Lemmas (2.1) and (2.2) to see that there exists a constant c such that $N_{\mathcal{C}} f \geq c\rho^D$, since

$N_{\mathcal{C}} f \leq \|H_D\| + \cdots + \|H_L\| \leq c_D \rho^D + \cdots + c_L \rho^L \leq (c_D + \cdots + c_L)\rho^D$ for constants $c_D, \ldots, c_L$.

Hence there exist constants k_1 and $k_2 > 0$ such that $k_1 \rho^D \leq N_{\mathcal{C}} f \leq k_2 \rho^D$.

Consider now a deformation $f_t = f + t\theta$ of f with $fil(\theta_i) > fil(f_i)$ and define $N_{\mathcal{C}} f_t := \sum_{i=1}^p (f_{ti})^{2\beta_i}$ where β_i is defined in $N_{\mathcal{C}} f$

Lemma 4.1. *Suppose that $N_{\mathcal{C}} f$ is A-isolated for some matrix A. If $f_t = f + t\theta$ is a deformation of f with $fil(\theta_i) > fil(f_i)$, there exist constants k_1 and $k_2 > 0$ and a neighbourhood V of 0 such that for all $x \in V$,*

$$k_1 \rho^{D_1} \leq N_{\mathcal{C}} f_t \leq k_2 \rho^{D_1}.$$

Proof: Since $N_{\mathcal{C}} f_t = N^*_{\mathcal{C}} f + t\Theta$, with $fil(\Theta) > fil(N_{\mathcal{C}} f)$.

We obtain $N_{\mathcal{C}} f \leq N_{\mathcal{C}} f_t + \|\Theta\|$, for all t with $0 \leq t \leq 1$.

Since $N_{\mathcal{C}} f = H_{D_1} + \cdots + H_{D_r}$ is A-isolated point, there exist constants k_1 and $k_2 > 0$ with $k_1 \rho^{D_1} \leq N_{\mathcal{C}} f \leq k_2 \rho^{D_1}$.

Therefore

$$k_1 \rho^{D_1} \leq N_{\mathcal{C}} f \leq N^*_{\mathcal{C}} f_t + \|\Theta\|$$

and as $fil(\Theta) > fil(N_{\mathcal{C}}f)$, $\lim_{x\to 0} \Theta/\rho^{D_1} = 0$ and this implies that

$$k_1\rho^{D_1} \leq N_{\mathcal{C}}f_t.$$

On the other hand, $N_{\mathcal{C}}f_t \leq N_{\mathcal{C}}f \leq k_2\rho^{D_1}$, and the result follows.

In order to show the main result of this section we fix a Newton polyhedron $\Gamma_+(A)$ with associated primitive integers v^j, and call

$$R = \max_j \max_i \left\{ \frac{M}{\ell(v^j)} v_i^j \right\}, \text{ and } r = \min_j \min_i \left\{ \frac{M}{\ell(v^j)} v_i^j \right\}.$$

We prove now the following:

Proposition 4.1. *Let $f_t(x) = f(x) + t\theta(x)$, with $\theta = (\theta_1, ..., \theta_p)$ be a deformation of a polynomial map germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$. Suppose that $N_{\mathcal{C}}f$ is A-isolated for some matrix A. Then $f_t = f + t\theta$ is C^ℓ-$\mathcal{C}$-trivial for all $t \in [0,1]$, if $fil(\theta_i) \geq d + \ell R + 1$, for all i and $\ell \geq 1$, with $d := \max\{fil(f_i)\}$.*

Proof: The C^ℓ-$\mathcal{C}$-triviality of f_t is shown by constructing map germs $V_i : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^p \times \mathbb{R}, 0$; $V_i = (V_{i1}, ..., V_{ip})$ of class C^ℓ, with $V_{ij}(x,0) = \delta_{ij}(x)$ in such a way that $\frac{\partial f_t}{\partial t} = \sum_{i=1}^p V_i(x,t)(f_{ti})$.

Since $\dfrac{\partial f_t}{\partial t} = \dfrac{\partial f_t}{\partial t} \cdot \left(\dfrac{\sum_{i=1}^p f_{ti}^{2\beta_i - 1} f_{ti}}{N_{\mathcal{C}}f_t} \right)$, define $W_i(x,t) = \dfrac{\partial f_t}{\partial t} \cdot f_{ti}^{2\beta_i - 1}$.

Hence $\frac{\partial f_t}{\partial t} = \sum\limits_{i=1}^{p} \left(W_i(x,t)/N_{\mathcal{C}}f_t\right)(f_{ti})$. If $B := mmc\{fil(f_j), j = 1, ..., p\}$, then

$$\begin{aligned} fil(W_i) &= \min_j\{fil(f_i^{2\beta_i - 1}) + fil(\theta_j)\} \geq \\ &\geq 2B - d + d + \ell R + 1 = \\ &= 2B + \ell R + 1, \ \forall i. \end{aligned}$$

Let $V : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0$ the vector field defined as: $(0, V_p, 0)$, where $V_p(x,y,t) = \sum_{i=1}^p (W_i(x,t)/N_{\mathcal{C}}f_t) y_i$.

¿From the Lemma (2.4), V is of class C^ℓ and the result follows by integrating the vector field V. ‖

Example 4.1. Let $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$; $f(x,y) = (xy + x^2y^2, x^{2(c+1)} + xy - y^{2c})$ with $c \geq 2$.

Consider $A = \begin{pmatrix} 2(c+1) & 0 & 1 \\ 0 & 2c & 1 \end{pmatrix}$ with associate control function

$$\rho(x,y) = (y^{4c} + x^2y^2 + x^{4(c+1)})^{\frac{1}{2}}.$$

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Here $m.c.m.\{\ell(v^1), \ell(v^2)\} = 2c(c+1)$, $R = 2c^2 + c$ and

$$\varphi(a,b) = \min\{(c+1)\langle(a,b),(2c-1,1)\rangle, c\langle(a,b),(1,2c+1)\rangle\}.$$

Since $N^*_{\mathcal{C}}f$ is A-isolated, we apply the Proposition 4.1 to obtain that $f_t(x,y) = (xy + x^2y^2 + tx^cy^c, x^{2(c+1)} + xy - y^{2c} + tx^{c-1}y^{c+1})$ is $C^{(c-2)}$-$\mathcal{C}$-trivial, for $c \geq 3$.

5. THE GROUP $\mathcal{K}$

To define the control function for the group $\mathcal{K}$ we use the control functions $N_{\mathcal{R}}f$ and $N_{\mathcal{C}}$ and consider two numbers a and b such that $fil([N_{\mathcal{R}}f]^a) = fil([N_{\mathcal{C}}f]^b)$ with respect to some Newton polyhedron $\Gamma_+(A)$. Then we define $N_{\mathcal{K}}f = [N_{\mathcal{R}}f]^a + [N_{\mathcal{C}}f]^b$.

If we suppose that $N_{\mathcal{K}}f$ is A-isolated we have the following:

Proposition 5.1. *Deformations $f_t = f + t\theta$ of f, with com $fil(\theta_i) \geq d + \ell R + 1, \forall i$, are C^ℓ-$\mathcal{K}$-trivial for all $t \in [0,1]$.*

Proof: Define $N^*_{\mathcal{K}}f_t := [N_{\mathcal{R}}f_t]^a + [N_{\mathcal{C}}f_t]^b$, then

$$\begin{aligned} N_{\mathcal{K}}f_t \cdot \frac{\partial f_t}{\partial t} &= [N_{\mathcal{R}}f_t]^a \cdot \frac{\partial f_t}{\partial t} + [N_{\mathcal{C}}f_t]^b \cdot \frac{\partial f_t}{\partial t} \\ &= [N_{\mathcal{R}}f_t]^{a-1} \cdot [df_t]_x(W_{\mathcal{R}}) + [N_{\mathcal{C}}f_t]^{b-1} \cdot \sum W_i(x,t)(f_{ti}) \\ &= [df_t]_x([N_{\mathcal{R}}f_t]^{a-1}W_{\mathcal{R}}) + \sum([N_{\mathcal{C}}f_t]^{b-1}W_i(x,t))(f_{ti}). \end{aligned}$$

Hence $\dfrac{\partial f_t}{\partial t} = [df_t]_x(\xi) + \sum(\eta_i)(f_{ti})$ with $\xi := \frac{[N_{\mathcal{R}}f_t]^{a-1}W_{\mathcal{R}}}{N_{\mathcal{K}}f_t}$ and $\eta_i := \dfrac{[N_{\mathcal{C}}f_t]^{b-1}W_i(x,t)}{N_{\mathcal{K}}f_t}$. Since

$$\begin{aligned} fil([N_{\mathcal{R}}f_t]^{a-1}W_{\mathcal{R}}) &\geq (a-1)\cdot 2\alpha + 2\alpha + \ell R + 1 \\ &= 2\alpha a + \ell R + 1 \text{ where } \alpha := m.m.c.\{fil(M_I)\}; \end{aligned}$$

$$\begin{aligned} fil([N_{\mathcal{C}}f_t]^{b-1}W_i) &\geq (b-1)\cdot 2B + 2B + \ell R + 1 \\ &= 2Bb + \ell R + 1 \text{ where } B := m.m.c.\{fil(f_i)\}; \end{aligned}$$

and

$$\begin{aligned} fil(N_{\mathcal{K}}f_t) &= fil([N_{\mathcal{R}}f_t]^a) = fil([N_{\mathcal{C}}f_t]^b) \\ &= 2\alpha a = 2Bb, \end{aligned}$$

we obtain that the vector fields ξ e $\eta = (\eta_1, ..., \eta_p)$ are of class C^ℓ and $f_t = f + t\theta$ is a C^ℓ-$\mathcal{K}$-trivial deformation of f. $\|$

EXAMPLE 5.1. Let $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0,\ f(x, y, z) = (x, xy + yz^2 \pm y^4)$. We have $df = \begin{bmatrix} 1 & 0 & 0 \\ y & x + z^2 \pm 4y^3 & 2yz \end{bmatrix}$, hence $M_{12} = x + z^2 \pm 4y^3$ and $M_{13} = 2yz$. Consider $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$. Here $R = 6$ and $\varphi(a, b, c) = \{6a + 3b + 3c, 6a + 2b + 4c\}$, hence $fil(f_1) = 6$ e $fil(f_2) = 8$.

For $\Theta = (0, z^5)$ we have $fil(z^5) = 15$, hence $f_t(x, y, z) = f(x, y, z) + t(0, z^5)$ is C^1-$\mathcal{K}$-trivial.

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