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$\mathcal{A}$-topological triviality of map germs and Newton filtrations.

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We apply the method of constructing controlled vector fields to give sufficient conditions for the $\mathcal{A}$-topological triviality of deformations of map germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $f_t(x) = f(x)+th(x)$, with $n \geq p$ or $n \leq 2p$. These conditions are given in terms of an appropriate choice of Newton filtrations for $\mathcal{O}_n$ and $\mathcal{O}_p$ and are for the $\mathcal{A}$-tangent space of the germ f.

For the case $n \geq p$, we follow the technique used by M. A. S. Ruas in her Ph.D. Thesis [5] and construct *control functions* in the target and in the source to obtain, via a partition of the unit, a unique control function. We use the control function of the target to give an estimate for the case $p \geq 2n$. Moreover, in this case we also show that if the the coordinates of the map germ satisfy a Newton non-degeneracy condition, deformations by terms of higher filtration are topologically trivial. May, 2004 ICMC-USP

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1. INTRODUCTION

The determinacy of topological triviality for families of map-germs is a fundamental subject in singularity theory. As we see in the articles of Damon (see [3] for example), the method of constructing controlled vector fields is a very powerful tool to compute the topological triviality. M. A. S. Ruas in her PhD. Thesis gives an explicit order such that the $\mathcal{A}$-topological structure of a homogeneous polynomial map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $n \geq p$, is preserved after higher order perturbations.

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In this paper we apply this method to give sufficient conditions for the $\mathcal{A}$-topological triviality of deformations of map germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ of type $f_t(x) = f(x) + th(x)$, with $n \geq p$ or $n \leq 2p$. These conditions are given in terms of an appropriate choice of Newton filtrations for $\mathcal{O}_n$ and $\mathcal{O}_p$ and are for the $\mathcal{A}$-tangent space of the germ f.

First we generalize the results of M. A. S. Ruas [5], by considering different Newton filtrations A_k for $\mathcal{O}_n$ and $B_{k'}$ for $\mathcal{O}_p$, these results are done for the case $n \geq p$. We construct *control functions* in the target and in the source to obtain, via a partition of the unit, a unique control function. We remark that in [5] these control functions are homogeneous, since they are associated to the usual filtration, done by the degree of monomials.

In the case $p \geq 2n$ we give an estimate in terms of the control function of the target. Moreover, if $p \geq 2n$, we apply the results of Gaffney in [4] to show that deformations by higher Newton filtration are $\mathcal{A}$-topologically trivial if the map germs satisfy a Newton non-degeneracy condition.

2. NEWTON FILTRATION AND CONTROL FUNCTIONS

To construct controlled vector fields that guarantee the topological triviality we define a convenient control function in terms of a fixed Newton polyhedron. First construct a control function in the target, denoted by ρ_{m} and a function in the source, denoted by ρ_{f}. When $n \geq p$, the control function ρ is defined from these, via a partition of the unity. For $p \geq 2n$, the control function used is ρ_{m}.

We fix a coordinate system $\mathbf{x}$ in $\mathbb{C}^n$, ($\mathbf{y}$ in $\mathbb{C}^p$) and identify the set $\mathcal{O}_\ell$ of holomorphic germs $g\colon (\mathbb{C}^\ell, 0) \to (\mathbb{C}, 0)$, $\ell = n$ or p, with the ring $\mathbb{C}[[x]]$ ($\mathbb{C}[[y]]$), of convergent power series.

We follow [1] and say that a subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a *Newton polyhedron* if there exist some $k_1, \ldots, k_r \in \mathbb{Q}^n_+$ such that Γ_+ is the convex hull in $\mathbb{R}^n_+$ of the set $\{k_i + v : v \in \mathbb{R}^n_+, i = 1, \ldots, r\}$ and Γ_+ intersects all the coordinate axis. Denote by Γ the union of the compact faces of Γ_+ and consider the *Newton filtration* of $\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \ldots$, by the ideals $\mathcal{A}_q = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq \phi_\Gamma^{-1}(q + \mathbb{N})\}$, for all $q \in \mathbb{N}$, here ϕ_Γ is the Newton function of Γ, see [1] for more details.

For a germ of function $g = \sum_k a_k x^k$, we denote $\mathrm{d}(g) = \max\{q : g \in \mathcal{A}_q\}$. For a map germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $g = (g_1, \ldots, g_p)$, call $D_i = \mathrm{d}(g_i)$ and consider $D_1 \leq D_2 \leq \ldots \leq D_p$.

Denote by M_I the determinant indexed by I, of the $p \times p$ minor of the matrix of the partial derivatives of g, $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots n\}$, with $i_1 < \ldots < i_p$, and call $L_I = \mathrm{d}(M_I)$.

Define the control function in the target, $\rho_{\mathrm{m}} : \mathbb{C}^p \to \mathbb{R}$ by

$$\rho_{\mathrm{m}}(y) = |y_1|^{2r_1} + |y_2|^{2r_2} + \ldots + |y_p|^{2r_p}$$

and the function in the source $\rho_{\mathrm{f}}(g) : \mathbb{C}^n \to \mathbb{R}$ by

$$\rho_{\mathrm{f}}(g)(x) = \sum |M_I|^{2\alpha_I}.$$

The numbers r_i, α_I and D are chosen as the smallest positive integers such that $D_i r_i = \alpha_I L_I = D$ for all i and I. Then $\mathrm{d}(\rho_{\mathrm{f}}(g)) = \min_I\{2\alpha_I . L_I\} = 2D$.

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The Newton filtration in $\mathcal{O}_p = \mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \dots$, is given by the Newton polyhedron, which has only one compact face, defined by the p tuple $w = (w_1, ..., w_p)$, with $w_i = R/r_i$, $R = m.c.m\{r_1, ..., r_p\}$.

For any monomial $y^\alpha \in \mathcal{O}_p$, we denote $fil_w(y^\alpha) = w_1\alpha_1 + \dots + w_p\alpha_p$ and call $B_{k'} = \{g \in \mathcal{O}_p; fil_w(g) \geq k'\}$. Hence $fil_w(\rho_m) = 2R$ and

$$\mathrm{d}(\rho_m \circ g) = fil(|g_1|^{2r_1} + |g_2|^{2r_2} + ... + |g_p|^{2r_p}) = 2.r_i.D_i = 2D.$$

Denote by $v^j = (v_1^j, \dots, v_n^j)$, $j = 1, 2, \dots, r$ the vertices of the Newton polyhedron $\Gamma_+(A_{2D})$, and define the control function

$$\rho_v(x_1, \dots, x_n) = \left(\sum_{j=1}^{r} x_1^{2v_1^j} \dots x_n^{2v_n^j} \right)^{\frac{1}{2}}.$$

We remark that the function ρ_{f} is not a control function for any Newton polyhedron.

Example: Let $g : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $g(x, y) = (xy, x^4 + y^5 + xy^2)$. Fix the Newton polyhedron $\Gamma_+(g_2)$ to obtain the control function in the target $\rho_{\mathrm{m}} : \mathbb{C}^2 \to \mathbb{R}$ $\rho_{\mathrm{m}}(y) = |y_1|^{16} + |y_2|^{10}$, and the function on the source $\rho_{\mathrm{f}}(g) : \mathbb{C}^2 \to \mathbb{R}$ defined by: $\rho_{\mathrm{f}}(g) = (5y^5 - 4x^4 + xy^2)^{10}$. The control function ρ_v associated to $\Gamma_+(g_2^{10})$ is $\rho_v(x, y) = (x^{80} + y^{100} + x^{20}y^{40})^{\frac{1}{2}}$.

3. $\mathcal{A}$-TOPOLOGICAL TRIVIALITY

Denote by $F : (\mathbb{C}^n \times \mathbb{C}, (0, 0)) \to (\mathbb{C}^p \times \mathbb{C}, (0, 0))$, $F(x, \lambda) = (f(x, \lambda), t)$ a one parameter unfolding of a finitely determined map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and call the family of map germs $f_\lambda(x) = f(x, \lambda)$ a deformation of the germ f.

A deformation $f_\lambda(x)$ is $\mathcal{A}$-topologically trivial if, for small values of λ, there exist two families of homeomorphisms $h_\lambda : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ and $k_\lambda : \mathbb{C}^p, 0 \to \mathbb{C}^p, 0$ such that $k_\lambda \circ f_\lambda \circ h_\lambda^{-1} = f_0$.

Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, be a finitely determined map germ satisfying

$$A_{2D+D_p}\theta_g \subseteq tg(A_{2D}\theta_n) + wg(B_{2R+1}\theta_p). \tag{1}$$

¿From the constructions done above we have the following:

Proposition 3.1.

1. If $n \geq p$, suppose that in a neighborhood V of 0 in $\mathbb{C}^n$, there exist constants α e β such that $\rho_{\mathrm{f}}(g(x)) \geq \beta\rho_v(x)$, for all $x \in V \cap \{x; \rho_{\mathrm{m}}(g(x)) < \alpha\rho_v(x)\}$.

2. If $p \geq 2n$, suppose that $\rho_m(g(x)) \geq c\rho_v(x)$, for all x in a neighborhood V of 0.

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Then deformations of $g_\lambda = g + \lambda h$ of g, with $fil(h_i) \geq D_p$, $\forall\ i = 1, \ldots, p$, are $\mathcal{A}$-topologically trivial for small values of λ.

In order to prove this Proposition we use the next Lemma, where it is shown that it is possible to extend the filtration condition of the equation (1), to the tangent space of an unfolding of the germ g.

We fix a finitely generated ideal I in $\mathcal{O}_p$, call m_1 the maximal ideal in $\mathcal{O}_1$ and $\tilde{A}_{2D+D_p}$ the ideal in $\mathcal{O}_{p+1}$ generated by the monomial λ and the ideal A_{2D+D_p}.

Lemma 3.2. *Let $G(x, \lambda) = (g_\lambda(x), \lambda)$ be un unfolding of $g_0(x) = g(x)$, such that $g_\lambda - g_0 \in m_1.A_{D_p}\theta_G$ and $|\lambda| < \epsilon$ for small values of ϵ. If the equation (1) holds, then*

$$A_{2D+D_p}\theta_G \subseteq tG(A_{2D}\theta_{n+1}) + wG(I.\theta_{p+1}).$$

Proof: Since

$$A_{2D+D_p}\theta_G = A_{2D+D_p}\theta_g + \lambda A_{2D+D_p}\theta_G \subseteq tg(A_{2D}\theta_n) + wg(I.\theta_p) + \lambda A_{2D+D_p}\theta_G,$$

and $tg(A_{2D}\theta_n) + wg(I.\theta_p) \subseteq tG(A_{2D}\theta_{n+1}) + wG(I.\theta_{p+1}) + \lambda A_{2D+D_p}\theta_G$ it follows that

$$A_{2D+D_p}\theta_G \subseteq tG(A_{2D}\theta_{n+1}) + wG(I.\theta_{p+1}) + \lambda A_{2D+D_p}\theta_G. \tag{2}$$

Let E be the finitely generated $\mathcal{O}_{n+1}$-modulo defined as

$$E = \frac{tG(A_{2D}\theta_{n+1}) + wG(I.\theta_{p+1}) + A_{2D+D_p}\theta_G}{tG(A_{2D}\theta_{n+1}) + wG(I.\theta_{p+1})}.$$

We remark that E is a $G^*(\mathcal{O}_{p+1})$-modulo and $(\lambda).E = E$ since

$$\begin{aligned}(\lambda).E &= \tfrac{tG(A_{2D}\theta_{n+1})+wG(I.\theta_{p+1})+(\lambda)[tG(A_{2D}\theta_{n+1})+wG(I.\theta_{p+1})+A_{2D+D_p}\theta_G]}{tG(A_{2D}\theta_{n+1})+wG(I.\theta_{p+1})}\\ &= \tfrac{tG(A_{2D}\theta_{n+1})+wG(I.\theta_{p+1})+A_{2D+D_p}\theta_G}{tG(A_{2D}\theta_{n+1})+wG(I.\theta_{p+1})} = E.\end{aligned}$$

Therefore if we show that E is finitely generated as $G^*(\mathcal{O}_{p+1})$-modulo we apply the lemma of Nakayama to obtain $E = 0$, or

$$A_{2D+D_p}\theta_G \subseteq tG(A_{2D}\theta_{n+1}) + wG(I.\theta_{p+1}).$$

We show now that E is finitely generated as $G^*(\mathcal{O}_{p+1})$- module.

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¿From the inequality

$$\frac{tG(A_{2D}\theta_{n+1})+wG(I.\theta_{p+1})+A_{2D+D_p}\theta_G}{tG(A_{2D}\theta_{n+1})+wG(I.\theta_{p+1})}\subseteq\frac{tG(A_{2D}\theta_{n+1})+A_{2D+D_p}\theta_G}{tG(A_{2D}\theta_{n+1})}+\frac{wG(I.\theta_{p+1})}{tG(A_{2D}\theta_{n+1})}$$

call E' the finitely generated $\mathcal{O}_{n+1}$-modulo

$$E'=\frac{tG(A_{2D}\theta_{n+1})+A_{2D+D_p}\theta_G}{tG(A_{2D}\theta_{n+1})}.$$

Since $\frac{wG(I.\theta_{p+1})}{tG(A_{2D}\theta_{n+1})}$ is finitely generated as a $G^*(\mathcal{O}_{p+1})$- modulo and the ideal I is finitely generated, we need to show that E' is finitely generated as a $G^*(\mathcal{O}_{p+1})$- modulo.

¿From the Theorem of Preparation of Malgrange, E' is a finitely generated $G^*(\mathcal{O}_{p+1})$-modulo if, and only if $\dim_{\mathbb{C}} \frac{E'}{G^*(m_{p+1})E'}<+\infty$.

$$\text{Write } \frac{E'}{G^*(m_{p+1})E'} = \frac{\frac{tG(A_{2D}\theta_{n+1})+A_{2D+D_p}\theta_G}{tG(A_{2D}\theta_{n+1})}}{\frac{G^*(m_{p+1})[tG(A_{2D}\theta_{n+1})+A_{2D+D_p}\theta_G]+tG(A_{2D}\theta_{n+1})}{tG(A_{2D}\theta_{n+1})}}$$

$$= \frac{tG(A_{2D}\theta_{n+1})+A_{2D+D_p}\theta_G}{tG(A_{2D}\theta_{n+1})+G^*(m_{p+1})[tG(A_{2D}\theta_{n+1})+A_{2D+D_p}\theta_G]}$$

and denote $S=A_{2D+D_p}\theta_G$ and $T=tG(A_{2D}\theta_{n+1})+G^*(m_{p+1})A_{2D+D_p}\theta_G$.

Since the quocient above is $\frac{T+S}{T}$, by the isomorphism theorem $\dfrac{T+S}{T}\cong\dfrac{S}{T\cap S}$.

¿From $tG(A_{2D}\theta_{n+1})+wG(I.\theta_{p+1})+\lambda A_{2D+D_p}\theta_G\subseteq tG(A_{2D}\theta_{n+1})+G^*(m_{p+1})\theta_G$ and by the equation (2) we conclude that $tG(A_{2D}\theta_{n+1})+G^*(m_{p+1})\theta_G\supseteq A_{2D+D_p}\theta_G$.

Multiplying by A_{2D+D_p}:

$$tG(A_{4D+D_p}\theta_{n+1})+G^*(m_{p+1})A_{2D+D_p}\theta_G\supseteq A_{4D+2D_p}\theta_G.$$

On the other hand,

$$tG(A_{4D+D_p}\theta_{n+1})+G^*(m_{p+1})A_{2D+D_p}\theta_G\supseteq \lambda A_{2D+D_p}\theta_G.$$

Hence, $dim_{\mathbb{C}}\frac{S}{T\cap S}\leq dim_{\mathbb{C}}\frac{A_{2D+D_p}\theta_G}{\tilde{A}_{2D+D_p}A_{2D+D_p}\theta_G}<+\infty$. $\|$

Proof of the Proposition 3.1: Let $G_\lambda(x,y) = (g_\lambda(x,y), \lambda)$ be an unfolding of g, with $g_\lambda(x,y) = g(x,y) + \lambda h(x,y)$ and $h \in A_{D_p}\theta_g$.

¿From the general hypotheses, $h\rho_{\rm m}(g) \in tg(A_{2D}\theta_n) + wg(B_{2R+1}\theta_p)$. Applying the Lemma above, there exist analytic vector fields $\xi \in A_{2D}\theta_{n+1}$ and $\eta \in B_{2R+1}\theta_{p+1}$ such that the above inclusion holds for deformations, i.e.:

$$h\rho_{\rm m}(g_\lambda) = tG(\xi) + \eta \circ G. \tag{3}$$

¿From the equation (3) we construct the vector field controlled by $\rho_{\rm m}$. Define ω in $(\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$ as:

$$\omega(y,\lambda) = \begin{cases} \dfrac{\eta(y,\lambda)}{\rho_{\rm m}(y)}, & y \neq 0 \\ \\ 0, & y = 0. \end{cases}$$

Since $fil_w(\eta) \geq 2R+1 = fil_w(\rho_{\rm m}) + 1$ we apply Lemmas (1) e (2) of [6] to conclude that the vector field ω is integrable.

Proof of the case $n \geq p$. In order to define the vector field controlled by the function $\rho_{\rm f}$, for each $I = \{i_1, i_2, \ldots, i_p\} \subset \{1, 2, \ldots, n\}$ write $\dfrac{\partial g_\lambda}{\partial \lambda} M_{I_\lambda} = tG(\gamma_I)$, with $\gamma_I = \sum \gamma_i \frac{\partial}{\partial x_i}$ and each γ_i is defined as

$$\begin{cases} \gamma_i = 0, & \text{if } i \notin I \\ \\ \gamma_{i_m} = \sum N_{ji_m} \left(\frac{\partial g_\lambda}{\partial \lambda}\right)_j, & \text{if } i_m \in I. \end{cases} \tag{4}$$

where N_{ji_m} is the $(p-1) \times (p-1)$ cofactor of $\frac{\partial g_j}{\partial x_{i_m}}$.

Since $\rho_{\rm f}(g_\lambda) = \sum |M_{I_\lambda}|^{2\alpha_I}$, we obtain

$$h\rho_{\rm f}(g_\lambda) = tG\left(\sum \gamma_I M_{I_\lambda}^{\alpha_I - 1} \overline{M_{I_\lambda}}^{\alpha_I}\right),$$

therefore $h = tG(\psi)$, with $\psi = \frac{\sum \gamma_I M_{I_\lambda}^{\alpha_I-1}\overline{M_{I_\lambda}}^{\alpha_I}}{\rho_{\rm f}(g_\lambda)}$. Denote $\gamma_{\mathcal{R}} = \sum \gamma_I M_{I_\lambda}^{\alpha_I - 1} \overline{M_{I_\lambda}}^{\alpha_I}$ to obtain $fil_{\Gamma_+}(\gamma_{\mathcal{R}}) = fil_{\Gamma_+}(\rho_{\rm f}(g_\lambda)) + r$, with $r = \min\limits_{i,k} \left\{ \frac{M}{\ell(v^k)} . v_i^k \right\}$.

The integrability of the vector field $\psi = \dfrac{\gamma_R}{\rho_{\rm f}(g_\lambda)}$, is guaranteed by the hypotheses of the following:

Lemma 3.3. There exist positive constants α_1, β and a neighborhood V of the origin in $\mathbb{C}^n$ such that

$$\rho_{\mathrm{f}}(g_\lambda(x)) \geq \alpha_1 \rho_v(x),\ \forall\ x \in\ V \cap \{\rho_{\mathrm{m}}(g_\lambda(x)) < \beta\rho_v(x)\}.$$

Proof: Since $g_\lambda = g + \lambda h$, and $fil(h) \geq fil(g)$, we have

$$\rho_{\mathrm{f}}(g_\lambda) \geq \rho_{\mathrm{f}}(g) - \lambda\theta(x,\lambda), \text{with }\ fil(\theta) \geq fil(\rho_{\mathrm{f}}(g)).$$

By hypotheses $\rho_{\mathrm{f}}(g) \geq \alpha\rho_v(x)$ for $x \in V \cap \{x; \rho_{\mathrm{m}}(g(x)) < \beta\rho_v(x)\}$ hence there exists a constant $c > 0$ such that $\lambda\theta(x,\lambda) \leq c\rho_v(x)$. Since $\rho_{\mathrm{m}}(g_\lambda(x)) < \rho_{\mathrm{m}}(g(x))$, for each $x \in V \cap \{x; \rho_{\mathrm{m}}(g_\lambda(x)) < \rho_{\mathrm{m}}(g(x)) < \alpha\rho_v(x)\}$, then

$$\begin{aligned} \rho_{\mathrm{f}}(g_\lambda(x)) &\geq \rho_{\mathrm{f}}(g(x)) - \lambda\theta(x,\lambda) \\ &\geq (\alpha - c)\rho_v(x) \\ &= \alpha_1\rho_v(x). \end{aligned}$$

||

To finish the proof ot the Proposition 3.1., consider the following partition of the unity.

Let $H = (V \times I) - (0 \times I)$, with $I = (-\epsilon, \epsilon)$ and the following sets

$F_1 = (\{(x,\lambda); g_\lambda(x) = 0\} - (0 \times \mathbb{C})) \cap H$, $F_2 = \{(x,\lambda); \rho_{\mathrm{m}}(g_\lambda(x)) \geq \alpha\rho_v(x)\} \cap H$,

$E_1 = \{(x,\lambda); \rho_{\mathrm{m}}(g_\lambda(x)) < \alpha_1\rho_v(x)\} \cap H$ and $E_2 = \{(x,\lambda); \rho_{\mathrm{m}}(g_\lambda(x)) < \alpha_2\rho_v(x)\} \cap H$,

with $\alpha_1 < \alpha < \alpha_2$.

We remark that F_1 anf F_2 are closed and disjoint from H.

Define $\zeta(x) = \zeta_1(x) + \zeta_2(x)$, a partition of the unity related to $\{E_2, (\overline{E_1})^c\}$.

$$\zeta_1(x,\lambda) = \begin{cases} 1, & \text{if } (x,\lambda) \in F_1 \\ 0, & \text{if } (x,\lambda) \in (E_2)^c \end{cases}$$

and

$$\zeta_2(x,\lambda) = \begin{cases} 1, & \text{if } (x,\lambda) \in F_2 \\ 0, & \text{if } (x,\lambda) \in \overline{E_1}. \end{cases}$$

Call $\nu_2(x,\lambda) = \begin{cases} \frac{\xi(x,\lambda)}{\rho_{\mathrm{m}}(g_\lambda(x))}, & \text{if } (x,\lambda) \in {F_1}^c \\ 0, & \text{if } (x,\lambda) \in F_1, \end{cases}$

where $\xi(x,\lambda)$ is given in equation (3), and define

$$\nu_1(x,\lambda)=\begin{cases}\frac{\gamma_{\mathcal{R}}}{\rho_{\mathrm{f}}(g_\lambda(x))}, & \text{if } (x,\lambda)\in {F_2}^c\\ \\ 0, & \text{if } (x,\lambda)\in F_2.\end{cases}$$

Since all functions defined above can be extended in such a way that they are zero at $0\times\lambda$, let ν be the vector field in $(\mathbb{C}^n\times\mathbb{C},0\times 0)$ defined as

$$\nu(x,\lambda)=\begin{cases}\zeta_1(x,\lambda)\nu_1(x,\lambda)+\zeta_2(x,\lambda)+\nu_2(x,\lambda), & \text{if } x\neq 0\\ \\ 0, & \text{if } x=0.\end{cases}$$

Then the vector field ν is continuous, integrable and

$$h=tG(\nu(x,\lambda))+w(G(x,\lambda)).$$

¿From the integral curve solutions of ν e ω we construct the germs of homeomorphisms

$$H:(\mathbb{C}^n\times\mathbb{C},0\times 0)\to(\mathbb{C}^n\times\mathbb{C},0\times 0),\quad H(x,\lambda)=(h(x,\lambda),\lambda),\ h(x,0)=x,$$

and

$$K:(\mathbb{C}^p\times\mathbb{C},0\times 0)\to(\mathbb{C}^p\times\mathbb{C},0\times 0),\ K(y,\lambda)=(k(y,\lambda),\lambda),\ k(y,0)=y$$

to obtain $K\circ G\circ H^{-1}=(g,id_{\mathbb{C}})$.

Therefore the two families of homeomorphisms $h_\lambda:\mathbb{C}^n,0\to\mathbb{C}^n,0$ and $k_\lambda:\mathbb{C}^p,0\to\mathbb{C}^p,0$ that give $k_\lambda\circ f_\lambda\circ h_\lambda^{-1}=f_0$ are $h_\lambda=h(x,\lambda)$ and $k_\lambda=k(x,\lambda)$. ||

Proof of the case $p\geq 2n$: From the equation (3) we have

$$h=tG\left(\frac{\xi}{\rho_{\mathrm{m}}(g_\lambda)}\right)+\frac{\eta\circ G}{\rho_{\mathrm{m}}(g_\lambda)}.$$

By the general hypotheses, $\xi\in A_{2D}$ and by (2) we have, $\rho_{\mathrm{m}}(g_\lambda)(x)\geq c\rho_v(x)$, therefore the vector field $\dfrac{\xi}{\rho_{\mathrm{m}}(g_\lambda)}$ is integrable and the homeomorphisms H and K are obtained as above. ||

4. THE NON-DEGENERATE CASE WHEN $P\geq 2N$

In the case $p\geq 2n$, we show the $\mathcal{A}$-topological triviality of a family $g_\lambda=g+\lambda h$ if the map germ h has a higher order filtration and the ideal generated by the coordinate functions g_i satisfies a Newton non-degeneracy condition.

First we recover the basic definitions needed for the Newton non-degeneracy condition.

Let $g = \sum_k a_k x^k$ in $\mathcal{O}_n$, denote $\operatorname{supp} g$ the set of points $k \in \mathbb{Z}^n$ with $a_k \neq 0$. If I is an ideal in $\mathcal{O}_n$, define $\operatorname{supp} I = \cup_{g \in I} \operatorname{supp} g$.

Fix an ideal I with finite colength in $\mathcal{O}_n$, i.e., $\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I} < \infty$, and consider its Newton polyhedron $\Gamma_+(I)$, the convex hull in $\mathbb{R}^n_+$ of the set $\{k + v : v \in \mathbb{R}^n_+,\ k \in \operatorname{supp}(I)\}$

For each compact face Δ of Γ, denote by $C(\Delta)$ the cone with vertex at the origin and passing through Δ and denote by $\mathcal{A}_\Delta$ the subring with unity of $\mathcal{O}_n$, $\mathcal{A}_\Delta = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq C(\Delta)\}$. For a germ $g \in \mathcal{O}_n$, denote by g_Δ the polynomial $g_\Delta = \sum a_k x^k$ with $k \in \operatorname{supp} g \cap \Delta$.

Definition 4.1. *An ideal I is Newton non-degenerate if there exists a system of generators $g_1, \ldots, g_s$ of I such that for each compact face $\Delta \subseteq \Gamma$, the ideal generated by the system $\{g_{1\Delta}, \ldots, g_{s\Delta}\}$ has finite colength in $\mathcal{A}_\Delta$.*

To show the main result of this section we fix some notation.

Since $p \geq 2n$, any map germ $g = (g_1, \ldots, g_p)$ is $\mathcal{A}$-finitely determined if, and only if, g is $\mathcal{L}$- finitely determined, here $\mathcal{L}$ denotes the $\mathcal{L}$-group of Mather.

In this case, we define the control function $\rho : \mathbb{C}^p \to \mathbb{R}$, by

$$\rho(y) = |y_1|^{2s_1} + |y_2|^{2s_2} + \ldots + |y_p|^{2s_p},$$

where the numbers $\{s_1, \ldots, s_p\}$ are integers, such that we can write

$$\frac{\partial g_\lambda}{\partial \lambda} \rho(g) \in T\mathcal{L}g.$$

Now we denote by $w^j = (w^j_1, \ldots, w^j_n)$, $j = 1, 2, \ldots, r$ the vertices of the Newton polyhedron fixed by $\rho(g)$, and define the control function

$$\rho_w(x_1, \ldots, x_n) = \left(\sum_{j=1}^{r} x_1^{2w_1^j} \ldots x_n^{2w_n^j} \right)^{\frac{1}{2}}.$$

Call I the ideal generated by the system $\{g_1^{2s_1}, \ldots, g_p^{2s_p}\}$.

Proposition 4.2. *Suppose that I is Newton non-degenerate. Then, deformations of g of type $g_\lambda = g + \lambda h$, with $fil(h_i) \geq fil(g_i)$, for $i = 1, \ldots, p$ are $\mathcal{A}$-topologically trivial.*

Proof: Since the map germ g is $\mathcal{L}$-finitely determined, there exists a vector field $\eta \in m_p\theta_p$ such that

$$\frac{\partial g_\lambda}{\partial \lambda}.\rho(g) = \eta \circ g.$$

To prove that the vector field $\dfrac{\eta \circ g}{\rho\mathrm{m}(g)}$ is integrable, we follow Gaffney in [4] p.482.

Then it is sufficient to show that there exists a constant $C > 0$ such that

$$\left| \frac{\partial g_\lambda}{\partial \lambda}(\lambda, y_1, \ldots, y_p) \right| \leq C|y_1, \ldots, y_p|,$$

or equivalently:

$$\left| \frac{\partial g_\lambda}{\partial \lambda}(\lambda, y_1, \ldots, y_p) \right| \leq C(|y_1|^{2s_1} + \ldots + |y_p|^{2s_p}) \tag{5}$$

And this inequality is equivalent to $\left| \frac{\partial g_\lambda}{\partial \lambda}(\lambda, x_1, \ldots, x_n) \right| \leq C\rho(g_\lambda)$.

For $g_\lambda = g + \lambda h$, we have $\rho(g_\lambda) \leq \rho(g) + \lambda\theta(x, \lambda)$, with $fil(\theta) \geq fil(\rho(g))$.

Since the ideal I is Newton non-degenerate, there exists a constant $C > 0$ such that $\rho_w(x) \leq C\rho(g)$.

Then, as $fil(h_i) \geq fil(g_i)$ we obtain $\left| \frac{\partial g_\lambda}{\partial \lambda}(\lambda, x_1, \ldots, x_n) \right| \leq \rho_w(x) \leq C\rho(g)$. ‖

An example in $\mathbb{C}^2 \to \mathbb{C}^4$

Let $k \geq 5$ be an odd integer with k, $k+1$, $k+2$, $k+3$ two by two primes, and $g : \mathbb{C}^2 \to \mathbb{C}^4$, the map germ $g = (g_1, g_2, g_3, g_4)$ with

$$\begin{aligned} g_1(x, y) &= \alpha_1 x^k + \alpha_2 y^k + a_1 x y^{k-2} + a_2 x^{k-2} y; \\ g_2(x, y) &= \beta_1 x^{k+2} + \beta_2 y^{k+2} + b_1 x y^k + b_2 x^k y; \\ g_3(x, y) &= \theta_1 x^{k+4} + \theta_2 y^{k+4} + c_1 x y^{k+2} + c_2 x^{k+2} y; \\ g_4(x, y) &= \gamma_1 x^{k+6} + \gamma_2 y^{k+6} + d_1 x y^{k+4} + d_2 x^{k+4} y \end{aligned}$$

We see in the example 2.1 of [2] that g is $\mathcal{A}$-finitely determined for generic values of α_i, β_i, θ_i and γ_i, with a_i, b_i, c_i e d_i being all distinct prime numbers.

To apply the Proposition 4.2 we fix the Newton polyhedron $\Gamma_+(g_1)$, and denote

$$s_1 = (k+2)(k+4)(k+6),\ s_2 = k(k+4)(k+6),\ s_3 = k(k+2)(k+6),\ s_4 = k(k+2)(k+4).$$

We define the control function

$$\rho\mathrm{m}(y_1, y_2, y_3, y_4) = |y_1|^{2s_1} + |y_2|^{2s_2} + |y_3|^{2s_3} + |y_4|^{2s_4}.$$

Since $fil(g_1) = k(k-1)$, $fil(g_2) = (k+2)(k-1)$, $fil(g_3) = (k+4)(k-1)$, and $fil(g_4) = (k+6)(k-1)$, we see that the ideal I generated by the system $\{g_1^{s_1}, g_2^{s_2}, g_3^{s_3}, g_4^{s_4}\}$ is Newton non-degenerate.

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Therefore we conclude that if $g_\lambda = g + \lambda h$ is an $\mathcal{A}$-topologically trivial deformation if $fil(h_1) \geq k(k-1)$, $fil(h_2) \geq (k-2)(k-1)$, $fil(h_3) \geq (k+4)(k-1)$ and $fil(h_4) \geq (k+6)(k-1)$.

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REFERENCES

1. C. Bivia-Ausina, T. Fukui & M. J. Saia, Newton filtrations, graded algebras and codimension of non-degenerated ideals. Math. Proc. Cambridge Phil. Soc., **133**, 1, (2002). 55–75.
2. J. W. Bruce, M. A. S. Ruas and M. J. Saia, A note on determinacy, Proc. Amer. Math. Soc., 115 (1992), 856–887.
3. J. Damon, Topological invariants of μ-constant deformations of complete intersection singularities, Quart. J. Math. Oxford, **2**, 40 (1989), 139–159.
4. T. Gaffney , $\mathcal{L}^0$-equivalence of maps, Mat. Proc. Cambridge Phil. Soc. **128**, (2000), 479–496.
5. M. A. S. Ruas, C^ℓ-determinação finita e aplicações, Tese de doutorado, ICMC, USP, (1983).
6. M. A. S. Ruas and M. J. Saia, C^ℓ-determinacy of weighted homogeneous germs, Hokkaido Math. Journal, **26**, (1998), 89–99.