

A general approximation scheme for attractors of abstract parabolic problems

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ABSTRACT: In this paper we consider semilinear problems of the form $u' = Au + f(u)$ where A generates an exponentially decaying compact analytic semigroup in a Banach space E and f is globally Lipschitz and bounded map from E^α into E ($E^\alpha = D((-A)^\alpha)$ with the graph norm). These assumptions ensure that the problem has a global attractor. Under a very general approximation scheme we prove that the dynamics of such problem behaves upper semicontinuously.

We also prove that, if all equilibrium solutions of this problem are hyperbolic, then there is an odd number of such equilibrium solutions. Additionally, if we impose that every global solution converges as $t \rightarrow \pm\infty$, (e.g. gradient systems), then we prove that under this approximation scheme the attractors also behave lower semicontinuously.

This general approximation scheme includes finite element method, projection and finite difference methods. The main assumption on the approximation is the compact convergence of resolvents which may be applied to many other problems not related to discretization. May, 2004 ICMC-USP

Keywords: Abstract parabolic equations, general approximation scheme, compact convergence of resolvents, upper and lower semicontinuity of attractors.

Mathematics Subject Classification: Primary 35B40; Secondary 35K55, 65P05

1. INTRODUCTION

In this paper we give rather general conditions under which the asymptotic dynamics (global attractors) of parabolic problems behave continuously with respect to perturbations of the equation. This is done from a functional analytic point of view on general approximation scheme. By continuity of attractors we understand upper and lower semicontinuity

* Research partially supported by CNPq grant # 300889/92-5

† Research partially supported by FAPESP grant # 02/10699-7, Russian Foundation for Basic Research (04-01-00026), grant NSC 91-2115-M-008-003 from the National Science Council of Taiwan

of attractors. Next we introduce some terminology to be able to present the results that we will prove in this paper.

Let X be a Banach space, Λ be a topological spaces and $\mathcal{A}_\lambda \subset X$, $\lambda \in \Lambda$. Denote by $\text{dist}(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}^+$ the metric induced by the norm in X .

DEFINITION 1.1. By upper and lower semicontinuity of the family of sets $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ at $\lambda = \lambda_0$ we understand the following

1. We say that $\{\mathcal{A}_\lambda\}$ is upper semicontinuous at λ_0 if $\sup_{x_\lambda \in \mathcal{A}_\lambda} \text{dist}(x_\lambda, \mathcal{A}) \xrightarrow{\lambda \rightarrow \lambda_0} 0$.
2. We say that $\{\mathcal{A}_\lambda\}$ is lower semicontinuous at λ_0 if $\sup_{x \in \mathcal{A}} \text{dist}(x, \mathcal{A}_\lambda) \xrightarrow{\lambda \rightarrow \lambda_0} 0$.

In order to prove upper and lower semicontinuity we employ the following result

LEMMA 1.1. Let $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ be as in Definition 1.1.

1. If any sequence $\{x_{\lambda_n}\}$ with $x_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow \lambda_0$, has a convergent subsequence with limit belonging to \mathcal{A} , then $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ is upper semicontinuous at λ_0 .

2. If \mathcal{A} is compact and for any $x \in \mathcal{A}$ there is a sequence $\{x_{\lambda_n}\}$ with $x_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow \lambda_0$, which converges to x , then $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ is lower semicontinuous at λ_0 .

Proof: i) If any sequence $\{x_{\lambda_n}\}$ with $x_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow \lambda_0$, has a convergent subsequence with limit belonging to \mathcal{A} , and $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ is not upper semicontinuous at λ_0 then, there are $\epsilon > 0$ and sequence $\{\lambda_n\}$ with $\lambda_n \rightarrow \lambda_0$ such that $\sup_{x \in \mathcal{A}_{\lambda_n}} \text{dist}(x, \mathcal{A}) \geq 2\epsilon$, $n \in \mathbb{N}$. Thus, for some $x_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, we have that $\text{dist}(x_{\lambda_n}, \mathcal{A}) \geq \epsilon$, $n \in \mathbb{N}$. This contradicts the fact that $\{x_{\lambda_n}\}$ has a subsequence which converges to an element of \mathcal{A} .

ii) If \mathcal{A} is compact and for any $x \in \mathcal{A}$ there is a sequence $\{x_{\lambda_n}\}$ with $x_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow \lambda_0$, which converges to x and $\{\mathcal{A}_\lambda\}$ is not lower semicontinuous at λ_0 then, there are $\epsilon > 0$ and sequence $\{\lambda_n\}$ with $\lambda_n \rightarrow \lambda_0$ such that $\sup_{x \in \mathcal{A}} \text{dist}(x, \mathcal{A}_{\lambda_n}) \geq 3\epsilon$, $n \in \mathbb{N}$. Thus, for some $x_{\lambda_n} \in \mathcal{A}$, we have that $\text{dist}(x_{\lambda_n}, \mathcal{A}_{\lambda_n}) \geq 2\epsilon$, $n \in \mathbb{N}$. Since \mathcal{A} is compact we may assume that $\{x_{\lambda_n}\}$ converges to some $x \in \mathcal{A}$ and that $\text{dist}(x, \mathcal{A}_{\lambda_n}) \geq \epsilon$, $n \in \mathbb{N}$. From our assumptions, there is a sequence $y_{\lambda_n} \in \mathcal{A}_{\lambda_n}$ such that $y_{\lambda_n} \rightarrow x$ and $0 \leftarrow \text{dist}(x, y_{\lambda_n}) \geq \text{dist}(x, \mathcal{A}_{\lambda_n}) \geq \epsilon$, which is a contradiction. \square

Next we introduce the notion of global attractors for continuous semigroups. For that we follow [31] (see also [9, 57, 69]). Let X be a Banach space, a *continuous semigroup* is a one parameter family $\{T(t) : t \geq 0\}$ of (nonlinear) operators such that

- (i) $T(0) = I$,
- (ii) $T(t+s) = T(t)T(s)$, $t, s \geq 0$,
- (iii) $\mathbb{R}^+ \times X \ni (t, x) \mapsto T(t)x \in X$ is continuous.

For each $x \in X$, the positive orbit $\gamma^+(x)$ through x is defined as $\gamma^+(x) = \{T(t)x : t \geq 0\}$. A backward solution through x is a continuous function $\phi : (-\infty, 0] \rightarrow X$ such that $\phi(0) = x$ and, for any $s \leq 0$, $T(t)\phi(s) = \phi(t+s)$ for $0 \leq t \leq -s$. A global solution

through x is a function $\phi : \mathbb{R} \rightarrow X$ such that $\phi(0) = x$, $s \in \mathbb{R}$ and $t \geq 0$ we have that $T(t)\phi(s) = \phi(t + s)$.

Backward or global solutions may not exist and its existence may depend on the choice of x . Also, when a backward solution exists, it may not be unique. Let the negative orbit through x be defined as

$$\gamma^-(x) = \cup_{t \geq 0} H(t, x),$$

where $H(t, x) = \{y \in X : \text{there is a backward solution through } x \text{ defined by } \phi : (-\infty, 0] \rightarrow X \text{ with } \phi(0) = x \text{ and } \phi(-t) = y\}$.

The complete orbit $\gamma(x)$ through x is defined as $\gamma(x) = \gamma^-(x) \cup \gamma^+(x)$. If B and C are subsets of X , we say that the set B attracts the set C under $T(t)$ if $\text{dist}(T(t)C, B) \rightarrow 0$ as $t \rightarrow \infty$. A set $S \subset X$ is said to be invariant under $T(\cdot)$ if, for any $x \in S$, there is a complete orbit $\gamma(x)$ through x such that $\gamma(x) \subset S$ or equivalently if $T(t)S = S$ for any $t \geq 0$.

DEFINITION 1.2. A set $\mathcal{A} \subseteq X$ is said to be a global attractor for semigroup $\{T(t) : t \geq 0\}$ if it is compact, invariant and attracts bounded subsets of X .

Let $\{T(t, \lambda) : t \geq 0\}$ be a family of semigroups. Assume that for each $\lambda \in \Lambda$ the semigroup $\{T(t, \lambda) : t \geq 0\}$ has a global attractors \mathcal{A}_λ .

To obtain the upper semicontinuity of attractors is a relatively simple matter being enough to prove that the union of the attractors is relatively compact in X and that the family of semigroups behaves continuously with respect to λ .

THEOREM 1.1. Assume that the set $\cup\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ is relatively compact in X and that for any $x_\lambda \xrightarrow{\lambda \rightarrow \lambda_0} x_{\lambda_0}$ in X ($x_\lambda \in \mathcal{A}_\lambda$, $\lambda \neq \lambda_0$) one has $\|T(t, \lambda)x_\lambda - T(t, \lambda_0)x_{\lambda_0}\| \rightarrow 0$ as $\lambda \rightarrow \lambda_0$ for any $t \in \mathbb{R}^+$. Then, $\{\mathcal{A}_\lambda\}_{\lambda \in \Lambda}$ is upper semicontinuous at $\lambda = \lambda_0$.

Proof: If $\{x_{\lambda_n}\}$ is a sequence with $x_{\lambda_n} \in \mathcal{A}_{\lambda_n}$, $\lambda_n \rightarrow \lambda_0$, we may assume, from the compactness of $\cup\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$, that x_{λ_n} is convergent to some $x_{\lambda_0} \in X$. By Lemma 1.1, the result is proved if we show that $x_{\lambda_0} \in \mathcal{A}_{\lambda_0}$. To show that $x_{\lambda_0} \in \mathcal{A}_{\lambda_0}$ it is enough to show that there is a complete bounded solution through x_{λ_0} . Let $\mathbb{R} \ni t \mapsto \phi_{\lambda_n}(t) \in X$ be a complete solution through x_{λ_n} . Since the union of the attractors is bounded we have that $\sup_{n \geq 1} \sup_{t \in \mathbb{R}} \|\phi_{\lambda_n}(t)\|_X < \infty$. From the convergence properties of the semigroups we have that there is a complete bounded solution through x_{λ_0} and the result is proved. \square

The upper semicontinuity has been the subject of study by a number of authors, among them we cite [1, 3, 5, 6, 8], [11] – [16], [30] – [37], [39, 41, 42], [45] – [48], [64] – [67].

On the other hand, the lower semicontinuity of attractors is a much more complicated matter. We will not attempt to describe this procedure in this degree of generality and will leave for a more specific case, still in this introduction. The pioneer work done in [44] established the procedure to obtain the lower semicontinuity of attractors for gradient semigroups (see Definition 6.2). In the sequel we cite the work [43] by the same authors. In the later the set of equilibria does not depend upon the parameter which considerably simplifies the problem. Other works used the reduction to finite dimension to obtain lower semicontinuity or even better continuity results as in [11, 13, 14, 15, 47, 66]. More recently

the work [5] brought new light into the subject, in this case the set of equilibria also depends upon the parameter and reduction to finite dimension is not available. The results in [5] are oriented in such a manner that the continuity of attractors is drawn from the continuity of the spectrum of the involved linear operators. Our approach resembles the steps described in [5] in a rather general abstract formulation.

The aim of the present paper is to consider approximation/perturbation of attractors. Actually, the problem of approximation/perturbation of attractors is naturally motivated by the following arguments. The semigroup $\{T(t) : t \geq 0\}$ usually comes from a differential equation that is a mathematical model for a phenomenon (physical, biological, economical, etc). One central question in the modelling is the stability under perturbations. In fact, in every step of modelling there is always a degree of uncertainty in the determination of the parameters of the mathematical model. Roughly speaking everything in the mathematical model is determined approximately. Therefore, it is crucial to determine whether the solutions depend continuously on all the parameters of the model. We tackle this question from the point of view of the asymptotic dynamics; that is, we will be satisfied with the proof that the attractors behave continuously on all the parameters of the mathematical model.

One subject for which approximation/perturbation techniques are crucial tools is numerical analysis. In the setting of present paper we consider just discretization in space and consider just convergence aspects, leaving the order of convergence and discretization in time for forthcoming papers. In fact we will concentrate on the continuity of attractor on a very general discretization scheme, which include finite element, projection and finite difference methods and may as well apply to many perturbations that are not concerned directly with discretization.

In [11] the authors study the continuity of attractors of a one dimensional parabolic problem in space with respect to discretization. Of course the proof of the results in [11] cannot be extended to higher dimensions. The spectral gap and the special convergence of eigenvalues and eigenfunctions cannot be extended to higher dimensional domains and enter in a crucial way in the proof of the results in [11].

In [32, 46, 52, 53] the authors also consider the continuity of dynamics with respect to discretization.

To state the main results of this paper we specialize to the class of problems considered here. Let E be a complex Banach space and $A : D(A) \subseteq E \rightarrow E$ be a closed linear operator with compact resolvent, such that

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{1 + |\lambda|} \text{ for any } \operatorname{Re} \lambda \geq 0. \quad (1.1)$$

In such situation $\theta(A) = \inf\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\} < 0$. Let $(-A)^\alpha$, $\alpha \in \mathbb{R}^+$, denote the fractional power operators (see [56, 49]) associated to A and denote by E^α the corresponding fractional power spaces; that is, $E^\alpha := D((-A)^\alpha)$ endowed with the graph norm $\|x\|_{E^\alpha} = \|(-A)^\alpha x\|_E$.

If A is as above and $0 \leq \alpha < 1$ is fixed, consider in the space E^α the following semilinear autonomous parabolic problem

$$\begin{aligned} u' &= Au + f(u), \quad t > 0, \\ u(0) &= u^0 \in E^\alpha, \end{aligned} \quad (1.2)$$

where $f(\cdot) : E^\alpha \subseteq E \rightarrow E$ is a globally Lipschitz, bounded and continuously Fréchet differentiable function.

Remark 1. 1. The fact that $-A$ is sectorial with compact resolvent enters in a crucial way to prove the continuity of the dynamics near equilibria through the fact that the linearization of the right hand side of (1.2) has only a finite number of eigenvalues with positive real part. This is the main difficulty to generalize such results to the case when A generates C_0 -semigroups. Of course, for a certain class of operators (as those coming from second order equations) the generalization may be possible but we will not consider them here.

Under the above assumptions the mild solution of (1.2) is defined for all $t \geq 0$ (see [49], [82]). Let $u(\cdot) = T(\cdot)u^0 : \mathbb{R}^+ \rightarrow E^\alpha$ be the only mild solution of (1.2). With this notation we have defined a family of nonlinear operators in E^α $\{T(t) : t \geq 0\}$ which is a continuous semigroup. It is well known that nonlinear semigroup $T(\cdot)$ is given by the Variation of Constants Formula

$$T(t)u^0 = \exp((t - t_0)A)T(t_0)u^0 + \int_{t_0}^t \exp((t - s)A)f(T(s)u^0)ds, \quad t \geq t_0. \quad (1.3)$$

Under all above assumptions it is not difficult to prove that

THEOREM 1.2 (Theorem 2.1). *The problem (1.2) has a global attractor \mathcal{A} in E^α .*

In the models that give rise to (1.2), the nonlinearity $f(\cdot)$ is not in general globally bounded and globally Lipschitz. Instead, it usually satisfies conditions that ensure local well posedness and existence of a global attractor. After obtaining some “nice” bounds on the attractor we can change the nonlinearity, without affecting the attractor, so that it becomes globally bounded and globally Lipschitz. This is seen in Example 6.1.

The equilibrium solutions of (1.2) are those solutions which are independent of time $u(t) = u, t \geq 0$; that is, the solutions of equation in E^α

$$Au + f(u) = 0. \quad (1.4)$$

We denote by \mathcal{E} the set of solutions to (1.4); that is, $\mathcal{E} = \{u \in E^\alpha : Au + f(u) = 0\}$.

About the solutions of (1.4) we can prove the following result

PROPOSITION 1.1. (see Proposition 2.1) *If every solution u^* of (1.2) satisfies $0 \notin \sigma(A + f'(u^*))$, then \mathcal{E} has $2k + 1$ elements for some non-negative integer k .*

This feature can be seen in [17], so called Chafee–Infante example, for which one can check that the number of equilibrium points is always odd.

DEFINITION 1.3. We say that a solution u^* of (1.4), i.e. $u^* \in \mathcal{E}$, is hyperbolic if the spectrum $\sigma(A + f'(u^*))$ is disjoint from the imaginary axis, i.e. $\sigma(A + f'(u^*)) \cap i\mathbb{R} = \emptyset$.

PROPOSITION 1.2. *If a solution u^* of (1.4) is hyperbolic then, u^* is isolated. If all solutions of u^* are hyperbolic then, every element in \mathcal{E} is isolated and there is an odd number of them.*

DEFINITION 1.4. The unstable manifold of the equilibrium set \mathcal{E} is the set $W^u(\mathcal{E}) = \{\eta \in E^\alpha : u(t, \eta) \text{ is defined for all } t \leq 0 \text{ and } u(t, \eta) \rightarrow \mathcal{E} \text{ as } t \rightarrow -\infty\}$. The unstable manifold of an equilibrium solution $u^* \in \mathcal{E}$ is defined by $W^u(u^*) = \{\eta \in E^\alpha : u(t, \eta) \text{ is defined for all } t \leq 0 \text{ and } u(t, \eta) \rightarrow u^* \text{ as } t \rightarrow -\infty\}$.

THEOREM 1.3. (see Theorem 6.1) *If A , α and f are as before and if $\{T(t) : t \geq 0\}$ is gradient (see Definition 6.2) then the attractor \mathcal{A} for (1.2) can be characterized as the unstable manifold of the equilibrium set \mathcal{E} ; that is, $\mathcal{A} = W^u(\mathcal{E})$. If in addition, each $u^* \in \mathcal{E}$ is isolated then \mathcal{E} is a finite set and $\mathcal{A} = \cup_{u^* \in \mathcal{E}} W^u(u^*)$, i.e. attractor consists of the union of the unstable manifolds of equilibrium solutions.*

Next we introduce the general discretization scheme and state the main results of the paper. The general approximation scheme [23] – [25], [68] – [77], can be described in the following way. Denote by \mathbb{N} the set of natural numbers and by \mathbb{N}' , \mathbb{N}'' infinite subsets of \mathbb{N} . Let E_n and E be Banach spaces, and let $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \rightarrow E_n, p_n \in B(E, E_n), n \in \mathbb{N}$, with the following property:

$$\|p_n x\|_{E_n} \rightarrow \|x\|_E \text{ as } n \rightarrow \infty \text{ for any } x \in E. \quad (1.5)$$

We will also use the operators $p_n^\alpha = (-A_n)^{-\alpha} p_n (-A)^\alpha \in B(E^\alpha, E_n^\alpha)$ with the same property (1.5), but for the spaces E^α, E_n^α . Operators A_n and A are supposed to be related with conditions (1.1), (A) and (B_1) .

DEFINITION 1.5. A sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be \mathcal{P} -convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \rightarrow 0$ as $n \rightarrow \infty$; we write this as $x_n \xrightarrow{\mathcal{P}} x$.

DEFINITION 1.6. A sequence of elements $\{x_n\}, x_n \in E_n, n \in \mathbb{N}$, is said to be \mathcal{P} -compact if for any $\mathbb{N}' \subseteq \mathbb{N}$ there exist $\mathbb{N}'' \subseteq \mathbb{N}'$ and $x \in E$ such that $x_n \xrightarrow{\mathcal{P}} x$, as $n \rightarrow \infty$ in \mathbb{N}'' .

DEFINITION 1.7. A sequence of closed linear operators $\{A_n\}$, $A_n \in \mathcal{C}(E_n)$, $n \in \mathbb{N}$, is said to be *compatible* with a closed linear operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}$, $x_n \in D(A_n) \subseteq E_n$, $n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ and $A_n x_n \xrightarrow{\mathcal{P}} Ax$. We write this as (A_n, A) are compatible.

Remark 1. 2. In a certain sense, a family $\{A_n\}$ is compatible with $A =: A_\infty$ the family $\{G(A_n)\}$ is lower semicontinuous at infinity (here $G(A_n)$ is the graph of A_n).

Consider in Banach spaces E_n also the family of parabolic problems

$$\begin{aligned} u_n' &= A_n u_n + f_n(u_n), \quad t > 0, \\ u_n(0) &= u_n^0 \in E_n^\alpha, \end{aligned} \tag{1.6}$$

where $u_n^0 \xrightarrow{\mathcal{P}^\alpha} u^0$, operators (A_n, A) are compatible, $f_n : E_n^\alpha \rightarrow E_n$ is globally bounded uniformly for $n \in \mathbb{N}$, globally Lipschitz continuous and continuously Fréchet differentiable.

Under the above assumptions the mild solution of (1.6) is defined for all $t \geq 0$ (see [49], [82]). Let $u_n(\cdot) = T_n(\cdot)u_n^0 : \mathbb{R}^+ \rightarrow E_n^\alpha$ be the only mild solution of (1.6). With this notation we have defined a family of nonlinear operators $\{T_n(t) : E_n^\alpha \rightarrow E_n^\alpha, t \geq 0\}$ which is a continuous semigroup, for each $n \in \mathbb{N}$. It is well known that nonlinear semigroup $T_n(\cdot)$ is given by the Variation of Constants Formula

$$T_n(t)u_n^0 = \exp((t - t_0)A_n)T_n(t_0)u_n^0 + \int_{t_0}^t \exp((t - s)A_n)f_n(T_n(s)u_n^0)ds, \quad t \geq t_0. \tag{1.7}$$

Under all above assumptions it can be proved (exactly as Theorem 2.1) that

THEOREM 1.4. *The problem (1.6) has a global attractor \mathcal{A}_n in E_n^α .*

Consider the associated elliptic problems in E_n^α

$$A_n u_n + f_n(u_n) = 0. \tag{1.8}$$

We denote by \mathcal{E}_n the set of solutions to (1.6); that is, $\mathcal{E}_n = \{u_n \in E_n^\alpha : A_n u_n + f_n(u_n) = 0\}$.

DEFINITION 1.8. A sequence of bounded linear operators $B_n \in B(E_n)$, $n \in \mathbb{N}$, is said to be *PP-convergent* to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}$, $x_n \in E_n$, $n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We write this as $B_n \xrightarrow{\mathcal{PP}} B$.

We have the following version of Trotter–Kato Theorem on the general approximation scheme.

THEOREM 1.5. [62] *Let operators A and A_n generate analytic C_0 -semigroups. The following conditions (A) and (B_1) are equivalent to condition (C_1) .*

(A) *Compatibility. There exists $\lambda \in [\cap_n \rho(A_n)] \cap \rho(A)$ such that the resolvents converge:*

$$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1};$$

(B_1) *Stability. There are some constants $M_2 \geq 1$ and ω_2 such that*

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_2}{|\lambda - \omega_2|}, \quad \operatorname{Re} \lambda > \omega_2, n \in \mathbb{N};$$

(C_1) *Convergence. For any finite $\mu > 0$ and some $0 < \theta < \frac{\pi}{2}$, we have*

$$\max_{\eta \in \Sigma(\theta, \mu)} \|\exp(\eta A_n) u_n^0 - p_n \exp(\eta A) u^0\|_{E_n} \rightarrow 0$$

as $n \rightarrow \infty$, whenever $u_n^0 \rightarrow u^0$. Here, $\Sigma(\theta, \mu) = \{z \in \Sigma(\theta) : |z| \leq \mu\}$ and $\Sigma(\theta) = \{z \in \mathbb{C} : |\arg z| \leq \theta\}$.

The above Theorem ABC gives a necessary and sufficient condition for convergence of mild solutions of (1.6) to mild solution of (1.2) in case $f_n \equiv 0, f \equiv 0$. In this trivial case under our above assumptions on A the attractor $\mathcal{A} = \{0\}$, since $\|\exp(tA)\| \leq Me^{-bt} \rightarrow 0$ as $t \rightarrow \infty$ and equation $Au + f(u) = 0$ has only one solution $u^* = 0$.

One may conjecture that from conditions (A) and (B_1) of Theorem 1.5, it follows also that $\|\exp(tA_n)\| \leq \tilde{M}e^{-\tilde{\omega}t}$ as $t \rightarrow \infty$ for some $\tilde{\omega} > 0$ and $\tilde{M} \geq 1$. In general it is not true as we can see from Example 6.2. If we also assume that $\Delta_{cc} \neq \emptyset$ and for $\lambda \in \Delta_{cc}$ resolvents $(\lambda I_n - A_n)^{-1}$ are compact, then this holds as a consequence of Lemma 6.7

In Example 6.3 we prove that the results in this paper holds for semilinear parabolic problems in bounded smooth domains of \mathbb{R}^2 when approximated using finite element method.

Next we state the main results in this paper. To this end we will need the following assumptions

$$[A1] \left\{ \begin{array}{l} \bullet \text{ Conditions (1.1) and } B_1 \text{ hold,} \\ \bullet \Delta_{cc} \neq \emptyset \text{ and for } \lambda \in \Delta_{cc} \text{ resolvents } (\lambda I_n - A_n)^{-1} \text{ are compact,} \\ \bullet f_n(x_n) \xrightarrow{\mathcal{P}} f(x) \text{ whenever } x_n \xrightarrow{\mathcal{P}^\alpha} x. \end{array} \right.$$

To prove (4.4) (which is used to prove Theorem 4.4), besides the assumptions [A1] we will need the following assumptions

$$[A2] \left\{ \begin{array}{l} \bullet f'_n(x_n) \xrightarrow{\mathcal{PP}} f'(x) \text{ whenever } x_n \xrightarrow{\mathcal{P}^\alpha} x \text{ and} \\ \bullet \text{ If } x_n^* \xrightarrow{\mathcal{P}^\alpha} x^*, \sup_{n \in \mathbb{N}} \sup_{\|y_n\|_{E_n^\alpha} \leq \rho} \|f'_n(y_n + x_n^*)\| < \infty, \text{ for some } \rho > 0. \end{array} \right.$$

THEOREM 1.6. (see Theorem 4.6) Assume that [A1] holds, then $T_n(t)u_n^0 \xrightarrow{\mathcal{P}^\alpha} T(t)u^0$ uniformly in $t \in [0, T]$, $T < \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}^\alpha} u^0$.

Remark 1. 3. We observe that compact convergence of resolvents and condition (1.1) are not enough to ensure that (B_1) is satisfied (see Example 6.4).

DEFINITION 1.9. Let $\mathcal{A}_n \subset E_n^\alpha$, $n \in \mathbb{N}$ and $\mathcal{A}_\infty = \mathcal{A} \subset E^\alpha$. Denote by $\text{dist}(\cdot, \cdot)$ the metric induced by the norm in E_n^α , $n \in \mathbb{N}$, i.e. $\text{dist}(x_n, y_n) = \|x_n - y_n\|_{E_n^\alpha}$.

1. We say that the family of sets $\{\mathcal{A}_n\}_{1 \leq n \leq \infty}$ is \mathcal{P}^α -upper semicontinuous at infinity if $\sup_{u_n \in \mathcal{A}_n} \text{dist}(u_n, p_n^\alpha \mathcal{A}) \xrightarrow{n \rightarrow \infty} 0$.
2. We say that the family of sets $\{\mathcal{A}_n\}_{1 \leq n \leq \infty}$ is \mathcal{P}^α -lower semicontinuous at infinity if $\sup_{u \in \mathcal{A}} \text{dist}(p_n^\alpha u, \mathcal{A}_n) \xrightarrow{n \rightarrow \infty} 0$.

THEOREM 1.7. (see Theorem 5.1) Let \mathcal{A} , \mathcal{A}_n be the attractors for (1.2) and (1.6), respectively and assume that [A1] holds. If $\{x_n^*\}$, $x_n^* \in \mathcal{A}_n$, \mathcal{P}^α -converges to some point x^* , then $x^* \in \mathcal{A}$ and the $\{\mathcal{A}_n\}$ is \mathcal{P}^α -upper semicontinuous at infinity.

THEOREM 1.8. (see Theorem 5.2) Assume that (1.4) has exactly m solutions u^1, \dots, u^m , that all of them are hyperbolic and that [A1], [A2] hold. Then, there exists n_0 such that for all $n \geq n_0$ the problem (1.8) has at least m solutions u_n^1, \dots, u_n^m which satisfy

$$\|u_n^i - p_n^\alpha u^i\|_{E_n^\alpha} \xrightarrow{n \rightarrow \infty} 0, \quad i = 1, \dots, m.$$

Furthermore, the sequence of unstable manifolds $\{W^u(u_n^k)\}_{n \in \mathbb{N}}$, $k = 1, \dots, m$, is \mathcal{P}^α -lower semicontinuous at infinity (here $W^u(u_\infty^k) =: W^u(u^k)$).

THEOREM 1.9. (see Theorem 5.6) Let $\mathcal{A} = \mathcal{A}_\infty$, \mathcal{A}_n be the attractors for (1.2) and (1.6), respectively. Assume \mathcal{A} can be written in the form (6.2) and that \mathcal{E} consists only of hyperbolic equilibria. Assume also that [A1], [A2] hold. If $x^* \in \mathcal{A}$, then there is a sequence of points $\{x_n^*\}$, with $x_n^* \in \mathcal{A}_n$, such that $x_n^* \xrightarrow{\mathcal{P}^\alpha} x^*$ and the $\{\mathcal{A}_n\}$ is \mathcal{P}^α -lower semicontinuous at infinity.

In what follows we describe the algorithm that we followed to show continuity (upper and lower semicontinuity) of attractors. As presented here, the problem (1.6) is a perturbation of the problem (1.2).

- The perturbation of the unbounded operator A may be very singular and the continuity of its approximation by A_n is given in terms of continuity of resolvent operators.
- Continuity of resolvents (plus some \mathcal{P} -compactness and stability) ensures the continuity (upper and lower) of the spectrum and of generalized eigenspaces.
- The continuity of resolvent operators (plus some stability) implies the continuity of linear semigroups. This is accomplished through a version of Trotter-Kato Theorem (Theorem 1.5) referred as Theorem ABC.
- Using the variation of constants formula (plus some compactness) we can obtain the continuity of nonlinear semigroups given in Theorem 4.6.
- The continuity of nonlinear semigroups (plus some uniform boundedness and \mathcal{P} -compactness) imply the upper semicontinuity of attractors.
- Assuming that all points in \mathcal{E} (solutions of (2.4)) are hyperbolic equilibrium we obtain continuity of the set of equilibria.
- The continuity of the set of equilibria implies the continuity of the linear unstable manifold of the linearization around equilibria. Using the continuity of the linear unstable manifolds we obtain the continuity of the nonlinear local (near the equilibrium) unstable manifolds.
- Assuming that the attractor for the problem (1.2) can be written in the form (6.2) and using the continuity of the nonlinear semigroups and the continuity of the local unstable manifolds (plus uniform boundedness and \mathcal{P} -compactness of the attractors) we obtain lower semicontinuity of attractors.

This paper is organized as follows. In Section 2 we establish the existence of attractors for (1.2), prove that if all equilibrium points for (1.2) are hyperbolic then there is an odd number of them, prove the existence of unstable manifolds in the way needed for the approximation and characterize the attractors that can be approximated lower semicontinuously. In Section 3 we establish the results on convergence of the spectrum (upper and lower semicontinuously). Section 4 is devoted to approximation of semiflows. Finally in Section 5 we prove the upper semicontinuity of attractors, the continuity of hyperbolic equilibria and unstable manifolds and the lower semicontinuity of attractors. We conclude the paper with an Appendix where we have placed basic results on attractors, examples and counterexamples and some basic results on convergence of resolvents and uniform estimates we needed throughout the paper.

Acknowledgement: This article has been written while the second author visited the *Departamento de Matemática* of the *Instituto de Ciências Matemáticas e de Computação/Universidade de São Paulo* supported by grant # 02/10699-7 from FAPESP. The author wishes to acknowledge the great hospitality of the people at this institution and the support from FAPESP.

2. ATTRACTORS FOR ABSTRACT PARABOLIC PROBLEMS

The results in this section will be proved under the following assumptions

- the operator A generates a compact analytic C_0 -semigroup with estimate $\|e^{tA}\| \leq Me^{-t\omega}$, $t \geq 0$, for some $\omega > 0$, (we note that this follows from (1.1)),

• function $f(\cdot) : E^\alpha \rightarrow E$ is a globally Lipschitz, bounded (i.e. $\exists K > 0$, such that $\sup_{u \in E^\alpha} \|f(u)\|_E \leq K$), and continuously Fréchet differentiable function.

2.1. Preliminaries

Let us show that the conditions imposed on A and $f(\cdot)$ in (1.2) are enough to guarantee that semigroup $\{T(t) : t \geq 0\}$ has a global attractor. Recall that for any set $B \subset E^\alpha$, the sets $\gamma^+(B) = \cup_{u^0 \in B} \gamma^+(u^0)$, $\gamma^-(B) = \cup_{u^0 \in B} \gamma^-(u^0)$, $\gamma(B) = \cup_{u^0 \in B} \gamma(u^0)$, are respectively, the positive orbit, the negative orbit, the complete orbit through B when they exist.

For any set $B \subset E^\alpha$, define the ω -limit set $\omega(B)$ of B and the α -limit set $\alpha(B)$ of B as

$$\omega(B) = \bigcap_{s \geq 0} \overline{\cup_{t \geq s} T(t)B}, \quad \alpha(B) = \bigcap_{s \geq 0} \overline{\cup_{t \geq s} H(t, B)}.$$

We start with some basic lemmas:

LEMMA 2.1. *If $S \subset E^\alpha$ is invariant under $\{T(t) : t \geq 0\}$ and $v \in S$, then there is a complete orbit through v .*

LEMMA 2.2. *A point v belongs to $\omega(B)$ if and only if there is a sequence $t_n \rightarrow +\infty$ and a sequence $v_n \in B$ such that $T(t_n)v_n \rightarrow v$.*

LEMMA 2.3. *For any bounded set $B \subset E^\alpha$ the set $\gamma^+(B)$ is bounded and $T(t)\gamma^+(B)$ is precompact for any $t > 0$.*

Proof: It is well known that $T(t) : E^\alpha \rightarrow E^\alpha$ is compact. Using the Variation of Constants Formula and standard estimates, it follows from the assumptions on A and $f(\cdot)$ that $\gamma^+(B)$ is bounded in E^α whenever B is bounded in E^α . Therefore, since $\gamma^+(B)$ is bounded, we have that $\{T(s)B : s \geq t\} = T(t)\gamma^+(B)$ is compact and the result follows. \square

LEMMA 2.4. *For any $u^0 \in E^\alpha$ the set $\omega(u^0)$ is non-empty, connected, compact, invariant and attracts u^0 .*

See [31] for a proof.

LEMMA 2.5. *Suppose that $u^0 \in E^\alpha$ is such that there is a negative orbit $\phi : (-\infty, 0] \rightarrow E^\alpha$ through u^0 and such that $\overline{\phi((-\infty, 0])}$ is compact. Define*

$$\alpha_\phi(u^0) = \{v \in E^\alpha : \exists t_n \rightarrow \infty \text{ such that } \phi(-t_n) \rightarrow v\}.$$

Then, $\alpha_\phi(u^0)$ is non-empty, connected, compact and invariant.

Proof: It is easy to see that $\alpha_\phi(u^0) = \bigcap_{t \geq 0} \overline{\phi((-\infty, t])}$ and from this it is immediate that $\alpha_\phi(u^0)$ is non-empty, compact and connected.

It remains to prove that $\alpha_\phi(u^0)$ is invariant. In fact, if $v \in \alpha_\phi(u^0)$, then there is a sequence $t_n \rightarrow +\infty$ such that $\phi(-t_n) \rightarrow v$. From the continuity of $T(t) : E^\alpha \rightarrow E^\alpha$ we obtain that $T(t)\phi(-t_n) = \phi(t - t_n) \rightarrow T(t)v$ and therefore $T(t)v \in \alpha_\phi(u^0)$. On the other hand, if $w \in \alpha_\phi(u^0)$, there is a sequence $t_n \rightarrow \infty$ (w.l.o.g. we can assume $t_n \geq t$, $n \in \mathbb{N}$) such that $\phi(-t_n) \rightarrow w$. Since $\{\phi(-t_n - t) : n \in \mathbb{N}\}$ is relatively compact, taking subsequences if necessary, there is a $z \in E^\alpha$ such that $\phi(-t_n - t) \rightarrow z$ and $z \in \alpha_\phi(u^0)$. It follows from the uniqueness of the limit that $T(t)z = w$. \square

Remark 2. 1. It follows immediately from the invariance that, if $u_0 \in E^\alpha$ and $\omega(u_0) = \{u^*\}$, then u^* is an equilibrium. Similar statements hold for $\alpha(u_0)$ and $\alpha_\phi(u_0)$.

LEMMA 2.6. *The orbit of any bounded set $B \subset E^\alpha$ is bounded and for each bounded set $B \subset E^\alpha$ there is an instant τ_B and a constant N (independent of B) such that*

$$\sup_{t \geq \tau_B} \sup_{w \in T(t)B} \|w\|_{E^\alpha} \leq N. \quad (2.1)$$

Furthermore,

$$\sup_{\substack{B \subset E^\alpha \\ B \text{ bounded}}} \sup_{v \in \omega(B)} \|v\|_{E^\alpha} \leq N. \quad (2.2)$$

Proof: The first statement follows trivially from the Variation of Constants Formula and (2.2) follows from (2.1) and from the definition of $\omega(B)$. It remains to prove (2.1). Given a bounded set $B \subset E^\alpha$ and $v \in B$ we have from (1.3) that

$$T(t)v = \exp(tA)v + \int_0^t \exp((t-s)A)f(T(s)v)ds, \quad t \geq 0. \quad (2.3)$$

From the hypothesis on $f(\cdot)$ and on A we have that there is a constant K such that $\|f(w)\|_E \leq K$ for all $w \in E^\alpha$ and

$$\|T(t)v\|_{E^\alpha} \leq Me^{-\beta t}\|v\|_{E^\alpha} + MK \int_0^t (t-s)^{-\alpha} e^{-\omega(t-s)} ds.$$

From the fact that v lies in a bounded subset of E^α we have that there exists τ_B such that

$$\|T(t)v\|_{E^\alpha} \leq 1 + MK \int_0^\infty \xi^{-\alpha} e^{-\omega\xi} d\xi =: N, \quad t \geq \tau_B.$$

Since N is independent of $B \subset E^\alpha$ the result follows. \square

LEMMA 2.7. *If $B \subset E^\alpha$ is a bounded set, then $\omega(B)$ is non-empty, compact, invariant, and attracts B under $\{T(t) : t \geq 0\}$. Furthermore, if B is connected, then $\omega(B)$ is connected.*

See [31] for a proof.

THEOREM 2.1. *If $B_N = \{u \in E^\alpha : \|u\|_{E^\alpha} \leq N\}$, then $\omega(B_N)$ is the global attractor for $\{T(t) : t \geq 0\}$.*

Proof. It is clear that $\omega(B_N)$ is compact, invariant, connected and attracts B_N . First note that for any bounded set B there is a τ_B such that $T(t)B \subset B_N$ for all $t \geq \tau_B$. Since B_N is attracted by $\omega(B_N)$ and since B_N attracts B the result follows. \square

2.2. Equilibrium points

Now we are going to describe the structure of the attractor for (1.2). We will start with the simplest elements from the attractor, the equilibrium solutions. The equilibrium solutions of (1.2) are those solutions which are independent of time $u(t) = u$, $t \geq 0$; that is, the solutions of equation

$$Au + f(u) = 0. \quad (2.4)$$

We say that a solution u^* of (2.4), i.e. equilibrium point u^* , is hyperbolic if the spectrum $\sigma(A + f'(u^*))$ is disjoint from the imaginary axis, i.e. $\sigma(A + f'(u^*)) \cap i\mathbb{R} = \emptyset$.

Let C be an open convex set in a Banach space F , and let $\mathcal{B} : \overline{C} \rightarrow F$ be a compact operator having no fixed points on the boundary ∂C of C . Then for the vector field $\mathcal{V}(x) = x - \mathcal{B}x$, the rotation (degree) $\gamma(I - \mathcal{B}; \partial C)$ being an integer-valued characteristic of this field is well defined. Let z^* be an isolated fixed point of the operator \mathcal{B} in the ball $B(z^*, r_0)$ of radius r_0 centered at a point z^* . Then $\gamma(I - \mathcal{B}; \partial B(z^*, r_0)) = \gamma(I - \mathcal{B}; \partial B(z^*, r))$ for $0 < r < r_0$. This common value of the rotation is called the index of the fixed point z^* and is denoted by $\text{ind}(z^*; I - \mathcal{B})$.

THEOREM 2.2. *If u^* is equilibrium point of (1.2) which satisfies $0 \notin \sigma(A + f'(u^*))$ then, u^* is an isolated equilibrium and $|\text{ind}(u^*, I + A^{-1}f'(u^*))| = 1$.*

Proof: Note that u^* is an equilibrium point of (1.2) if and only if it is a fixed point of the compact operator $-A^{-1}f(\cdot) : E^\alpha \rightarrow E^\alpha$. Also, $0 \notin \sigma(A + f'(u^*))$ if and only if $1 \notin \sigma(-A^{-1}f'(u^*))$. Now the proof is a direct consequence of Theorem 21.6 in [55]. \square

COROLLARY 2.1. *If u^* is a hyperbolic equilibrium point of (1.2) then, u^* is an isolated equilibrium and $|\text{ind}(u^*, I + A^{-1}f'(u^*))| = 1$.*

PROPOSITION 2.1. *If all points in \mathcal{E} are isolated then, there is only a finite number of them. If $0 \notin \sigma(A + f'(u^*))$ for each $u^* \in \mathcal{E}$ then, \mathcal{E} is a finite set with an odd number of elements.*

Proof. First we observe that since $f(\cdot) : E^\alpha \rightarrow E$ is bounded we have that all solutions of (2.4) satisfies

$$Iu + A^{-1}f(u) = 0 \quad (2.5)$$

and therefore $\|u\|_{E^1} \leq \|f(u)\|_E \leq K$. So if we consider the ball of radius more than $\|A^{-1+\alpha}\|K$, then the operator $-A^{-1}f(\cdot)$ maps the ball $B(0, \|A^{-1+\alpha}\|K) \subset E^\alpha$ into itself. By Schauder fixed point Theorem (see Theorem 21.5 [55]) $\gamma(I+A^{-1}f(\cdot), \partial B(0, \|A^{-1+\alpha}\|K)) = 1$, and there is at least one fixed point u^* for $-A^{-1}f(\cdot)$ in $B(0, \|A^{-1+\alpha}\|K)$; that is,

$$Iu^* + A^{-1}f(u^*) = 0 \text{ with } u^* \in B(0, \|A^{-1+\alpha}\|K).$$

Since the operator $-A^{-1}f(\cdot) : E^\alpha \rightarrow E^\alpha$ is compact we have that the set $\mathcal{E} = \{u : Au + f(u) = 0\}$ is compact in E^α . Moreover, by Theorem 2.2 any fixed point u^* is isolated and $|\text{ind}(u^*, I + A^{-1}f(\cdot))| = 1$. If the number of the fixed points is infinite, i.e. we have a sequence $\{u_i^*\}_{i=1}^\infty$, then the sequence $-A^{-1}f(u_i^*) = u_i^* \rightarrow u_\infty^*$ converges on some subsequence $i \in \mathbb{N}' \subset \mathbb{N}$, which is a contradiction with the fact that each fixed point u_∞^* is isolated. So the number of the equilibrium points is finite. Now by Theorem 20.6 in [55]

$$1 = \gamma(I + A^{-1}f(\cdot), \partial B(0, \|A^{-1}\|K)) = \sum_{i=1}^d \text{ind}(u_i^*, I + A^{-1}f(\cdot))$$

and therefore the number $d = 2k + 1$ for some integer $k \geq 0$. \square

2.3. Local unstable manifolds

Next we show that there are many other solutions which are in the attractor besides the stationary solutions. It is easy to see that every mild solution of (1.2) remains bounded for any $u^0 \in E^\alpha$; that is, $\sup_{t \geq 0} \|u(t, u^0)\|_{E^\alpha} < \infty$. If we show that there is a global solution $u(\cdot, u^0) : \mathbb{R} \rightarrow E^\alpha$ through u_0 which is bounded, then it must lie in the attractor \mathcal{A} . So it is enough to exhibit a bounded backward solution through u_0 to obtain a solution in the attractor. We define the unstable set of $u^* \in \mathcal{E}$ as $W^u(u^*) = \{\eta \in E^\alpha : \text{there is a backward solution } u(t, \eta) \text{ through } \eta \text{ such that } u(t, \eta) \rightarrow u^* \text{ as } t \rightarrow -\infty\}$.

Before we give more details about how to construct $W^u(u^*)$ in a neighborhood of u^* , let us consider (1.2) locally near u^* . If we consider the change of variables $v(\cdot) = u(\cdot) - u^*$ in the problem (1.2), add and subtract $f'(u^*)v(\cdot)$ to its right hand side we obtain

$$\begin{aligned} v'(t) &= (A + f'(u^*))v(t) + f(v(t) + u^*) - f(u^*) - f'(u^*)v(t), \\ v(0) &= u^0 - u^* = v^0. \end{aligned} \quad (2.6)$$

In this equation, for v very small, the part $f(v + u^*) - f(u^*) - f'(u^*)v$ becomes small when compared to v . It is natural then to consider what happens when we neglect the nonlinearity; that is, what happens to the Cauchy problem

$$\begin{aligned} v'(t) &= (A + f'(u^*))v(t), \\ v(0) &= v^0. \end{aligned} \quad (2.7)$$

Just to say a word about the operator $A + f'(u^*)$, the operator $f'(u^*) \in B(E^\alpha, E)$ and therefore the operator $A + f'(u^*)$ is defined in $D(A)$, has compact resolvent and it is the generator of an analytic C_0 -semigroup. It is well-known that such generator has the spectrum consisting of isolated eigenvalues with finite dimensional generalized eigenspaces.

The part σ^+ of the spectrum of operator $A + f'(u^*)$ to the right of the imaginary axis consists of a finite number of eigenvalues with finite multiplicity (see Figure 4). Let $U(\sigma^+) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda > 0\}$ be an open connected neighborhood of σ^+ which has a closed rectifiable curve, $\partial U(\sigma^+)$, as boundary. We decompose E^α using the projection

$$Q(\sigma^+) := Q(\sigma^+, A + f'(u^*)) := \frac{1}{2\pi i} \int_{\partial U(\sigma^+)} \left(\zeta I - (A + f'(u^*)) \right)^{-1} d\zeta \quad (2.8)$$

defined by σ^+ . Also, for some positive $M, \beta > 0$, because of analyticity of semigroup, i.e. from the property $\omega(A) = \theta(A)$, we have that

$$\begin{cases} \|\exp(t(A + f'(u^*)))z\|_{E^\alpha} \leq M e^{-\beta t} \|z\|_{E^\alpha}, & t \geq 0, \\ \|\exp(t(A + f'(u^*)))z\|_{E^\alpha} \leq M t^{-\alpha} e^{-\beta t} \|z\|_E, & t > 0, \\ \|\exp(t(A + f'(u^*)))v\|_{E^\alpha} \leq M e^{\beta t} \|v\|_E, & t \leq 0 \end{cases} \quad (2.9)$$

for all $v \in Q(\sigma^+)E^\alpha$ and $z \in (I - Q(\sigma^+))E^\alpha$. Note that $\|\exp(t\tilde{A})z\|_{E^\alpha} = \|(-A)^\alpha \exp(t\tilde{A})z\| = \|(-A)^\alpha \tilde{A}^{-\alpha} (-\tilde{A})^\alpha \exp(t\tilde{A})z\|$ and since $(-A)^\alpha (-\tilde{A})^{-\alpha}$ is bounded we get inequalities (2.9). We also note also that $Q(\sigma^+)E^\alpha$ and $Q(\sigma^+)E$ are the same finite dimensional subspace of E and that in this sense the E^α -norm is equivalent to the E -norm in it.

For $v_0 \in Q(\sigma^+)E^\alpha$, the solution $v(t, v_0)$ of (2.7) exists for all negative time and $v(t, v_0) \rightarrow 0$ as $t \rightarrow -\infty$ and $v(t) + u^* \rightarrow u^*$ as $t \rightarrow -\infty$. That serves as inspiration to what we want to do. When we perturb (2.7) with a very small nonlinearity we should observe solutions of (2.6) that exist for all negative time. Of course, the initial data for which such solutions exist will no longer be in $Q(\sigma^+)E^\alpha$, but in a nonlinear manifold near it (see Figure 1).

THEOREM 2.3. *Assume that u^* is a hyperbolic equilibrium for (1.2), which is an isolated solution of (2.4). Then, there exist a Lipschitz continuous function $S : \{\eta \in Q(\sigma^+)E^\alpha : \|\eta\|_{E^\alpha} \leq \delta\} \rightarrow (I - Q(\sigma^+))E^\alpha$ such that, for suitably small $\delta > 0$, the local unstable manifold $W_\delta^u(u^*)$ is given by*

$$W_\delta^u(u^*) = \{(\eta, S(\eta)) + u^* : \eta \in Q(\sigma^+)E^\alpha, \|\eta\|_{E^\alpha} \leq \delta\}$$

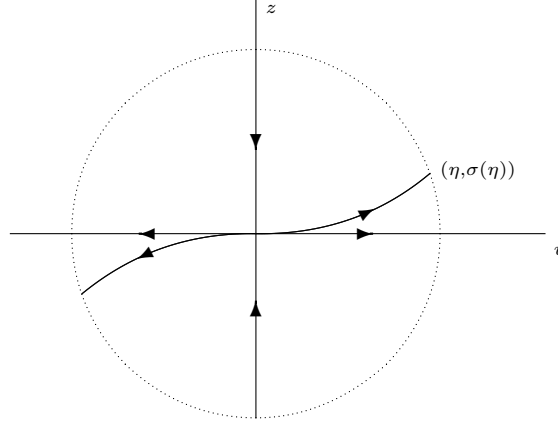


Figure 1

Proof: Rewriting (1.2) for $w(t) = u(t) - u^*$ to deal with the neighborhood of $u^* \in \mathcal{E}$ we arrive at

$$w'(t) = (A + f'(u^*))w(t) + f(w(t) + u^*) - f(u^*) - f'(u^*)w(t). \quad (2.10)$$

Now we decompose the equation (2.10) in the following way. If $w(\cdot)$ is a mild solution of (2.10) we write $v(t) = Q(\sigma^+)w(t)$ and $z(t) = (I - Q(\sigma^+))w(t)$. Hence, if B denotes the restriction of $(A + f'(u^*))$ to $Q(\sigma^+)E^\alpha$ and by \tilde{A} the restriction of $(A + f'(u^*))$ to $(I - Q(\sigma^+))E^\alpha$, we have that

$$\begin{aligned} v'(t) &= Bv(t) + H(v(t), z(t)), \\ z'(t) &= \tilde{A}z(t) + G(v(t), z(t)), \end{aligned} \quad (2.11)$$

where

$$H(v, z) = Q(\sigma^+)f(v(t) + z(t) + u^*) - Q(\sigma^+)f(u^*) - Q(\sigma^+)f'(u^*)(v(t) + z(t))$$

and

$$G(v, z) = (I - Q(\sigma^+))f(v(t) + z(t) + u^*) - (I - Q(\sigma^+))f(u^*) - (I - Q(\sigma^+))f'(u^*)(v(t) + z(t)).$$

Hence, we have that, at $(0, 0)$ the functions H and G are zero with zero derivatives at $(0, 0)$.

Now we will show that the equilibrium solution u^* of (2.11) has a local unstable manifold which is given by

$$W_\delta^u(u^*) = \{(\eta, S(\eta)) + u^* : \eta \in Q(\sigma^+)E^\alpha, \|\eta\|_{E_\alpha} \leq \delta\}.$$

A backward solution $(v(t), z(t))$ on the unstable manifold must go to zero as $t \rightarrow -\infty$ and in particular it must stay bounded. Since

$$z(t) = \exp((t - t_0)\tilde{A})z(t_0) + \int_{t_0}^t \exp((t - s)\tilde{A}) G(v(s), z(s)) ds,$$

making $t_0 \rightarrow -\infty$, we have that

$$z(t) = \int_{-\infty}^t \exp((t - s)\tilde{A}) G(v(s), z(s)) ds.$$

The mild solution of (2.11) is a vector-function $w(t) = \begin{pmatrix} v(t) \\ z(t) \end{pmatrix}$, which satisfies equation

$$w(t) = \Theta(\eta, w(\cdot))(t) \equiv \begin{pmatrix} \exp((t - \tau)B)\eta + \int_{\tau}^t \exp((t - s)B) H(w(s)) ds \\ \int_{-\infty}^t \exp((t - s)\tilde{A}) G(w(s)) ds \end{pmatrix}. \quad (2.12)$$

Now, let us consider (2.12) in the space

$$\begin{aligned} \Upsilon &= C_0((-\infty, \tau]; Q(\sigma^+)E^\alpha \times (I - Q(\sigma^+))E^\alpha) \\ &= \left\{ \begin{pmatrix} v(\cdot) \\ z(\cdot) \end{pmatrix} \in C((-\infty, \tau]; Q(\sigma^+)E^\alpha \times (I - Q(\sigma^+))E^\alpha) : \lim_{t \rightarrow -\infty} \begin{pmatrix} v(t) \\ z(t) \end{pmatrix} = 0 \right\}. \end{aligned}$$

with the sup norm. Let us show that $\Theta(\eta, \cdot) : \Upsilon \rightarrow \Upsilon$.

Take $w(\cdot) \in \Upsilon$. It is easy to see that $\Theta(\eta, w(\cdot))$ is a continuous function. It remains to show that $\lim_{t \rightarrow -\infty} \Theta(\eta, w(\cdot))(t) = 0$.

Given $\epsilon > 0$, there exists $t_\epsilon \in (-\infty, 0)$ such that $\|H(w(s))\|_{Q(\sigma^+)E} \leq \epsilon$ for all $s \leq t_\epsilon$. From (2.9) it follows that $\lim_{t \rightarrow -\infty} e^{(t-\tau)B}\eta = 0$ for any $\eta \in Q(\sigma^+)E^\alpha$ and, for $t \leq t_\epsilon$,

$$\begin{aligned} \left\| (-A)^\alpha \int_{\tau}^t e^{(t-s)B} H(w(s)) ds \right\| &\leq C \left\| \int_{\tau}^t e^{(t-s)B} H(w(s)) ds \right\| \\ &\leq C \left\| e^{(t-t_\epsilon)B} \int_{\tau}^{t_\epsilon} e^{(t_\epsilon-s)B} H(w(s)) ds + \int_t^{t_\epsilon} e^{(t-s)B} H(w(s)) ds \right\| \\ &\leq CM^2 e^{\beta(t-t_\epsilon)} \int_{\tau}^{t_\epsilon} e^{\beta(t_\epsilon-s)} K ds + M\epsilon \int_t^{t_\epsilon} e^{\beta(t-s)} ds \\ &\leq CM^2 e^{\beta(t-t_\epsilon)} \frac{K}{\beta} + \epsilon \frac{M}{\beta}. \end{aligned}$$

Hence, given $\epsilon > 0$,

$$0 \leq \limsup_{t \rightarrow -\infty} \left\| (-A)^\alpha \int_{\tau}^t \exp((t - s)B) H(w(s)) ds \right\| \leq \epsilon \frac{M}{\beta}.$$

This shows that the first coordinate of $\Theta(\eta, w(\cdot))(t)$ goes to zero as $t \rightarrow -\infty$. It is easy to see that the second coordinate of $\Theta(\eta, w(\cdot))(t)$ also goes to zero as $t \rightarrow -\infty$.

If we put $\eta = \eta_0 = 0$ and $w(t) = w_0(t) = \begin{pmatrix} v_0(t) \\ z_0(t) \end{pmatrix} = 0, t \in (-\infty, \tau]$, then $w_0(t) = \Theta(0, w_0(\cdot))(t)$ and operator $\Theta(\cdot, \cdot)$ is continuous in both arguments. Moreover, Fréchet derivative $\Theta'_w(\eta_0, w_0) : \Upsilon \rightarrow \Upsilon$ is given by

$$(\Theta'_w(\eta_0, w_0)h)(t) = \begin{pmatrix} \int_{-\infty}^t e^{(t-s)\tilde{A}} H'_v(w_0(s)) h^1(s) ds + \int_{-\infty}^t e^{(t-s)\tilde{A}} H'_z(w_0(s)) h^2(s) ds \\ \int_{-\infty}^t e^{(t-s)\tilde{A}} G'_v(w_0(s)) h^1(s) ds + \int_{-\infty}^t e^{(t-s)\tilde{A}} G'_z(w_0(s)) h^2(s) ds \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$t \leq \tau$, for any $h = \begin{pmatrix} h^1(t) \\ h^2(t) \end{pmatrix} \in \Upsilon$, since functions G, H have derivatives equal to 0 at zero.

Therefore continuous operator $(I - \Theta'_w(\eta_0, w_0))^{-1}$ exists. By Theorem 54.2 from [55] it follows that there are $\delta, \rho > 0$ such that for $\|\eta - \eta_0\|_{Q(\sigma^+)E^\alpha} \leq \delta$ equation (2.12) has in the ball $\|w - w_0\|_\Upsilon \leq \rho$ unique solution $w(\cdot)$, which depends on η continuously.

From this it follows that the unstable manifold is the graph of the function

$$S : \{\eta \in Q(\sigma^+)E^\alpha : \|\eta\|_{E^\alpha} \leq \delta\} \rightarrow (I - Q(\sigma^+)E^\alpha)$$

defined by

$$S(\eta) = z(\tau) = \int_{-\infty}^{\tau} \exp((\tau - s)\tilde{A}) G(w(s)) ds.$$

It is clear from the above reasoning that S is a continuous function and it is easy to see that S is Lipschitz continuous. \square

3. APPROXIMATION OF SPECTRUM OF LINEAR OPERATORS

The notions of *stable* and *regular* convergence play the most important role in approximations of equation $Bx = y$ and approximations of spectrum of an operator B . These notions are used in many different areas of numerical analysis (see [2, 21, 26, 27, 74, 77]).

3.1. Stable and Regular convergences

DEFINITION 3.1. A sequence of operators $\{B_n\}, B_n \in B(E_n), n \in \mathbb{N}$, is said to be stably convergent to an operator $B \in B(E)$ iff $B_n \xrightarrow{\mathcal{PP}} B$ and $\|B_n^{-1}\|_{B(E_n)} = O(1), n \rightarrow \infty$. We will write this as: $B_n \xrightarrow{\mathcal{PP}} B$ stably.

DEFINITION 3.2. A sequence of operators $\{B_n\}, B_n \in B(E_n)$, is called regularly convergent to the operator $B \in B(E)$ iff $B_n \xrightarrow{\mathcal{PP}} B$ and the following implication holds:

$$\|x_n\|_{E_n} = O(1) \ \& \ \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact} \implies \{x_n\} \text{ is } \mathcal{P}\text{-compact}.$$

We write this as: $B_n \xrightarrow{\mathcal{PP}} B$ regularly.

Let $\Lambda \subseteq \mathbb{C}$ be some open connected set, and let $B \in B(E)$. For an isolated point $\lambda \in \sigma(B)$, the corresponding maximal invariant space (or generalized eigenspace) will be denoted by $\mathcal{W}(\lambda; B) = Q(\lambda)E$, where $Q(\lambda) = \frac{1}{2\pi i} \int_{|\zeta-\lambda|=\delta} (\zeta I - B)^{-1} d\zeta$ and δ is small enough so that there are no points of $\sigma(B)$ in the disc $\{\zeta : |\zeta - \lambda| \leq \delta\}$ different from λ . The isolated point $\lambda \in \sigma(B)$ is a *Riesz point* of B if $\lambda I - B$ is a Fredholm operator of index zero and $Q(\lambda)$ is of finite rank. Denote by

$$\mathcal{W}(\lambda, \delta; B_n) = \bigcup_{\substack{\lambda_n \in \sigma(B_n), \\ |\lambda_n - \lambda| < \delta}} \mathcal{W}(\lambda_n, B_n).$$

It is clear that $\mathcal{W}(\lambda, \delta; B_n) = Q_n(\lambda)E_n$, where

$$Q_n(\lambda) = \frac{1}{2\pi i} \int_{|\zeta-\lambda|=\delta} (\zeta I - B_n)^{-1} d\zeta.$$

The following theorem state the complete picture of the approximation of the spectrum.

THEOREM 3.1. [74] Assume that $L_n(\lambda)$ and $L(\lambda)$ are Fredholm operators of index zero for all $\lambda \in \Lambda$. Suppose that $L_n(\lambda) \rightarrow L(\lambda)$ regularly for any $\lambda \in \Lambda$ and $\rho(B) \cap \Lambda \neq \emptyset$. Then

- (i) for any $\lambda_0 \in \sigma(B) \cap \Lambda$, there exists a sequence $\{\lambda_n\}, \lambda_n \in \sigma(B_n), n \in \mathbb{N}$, such that $\lambda_n \rightarrow \lambda_0$ as $n \rightarrow \infty$;
- (ii) if for some sequence $\{\lambda_n\}, \lambda_n \in \sigma(B_n), n \in \mathbb{N}$, one has $\lambda_n \rightarrow \lambda_0 \in \Lambda$ as $n \rightarrow \infty$, then $\lambda_0 \in \sigma(B)$;
- (iii) for any $x \in \mathcal{W}(\lambda_0, B)$, there exists a sequence $\{x_n\}, x_n \in \mathcal{W}(\lambda_0, \delta; B_n), n \in \mathbb{N}$, such that $x_n \rightarrow x$ as $n \rightarrow \infty$;
- (iv) there exists $n_0 \in \mathbb{N}$ such that $\dim \mathcal{W}(\lambda_0, \delta; B_n) = \dim \mathcal{W}(\lambda_0, B)$ for all $n \geq n_0$;
- (v) any sequence $\{x_n\}, x_n \in \mathcal{W}(\lambda_0, \delta; B_n), n \in \mathbb{N}$, with $\|x_n\|_{E_n} = 1$ is \mathcal{P} -compact and any limit point of this sequence belongs to $\mathcal{W}(\lambda_0, B)$.

The Theorem 3.1 gives us necessary and sufficient conditions for the equality

$$\dim \mathcal{W}(\lambda_0, \delta; B_n) = \dim \mathcal{W}(\lambda_0, B), \quad \forall n \geq n_0,$$

which is needed in the approximation of unstable manifolds.

3.2. Regions of convergence

Theorem 3.1 have been generalized to the case of closed operators in [79] by using the following notions introduced by T. Kato [51].

DEFINITION 3.3. The region of stability $\Delta_s = \Delta_s(\{A_n\})$, $A_n \in \mathcal{C}(B_n)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \rho(A_n)$ for almost all n and such that the sequence $\{\|(\lambda I_n - A_n)^{-1}\|\}_{n \in \mathbb{N}}$ is bounded. The region of convergence $\Delta_c = \Delta_c(\{A_n\})$, $A_n \in \mathcal{C}(E_n)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that $\lambda \in \Delta_s(\{A_n\})$ and such that the sequence of operators $\{(\lambda I_n - A_n)^{-1}\}_{n \in \mathbb{N}}$ is \mathcal{PP} -convergent to some operator $S(\lambda) \in B(E)$.

DEFINITION 3.4. A sequence of operators $\{K_n\}$, $K_n \in \mathcal{C}(E_n)$, is called regularly compatible with an operator $K \in \mathcal{C}(E)$ if (K_n, K) are compatible and, for any bounded sequence $\|x_n\|_{E_n} = O(1)$ such that $x_n \in D(K_n)$ and $\{K_n x_n\}$ is \mathcal{P} -compact, it follows that $\{x_n\}$ is \mathcal{P} -compact, and the \mathcal{P} -convergence of $\{x_n\}$ to some x and that of $\{K_n x_n\}$ to some y as $n \rightarrow \infty$ in $\mathbb{N}' \subseteq \mathbb{N}$ imply that $x \in D(K)$ and $Kx = y$.

DEFINITION 3.5. The region of regularity $\Delta_r = \Delta_r(\{A_n\}, A)$, is defined as the set of all $\lambda \in \mathbb{C}$ such that (K_n, K) are regularly compatible, where $K_n = \lambda I - A_n$ and $K = \lambda I - A$.

The relationships between these regions are given by the following statement.

PROPOSITION 3.1. [79] Suppose that $\Delta_c \neq \emptyset$ and $\mathcal{N}(S(\lambda)) = \{0\}$ at least for one point $\lambda \in \Delta_c$ so that $S(\lambda) = (\lambda I - A)^{-1}$. Then (A_n, A) are compatible and

$$\Delta_c = \Delta_s \cap \rho(A) = \Delta_s \cap \Delta_r = \Delta_r \cap \rho(A).$$

It is shown in [79] that the conditions (A_n, A) are compatible, $\lambda I_n - A_n$ and $\lambda I - A$ are Fredholm operators with index zero for any $\lambda \in \Lambda$ and $\rho(A) \cap \Lambda \neq \emptyset$ imply (i)–(iv) of Theorem 3.1 when $\rho(A) \cap \Lambda \subseteq \Delta_s$ and imply (i)–(v) of Theorem 3.1, when $\Lambda \subseteq \Delta_r$.

DEFINITION 3.6. A Riesz point $\lambda_0 \in \sigma(A)$ is said to be strongly stable in Kato's sense if $\dim \mathcal{W}(\lambda_0, \delta; B_n) \leq \dim \mathcal{W}(\lambda_0, B)$ for all $n \geq n_0$.

THEOREM 3.2. [79] The Riesz point $\lambda_0 \in \sigma(A)$ is strongly stable in Kato's sense iff $\lambda_0 \in \Lambda \cap \Delta_r \cap \sigma(A)$.

We consider now the important class of operators which have compact resolvents, this is say our case of problem (1.2). In this case, it is natural to consider approximate operators which “preserve” this property.

DEFINITION 3.7. A sequence of operators $\{B_n\}$, $B_n : E_n \rightarrow E_n$, $n \in \mathbb{N}$, converges compactly to an operator $B : E \rightarrow E$ if $B_n \xrightarrow{\mathcal{PP}} B$ and the following compactness condition holds:

$$\|x_n\|_{E_n} = O(1) \implies \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

DEFINITION 3.8. The region of compact convergence of resolvents, $\Delta_{cc} = \Delta_{cc}(A_n, A)$, where $A_n \in \mathcal{C}(E_n)$ and $A \in \mathcal{C}(E)$ is defined as the set of all $\lambda \in \Delta_c \cap \rho(A)$ such that $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ compactly.

THEOREM 3.3. Assume that $\Delta_{cc} \neq \emptyset$. Then for any $\zeta \in \Delta_s$ the following implication holds:

$$\|x_n\|_{E_n} = O(1) \ \& \ \|(\zeta I_n - A_n)x_n\|_{E_n} = O(1) \implies \{x_n\} \text{ is } \mathcal{P}\text{-compact.} \quad (3.1)$$

Conversely, if for some $\zeta \in \Delta_c \cap \rho(A)$ implication (3.1) holds, then $\Delta_{cc} \neq \emptyset$.

Proof: Let $(\mu I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\mu I - A)^{-1}$ compactly for some $\mu \in \Delta_{cc}$. Then for $\|x_n\|_{E_n} = O(1)$ and $\|(\zeta I_n - A_n)x_n\|_{E_n} = O(1)$, from the Hilbert identity

$$(\zeta I_n - A_n)^{-1} - (\mu I_n - A_n)^{-1} = (\mu - \zeta)(\zeta I_n - A_n)^{-1}(\mu I_n - A_n)^{-1}, \quad (3.2)$$

we obtain $x_n = (\mu I_n - A_n)^{-1}(\zeta I_n - A_n)x_n - (\zeta - \mu)(\mu I_n - A_n)^{-1}x_n$, and it follows that $\{x_n\}$ is \mathcal{P} -compact. Conversely, let implication (3.1) hold for some $\zeta_0 \in \Delta_c \cap \rho(A)$. We show that $\zeta_0 \in \Delta_{cc}$. Taking a bounded sequence $\{y_n\}$, $n \in \mathbb{N}$, we obtain $\|(\zeta_0 I_n - A_n)^{-1}y_n\|_{E_n} = O(1)$ for $n \in \mathbb{N}$. Let us apply implication (3.1) to the sequence $x_n = (\zeta_0 I_n - A_n)^{-1}y_n$. It is easy to see that $\{x_n\}$ is \mathcal{P} -compact. Hence $\zeta_0 \in \Delta_{cc}$. \square

COROLLARY 3.1. Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_{cc} = \Delta_c \cap \rho(A)$.

Proof: It is clear that $\Delta_{cc} \subseteq \Delta_c \cap \rho(A)$. To prove that $\Delta_{cc} \supseteq \Delta_c \cap \rho(A)$, let us consider the Hilbert identity (3.2). Now let $\mu \in \Delta_{cc}$. Then $\mu \in \Delta_{cc} \cap \Delta_c \cap \rho(A)$. Hence, for every $\zeta \in \Delta_c \cap \rho(A)$ and for any bounded sequence $\{x_n\}$, $n \in \mathbb{N}$, the sequence $\{(\zeta I_n - A_n)^{-1}x_n\}$ is \mathcal{P} -compact. \square

THEOREM 3.4. Assume that $\Delta_{cc} \neq \emptyset$. Then $\Delta_r = \mathbb{C}$.

Proof: Take any point $\lambda_1 \in \mathbb{C}$. We have to show that $(\lambda_1 I_n - A_n, \lambda_1 I - A)$ are regularly compatible. Assume that $\|x_n\|_{E_n} = O(1)$ and that $\{(\lambda_1 I_n - A_n)x_n\}$ is \mathcal{P} -compact. To show that $\{x_n\}$ is \mathcal{P} -compact, we take $\mu \in \Delta_{cc}$. Assume now that $\|x_n\| \leq \text{constant}$ and $(\lambda_1 I_n - A_n)x_n \xrightarrow{\mathcal{P}} y$, as $n \rightarrow \infty$ in $\mathbb{N}' \subseteq \mathbb{N}$. Applying operator $(\mu I_n - A_n)^{-1}$ one gets first

that $\{x_n\}$ is \mathcal{P} -compact and then assuming $x_n \xrightarrow{\mathcal{P}} x$ as $n \in \mathbb{N}'' \subset \mathbb{N}'$ one obtains

$$\begin{aligned} (\mu I_n - A_n)^{-1}(\lambda_1 I_n - A_n)x_n &= (\lambda_1 - \mu)(\mu I_n - A_n)^{-1}x_n + x_n \xrightarrow{\mathcal{P}} \\ &\xrightarrow{\mathcal{P}} (\mu I - A)^{-1}y = (\lambda_1 - \mu)(\mu I - A)^{-1}x + x \end{aligned}$$

as $n \rightarrow \infty$ in $\mathbb{N}' \subseteq \mathbb{N}$. Then it follows that $x \in D(A)$ and $(\lambda_1 I - A)x = y$. \square

From the Theorem 3.4 it follows that in case $\Delta_{cc} \neq \emptyset$ all the assumptions of Theorem 3.1 are satisfied and a complete picture of approximation of spectrum is valid.

4. DISCRETIZATION OF SEMIGROUPS

As we will see later, most of the results on approximation of attractors follow from convergence of linear operators and families of operators.

The general approximation scheme is described in the Introduction. We just mention that from (1.5) and the Uniform Boundedness Principle (see [83]) one has

LEMMA 4.1. *There is a constant $C \geq 1$ such that $\|p_n\|_{L(E, E_n)} \leq C$, $\forall n \in N$.*

Let us recall that each mild solution of problem (1.2) satisfies the equation

$$u(t) = (Ku)(t) \equiv \exp(tA)u^0 + \int_0^t \exp((t-s)A)f(u(s)) ds, \quad t \in [0, T], T < \infty. \quad (4.1)$$

4.1. Compact convergence of some operator families

We know that all solutions of (1.6) are globally defined, hence $t \mapsto T_n(t)(\cdot)$ is a continuous nonlinear semigroup for each $n \in N$. Consider the operator equations

$$u_n(t) = (K_n u_n)(t) \equiv \exp(tA_n)u_n^0 + \int_0^t \exp((t-s)A_n)f_n(u_n(s)) ds, \quad t \in [0, T], T < \infty. \quad (4.2)$$

Consider the operators $K : F \rightarrow F$ and $K_n : F_n \rightarrow F_n$ defined by (4.1) and (4.2) where

$$F = C([0, T]; E^\alpha) \equiv \{u(t) : \|u\|_F = \max_{t \in [0, T]} \|u(t)\|_{E^\alpha} < \infty\}$$

and

$$F_n = C([0, T]; E_n^\alpha) \equiv \{u_n(t) : \|u_n\|_{F_n} = \max_{t \in [0, T]} \|u_n(t)\|_{E_n^\alpha} < \infty\}.$$

THEOREM 4.1. *Assume that the operator A generates analytic and compact C_0 -semigroup. Then operator K is compact.*

Proof: The operator K defined by (4.1) is compact in F . Indeed, we obtain that the operator

$$\mathcal{F}_\epsilon(u^k)(t) = \exp(\epsilon A) \int_0^{t-\epsilon} (-A)^\alpha \exp((t-s-\epsilon)A) f(u^k(s)) ds$$

maps any bounded set of functions $\{u^k(\cdot)\} \subset F$, into a compact set in E^α for any $0 < \epsilon < t$.

Since

$$\int_{t-\epsilon}^t \|(-A)^\alpha \exp((t-s)A) f(u^k(s))\|_{E_n} ds \leq C\epsilon^{1-\alpha}$$

we see that $\|\mathcal{F}_\epsilon(u^k)(t) - \mathcal{F}(u^k)(t)\| \leq C\epsilon^{1-\alpha}$ for any $t \in (0, T]$, where

$$\mathcal{F}(u^k)(t) = \int_0^t (-A)^\alpha \exp((t-s)A) f(u^k(s)) ds$$

and $0 < \epsilon < t$. Then it follows that the operator $\mathcal{F}(\cdot)(t) : F \rightarrow E^\alpha$ is compact for $t > 0$, fixed. For $t = 0$, the operator $\mathcal{F}(\cdot)(0) = 0$ is also compact. Moreover, the set of functions $\{F_k(\cdot)\}$, $F_k(t) = \mathcal{F}(u^k)(t)$, $t \in [0, T]$, is an equibounded and equicontinuous family, since for $0 < t_1 < t_2$, we obtain

$$\|F_k(t_2) - F_k(t_1)\|_{E^\alpha} \leq C \left(\int_0^{t_1} \|\exp((t_2-s)A) - \exp((t_1-s)A)\|_{E^\alpha} ds + |t_2 - t_1|^{1-\alpha} \right),$$

and $\exp(\cdot A)$ is uniformly continuous in $t > 0$. By the generalized Arzela–Ascoli theorem, the same way as in Theorem 4.2, it follows that operator K is compact. \square

Consider the sequence of operators $P_n : F \rightarrow F_n$ defined by $(P_n u)(t) = p_n^\alpha(u(t))$, $t \in [0, T]$. With this we can talk about \mathcal{PP} –convergence of K_n to K .

THEOREM 4.2. *Let the condition [A1] be satisfied and assume that $u_n^0 \xrightarrow{\mathcal{P}^\alpha} u^0$. Then the operators $K_n \xrightarrow{\mathcal{PP}} K$ compactly.*

Proof: To show that $K_n \xrightarrow{\mathcal{PP}} K$ compactly, we assume that $\|u_n\|_{F_n} = O(1)$. We know from Theorem 1.5 that

$$\sup_{t \in [0, T]} \|\exp(tA_n)u_n^0 - p_n^\alpha \exp(tA)u^0\|_{E_n^\alpha} \rightarrow 0$$

whenever $\|u_n^0 - p_n^\alpha u^0\|_{E_n^\alpha} \rightarrow 0$ which means that $\sup_{t \in [0, T]} \|\exp(\cdot A_n)u_n^0 - P_n \exp(\cdot A)u^0\|_{F_n^\alpha} \rightarrow 0$. This ensures that the family $\{\exp(tA_n)u_n^0\}$ is equicontinuous. The condition $\Delta_{cc} \neq \emptyset$ implies (see Proposition 6.1) that $\exp(tA_n) \rightarrow \exp(tA)$ compactly for each $t \in [0, T]$.

Next we show that $\{K_n u_n\}$ is \mathcal{P} –compact by the generalized Arzela–Ascoli theorem. To show this, we verify the vanishing of the noncompactness measure $\mu(\{(K_n u_n)(t)\}) = 0$ for all $t \in [0, T]$. Let us consider the relation

$$(-A_n)^\alpha (K_n u_n)(t) = (-A_n)^\alpha \exp(tA_n)u_n^0 + \psi_n^\tau(t) + \varphi_n^\tau(t),$$

where

$$\psi_n^\tau(t) = \exp(\tau A_n) \int_0^{t-\tau} (-A_n)^\alpha \exp((t-s-\tau)A_n) f_n(u_n(s)) ds,$$

$$\varphi_n^\tau(t) = \int_{t-\tau}^t (-A_n)^\alpha \exp((t-s)A_n) f_n(u_n(s)) ds.$$

By virtue of the boundedness of $\|f_n(u_n(\cdot))\|_{F_n}$, we can choose the term $\|\varphi_n^\tau(\cdot)\|_{F_n}$ sufficiently small with τ small enough and $\mu(\{\psi_n^\tau\}) = 0$. The sequence $\{(-A)^\alpha \exp(tA_n) u_n^0\}$ is also \mathcal{P} -compact.

It is clear that $\{K_n u_n\}$ is uniformly bounded in n . To show equicontinuity one can write for $t_1 \leq t_2$

$$\begin{aligned} & \|(-A_n)^\alpha \int_0^{t_2} \exp((t_2-s)A_n) f_n(u_n(s)) ds - (-A_n)^\alpha \int_0^{t_1} \exp((t_1-s)A_n) f_n(u_n(s)) ds\|_{E_n} \\ & \leq \left\| \int_{t_1}^{t_2} (-A_n)^\alpha \exp((t_2-s)A_n) f_n(u_n(s)) ds \right\|_{E_n} \\ & + \left\| \int_0^{t_1} (-A_n)^\alpha \left(\exp((t_2-s)A_n) - \exp((t_1-s)A_n) \right) f_n(u_n(s)) ds \right\|_{E_n}. \end{aligned}$$

Exactly as in Theorem 4.1, the first term can be bound by $C(t_2 - t_1)^{1-\alpha}$. To estimate the second term we proceed as follows. First we rewrite it as

$$\left\| \int_0^{t_1} (-A_n)^{-\epsilon} \left(\exp((t_2-t_1)A_n) - I \right) (-A_n)^{\alpha+\epsilon} \exp((t_1-s)A_n) f_n(u_n(s)) ds \right\|_{E_n}$$

for $1 - \alpha > \epsilon > 0$. Now it is easy to see that the above term is bound by $C|t_2 - t_1|^\epsilon$. This concludes the proof of the result. \square

Let $M, \omega > 0$ and

$$\Omega^{M,\omega} = \{(v, z) \in C_0((-\infty, 0]; Q(\sigma^+)E^\alpha \times (I - Q(\sigma^+)E^\alpha) : \|v(t)\|_{E^\alpha}, \|z(t)\|_{E^\alpha} \leq M e^{\omega t}, t \leq 0\}.$$

One can show right from (2.12) that its solution belongs to $\Omega^{M,\omega}$.

THEOREM 4.3. *Assume that resolvent $(\lambda I - A)^{-1}$ is compact for some $\lambda \in \rho(A)$. Then the operator $\Theta(\eta, \cdot) : \Omega^{M,\omega} \subset \Upsilon \rightarrow \Upsilon$ is compact.*

Proof. Let us take the sequence of bounded functions $\{v^k(\cdot)\}, \{z^k(\cdot)\}, v^k, z^k \in \Omega^{M,\omega}, k \in \mathbb{N}$. Then the sequence $\{\Theta(\eta, (v^k, z^k))\}$ is compact in Υ . Indeed, one can easily check compactness for any fixed $t < 0$ and uniform boundedness. To check the compactness of $\{\Theta(\eta, (v^k, z^k))\}$ in Υ we note that H is continuous with $H(0, 0) = 0$ and that (from the definition of $\Omega^{M,\omega}$) given $\epsilon > 0$ there is a $t_\epsilon > 0$ such that

$$\|(-A)^\alpha \int_{-\infty}^{t-t_\epsilon} \exp((t-s)(A + f'(u^*))) Q(\sigma^+) H(v^k(s), z^k(s)) ds\| \leq \epsilon, \quad \forall k \in \mathbb{N}.$$

The rest of the proof follows from equicontinuity of

$$(-A)^\alpha \exp((t - \tau)B)\eta + (-A)^\alpha \int_\tau^t e^{(t-s)(A+f'(u^*))} Q(\sigma^+) H(v^k(s), z^k(s)) ds$$

and of

$$(-A)^\alpha \int_{t-t_\epsilon}^t \exp((t-s)(A+f'(u^*))) (I - Q(\sigma^+)) G(v^k(s), z^k(s)) ds$$

(obtained as in Theorem 4.2) and the Arzela-Ascoli Theorem. \square

The same way as we introduced operator $\Theta(\eta, \cdot) : \Upsilon \rightarrow \Upsilon$ in (2.12) we can introduce the operator $\Theta_n(\eta_n, u_n) : \Upsilon_n \rightarrow \Upsilon_n$. We assume that $\eta_n \xrightarrow{\mathcal{P}^\alpha} \eta$ or even $\eta_n = p_n^\alpha \eta$. Rewriting (1.6) for $w_n(t) = u_n(t) - u_n^*$ to deal with the neighborhood of u_n^* we arrive at

$$w_n'(t) = A_n w_n(t) + f_n'(u_n^*) w_n(t) + f_n(w_n(t) + u_n^*) - f_n(u_n^*(t)) - f_n'(u_n^*) w_n(t). \quad (4.3)$$

If $Q_n(\sigma_n^+)$ is as in Proposition 5.5 and $W_n(\sigma_n^+) = Q_n(\sigma_n^+) E_n^\alpha$, then $W_n(\sigma_n^+)$ is isomorphic to $W = Q(\sigma^+) E^\alpha$ through an isomorphism $\mathfrak{Q}_n = Q_n(\sigma_n^+) p_n^\alpha : W \rightarrow W_n(\sigma_n^+)$, which is bounded with bounded inverse \mathfrak{Q}_n^{-1} and the norms of \mathfrak{Q}_n and \mathfrak{Q}_n^{-1} are uniformly bounded in $n \geq n_0$. Recall that $Q_n(\sigma_n^+)$ are projectors such that $Q_n(\sigma_n^+) \xrightarrow{\mathcal{P}^\alpha} Q(\sigma^+)$ compactly for $Q(\sigma^+)$ from Section 2.3. We identify all spaces $W_n(\sigma_n^+)$ through these isomorphisms to the fixed space W .

Now we decompose the equation (4.3) in the following way. If w_n is a solution to (4.3) we write

$$w_n(t) = v_n(t) + z_n(t), \quad \text{where } v_n(t) = Q_n(\sigma_n^+) w_n(t), \quad z_n(t) = (I_n - Q_n(\sigma_n^+)) w_n(t).$$

Hence, we have

$$v_n' = (A_n + f_n'(u_n^*)) v_n(t) + Q_n(\sigma_n^+) \left(f_n(v_n + z_n + u_n^*) - f_n(u_n^*) - f_n'(u_n^*) (v_n(t) + z_n(t)) \right)$$

and

$$z_n' = (A_n + f_n'(u_n^*)) z_n + (I_n - Q_n(\sigma_n^+)) \left(f_n(v_n + z_n + u_n^*) - f_n(u_n^*) - f_n'(u_n^*) (v_n + z_n) \right).$$

We write

$$H_n(v_n, z_n) = Q_n(\sigma_n^+) \left(f_n(v_n + z_n + u_n^*) - f_n(u_n^*) - f_n'(u_n^*) (v_n + z_n) \right)$$

and

$$G_n(v_n, z_n) = (I_n - Q_n(\sigma_n^+)) \left(f_n(v_n + z_n + u_n^*) - f_n(u_n^*) - f_n'(u_n^*) (v_n + z_n) \right).$$

Hence, we have that, $H_n(0, 0) = 0, G_n(0, 0) = 0$. If $u_n^* \xrightarrow{\mathcal{P}^\alpha} u^*$, with u_n^* and u^* as before, it follows from our assumptions that given $\rho > 0$ there exist $n_0 > 0$ and $\delta > 0$ such that if $\|v_n\|_{E_n^\alpha} + \|z_n\|_{E_n^\alpha} < \delta$ and $n \geq n_0$ we have

$$\begin{aligned} \|H_n(v_n, z_n)\|_{E_n} &\leq \rho, \\ \|G_n(v_n, z_n)\|_{E_n} &\leq \rho, \end{aligned} \quad (4.4)$$

The fact that we can choose ρ and δ uniformly for $n \geq n_0$ satisfying the inequalities above is the key point to obtain that the local unstable manifolds are defined in a small “neighborhood” of the equilibrium point u_n uniformly for $n \geq n_0$. Then, equation (4.3) can be rewritten in the following form

$$\begin{aligned} v_n'(t) &= B_n(\sigma_n^+)v_n(t) + H_n(v_n(t), z_n(t)), \\ z_n'(t) &= \tilde{A}_n z_n(t) + G_n(v_n(t), z_n(t)), \end{aligned} \quad (4.5)$$

where $B_n = (A_n + f_n'(u_n^*))Q_n$, $\tilde{A}_n = (A_n + f_n'(u_n^*))(I_n - Q_n(\sigma_n^+))$ and H_n, G_n satisfy (4.4).

THEOREM 4.4. *If [A1] and [A2] hold then, the operators $\Theta_n(\eta_n, \cdot) : \Omega_n^{M, \omega} \subset \Upsilon_n \rightarrow \Upsilon_n$, converge compactly to $\Theta(\eta, \cdot) : \Omega^{M, \omega} \subset \Upsilon \rightarrow \Upsilon$.*

Proof. We recall that

$$\|u_n(\cdot)\|_{\Upsilon_n} = \left\| \begin{pmatrix} v_n(\cdot) \\ z_n(\cdot) \end{pmatrix} \right\|_{\Upsilon_n} = \sup_{-\infty < t \leq 0} \left\| \begin{pmatrix} v_n(t) \\ z_n(t) \end{pmatrix} \right\|_{E_n^\alpha}, \quad \lim_{t \rightarrow \infty} \left\| \begin{pmatrix} v_n(t) \\ z_n(t) \end{pmatrix} \right\|_{E_n^\alpha} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By \mathcal{P} -convergence $\Upsilon_n \ni u_n(\cdot) \xrightarrow{\mathcal{P}} u(\cdot) \in \Upsilon$ we mean that $\|u_n(\cdot) - P_n u(\cdot)\|_{\Upsilon_n} \rightarrow 0$ where $P_n(u(\cdot))(t) = p_n^\alpha u(t)$, $t \leq 0$. To prove that boundedness of the sequence $\{\|u_n(\cdot)\|_{\Upsilon_n}\}$ implies \mathcal{P} -compactness of the sequence $\{\Theta_n(\eta_n, u_n(\cdot))\}$ we apply the generalized Arzela-Ascoli Theorem to the sequences

$$\begin{aligned} &\left\{ (-A_n)^\alpha \exp((t-\tau)B_n)\eta_n + (-A_n)^\alpha \int_\tau^t \exp((t-s)B_n)H_n(v_n^k(s), z_n^k(s))ds \right\}, \\ &\left\{ (-A_n)^\alpha \int_{-\infty}^t \exp((t-s)\tilde{A}_n)G_n(v_n^k(s), z_n^k(s))ds \right\}. \end{aligned}$$

For that we proceed exactly as in Theorem 4.3 using (4.4). \square

4.2. Approximation of semiflows

We will need the following result

THEOREM 4.5. [29] *Assume that the operator A satisfies (1.1), and let K be the operator given by formula (4.1). If $u^*(\cdot)$ is a unique mild solution of problem (1.2), then $\text{ind}(u^*(\cdot); I - K) = 1$.*

THEOREM 4.6. *Assume that condition [A1] is satisfied. Then $T_n(t)u_n^0 \xrightarrow{\mathcal{P}^\alpha} T(t)u^0$ uniformly in $t \in [0, T]$, $T < \infty$, whenever $u_n^0 \xrightarrow{\mathcal{P}^\alpha} u^0$.*

Proof. We know (see Theorem 4.5) that $\text{ind}(u^*, I - K) = 1$. By Theorem 4.2 operators $K_n \xrightarrow{\mathcal{P}^\alpha} K$ compactly, so according to Theorem 3 from [77] one has $\gamma(I_n - K_n, \partial\Omega_n) = 1$ for $n \geq n_0$. This means that there is a sequence $\{u_n^*\}, n \in \mathbb{N}$, which is a solution of problems (4.2) and $\{u_n^*\}, u_n^* \in F_n$, is compact; that is one can write $u_n^*(t) \xrightarrow{\mathcal{P}^\alpha} u^*(t)$ uniformly in $t \in [0, T]$ as $n \rightarrow \infty$.

5. APPROXIMATION OF ATTRACTORS

Next we intend to obtain that the attractors behave continuously with respect to parameters under rather general conditions.

To prove upper or lower semicontinuity of attractors we use the following result

LEMMA 5.1. *Let $\mathcal{A} \subset E^\alpha$ and $\mathcal{A}_n \subset E_n^\alpha$, $n \in \mathbb{N}$.*

1.If any sequence $\{u_n\}$ with $u_n \in \mathcal{A}_n$ has a \mathcal{P}^α -convergent subsequence with limit belonging to \mathcal{A} , then $\{\mathcal{A}_n\}$ is \mathcal{P}^α -upper semicontinuous at infinity.

2.If \mathcal{A} is compact and for any $u \in \mathcal{A}$ there is a sequence $\{u_n\}$ with $u_n \in \mathcal{A}_n$, which \mathcal{P}^α -converges to u , then $\{\mathcal{A}_n\}$ is \mathcal{P}^α -lower semicontinuous at infinity.

The proof of this lemma is similar to the proof of Lemma 1.1.

5.1. Upper Semicontinuity of Attractors

In this subsection we obtain that the family of attractors $\{\mathcal{A}_n\}$ is upper-semicontinuous. In order to do that we use the results on continuity of nonlinear semiflows (Subsection 4.2) and the following lemma.

LEMMA 5.2. *Assume that the first two conditions in [A1] are satisfied. Given $\alpha \leq \gamma < 1$, there is a constant N_γ independent of n such that*

$$\sup_{n \in \mathbb{N}} \sup_{u_n \in \mathcal{A}_n} \|u_n\|_{E_n^\gamma} \leq N_\gamma.$$

Furthermore, any sequence $\{u_n\}, u_n \in \mathcal{A}_n$, is \mathcal{P}^α -compact.

Proof. The first part of the lemma follows immediately from the invariance of the attractor and of the variation of constants formula. For the second part we only need to apply Lemma 6.8. \square

Remark 5. 1. We know by definition that any attractor \mathcal{A}_n is a compact set. The above lemma is stating that the set of the attractors \mathcal{A}_n is collectively \mathcal{P}^α -compact.

THEOREM 5.1. *Let $\mathcal{A}, \mathcal{A}_n$ be the attractors for (1.2) and (1.6), respectively and assume that [A1] is satisfied. If $\{x_n^*\}, x_n^* \in \mathcal{A}_n$, \mathcal{P}^α -converges to some point x^* , then $x^* \in \mathcal{A}$ and the $\{\mathcal{A}_n\}$ is \mathcal{P}^α -upper semicontinuous at infinity.*

Proof. Since $x_n^* \in \mathcal{A}_n$, through x_n^* there is a global solution $u(\cdot, x_n^*) : \mathbb{R} \rightarrow E_n^\alpha$ of (1.6). For any $k \in \mathbb{N}$, from Lemma 5.2, $\{u_n(-k, x_n^*)\}$ has a convergent subsequence. From the fact that $u_n(s, x_n^*) = T(s - k)u_n(-k, x_n^*)$, $s \in [0, 2k]$ and from Theorem 4.6, there is a subsequence of $\{u_n(\cdot, x_n^*)\}$ which converges uniformly in $[-k, k]$. Using the Cantor diagonal procedure we can construct a global solution $u(\cdot, x^*)$ through x^* . It is easy to see that $u(\cdot, x^*)$ is bounded. From the fact that there is a bounded global solution through x^* it follows that $x^* \in \mathcal{A}$. \square

5.2. Approximation of the Set of Equilibria

Let us consider in Banach spaces E_n^α the following family of nonlinear problems (1.8).

PROPOSITION 5.1. *Assume that condition [A1] is satisfied and that the problems (1.8) have solutions $\{u_n^*\}$, $n \in \mathbb{N}$. Then, taking subsequences if necessary, there is a solution u^* of (2.4) such that $\|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. It is clear that

$$u_n^* + A_n^{-1} f_n(u_n^*) = 0.$$

Since $A_n^{-1} \xrightarrow{\mathcal{PP}} A^{-1}$ compactly by Lemma 6.8 and $f_n(\cdot)$ are bounded uniformly in $n \in \mathbb{N}$, we have that there is a subsequence, which we again denote by u_n^* and a point u^* such that $u_n^* \xrightarrow{\mathcal{P}^\alpha} u^*$. It follows from the continuity properties of $f_n(\cdot)$ and from convergence of resolvents $A_n^{-1} \xrightarrow{\mathcal{PP}} A^{-1}$ that

$$u^* + A^{-1} f^*(u^*) = 0,$$

which is equivalent to say that u^* is a solution of (2.4). \square

PROPOSITION 5.2. *Assume that [A1] holds and that u^* is hyperbolic solution of (2.4). Then there is n_0 and $\delta > 0$ such that for $n \geq n_0$ the equations (1.8) have at least one solution u_n^* in $\{w_n : \|w_n - p_n^\alpha u^*\|_{E_n^\alpha} \leq \delta\}$. Furthermore, $\|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof: As in Corollary 2.1 there is a ball $B(u^*, \delta)$ such that there are no other fixed points in it except u^* , and we get $|\text{ind}(u^*, I + A^{-1} f(\cdot))| = 1$. The compact convergence $A_n^{-1} f_n(\cdot) \xrightarrow{\mathcal{PP}} A^{-1} f(\cdot)$ follows from boundedness (uniformly for $n \in \mathbb{N}$), \mathcal{P} -convergence of f_n to f and compact convergence $A_n^{-1} \xrightarrow{\mathcal{PP}} A^{-1}$. Now compact convergence $A_n^{-1} f_n(\cdot) \rightarrow A^{-1} f(\cdot)$ implies (see [77], Theorem 3) that $\text{ind}(u^*, I + A^{-1} f) = \gamma(I_n + A_n^{-1} f_n, \partial B(p_n u^*, \delta))$, $n \geq n_0$, which means that there is at least one fixed point u_n^* exists in any ball $B(p_n u^*, \delta)$, $n \geq n_0$. This sequence $\{u_n^*\}$ is \mathcal{P}^α -convergent to u^* . \square

In case we have information on differentiability of corresponding functions the result could be sharper.

PROPOSITION 5.3. *Assume that problem (2.4) has a solution u^* , which is hyperbolic point. Assume that [A1] and [A2] hold and that, for any $\rho > 0$, there exists $\delta > 0$ such that $\sup_{n \in \mathbb{N}} \sup_{\|w_n\|_{E_n^\alpha} \leq \delta} \|f'_n(w_n + p_n^\alpha u^*) - f'_n(p_n^\alpha u^*)\|_{B(E_n^\alpha, E_n)} \leq \rho$. Then, there exists n_0 and $\delta > 0$ so that problem (1.8) has exactly one solution, u_n^* , in $\{w_n : \|w_n - p_n^\alpha u^*\|_{E_n^\alpha} \leq \delta\}$ for each $n \geq n_0$. Furthermore, $\|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. We want to prove that in a neighborhood of the solution u^* of (2.4) there is a unique solution u_n^* of (1.8). From the above assumptions on $f'_n(\cdot)$, there is a $\delta > 0$ such that $\|u_n - p_n^\alpha u^*\|_{E_n^\alpha} < \delta$ implies $C\|f'_n(u_n) - f'_n(p_n^\alpha u^*)\|_{B(E_n^\alpha, E_n)} < \frac{1}{2}$.

From Proposition 5.5, $\sigma(A_n + f'_n(p_n^\alpha u^*))$ is disjoint from the imaginary axis and there is a constant $M > 0$ such that

$$\|(A_n + f'_n(p_n^\alpha u^*))^{-1}\|_{B(E_n, E_n^\alpha)} \leq C.$$

If u_n is a solution to (1.8), then

$$\begin{aligned} 0 &= A_n u_n + f'_n(p_n^\alpha u^*) u_n + f_n(u_n) - f'_n(p_n^\alpha u^*) u_n \\ &= (A_n + f'_n(p_n^\alpha u^*)) \left(u_n + (A_n + f'_n(p_n^\alpha u^*))^{-1} (f_n(u_n) - f'_n(p_n^\alpha u^*) u_n) \right). \end{aligned}$$

Since $(A_n + f'_n(p_n^\alpha u^*))$ is invertible, to show that u_n is a solution to (1.8) is equivalent to show that it is a fixed point of mapping

$$\Phi_n(u_n) = -(A_n + f'_n(p_n^\alpha u^*))^{-1} (f_n(u_n) - f'_n(p_n^\alpha u^*) u_n).$$

It is clear from the convergence of the resolvent and the continuity properties of $f_n(\cdot)$ and $f'_n(p_n^\alpha u^*)$ that

$$\Phi_n(p_n^\alpha u^*) \rightarrow -(A + f'(u^*))^{-1} (f(u^*) - f'(u^*) u^*) = u^*. \quad (5.1)$$

Next we show that, for suitably small $\delta > 0$ and $n \geq n_0$, Φ_n is a contraction in the from the closed ball $\bar{B}(p_n^\alpha u^*, \delta)$ into itself. First, we show that Φ_n is a contraction (uniformly in $n \in N$); that is,

$$\begin{aligned} \|\Phi_n(u_n) - \Phi_n(v_n)\|_{E_n^\alpha} &= \|(A_n + f'_n(p_n^\alpha u^*))^{-1} \left((f_n(u_n) - f_n(v_n) - f'_n(p_n^\alpha u^*)(u_n - v_n)) \right)\|_{E_n^\alpha} \\ &\leq \|(A_n + f'_n(p_n^\alpha u^*))^{-1}\|_{B(E_n, E_n^\alpha)} \|f_n(u_n) - f_n(v_n) - f'_n(p_n^\alpha u^*)(u_n - v_n)\|_{E_n} \\ &\leq C \left\| \left(f'_n(su_n + (1-s)v_n) - f'_n(p_n^\alpha u^*) \right) (u_n - v_n) \right\|_{E_n} \\ &\leq C \|f'_n(su_n + (1-s)v_n) - f'_n(p_n^\alpha u^*)\|_{B(E_n^\alpha, E_n)} \|u_n - v_n\|_{E_n^\alpha} \\ &\leq \frac{1}{2} \|u_n - v_n\|_{E_n^\alpha}, \end{aligned}$$

for some $0 \leq s \leq 1$ and $n \geq n_0$. To show that Φ_n maps $B(p_n u^*, \delta)$ into itself we observe that if $u_n \in B(p_n u^*, \delta)$, then

$$\begin{aligned} \|\Phi_n(u_n) - p_n u^*\|_{E_n^\alpha} &\leq \|\Phi_n(u_n) - \Phi_n(p_n u^*)\|_{E_n^\alpha} + \|\Phi_n(p_n u^*) - p_n u^*\|_{E_n^\alpha} \\ &\leq \frac{\delta}{2} + \|\Phi_n(p_n u^*) - p_n u^*\|_{E_n^\alpha}. \end{aligned}$$

It follows from (5.1) that there is a n_0 such that

$$\|\Phi_n(u_n) - p_n u^*\|_{E_n^\alpha} \leq \frac{3\delta}{4} \text{ as } n \geq n_0.$$

From that we obtain that if $u_n \in \bar{B}(p_n^\alpha u^*, \delta)$, then

$$\|\Phi_n(u_n) - p_n^\alpha u^*\|_{E_n^\alpha} \leq \delta$$

and therefore $\Phi_n : B(p_n u^*, \delta) \rightarrow B(p_n u^*, \delta)$ is a contraction for all $n \geq n_0$. This shows that there is a unique fixed point of (1.8) in the ball $B(p_n^\alpha u^*, \delta)$.

To show that $u_n^* \xrightarrow{P^\alpha} u^*$ we proceed in the following manner

$$\|p_n^\alpha u^* - u_n^*\|_{E_n^\alpha} = \|\Phi_n(u_n^*) - p_n u^*\|_{E_n^\alpha} \leq \|\Phi_n(u_n^*) - \Phi_n(p_n u^*)\|_{E_n^\alpha} + \|\Phi_n(p_n u^*) - p_n u^*\|_{E_n^\alpha}.$$

From the fact that Φ_n is a uniform contraction in E_n^α we have that

$$\|p_n^\alpha u^* - u_n^*\|_{E_n^\alpha} \leq 2\|\Phi_n(p_n^\alpha u^*) - p_n^\alpha u^*\|_{E_n^\alpha}.$$

From (5.1) we have the desired convergence and the result follows. \square

As an immediate consequence of Proposition 5.2, we have

THEOREM 5.2. *Assume that conditions [A1] and [A2] hold and that (2.4) has exactly m solutions u^1, \dots, u^m , all of them hyperbolic. Then, there exists n_0 such that for all $n \geq n_0$ problems (1.8) has at least m solutions u_n^1, \dots, u_n^m for each $n \geq n_0$. Moreover, we have*

$$\|u_n^k - p_n^\alpha u^k\|_{E_n^\alpha} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } k = 1, \dots, m.$$

Proof: By Proposition 5.2, in a neighborhood of u^k there is at least one solution of (1.8) which converges to u^k . This proves the result. \square

THEOREM 5.3. *Assume that conditions [A1] and [A2] hold and that (2.4) has exactly m solutions u^1, \dots, u^m , all of them hyperbolic. Assume that, for any $\rho > 0$, there exists $\delta > 0$ such that $\sup_{n \in \mathbb{N}} \sup_{\|w_n\|_{E_n^\alpha} \leq \delta} \|f'_n(w_n + p_n^\alpha u^k) - f'_n(p_n^\alpha u^k)\|_{B(E_n^\alpha, E_n)} \leq \rho$, $1 \leq k \leq m$. Then, there exists n_0 such that for all $n \geq n_0$ problems (1.8) has exactly m solutions u_n^1, \dots, u_n^m . Moreover, we have*

$$\|u_n^k - p_n^\alpha u^k\|_{E_n^\alpha} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ for any } k = 1, \dots, m.$$

Proof: By Proposition 5.1 we have that for any sequence $\{u_n^*\}$ of solutions to (1.8) must converge to the set of equilibria (2.4). But by Proposition 5.3, in a neighborhood of u^k there is only one solution of (1.8) which converges to u^k . This proves the result. \square

5.3. Approximation of Unstable Manifolds

Recall that we are assuming that all equilibrium solutions of (1.2) are hyperbolic; that is, assume that each solution u^* of (2.4) is such that the operator $A + f'(u^*)$ has spectrum $\sigma(A + f'(u^*))$ disjoint from the imaginary axis.

PROPOSITION 5.4. *Let A be a sectorial operator with compact resolvent and let $f'(u^*) \in L(E^\alpha, E)$. Then the operator $(A + f'(u^*))$ is sectorial with compact resolvent.*

Proof: The result follows immediately from Theorems 1.3.2 and 1.4.8 in [49] or [81]. \square

From the above proposition, $\sigma^+ = \sigma((A + f'(u^*))) \cap \{\lambda \in \mathbb{C} : \text{Re}\lambda > 0\}$ consists of a finite number (possibly zero) of eigenvalues with finite multiplicity and there is a $\beta > 0$ and a finite rank projection $Q(\sigma^+)$ such that (2.9) holds.

PROPOSITION 5.5. *Assume that [A1] and [A2] hold. Let $u_n^* \xrightarrow{\mathcal{P}^\alpha} u^*$ with u_n^* be solution of (1.8) and u^* be solution of (2.4). Then there is a $n \geq n_0$ such that $\sigma(A_n + f'_n(u_n^*))$ does not intersect the imaginary axis for $n \geq n_0$ and $\|(A_n + f'_n(u_n^*))^{-1}\| \leq C$ with C independent of n . Furthermore, if $Q_n(\sigma_n^+)$ denotes the projection defined by the spectral set $\sigma_n^+ = \{\lambda \in \sigma(A_n + f'_n(u_n^*)) : \text{Re}\lambda > 0\}$, then $Q_n(\sigma_n^+)$ converges compactly to $Q(\sigma^+)$ as $n \rightarrow \infty$ (hence, $\dim(Q_n(\sigma_n^+)E_n^\alpha) = \dim(Q(\sigma^+)E^\alpha)$ for $n \geq n_0$) and the family of sets $\sigma_n^+ \rightarrow \sigma^+$ in the sense of (i) – (v) of Theorem 3.1.*

Proof. It follows from Lemma 6.10 and Theorem 3.1.

THEOREM 5.4. *Let u^* be hyperbolic equilibrium of (1.2). Assume that [A1] and [A2] hold. Then, for sufficiently small $\delta > 0$, (2.12) has a solution $\zeta^*(\cdot)$. Furthermore, there is a $n_0 \in \mathbb{N}$ such that the equation $\zeta_n(t) = \Theta_n(\eta_n, \zeta_n(t))$ has at least one solution $\zeta_n^*(\cdot)$ for $n \geq n_0$ and $\zeta_n^*(t) \rightarrow \zeta^*(t)$ uniformly in $t \in (-\infty, 0]$ as $n \rightarrow \infty$.*

Proof. We know that $|\text{ind}(\zeta_0, I - \Theta)| = 1$ for $\zeta_0 = 0$. According to Theorem 54.1 [55] there exist $\rho, \delta > 0$ such that $|\text{ind}(\zeta, I - \Theta)| = 1$ for $\|\eta - 0\| \leq \delta, \|\zeta - 0\| \leq \rho$. By Theorem 4.4 operators $\Theta_n \rightarrow \Theta$ compactly, so $\gamma(I_n - \Theta_n, \partial\Omega_n^{M,\omega}) = \text{ind}(\zeta, I - \Theta)$ and therefore equation $\zeta_n(t) = \Theta_n(\eta_n, \zeta_n(t))$ has at least one solution ζ_n^* at any set $\Omega_n^{M,\omega}$ for $n \geq n_0$. This sequence $\{\zeta_n^*\}$ is discretely compact and $\zeta_n^* \rightarrow \zeta^*$ as $n \rightarrow \infty$.

PROPOSITION 5.6. *Assume that [A1] and [A2] hold and that u^* is a hyperbolic equilibrium for (1.2). Let $W^u(u^*)$ be the unstable manifold of u^* . Let $\delta > 0$ be such that u^* is the only solution of (2.4) in $\{w : \|w - u^*\|_{E^\alpha} \leq \delta\}$. By Proposition 5.2 there is a $n_0 \in \mathbb{N}$ such that (1.8) has at least one solution u_n^* in $\{w_n : \|w_n - p_n^\alpha u^*\|_{E_n^\alpha} \leq \delta\}$. Furthermore, $\|u_n^* - p_n^\alpha u^*\|_{E_n^\alpha} \rightarrow 0$ as $n \rightarrow \infty$. Let $W^u(u_n^*)$ be the unstable manifold of u_n^* . Then, the family $\{W_n^u(u_n^*)\}$ is lower semicontinuous at infinity ($W^u(u_\infty^*) = W^u(u^*)$).*

Proof: As we have seen in Lemma 5.1 it is enough to prove that given a point $x \in W^u(u^*)$ there is a sequence $\{x_n\}$ with $x_n \in W^u(u_n^*)$ which satisfies $\|x_n - p_n^\alpha x\|_{E_n^\alpha} \rightarrow 0$. Let $u(t, x)$ be a backward solution of (1.2) through x with $u(t, x) \rightarrow u^*$ as $t \rightarrow -\infty$. Let $\tau > 0$ be such that $u(t, x) \in \{w : \|w - u^*\|_{E^\alpha} \leq \delta\}$, $t \leq -\tau$, and let $\zeta^*(\cdot) = u(\cdot - \tau, x)$ be the solution of (2.12). By Theorem 5.4 there is a sequence $\{\zeta_n^*(\cdot)\}$ such that $\zeta_n^*(t) \rightarrow u_n^*$ as $t \rightarrow -\infty$ and $\zeta_n^*(\cdot) \rightarrow \zeta^*(\cdot)$ uniformly in $(-\infty, 0]$. It follows from Theorem 4.6 that $W^u(u_n^*) \ni T_n(\tau)(\zeta_n(0)) \xrightarrow{\mathcal{P}} T(\tau)u(-\tau, x) = x$. \square

As an immediate consequence of this Proposition 5.6, we have

THEOREM 5.5. *Assume that [A1] and [A2] hold and that problem (2.4) has exactly m solutions u^1, \dots, u^m all of them hyperbolic. Then problem (1.8) has at least m solutions u_n^1, \dots, u_n^m with $\|u_n^k - p_n^\alpha u^k\|_{E_n^\alpha} \rightarrow 0$. If $W^u(u_n^k)$, $k = 1, \dots, m$, denotes the unstable manifold of u_n^k and $W^u(u_\infty^k) = W^u(u^k)$ denotes the unstable manifold of u^k we have that the family $\{W^u(u_n^k)\}$ is lower semicontinuous at infinity for each $k = 1, \dots, m$.*

5.4. Lower Semicontinuity of Attractors

THEOREM 5.6. *Let \mathcal{A} , \mathcal{A}_n be the attractors for (1.2) and (1.6), respectively. Assume that the attractor \mathcal{A} of (1.2) can be written in the form (6.2) and that all points in \mathcal{E} are hyperbolic equilibrium. If $x^* \in \mathcal{A}$, then there is a sequence of points $\{x_n\}$, with $x_n^* \in \mathcal{A}_n$, such that $x_n^* \xrightarrow{\mathcal{P}^\alpha} x^*$ and the $\{\mathcal{A}_n\}$ is \mathcal{P}^α -lower semicontinuous at infinity.*

Proof: If $x^* \in \mathcal{A}$, then it is on the unstable manifold of some equilibrium point. Following back in time by some trajectory through x^* one can come, in time $-T < 0$, to a point y in the neighborhood of some equilibrium point, which as we know by Theorem 5.4 is approximated by points $y_n \in \mathcal{A}_n$. Then, $T_n(t)y_n \in \mathcal{A}_n$ approximates x^* and the result now follows from Theorem 4.6. \square

6. APPENDIX

6.1. Additional Results on Attractors and its Characterization

Let X be a Banach space and denote by $\text{dist}(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}^+$ the metric induced by the norm in X .

DEFINITION 6.1. An equilibrium point for the semigroup $\{T(t) : t \geq 0\}$ is an element $x \in X$ such that $T(t)x = x$ for all $t \geq 0$. The set of all equilibrium points for the semigroup $\{T(t) : t \geq 0\}$ will be denoted by \mathcal{E} .

It is clear that if the semigroup $\{T(t) : t \geq 0\}$ has a global attractor \mathcal{A} , then \mathcal{E} is compact and $\mathcal{E} \subset \mathcal{A}$. In this case, if all equilibrium points are isolated, then there is only a finite number of them.

We now specialize to a class of equations for which we can describe the structure of attractor. We assume that any bounded global solution $\phi(\cdot) : \mathbb{R} \rightarrow X$ of (1.2) has limits

$$\phi(t) \rightarrow \phi_{-\infty}^* \text{ as } t \rightarrow -\infty \text{ and } \phi(t) \rightarrow \phi_{+\infty}^* \text{ as } t \rightarrow \infty. \tag{6.1}$$

In this case $\phi_{+\infty}^*$ and $\phi_{-\infty}^*$ are in \mathcal{E} ($\phi_{\pm\infty}^* \xleftarrow{t \rightarrow \pm\infty} \phi(t+s) = T(s)\phi(t) \xrightarrow{t \rightarrow \pm\infty} T(s)\phi_{\pm\infty}^*, s \geq 0$) and the attractor has a representation $\mathcal{A} = \cup_{x^* \in \mathcal{E}} W^u(x^*)$ (note that \mathcal{E} may be an infinite set). Next we prove that this is observed for gradient semigroup with isolated equilibria (in this case \mathcal{E} will be a finite set).

DEFINITION 6.2. We say that $\{T(t) : t \geq 0\}$ is gradient if there is a continuous function $V : X \rightarrow \mathbb{R}$ such that function $\mathbb{R}^+ \ni t \mapsto V(T(t)x) \in \mathbb{R}$ is non-increasing and $\mathcal{E} = \{x \in X : V(T(t)x) = V(x), \forall t \in \mathbb{R}\}$.

We observe that our general assumption (6.1) admits the case $\phi_{-\infty}^* = \phi_{+\infty}^*$, which is not allowed in case of gradient systems.

LEMMA 6.1. Assume that $\{T(t) : t \geq 0\}$ is a gradient semigroup for which any element of \mathcal{E} is isolated. Suppose that $x \in X$ is such that $\overline{\gamma^+(x)}$ is compact, $\omega(x) = \{x^*\}$ for some $x^* \in \mathcal{E}$. As a consequence of that we conclude that $T(t)x \rightarrow x^*$ as $t \rightarrow \infty$.

Proof: First note that if $y \in \omega(x)$ then there is a sequence $t_n \rightarrow \infty$ such that $T(t_n)x \rightarrow y$. Since $V(T(t)x)$ is decreasing and bounded below we have that there is an $\ell \in \mathbb{R}$ such that $V(T(t)x) \rightarrow \ell$ as $t \rightarrow \infty$. It is easy to see that $V(T(t)y) = \ell$ for all $t \in \mathbb{R}$. From this it follows that $y \in \mathcal{E}$. Since the set \mathcal{E} consist only of a finite number of points and since $\omega(x)$ is connected we have that $\omega(x) = \{x^*\}$ for some $x^* \in \mathcal{E}$. \square

Our results on lower semicontinuity of attractors require that the attractor \mathcal{A} of (1.2) is characterized in the form

$$\mathcal{A} = \bigcup_{x^* \in \mathcal{E}} W^u(x^*) \tag{6.2}$$

with \mathcal{E} being a finite set consisting only of hyperbolic equilibria of (1.2). Other results on upper semicontinuity of attractor or on continuity of the set of equilibria do not require such special characterization. Let us also note that we do not require that the approximation problems (1.6) have attractors in the form $\mathcal{A}_n = \bigcup_{x_n^* \in \mathcal{E}_n} W^u(x_n^*)$ while in [50, 54] gradient structure is used.

DEFINITION 6.3. We say that an invariant set Γ is asymptotically stable if there is a neighborhood V of Γ such that $\text{dist}(T(t)y, \Gamma) \rightarrow 0$ as $t \rightarrow \infty$, for any $y \in V$.

In fact, in the proof of lower semicontinuity of attractors, we may assume that the attractor is the union of unstable manifolds of hyperbolic equilibria $W^u(x_i^*), 1 \leq i \leq n^*$,

together with asymptotically stable invariant set Γ ; that is,

$$\mathcal{A} = \bigcup_{1 \leq i \leq n^*} W^u(x_i^*) \cup \Gamma$$

which can be approximated (upper and lower semicontinuously). As an example, under the same assumptions on A and $f(\cdot)$ we can approximate uniformly stable periodic orbits of (1.2), see [10, 28].

Remark 6. 1. In Chafee–Infante example [17] the attractor is the union of unstable manifolds of equilibrium points.

The next result implies that if there is a negative orbit through x which is bounded, then it must converge to an element $x^* \in \mathcal{E}$ as $t \rightarrow -\infty$.

LEMMA 6.2. *Assume that $\{T(t) : t \geq 0\}$ is a gradient semigroup for which any element of \mathcal{E} is isolated. Suppose that $x \in X$ is such that there is a negative orbit $\phi : (-\infty, 0] \rightarrow X$ through x and such that $\overline{\phi((-\infty, 0])}$ is compact. Define*

$$\alpha_\phi(x) = \{y \in X : \exists t_n \rightarrow \infty \text{ such that } \phi(-t_n) \rightarrow y\}.$$

Then, there is a $x^ \in \mathcal{E}$ such that $\alpha_\phi(x) = \{x^*\}$. As a consequence of that we conclude that $\phi(t) \rightarrow x^*$ as $t \rightarrow -\infty$.*

Proof. First note that if $y \in \alpha_\phi(x)$, then there is a sequence $t_n \rightarrow \infty$ such that $\phi(-t_n) \rightarrow y$. Since $V(\phi(-t))$ is increasing and bounded above we have that there is an $r \in \mathbb{R}$ such that $V(\phi(-t)) \rightarrow r$ as $t \rightarrow \infty$. It is easy to see that $V(T(t)y) = r$ for all $t \in \mathbb{R}$. From this it follows that $y \in \mathcal{E}$. Since the set \mathcal{E} consist only of a finite number of points and since $\alpha_\phi(x)$ is connected we have that $\alpha_\phi(x) = \{x^*\}$ for some $x^* \in \mathcal{E}$. Furthermore, for any sequence $t_n \rightarrow \infty$, $\{\phi(-t_n)\}$ has a subsequence which converges to x^* and therefore $\phi(t) \rightarrow x^*$ as $t \rightarrow -\infty$. \square

LEMMA 6.3. *Let S be a relatively compact invariant subset of X under $\{T(t) : t \geq 0\}$. If $y \in S$ such that (6.1) is satisfied, then there are $x^+, x^- \in \mathcal{E}$ such that $y \in W^u(x^+)$ and $y \in W^s(x^-)$.*

Proof: Let $y \in S$. Since S is invariant, there is a complete orbit $\mathbb{R} \ni t \mapsto T(t)y \in S$ through y . Since \bar{S} is compact this complete orbit is compact. The result now follows from the previous lemmas. \square

From this lemma one gets the following result.

THEOREM 6.1. *Assume that the semigroup $\{T(t) : t \geq 0\}$ is a gradient semigroup for which any element of \mathcal{E} is isolated. If $\{T(t) : t \geq 0\}$ has a global attractor \mathcal{A} then, $\mathcal{A} = \bigcup_{x^* \in \mathcal{E}} W^u(x^*)$.*

6.2. Examples and Counterexamples

EXAMPLE 6.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Consider the second order strongly elliptic operator

$$Lu = \sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + \sum_{j=1}^n b_j(x)u_{x_j} + (c(x) + \nu)u, \tag{6.3}$$

where the coefficients a_{ij}, b_j, c are smooth. Consider the associated parabolic problem

$$\begin{aligned} u_t(t, x) &= Lu(t, x) + f(u(t, x)), \quad t > 0, \quad x \in \Omega, \\ u(t, x) &= 0, \quad t > 0, \quad x \in \partial\Omega, \\ u(0, x) &= u^0(x) \in H_0^1(\Omega). \end{aligned} \tag{6.4}$$

Let $E = L^2(\Omega)$ and define the operator $A : D(A) \subset E \rightarrow E$ by $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $Au = Lu$ for all $u \in D(A)$. It is well known that A generates an analytic and compact C_0 -semigroup $\{\exp(tA) : t \geq 0\}$. Assume that ν is chosen such that the spectrum of A is located to the left of the imaginary axis. Then, we can define the fractional powers $(-A)^\alpha$ of $-A$ as before. It is well known that $E^1 = D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and $E^{\frac{1}{2}} = H_0^1(\Omega)$.

Concerning the nonlinear term $f(\cdot)$, it was shown in [4], [7] that under some growth conditions, the problem (6.4) is locally well posed in $E^{\frac{1}{2}}$. These growth restrictions are expressed as follows:

(G)_E: Let $f(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^2(\mathbb{R}, \mathbb{R})$ function and assume one of the following

- i) $n = 1$,
- ii) $n = 2$ and f satisfies, for every $\eta > 0$, that there exists $c_\eta > 0$ such that

$$|f(x, u) - f(x, v)| \leq c_\eta (e^{\eta|u|^{\frac{n}{n-1}}} + e^{\eta|v|^{\frac{n}{n-1}}})|u - v|, \tag{6.5}$$

- iii) $n > 2$ and f satisfies

$$|f(x, u) - f(x, v)| \leq c|u - v|(|u|^{\rho-1} + |v|^{\rho-1} + 1), \tag{6.6}$$

with exponent ρ such that

$$1 < \rho \leq \frac{n + 2}{n - 2}.$$

It is also known (see [4],[7]) that, under these assumptions, these solutions are classical. If in addition to the assumptions above we assume that

$$\limsup_{|u| \rightarrow \infty} \frac{f(u)}{u} < 0, \tag{6.7}$$

then all solutions of (6.4) are globally defined. In this case we have, for each $u^0 \in E^{\frac{1}{2}}$ a globally defined solution $t \mapsto u(t, u^0) \in E^{\frac{1}{2}}, t \geq 0$.

Under the above hypotheses, the nonlinear semigroup $\{T(t, \cdot) : t \geq 0\}$ associated to (6.4) has a global attractor \mathcal{A} in $E^{\frac{1}{2}}$ and the attractor \mathcal{A} satisfies (see [8])

$$\sup_{u \in \mathcal{A}} \|u\|_{L^\infty(\Omega)} < \infty. \quad (6.8)$$

This bound enable us to cut the nonlinearity $f(\cdot)$ in such a way that it becomes bounded with bounded derivatives up to second order, but the problem with such changed nonlinearity has the same attractor. After these considerations we may assume, without loss of generality, that $f(\cdot)$ is bounded with bounded derivatives up to second order.

If we change L to $Lu = \sum_{i,j=1}^n (a_{ij}(x)u_{x_i})_{x_j}$ and ask that $(a_{ij})_{1 \leq i,j \leq n}$ is symmetric, then the semigroup associated to (6.4) is gradient (see Definition 6.2).

EXAMPLE 6.2. Discrete convergence of resolvents (without condition $\Delta_{cc} \neq \emptyset$) is not enough to get uniform estimates (and also the statement of Theorem 6.6), which is clear from the following example noted to us by Professor Cesar Palencia (private communication). Let us consider a Banach space $E = l_2$, the Hilbert space formed by all the square summable sequences, i.e $x = (x_1, x_2, \dots)$, $\sum_{k=1}^\infty |x_k|^2 < \infty$. Then, the operator

$$A = \text{diag}(-\alpha_1, -\alpha_2, -\alpha_3, \dots)$$

with $1 = \alpha_1 \leq \alpha_2 \leq \dots$ generates on E an analytic C_0 -semigroup with $\|e^{tA}\| = e^{-t}$, $t \geq 0$, i.e. it is exponentially decreasing semigroup. The spectrum $\sigma(A) = \{-\alpha_1, -\alpha_2, -\alpha_3, \dots\}$.

One could consider approximation of A by the operators

$$A_n = \text{diag}(-\alpha_1, -\alpha_2, \dots, -\alpha_n, M_n, 0, 0, \dots),$$

where $M_n = -\omega I_n + L_n$, with $0 < \omega < 1$, is $n \times n$ -matrix and L_n has got all of its entries equal to zero, except the first upper diagonal which is formed by ones. The spectrum of M_n reduces to $-\omega$. Moreover, since

$$\|\exp(tM_n)\| \leq e^{-\omega t} e^{\|L_n\|t} \leq e^{(1-\omega)t},$$

it is clear that, on finite intervals of $t \in [0, T]$, we have the strong convergence of the semigroups generated by

$$A_n = \text{diag}(-\alpha_1, -\alpha_2, \dots, -\alpha_n, M_n, 0, 0, \dots)$$

to $\exp(tA) = \text{diag}(e^{-\alpha_1 t}, e^{-\alpha_2 t}, e^{-\alpha_3 t}, \dots)$. In the mean time, the $(1, n)$ element of $\exp(tM_n)$ is nothing but $e^{-\omega t} \frac{t^{n-1}}{(n-1)!}$ so that, by Stirling formula

$$\|e^{(n-1)M_n}\| \geq e^{-\omega(n-1)} \frac{(n-1)^{n-1}}{(n-1)!} \sim \frac{e^{(n-1)(1-\omega)}}{(2\pi n)^{1/2}} \text{ as } n \rightarrow \infty,$$

and since $0 < \omega < 1$ we have then

$$\sup\{\|e^{(n-1)M_n}\| : n \geq 1\} = +\infty.$$

It is necessary to stress that in this example $\operatorname{Re} \sigma(A_n) \leq -\omega$, one has uniform convergence $\exp(tA_n) \rightarrow \exp(tA)$ in $t \in [0, T]$ for any finite $T > 0$, and $\|\exp(tA)\| \leq e^{-t}, t \geq 0$, but in the mean time $\|\exp(t_n A_n)\| \rightarrow \infty$ as $t_n = n - 1 \rightarrow \infty$.

So, this counterexample shows that it can happen $\|\exp(t_n A_n)\| \rightarrow \infty$ as $t_n \rightarrow \infty$. Since splitting of the solutions of the problem (1.2) in the neighborhood of any equilibrium point involves (see (6.21)) estimates like $\|\exp(t\bar{A}_n)\| \leq \bar{M}e^{-\bar{\omega}t}, t \geq 0$, with some $\bar{M} \geq 1, \bar{\omega} > 0$, for operator \bar{A}_n , which is the restriction of A_n on the corresponding subspaces, one needs to put on (A_n, A) additional assumption besides the discrete convergence of resolvents (A) and stability (B_1) .

EXAMPLE 6.3. Let A, E and $E^{\frac{1}{2}}$ be as in Example 6.1. It is well known (see [51]) that A can be associated one-to-one with a sesquilinear form $\sigma : E^{\frac{1}{2}} \times E^{\frac{1}{2}} \rightarrow \mathbb{C}$ such that

$$|\sigma(u, v)| \leq c_1 \|u\|_{E^{\frac{1}{2}}} \|v\|_{E^{\frac{1}{2}}}, \quad u, v \in E^{\frac{1}{2}},$$

$$\operatorname{Re} \sigma(u, u) \geq c_2 \|u\|_{E^{\frac{1}{2}}}, \quad u \in E^{\frac{1}{2}},$$

$$\sigma(u, v) = \langle -Au, v \rangle, \quad u \in D(A), v \in E^{\frac{1}{2}}.$$

If $\Omega \subset \mathbb{R}^2$ is a convex polygon consider a subdivision of it which consists of a regular triangulation where each triangle has largest diameter equal to h . In this case denote by S_h the set of all functions in $E^{\frac{1}{2}}$ which are linear in each triangular element.

So we are given a family S_h of finite dimensional subspaces of $H_0^1(\Omega)$ with the standard (see [70]) approximation property

$$\inf_{\chi \in S_h} (\|v - \chi\|_E + h\|v - \chi\|_{E^{1/2}}) \leq Ch^2 \|v\|_{E^1} \quad \text{for } v \in H^2(\Omega) \cap H_0^1(\Omega).$$

We also introduce analogy of our connecting mappings $\{p_n^\alpha\}$ as $\tilde{u}_h = \tilde{P}_h u$, i.e. the projection of $u \in E^{\frac{1}{2}}$ on $E_h^{\frac{1}{2}}$ with respect to the $H^1(\Omega)$ inner product and we have

$$\|u - \tilde{u}^h\|_{E^{\frac{1}{2}}} \leq Ch \|u\|_{E^1} \quad \text{and} \quad \|u - \tilde{u}^h\|_E \leq Ch^2 \|u\|_{E^1}, \quad u \in E^1$$

(see [19, 20]). Incidentally we stand $P_h u$ for the projection of $u \in E$ on $S_h = E_h^{\frac{1}{2}}$ with respect to the $L^2(\Omega)$ inner product, these operators play the role of connecting mappings $\{p_n\}$. In this framework, our finite element approximation $A_h : S_h \rightarrow S_h$ of A is defined by

$$\langle -A_h \phi_h, \psi_h \rangle = \sigma(\phi_h, \psi_h), \quad \phi_h, \psi_h \in E_h^{\frac{1}{2}}.$$

In other words, A_h is the operator associated with the sesquilinear form $\sigma_h(\cdot, \cdot)$ which is the restriction of $\sigma(\cdot, \cdot)$ to $E_h^{\frac{1}{2}} \times E_h^{\frac{1}{2}}$. With such setting one can prove that there exist a constant C and an acute angle θ such that for any $x \in E$ and $z \in \{z \in \mathbb{C} : \theta \leq |\arg z| \leq \pi\}$ we have for instance

$$\|(zI - A)^{-1}x - (zI_h - A_h)^{-1}P_h x\|_E \leq Ch^2 \|x\|_E.$$

This estimate shows that in such example one has \mathcal{P} -convergence with uniform convergence of the resolvents. Since our resolvent $(\lambda I - A)^{-1}$ is compact for some λ , then because of inequality (where $\mu(\cdot)$ is measure of noncompactness)

$$\mu((zI_h - A_h)^{-1}x_h) \leq \mu((zI - A)^{-1}x_h) + \|(zI - A)^{-1}x_h - (zI_h - A_h)^{-1}x_h\|_E \leq Ch^2 \|x_h\|_E$$

resolvents converge compactly as $h \rightarrow 0$. So we will use compact convergence property as one of the basic assumption on general approximation scheme to get the estimates like (6.21).

Let $u_h(t) : [0, \infty) \rightarrow E_h^{\frac{1}{2}}$ be the solution of

$$\langle u_h'(t), \phi_h \rangle + \sigma(u_h(t), \phi_h) = 0, \quad \phi_h \in E_h^{\frac{1}{2}}, \quad (6.9)$$

and $u_h(0) = u_h^0$. Then we can write instead of (6.9)

$$u_h'(t) = A_h u_h(t), \quad u_h(0) = u_h^0 \in E_h^{\frac{1}{2}}, \quad (6.10)$$

and we also have $u_h(t) = \exp(tA_h)u_h^0$. Moreover, for $x \in E^{\frac{1}{2}}$ we have (see [19]) for example that

$$\|\exp(tA)x - \exp(tA_h)\tilde{P}_h x\|_{E^{\frac{1}{2}}} \leq Ch t^{-\frac{1}{2}} \|x\|_{E^{\frac{1}{2}}}.$$

With this we conclude that the results in this paper holds for (6.4), with $n = 2$, when we use finite element method to approximate it.

For the case of finite difference methods approximation of eigenvalues and regular convergence of operators (where the notion of discrete compactness play crucial role) has been considered in [78]. For more references concerning discretization of problems like (1.2), see for example, [10, 28, 58, 59, 61, 63].

EXAMPLE 6.4. The condition (B_1) and compact convergence of resolvents are independent in the following sense. Let A be as in Example 6.2, with $\alpha_n = n^2, n \in \mathbb{N}$, but we approximate it with A_n in the form

$$A_n = \text{diag}(-\alpha_1, -\alpha_2, -\alpha_3, \dots, -\alpha_{n-1}, \alpha_n, -\alpha_{n+1}, \dots),$$

one can see that resolvents are compact and $A_n^{-1} \rightarrow A^{-1}$ compactly. But, since $\alpha_n \rightarrow +\infty$, the estimate (B_1) does not hold. So this means that compact convergence of resolvents is not enough to ensure that (B_1) is satisfied.

6.3. Compact Convergence and Uniform Estimates

In this subsection we present some basic results on compact convergence and its consequences

6.3.1. Basic Results on Compact Convergence

Recall that $B \in B(E)$ is called a Fredholm operator if $\mathcal{R}(B)$ is closed, $\dim \mathcal{N}(B) < \infty$ and $\text{codim } \mathcal{R}(B) < \infty$, the index of B is defined as $\text{ind } B = \dim \mathcal{N}(B) - \text{codim } \mathcal{R}(B)$.

THEOREM 6.2. [77] *For $B_n \in B(E_n)$ and $B \in B(E)$ the following conditions are equivalent*

- (i) $B_n \xrightarrow{\mathcal{PP}} B$ regularly, B_n are Fredholm operators of index 0 and $\mathcal{N}(B) = \{0\}$;
- (ii) $B_n \rightarrow B$ stably and $\mathcal{R}(B) = E$;
- (iii) $B_n \rightarrow B$ stably and regularly;
- (iv) if one of conditions (i)–(iii) holds, then there exist $B_n^{-1} \in B(E_n)$, $B^{-1} \in B(E)$, and $B_n^{-1} \xrightarrow{\mathcal{PP}} B^{-1}$ regularly and stably.

This theorem admits an extension to the case of closed operators $B \in \mathcal{C}(E)$, $B_n \in \mathcal{C}(E_n)$ [79].

LEMMA 6.4. *Assume that $B_n \xrightarrow{\mathcal{PP}} B$ compactly and operators B, B_n are compact. Assume also that $\mathcal{N}(I + B) = \{0\}$. Then there exist an $n_0 > 0$ and $M > 0$ such that*

$$\|(I_n + B_n)^{-1}\|_{B(E_n)} \leq M, \quad \text{for } n \geq n_0. \tag{6.11}$$

Proof: Since the compact convergence $B_n \xrightarrow{\mathcal{PP}} B$ implies that $I_n + B_n \xrightarrow{\mathcal{PP}} I + B$ regularly, the result follows from Theorem 6.2.

LEMMA 6.5. *Let $\Delta_{cc} \neq \emptyset$. Then for any $\lambda \in \rho(A)$, there is a $n_\lambda > 0$ such that $\lambda \in \rho(A_n)$ for all $n \geq n_\lambda$ and there is a constant $M_\lambda > 0$ such that*

$$\|(\lambda I_n - A_n)^{-1}\| \leq M_\lambda, \quad \text{for } n \geq n_\lambda.$$

Furthermore, $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ compactly.

Proof. The result follows from Lemma 6.4 and Corollary 3.1.

THEOREM 6.3. [74] *Let $B \in B(E)$, $B_n \in B(E_n)$. The following conditions are equivalent:*

- (i) operators $B_n \xrightarrow{\mathcal{PP}} B$ as $n \rightarrow \infty$;
(ii) $\|B_n\| \leq \text{constant}$, $n \in N$, and $\|B_n p_n x - p_n Bx\| \rightarrow 0$ for any $x \in E$ as $n \rightarrow \infty$;
(iii) $\|B_n\| \leq \text{constant}$, $n \in N$, and $\|B_n x_n - p_n Bx\| \rightarrow 0$ for any $x \in E$ as $n \rightarrow \infty$, whenever $x_n \rightarrow x \in E$.

LEMMA 6.6. Assume that $\Delta_{cc} \neq \emptyset$ and resolvents $(\lambda I - A)^{-1}$, $(\lambda I_n - A_n)^{-1}$ are compact. Let Λ be a compact subset of $\rho(A)$. Then there is a constant $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$ and

$$\sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(\lambda I_n - A_n)^{-1}\| < \infty. \quad (6.12)$$

Furthermore, for any $u \in E$

$$\sup_{\lambda \in \Lambda} \|(\lambda I_n - A_n)^{-1} p_n u - p_n (\lambda I - A)^{-1} u\| \rightarrow 0. \quad (6.13)$$

Proof. Let us first prove that there is a $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$. Suppose that this is not the case, then there are sequences $k_n \rightarrow \infty$, $\{\lambda_{k_n}\} \in \Lambda$ such that λ_{k_n} is an eigenvalue of A_{k_n} . Since Λ is compact we may assume that there is a $\bar{\lambda} \in \Lambda$ such that $\lambda_{k_n} \rightarrow \bar{\lambda}$. It follows from Theorem 3.1 (ii), that $\bar{\lambda} \in \sigma(A)$ which is a contradiction.

To prove (6.12), it is enough to prove that

$$\sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(I - \lambda A_n^{-1})^{-1}\| < \infty.$$

Assuming in contradiction that this is not the case; that is, assume that there are sequences $k_n \rightarrow \infty$, $\lambda_{k_n} \in \Lambda$ (which we may assume convergent to $\bar{\lambda} \in \Lambda$) such that

$$\|(I - \lambda_{k_n} A_{k_n}^{-1})^{-1}\| \rightarrow \infty.$$

Since $\lambda_{k_n} A_{k_n}^{-1}$ converges compactly to $\bar{\lambda} A^{-1}$ this is in contradiction with Lemma 6.4.

It remains to prove (6.13). Once again, we prove it by contradiction. Assume that there are sequences $k_n \rightarrow \infty$, $\Lambda \ni \lambda_{k_n} \rightarrow \bar{\lambda} \in \Lambda$ and $\epsilon > 0$ such that

$$\|(\lambda_{k_n} I_{k_n} - A_{k_n})^{-1} p_{k_n} u - p_{k_n} (\lambda_{k_n} I - A)^{-1} u\| \geq \epsilon. \quad (6.14)$$

But this contradicts according to Theorem 6.3 to $\|(\lambda I_n - A_n)^{-1} p_n u - p_n (\lambda I - A)^{-1} u\| \rightarrow 0$ for $\lambda = \bar{\lambda}$ and estimate (6.12).

Next we show that the compact convergence of A_n^{-1} to A^{-1} ensures that the resolvent of A_n contains a fixed sector for almost all n . We observe that no information about bounds on the resolvent are provided. Let us denote $\Sigma_{\phi, \bar{\omega}} = \{\lambda \in \mathbb{C} : |\arg(\lambda - \bar{\omega})| < \pi - \phi\}$.

LEMMA 6.7. Assume that $\Delta_{cc} \neq \emptyset$, for $\lambda \in \Delta_{cc}$ resolvents $(\lambda I_n - A_n)^{-1}$ are compact and conditions (1.1), (B_1) are satisfied. Then there are a sector $\Sigma_{\bar{\phi}, -\omega}$, $\phi < \bar{\phi} < \pi/2$, $0 < \omega < \tilde{\omega}$, a constant $n_\omega > 0$ and a constant M_ω such that

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M_\omega}{|\lambda - \omega|}, \quad \forall \lambda \in \Sigma_{\bar{\phi}, -\omega} \text{ as } n \geq n_\omega. \tag{6.15}$$

Proof. Let $\omega < \omega_1 < \tilde{\omega}$ and consider the region $\Lambda = \{\lambda \in \mathbb{C} : \text{Re}\lambda \geq -\omega_1 \text{ and } \lambda \notin \Sigma_{\phi, \tilde{\omega}}\}$. Then Λ (see Figure 2) is a compact set and $\Lambda \subset \rho(A)$. It follows from Lemma 6.6 that there is a constant $n_\Lambda > 0$ such that $\Lambda \subset \rho(A_n)$ for all $n \geq n_\Lambda$ and

$$\sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(\lambda I_n - A_n)^{-1}\| < \infty. \tag{6.16}$$

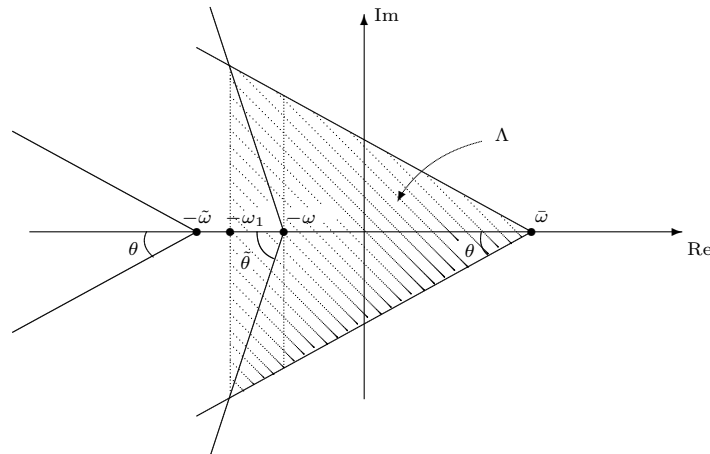


Figure 2

If we take $\bar{\phi} < \pi/2$ such that $\Sigma_{\bar{\phi}, \omega} \cap \{\lambda \in \Sigma_{\phi, \tilde{\omega}} : \text{Re}\lambda < \omega_1\} = \emptyset$ we have that $\Sigma_{\bar{\phi}, \omega} \subset \rho(A_n)$ for all $n \geq n_\Lambda$. The estimate (6.15) follows easily from (6.16) and from estimate (B_1) . \square

PROPOSITION 6.1. Let conditions (A) and (B_1) be satisfied. Then the compact convergence of resolvents $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ is equivalent to the compact convergence of analytic C_0 -semigroups $\exp(tA_n) \xrightarrow{\mathcal{PP}} \exp(tA)$ for any $t > 0$.

Proof: First, let us show that the compact convergence of resolvents $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ is equivalent to the compact convergence of C_0 -semigroups $\exp(tA_n) \rightarrow \exp(tA)$

for any $t > 0$. Let $\|x_n\| = O(1)$. Then from the estimate $\|A_n \exp(tA_n)\| \leq \frac{M}{t} e^{\omega t}$, we obtain the boundedness of the sequence $\{(A_n - \lambda I_n) \exp(tA_n)x_n\}$. Because of Theorem 3.3 and the compact convergence of resolvents, we obtain the compactness of the sequence $\{\exp(tA_n)x_n\}$.

The necessity will be proved if for the measure of noncompactness $\mu(\cdot)$ we establish that $\mu(\{(\lambda I_n - A_n)^{-1}x_n\}) = 0$ for $\|x_n\| = O(1)$. We have

$$\mu(\{(\lambda I_n - A_n)^{-1}x_n\}) = \mu\left(\int_0^\infty e^{-\lambda t} \exp(tA_n)x_n dt\right) \leq \mu\left(\int_0^q e^{-\lambda t} \exp(tA_n)x_n dt\right) +$$

$$\mu\left(\int_q^\infty e^{-\lambda t} \exp(tA_n)x_n dt\right) + \mu\left(\exp(\epsilon A_n) \int_q^Q e^{-\lambda t} \exp((t - \epsilon)A_n)x_n dt\right).$$

Two first terms can be made less than ϵ by the choice of q, Q . The last term is equal to zero because of the compact convergence of semigroups $\exp(\epsilon A_n) \xrightarrow{\mathcal{PP}} \exp(\epsilon A)$ for any $0 < \epsilon < q$ and boundedness of the sequence $\{\int_q^Q e^{-\lambda t} \exp((t - \epsilon)A_n)x_n dt\}$. \square

6.3.2. Compact Convergence of Fractional Powers

Here we obtain the compact convergence of negative fractional powers assuming compact convergence of resolvents. Such results are essential to obtain the \mathcal{P}^α -compactness of the union of the attractors in Lemma 5.2.

LEMMA 6.8. *Let A_n, A be such that conditions (1.1), (B_1) and $\Delta_{cc} \neq \emptyset$ are satisfied. Then for any $0 < \theta < 1$ one has $A_n^{-\theta} \xrightarrow{\mathcal{PP}} A^{-\theta}$ compactly.*

Proof. Let $\Sigma_{\bar{\phi}, -\omega}$ be as in Lemma 6.7 and Ω be the boundary of $\Sigma_{\bar{\phi}, -\omega}$. Since

$$(-A_n)^{-\theta} = \frac{1}{2\pi i} \int_{\Omega} (-\lambda)^{-\theta} (\lambda I_n - A_n)^{-1} d\lambda.$$

From Lemma 6.7, the above integral is absolutely convergent uniformly for $n \geq n_\omega$. Therefore, given $\epsilon > 0$, one can divide contour $\Omega = \Omega_1^\epsilon \cup \Omega_2^\epsilon$ in such way that Ω_1^ϵ is bounded and the integral

$$\frac{1}{2\pi i} \int_{\Omega_2^\epsilon} \|(-\lambda)^{-\theta} (\lambda I_n - A_n)^{-1}\| d\lambda \leq \epsilon, \quad n \geq n_\omega.$$

As for the integral over Ω_1^ϵ we rewrite it as

$$B_n := \frac{A_n^{-1}}{2\pi i} \int_{\Omega_1^\epsilon} (-\lambda)^{-\theta} \lambda (\lambda I_n - A_n)^{-1} d\lambda$$

we observe that $\|(-\lambda)^{-\theta} \lambda (\lambda I_n - A_n)^{-1}\|$ is bounded uniformly for $n \geq n_\omega$. From the fact that $A_n^{-1} \rightarrow A^{-1}$ compactly it follows that $B_n \rightarrow B$ compactly. Let μ be a measure of non-compactness. Now, taking any sequences $n \rightarrow \infty$ and $\{u_n\}$, $u_n \in E_n$, $\|u_n\| = 1$, we obtain that

$$\mu(\{(-A_n)^{-\theta} u_n\}) \leq \mu(\{B_n u_n\}) + \mu\left(\left\{\frac{1}{2\pi i} \int_{\Omega_2^c} (-\lambda)^{-\theta} (\lambda I_n - A_n)^{-1} d\lambda u_n\right\}\right) \leq \epsilon.$$

It follows that, given $\{u_n\}$, with $u_n \in E_n$, $n \rightarrow \infty$, $\|u_n\| = 1$, $\mu(\{(-A_n)^{-\theta} u_n\}) = 0$ and that $(-A_n)^{-\theta} \rightarrow (-A)^{-\theta}$ compactly.

To see that $(-A_n)^{-\theta} \xrightarrow{\mathcal{PP}} (-A)^{-\theta}$ we use that $(\lambda I_n - A_n)^{-1} \xrightarrow{h \rightarrow 0} (\lambda I - A)^{-1}$ for all $\lambda \in \Omega$ and the Dominated Convergence Theorem. \square

Now we consider a relatively bounded perturbation of A_n and A in the following manner. We consider perturbations satisfying

(L) $D_n \in B(E_n^\alpha, E_n)$, $D \in B(E^\alpha, E)$ such that D_n, D are compatible.

We assume the following hyperbolicity condition

(H) $\sigma(A + D) \cap \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda = 0\} = \emptyset$.

It is clear that if resolvent of A is compact, then $A + D$ has compact resolvent.

LEMMA 6.9. *Assume that $\Delta_{cc} \neq \emptyset$ and conditions (1.1), (B_1) are satisfied. Then, for any $0 \leq \theta < 1$ operators $D_n \in B(E_n^\alpha, E_n)$, such that $D_n \rightarrow D$ in the sense of the spaces E_n^α, E_n and E^α, E , then one has $(-A_n)^\theta (A_n + D_n)^{-1} \xrightarrow{\mathcal{PP}} (-A)^\theta (A + D)^{-1}$ compactly.*

Proof: To prove the result chose positive numbers r_1 and r_2 such that $1 > \alpha + r_1 \geq \theta$ and $r_2 = 1 - \alpha - r_1$ and note that

$$(-A_n)^\theta (A_n + D_n)^{-1} = (-A_n)^{-(\alpha+r_1-\theta)} (-I_n + (-A_n)^{-r_2} D_n (-A_n)^{-\alpha-r_1})^{-1} (-A_n)^{-r_2}$$

Since $\{(-A_n)^{-r_2} D_n (-A_n)^{-\alpha-r_1}\}$ converge compactly, the result now follows from Lemma 6.8 and from Lemma 6.4. \square

6.3.3. Convergence of Linearized Semigroups

The next result ensures a stability condition.

LEMMA 6.10. *Let A_n, A be such that conditions (1.1), (B_1) , $\Delta_{cc} \neq \emptyset$, (L) and (H) are satisfied. Then, there is a sector $\Sigma_{\phi_d, \omega_d}$, $\phi_d < \pi/2$, $\omega_d \in \mathbb{R}$, a constant $n_d > 0$ and a constant M_d such that*

$$\|(\lambda I_n - (A_n + f'_n(u_n^*)))^{-1}\| \leq \frac{M_d}{|\lambda - \omega_d|}, \text{ for any } \lambda \in \Sigma_{\phi_d, \omega_d} \text{ and for all } n \geq n_d. \quad (6.17)$$

Proof. It follows from Lemma 6.7 and from the Moment Inequality that for $\lambda \in \Sigma_{\bar{\phi},0}$

$$\|(-A_n)^\alpha(\lambda I_n - A_n)^{-1}\| \leq \frac{M}{|\lambda|^{1-\alpha}}, \quad n \geq n_\omega.$$

Since $D_n(-A_n)^{-\alpha}$ is bounded uniformly in n and

$$(\lambda I_n - A_n - D_n) = (I_n - D_n(-A_n)^{-\alpha}(-A_n)^\alpha(\lambda I_n - A_n)^{-1})(\lambda I_n - A_n)$$

we have that there is a $R > 0$ such that for $|\lambda| > R$, $\lambda \in \Sigma_{\bar{\phi},0}$, we have that the operator $(I_n - D_n(-A_n)^{-\alpha}(-A_n)^\alpha(\lambda I_n - A_n)^{-1})$ is invertible and

$$\|(I_n - D_n(-A_n)^{-\alpha}(-A_n)^\alpha(\lambda I_n - A_n)^{-1})^{-1}\| \leq 2.$$

From that we obtain the desired result (see Figure 3) for any $\omega_d > R$ with θ_d suitably chosen such that $\Sigma_{\theta_d,\omega_d} \subset \{\lambda \in \Sigma_{\bar{\theta},0} : |\lambda| > R\}$. \square

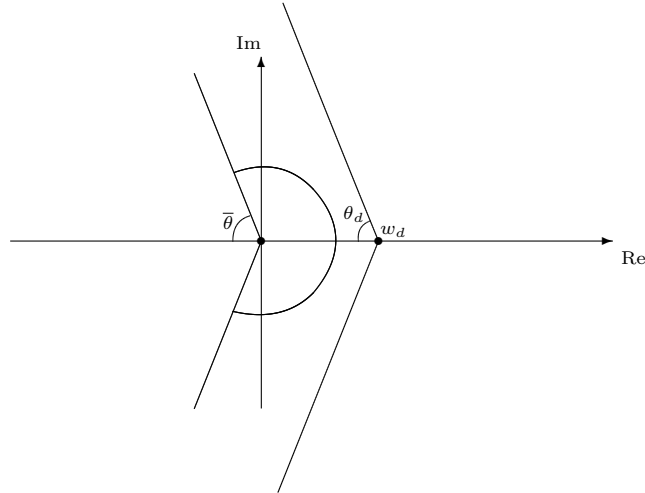


Figure 3

Observing that, from Lemma 6.9, $(A_n + f'_n(u_n^*))^{-1}$ converges compactly to $(A + f'(u^*))^{-1}$ and proceeding exactly as in Lemma 6.6 we obtain the following result

LEMMA 6.11. *Let A_n, A be such that conditions (1.1), (B_1) , $\Delta_{cc} \neq \emptyset$, (L) and (H) are satisfied. Let Λ be a compact subset of $\rho((A + f'(u^*)))$. Then, there is a constant $n_\Lambda > 0$ such that $\Lambda \subset \rho((A_n + f'_n(u_n^*)))$ for all $n \geq n_\Lambda$ and*

$$\sup_{\substack{\lambda \in \Lambda \\ n \geq n_\Lambda}} \|(\lambda I_n - (A_n + f'_n(u_n^*)))^{-1}\| < \infty. \quad (6.18)$$

Furthermore, for any $u \in E$

$$\sup_{\lambda \in \Lambda} \|(\lambda I_n - (A_n + f'_n(u_n^*)))^{-1} p_n u - (\lambda I - (A + f'(u^*)))^{-1} u\| \rightarrow 0. \quad (6.19)$$

The following result can be proved exactly as Theorem 3.1 and Lemma 6.7.

Let $Q(\sigma^+)$ be as in Definition 2.8 and denote by $W(\sigma^+, A + f'(u^*))$ the subspace $Q(\sigma^+)E^\alpha$. Similarly, if $Q_n(\sigma_n^+)$ is as in Proposition 5.5, we denote by $W_n(\sigma_n^+, A_n + f'_n(u_n^*))$ the subspace $Q_n(\sigma_n^+)E_n^\alpha$,

THEOREM 6.4. *Let A_n, A be such that conditions (1.1), (B_1) , $\Delta_{cc} \neq \emptyset$, (L) and (H) are satisfied. Then the following statements hold :*

(i) *there exists $n_0 > 0$ such that $\dim W_n(\sigma_n^+, (A_n + f'_n(u_n^*))) = \dim W(\sigma^+, (A + f'(u^*)))$ for all $n \geq n_0$;*

(ii) *for each $u \in W(\sigma^+, (A + f'(u^*)))$, there is a sequence $\{u_n\}$, with $u_n \in W_n(\sigma_n^+, (A_n + f'_n(u_n^*)))$, such that $u_n \xrightarrow{\mathcal{P}^\alpha} u$;*

(iii) *any sequence, $\{u_n\}_{n \in \mathbb{N}}$, with $u_n \in W_n(\sigma_n^+, (A_n + f'_n(u_n^*)))$, $n \in \mathbb{N}$, with $\|u_n\|_{E_n} = 1$ has a convergent subsequence and any limit point of this sequence belongs to $W(\sigma^+, (A + f'(u^*)))$;*

(iv) *there is a $\beta > 0$ and $n_\beta > 0$ such that $\sigma((A_n + f'_n(u_n^*))) \cap \{\lambda \in \mathbb{C} : |\operatorname{Re} \lambda| \leq \beta\} = \emptyset$, $n \geq n_\beta$;*

(v) *there are $\pi/2 > \phi_\beta > 0$, $\beta > 0$ and $n_\beta > 0$ such that*

$$\Sigma_{\phi_\beta, -\beta} \subset \rho \left((A_n + f'_n(u_n^*))|_{(I - Q_n(\sigma_n^+, A_n + f'_n(u_n^*)))E_n} \right) \parallel \leq \frac{M_\beta}{|\lambda - \beta|},$$

$n \geq n_\beta$ and

$$\left\| \left(\lambda - (A_n + f'_n(u_n^*))|_{(I - Q_n(\sigma_n^+, A_n + f'_n(u_n^*)))E_n} \right)^{-1} \right\| \leq \frac{M_\beta}{|\lambda - \beta|}, \quad \forall \lambda \in \Sigma_{\phi_\beta, -\beta}. \quad (6.20)$$

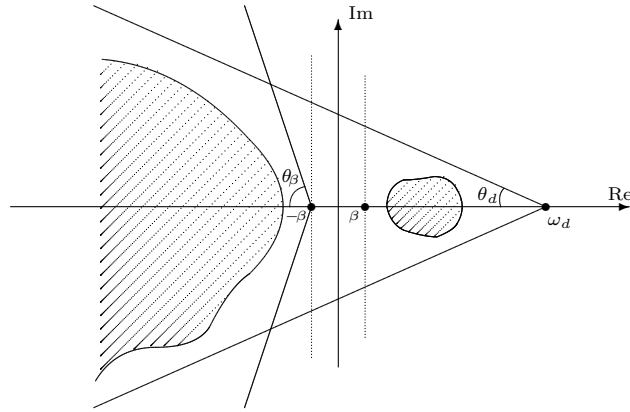


Figure 4

6.3.4. *Some Uniform Estimates*

It is well known that, if condition (1.1) is satisfied, for each $\omega < \tilde{\omega}$ there is a constant $M_\omega \geq 1$ such that

$$\|t^\alpha \exp(tA)\|_{B(E, E^\alpha)} \leq M_\omega e^{-\omega t}, \quad \forall t \geq 0.$$

From Lemma 6.7, the estimate in (1.1) holds uniformly for the semigroups generated by A_n , for n large enough. From this we obtain the following result

THEOREM 6.5. *Assume that $\Delta_{cc} \neq \emptyset$ and conditions (1.1), (B_1) , (L) , (H) are satisfied. Then, for each $\omega < \tilde{\omega}$, there are constants, n_0 and M_ω such that*

$$\|t^\alpha \exp(tA_n)\|_{B(E_n, E_n^\alpha)} \leq M_\omega e^{-\omega t}, \quad \forall t \geq 0, \quad 0 \leq \alpha \leq 1, \quad n \geq n_0.$$

As a consequence of Theorem 6.4 we have that

THEOREM 6.6. *Assume that $\Delta_{cc} \neq \emptyset$ and conditions (1.1), (B_1) , (L) , (H) are satisfied. Then, there are constants, n_β and M_β such that*

$$\begin{aligned} \|t^\alpha \exp(t(A_n + f'_n(u_n^*))) (I_n - Q_n(\sigma_n^+))\|_{B(E_n, E_n^\alpha)} &\leq M_\beta e^{-\beta t}, \quad \forall t \geq 0, \quad 0 \leq \alpha \leq 1, \\ \|\exp(t(A_n + f'_n(u_n^*))) Q_n(\sigma_n^+)\|_{B(E_n)} &\leq M_\beta e^{\beta t}, \quad \forall t \leq 0, \end{aligned} \quad (6.21)$$

$n \geq n_\beta$.

REFERENCES

1. E. A. M. Abreu and A.N. Carvalho, Lower Semicontinuity of Attractors for Parabolic Problems with Dirichlet Boundary Conditions in Varying Domains, *Notas do ICMC/USP - Série Matemática # 148*.
2. P. M. Anselone, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, Prentice-Hall Inc., Englewood Cliffs, N. J. (1971). With an appendix by Joel Davis, Prentice-Hall Series in Automatic Computation.
3. J. M. Arrieta, *Spectral behavior and upper semicontinuity of attractors*, Fiedler, B. (ed.) et al., International conference on differential equations. Proceedings of the conference, Equadiff '99, Berlin, Germany, August 1-7, 1999. Vol. 1. Singapore: World Scientific. 615-621, 2000 (English).
4. J. M. Arrieta and A. N. Carvalho, Abstract Parabolic Problems with Critical Nonlinearities and Applications to Navier-Stokes and Heat Equations, *Transactions of the American Mathematical Society*, **352** pp. 285-310, (2000).
5. J. M. Arrieta and A. N. Carvalho, Spectral convergence and nonlinear dynamics of reaction-diffusion equations under perturbations of the domain, *Journal of Differential Equations*, **199**(1), pp. 143-178, (2004).
6. J. M. Arrieta; A.N. Carvalho and A. Rodriguez-Bernal, Upper Semicontinuity of Attractors for Parabolic Problems with Localized Large Diffusion and Nonlinear Boundary Conditions, *Journal of Differential Equations* **168** no. 1, pp. 33-59, (2000).
7. J. M. Arrieta, A. N. Carvalho, A. Rodríguez-Bernal, Parabolic Problems with Nonlinear Boundary Conditions and Critical Nonlinearities, *Journal of Differential Equations* **156** (2) pp. 376-406, (1999).
8. J.M. Arrieta, A.N. Carvalho and A. Rodriguez-Bernal, Attractors Parabolic problems with Nonlinear Boundary Conditions: Uniform Bounds, *Communications in Partial Differential Equations*, **25** 1-2 pp. 1-37, (2000).
9. A.V. Babin and M.I. Vishik, *Attractors of evolution equations. Transl. from the Russian by A. V. Babin.*, Studies in Mathematics and its Applications. 25. Amsterdam etc.: North- Holland. x, 532 p., 1992 (English).
10. N. A. Bobylev, J. K. Kim, S. K. Korovin, and S. Piskarev, "Semidiscrete approximations of semilinear periodic problems in Banach spaces," *Nonlinear Anal.*, **33**, No. 5, 473-482 (1998).
11. S. M. Bruschi; A. N. Carvalho and J. G. Ruas-Filho, The dynamics of a one-dimensional parabolic problem versus the dynamics of its discretization, *Journal of Differential Equations* **168** no. 1 pp. 67-92, (2000).
12. A. N. Carvalho, Spatial Homogeneity in Damped Hyperbolic Equations, *Dynamic Systems and Applications*, **1**(3) pp. 221-250, (1992).
13. A. N. Carvalho, Infinite Dimensional Dynamical Systems Described by ODE, *Journal of Differential Equations*, **116**(2) pp. 338-404, (1995).
14. A. N. Carvalho and J. K. Hale, Large Diffusion with Dispersion, *Nonlinear Analysis TMA*, **17** (12) pp. 1139-1151, (1991).
15. A. N. Carvalho and A. L. Pereira, A scalar parabolic equation whose asymptotic behavior is dictated by a system of ordinary differential equations, *Journal of Differential Equations*, **112** (1) pp. 81-130, (1994).
16. A. N. Carvalho and L. A. F. Oliveira, Delay-Partial Differential Equations with Some Large Diffusion, *Nonlinear Analysis-Theory Methods and Applications*, **22** (9) pp. 1057-1095, (1994).
17. N. Chafee and E. F. Infante, A bifurcation problem for a nonlinear partial differential equation of parabolic type, *Applicable Analysis*, **4** pp. 17-31, (1974).
18. J. W. Cholewa, T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge, 2000.
19. H. Fujita and A. Mizutani, "On the finite element method for parabolic equations. I. Approximation of holomorphic semigroups," *J. Math. Soc. Japan*, **28**, No. 4, 749-771 (1976).

20. H. Fujita and T. Suzuki, "Evolution problems," In: *Handbook of Numerical Analysis*, Vol. II, North-Holland, Amsterdam (1991), pp. 789–928.
21. R. D. Grigorieff, "Zur Theorie linearer approximationsregulärer Operatoren. I, II," *Math. Nachr.*, **55**, 233–249, 251–263 (1973).
22. R. D. Grigorieff, "Diskrete Approximation von Eigenwertproblemen. I. Qualitative Konvergenz," *Numer. Math.*, **24**, No. 4, 355–374 (1975).
23. R. D. Grigorieff, "Diskrete Approximation von Eigenwertproblemen. II. Konvergenzordnung," *Numer. Math.*, **24**, No. 5, 415–433 (1975).
24. R. D. Grigorieff, "Über diskrete Approximationen nichtlinearer Gleichungen 1. Art.," *Math. Nachr.*, **69**, 253–272 (1975).
25. R. D. Grigorieff, "Diskrete Approximation von Eigenwertproblemen. III. Asymptotische Entwicklungen," *Numer. Math.*, **25**, No. 1, 79–97 (1975/76).
26. R. D. Grigorieff, Zur Charakterisierung linearer approximationsregulärer Operatoren," In: *Mathematical Papers Given on the Occasion of Ernst Mohr's 75th Birthday*, Tech. Univ. Berlin, Berlin (1985), pp. 63–77.
27. R. D. Grigorieff and H. Jeggle, "Approximation von Eigenwertproblemen bei nichtlinearer Parameterabhängigkeit," *Manuscripta Math.*, **10**, 245–271 (1973).
28. D. Guidetti, B. Karasozen and S. Piskarev. Approximation of abstract differential equations. *Journal of Mathematical Sciences*, 2004, Vol. 122, N 2, p. 3013–3054.
29. I. N. Gurova, "On a certain topological method for studying difference schemes," *Dokl. Akad. Nauk SSSR*, **248**, No. 25–28 (1979).
30. J. Hale, "Large diffusivity and asymptotic behavior in parabolic systems", *J. Math. Anal. and Appl.* **118** pp. 455-466 (1986).
31. J.K. Hale. *Asymptotic Behavior of Dissipative Systems*, Mathematics Surveys and Monographs **25**, American Mathematical Society (1989).
32. J. K. Hale, *Numerical dynamics*, Chaotic numerics (Geelong, 1993), Amer. Math. Soc., Providence, RI, 1994, pp. 1–30.
33. J. K. Hale, G. Raugel, Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equation, *Journal of Differential Equations*, **73** pp. 197-214, (1988).
34. J. K. Hale and G. Raugel, Regularity, determining modes and Galerkin methods, *J. Math. Pures Appl.*, **82** (9) pp. 1075-1136, (2003)
35. J. K. Hale and G. Raugel, A reaction-diffusion equation on a thin L -shaped domain, *Proc. Roy. Soc. Edinburgh Sect. A*, **125** (2) pp. 283–327, (1995).
36. J. K. Hale and G. Raugel, Limits of semigroups depending on parameters, *Resenhas*, **1** (1) pp. 1-45, (1993).
37. J. K. Hale and G. Raugel, Attractors and convergence of PDE on thin L -shaped domain, *Progress in partial differential equations: the Metz surveys, 2* (1992), pp. 149–171, Longman Sci. Tech., Harlow, (1993).
38. J. K. Hale and G. Raugel, Attractors for dissipative evolutionary equations, *International Conference on Differential Equations, Vol. 1, 2 (Barcelona, 1991)*, pp. 3-22, World Sci. Publishing, River Edge, NJ, (1993).
39. J. K. Hale and G. Raugel, Reaction-diffusion equation on thin domains, *J. Math. Pures Appl.* **71** (1) pp. 33-95, (1992).
40. J. K. Hale and G. Raugel, Convergence in gradient-like systems with applications to PDE, *Z. Angew. Math. Phys.* **43** (1), (1992).
41. J. K. Hale and G. Raugel, Partial differential equations on thin domains, *Differential equations and mathematical physics (Birmingham, AL, 1990)*, 63–97, Academic Press, Boston, MA (1992).

42. J. K. Hale and G. Raugel, A damped hyperbolic equation on thin domains, *Transactions of the American Mathematical Society*, **329** (1) pp. 185-219, (1992).
43. J. K. Hale and G. Raugel, Lower semicontinuity of the attractor of a singularly perturbed hyperbolic equation, *Journal of Dynamics and Differential Equations*, **2** (1) pp. 19-67, (1990).
44. J. K. Hale and G. Raugel, Lower semicontinuity of attractors of gradient systems and applications, *Ann. Mat. Pura Appl.*, **154** pp. 281-326, (1989).
45. J. K. Hale and G. Raugel, Upper semicontinuity of the attractor for a singularly perturbed hyperbolic equations, *Journal of Differential Equations* **73** (2) pp 197-214, (1988).
46. J. K. Hale, X.-B. Lin and G. Raugel, Upper semicontinuity of attractors for approximations of semigroups and partial differential equations, *Math. Comp.*, **50** pp. 89-123, no. 181, (1988)
47. J. Hale, C. Rocha, Varying boundary conditions and large diffusivity, *J. Math Pures et Appl.*, **66** pp. 139-158, (1987).
48. J. K. Hale and K. Sakamoto, Shadow Systems and Attractors in Reaction-Diffusion Equations, *Applicable Analysis*, **32** pp. 287-303, (1989).
49. D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin (1981).
50. L.V. Kapitanskii and I.N. Kostin, Attractors of nonlinear evolution equations and their approximations. (Russian) *Algebra i Analiz* **2**(1) (1990), 114-140; translation in *Leningrad Math. J.* **2**(1) (1991), 97-117.
51. T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics. Springer-Verlag, Berlin (1995).
52. S. N. S. Khalsa, Finite element approximation of a reaction diffusion equation, *Quart. Appl. Math.* **44** (1986), 375-386.
53. P. E. Kloeden and J. Lorenz, Stable attracting sets in dynamical system and their one-step discretizations, *SIAM J. Numer. Anal.* **23** (1986), 986-995.
54. A.A. Kornev Approximation of attractors of semidynamical system. *Math. Sbornik*, 2001, 192: 10, p.1435-1450.
55. M. A. Krasnosel'skii, P.P. Zabreiko. *Geometrical methods of nonlinear analysis*. Translated from the Russian by Christian C. Fenske. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 263. Springer-Verlag, Berlin, 1984. xix+409 pp.
56. S. G. Krein, *Linear Differential Equations in Banach Space*, American Mathematical Society, Providence, R.I., (1971).
57. O. Ladyzhenskaya. *Attractors for semigroups and evolution equations*. Lezioni Lincee. [Lincei Lectures] Cambridge University Press, Cambridge, (1991).
58. S. Larsson, S. Yu. Pilyugin, "Numerical shadowing near the global attractor for a semilinear parabolic equation," *Preprint* 21, Department of Mathematics, Chalmers University of Technology (1998).
59. S. Larsson, J. M. Sanz-Serna, "The behavior of finite element solutions of semilinear parabolic problems near stationary points," *SIAM J. Numer. Anal.*, **31**, No. 4, 1000-1018 (1994).
60. O. Nevanlinna, G. Vainikko, "Limit spectrum of discretely converging operators," *Numer. Funct. Anal. Optim.*, **17**, Nos. 7-8, 797-808 (1996).
61. A. Ostermann, C. Palencia. "Shadowing for nonautonomous parabolic problems with applications to long-time error bounds." *SIAM J. Numer. Anal.* **37** (2000), no. 5, 1399-1419.
62. S. I. Piskarev, "Approximation of holomorphic semigroups," *Tartu Riikl. Ül. Toimetised*, **492**, 3-14 (1979).
63. S. I. Piskarev, "Convergence of difference schemes in the solution of nonlinear parabolic equations," *Mat. Zametki*, **44**, No. 1, 112-123 (1988).
64. M. Prizzi and K. P. Rybakowski, The effect of domains squeezing upon the dynamics of reaction-diffusion equations, *Journal of Differential Equations*, **173** (2) pp. 271-320, (2001).

65. G. Raugel, *Dynamics of partial differential equations on thin domains.*, Johnson, Russell (ed.), Dynamical systems. Lectures given at the 2nd session of the Centro Internazionale Matematico Estivo (CIME) held in Montecatini Terme, Italy, June 13-22, 1994. Berlin: Springer-Verlag. Lect. Notes Math. 1609, 208-315 , 1995 (English).
66. S. M. Oliva Filho, Reaction-diffusion systems on domains with thin channels, *Journal of Differential Equations*, **123** (2), pp. 437-479, (1995).
67. J. Solà-Morales and M. València, *Trend to spatial homogeneity for solutions to semilinear damped wave equations.*, Proc. R. Soc. Edinb., Sect. A **105**, pp. 117-126, (1987).
68. F. Stummel, "Diskrete Konvergenz linearer Operatoren. III," In: *Linear Operators and Approximation (Proc. Conf., Oberwolfach, 1971)*, Internat. Ser. Numer. Math., Vol. 20. Birkhäuser, Basel (1972), pp. 196-216.
69. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer Verlag, New York, (1988).
70. V. Thomée, "Finite element methods for parabolic problems – some steps in the evolution," In: *Finite element methods (Jyväskylä, 1993)*, Vol. 164 of *Lecture Notes in Pure and Appl. Math.*, Dekker, New York (1994), pp. 433-442.
71. H. F. Trotter, "Approximation of semigroups of operators," *Pacific J. Math.*, **8**, 887-919 (1958).
72. H. F. Trotter, "Approximation and perturbation of semigroups," In: *Linear Operators and Approximation, II (Proc. Conf., Oberwolfach Math. Res. Inst., Oberwolfach, 1974)*, Internat. Ser. Numer. Math., Vol. 25. Birkhäuser, Basel (1974), pp. 3-21.
73. T. Ushijima, "Approximation theory for semigroups of linear operators and its application to approximation of wave equations," *Jpn. J. Math. (N.S.)*, **1**, No. 1, 185-224 (1975/76).
74. G. Vainikko, *Funktionalanalysis der Diskretisierungsmethoden*, B. G. Teubner Verlag, Leipzig (1976).
75. G. Vainikko, "Über die Konvergenz und Divergenz von Näherungsmethoden bei Eigenwertproblemen," *Math. Nachr.*, **78**, 145-164 (1977).
76. G. Vainikko, "Über Konvergenzbegriffe für lineare Operatoren in der numerischen Mathematik," *Math. Nachr.*, **78**, 165-183 (1977).
77. G. Vainikko, "Approximative methods for nonlinear equations (two approaches to the convergence problem)," *Nonlinear Anal.*, **2**, 647-687 (1978).
78. G. Vainikko, "Foundations of finite difference method for eigenvalue problems," In: *The Use of Finite Element Method and Finite Difference Method in Geophysics (Proc. Summer School, Liblice, 1977)*, Česk. Akad. Věd, Prague (1978), pp. 173-192.
79. G. Vainikko, S. Piskarev, "Regularly compatible operators," *Izv. Vuzov, Mat.*, **10**, 25-36 (1977).
80. V. V. Vasilev, S. G. Krein, and S. Piskarev, "Operator semigroups, cosine operator functions, and linear differential equations," In: *progress in Science and Technology, Seria on Mathematical Analysis*, Vol. 28 [in Russian]. All-Union Institute for Scientific and Technical Information (VINITI), Akad. Nauk SSSR, Moscow (1990), pp. 87-202. Engl. transl.: *J. Soviet Math.* **54**, No. 4, 1042-1129 (1991).
81. V.V. Vasiliev, S.I. Piskarev, *Differential equations in Banach spaces I. Semigroup theory*. Moscow State University Publishing House (1996) [in Russian].
82. V.V. Vasiliev, S.I. Piskarev, "Differential equations in Banach spaces II. Cosine-operator functions," *Journal of Mathematical Sciences*, 2004, Vol. 122, N 2, p. 3055-3174.
83. K. Yosida. *Functional analysis*. Die Grundlehren der Mathematischen Wissenschaften, Band 123 Academic Press, Inc., New York; Springer-Verlag, Berlin 1965.