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Transversality of Stable and Unstable Manifolds for Parabolic Problems Arising in Composite Materials

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In this paper we study one dimensional parabolic problems that arise from composite materials. We show that the eigenvalues and eigenfunctions of the associated linear unbounded operator have the Sturm-Liouville property and the non increase of the lap number along the solutions. These results are used to show that the stable and unstable manifolds of equilibrium points are transversal. October, 2003 ICMC-USP

1. INTRODUCTION

In this paper we deal with equations that appear as limit of second-order parabolic problems when the diffusion coefficient becomes large in a subregion which is interior to the physical domain of the differential equation. This situation can be found, for example, in composite materials, where the heat diffusion properties can change significantly from one part of the region to another.

Let $\Omega = (0,1)$, ε a positive parameter, m a positive integer and $\Omega_0 = \cup_{i=1}^{m}\Omega_{0,i}$ be a subset of Ω, where $\Omega_{0,i} = (a_i, b_i) \subset \Omega$ with $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ for $i \neq j$, and $\bar{\Omega}_0 \subset (0,1)$. We denote $\Omega_1 = \Omega \setminus \bar{\Omega}_0$. We also set $b_0 = 0$ and $a_{m+1} = 1$.

The diffusion coefficients p_ε, $0 < \varepsilon \leq \varepsilon_0$, are assumed to be smooth functions in $(0,1)$ satisfying

$$p_\varepsilon(x) \to \begin{cases} p(x) \text{ uniformly on } \Omega_1 \\ \infty \text{ uniformly on compact subsets of } \ \Omega_0 \end{cases} \tag{1.1}$$

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as $\varepsilon \to 0$. For $c \in \mathrm{C}^1(\Omega)$ and smooth nonlinearities f, $g : \mathbb{R} \to \mathbb{R}$ consider the family of parabolic equations

$$\begin{cases} u_t^\varepsilon = (p_\varepsilon(x)u_x^\varepsilon)_x + c(x)u^\varepsilon + f(u^\varepsilon) \text{ on } (0,1) \\ \frac{\partial u_\varepsilon}{\partial \vec{n}_\varepsilon} + b(x)u^\varepsilon = g(u^\varepsilon) \text{ for } x \in \{0,1\} \end{cases} \qquad (\mathrm{P}_\varepsilon)$$

where $0 < \varepsilon \leq \varepsilon_0$ and $\frac{\partial u_\varepsilon}{\partial \vec{n}_\varepsilon} = p_\varepsilon(x)\langle u_x^\varepsilon, \vec{n}_\varepsilon \rangle$.

Physically, large diffusion will imply a rapid redistribution of the spatial inhomogeneities. For this reason, we expect that, for small values of ε the solution of the problem (P_ε) should be approximately constant on each $\Omega_{0,i}$. To obtain the limit equation, as ε goes to zero, suppose u^ε converges to some function u, in some sense, and that u takes a time dependent spatially constant value on $\Omega_{0,i}$ for each $i = 1, \dots, m$. Denote this value by $u_{\Omega_{0,i}}(t)$ for each $i = 1, \dots, m$. Then, the limit equation is given by

$$\begin{cases} u_t = (p(x)u_x)_x + c(x)u + f(u), \text{ in } \Omega_1 \\ \dot{u}_{\Omega_{0,i}}(t) = \frac{1}{b_i - a_i}[p(b_i)u_x(b_i^+, t) - p(a_i)u_x(a_i^-, t)] + \hat{c}_i u_{\Omega_{0,i}}(t) + f(u_{\Omega_{0,i}}(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, \dots, m \\ \frac{\partial u}{\partial \vec{n}} + b(x)u = g(u), \text{ for } x \in \{0,1\} \end{cases} \qquad (\mathrm{P}_0)$$

where $\hat{c}_i = \frac{1}{b_i - a_i}\int_{a_i}^{b_i} c(x)dx$ for each $i = 1, \dots, m$ and $\frac{\partial u}{\partial \vec{n}} = p(x)\langle u_x, \vec{n} \rangle$.

This type of equation, in the n-dimensional case, was extensively studied by Arrieta, Carvalho and Rodríguez-Bernal in [3]. We will use their notation here. Let

$$\mathrm{L}^2_{\Omega_0}(0,1) = \{u \in \mathrm{L}^2(0,1), u \text{ is constant on } \Omega_{0,i},\ i = 1, \dots, m\}$$

and

$$\mathrm{H}^1_{\Omega_0}(0,1) = \{u \in \mathrm{H}^1(0,1), u \text{ is constant on } \Omega_{0,i},\ i = 1, \dots, m\}.$$

Henry [5] and Angenent [2] showed that for one dimensional reaction-diffusion equations

$$\begin{aligned} &u_t = u_{xx} + f(x, u, u_x) \\ &\alpha u_x(0) + \beta u(0) = 0 \\ &\gamma u_x(1) + \delta u(1) = 0 \end{aligned}$$

the stable and unstable manifolds of hyperbolic equilibrium points intersect transversally for every reaction function f.

In this paper we show that this result is also true for equations, like (P_0), that are limit of parabolic problems when the diffusion satisfies (1.1).

The paper is organized as follows. In section 2 we present the Sturm-Liouville theory for equations of the form of (P_0). In the proofs we only point out to the differences of ours results when compared with the proofs of the usual Sturm-Liouville theory. We also

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prove that the number of zeros of the solution $u(\cdot,t)$ of (P_0) decreases as time increases. To obtain this we establish a minimum principle for equations of the type (P_0).

In section 3 we study the asymptotic behavior of solutions of the linearization of (P_0). We obtain the behavior of theses solutions in terms of the eigenvalues and eigenfunctions of the operator $L(t)$ that comes from this linearization of (P_0).

In section 4 we obtain a description of the tangent spaces of the stable and unstable manifolds involving the asymptotic behavior of some solutions of the linear equation of (P_0) and of the adjoint equation. In order to obtain this asymptotic behavior we will rely on some lemmas due to Henry [5]. Using the description of the tangent spaces of the stable and unstable manifolds of the hyperbolic equilibrium points we can prove that they intersect transversally.

We intend to use the transversality of these manifolds in a future work to obtain the topological equivalence of the flows on the attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ of equations (P_ε) and (P_0), respectively.

2. STURM-LIOUVILLE PROPERTIES AND NON INCREASE OF THE LAP NUMBER

In our proof of the transversality of the invariant manifolds, the main arguments will be the Sturm-Liouvile property obtained and the decreasing of numbers of zeros of the solution $u(\cdot,t)$ of (P_0) as time increases. Since we work with functions that are constant on intervals we use the following definition of a zero of a function.

DEFINITION 2.1. *If φ is a function defined in $(0,1)$, constant in each interval (a_i,b_i), $i=1,\dots,m$, we say that z is a zero of φ if $z\in\Omega_1$ and $\varphi(z)=0$ or if $z=[a_i,b_i]$, for some $i=1,\dots,m$ and $\varphi(x)=0$ for all $x\in[a_i,b_i]$.*

The following result shows that the operator that appears in equation (P_0) has the Sturm-Liouville properties. This result will used throughout the paper.

THEOREM 2.1. *Suppose p is a continuous real valued continuous function with $p(x)>0$ in $(0,1)$, differentiable in $\bar\Omega_1$ and g is a C^1-function of x and λ, decreasing for λ in $(-\infty,+\infty)$. Let*

$$D=\sup_{x\in\overline{\Omega}_1} p(x)>0,\qquad d=\inf_{x\in\overline{\Omega}_1} p(x)>0,$$
$$\eta(\lambda)=\sup_{x\in\overline{\Omega}_1} g(x,\lambda),\qquad \xi(\lambda)=\inf_{x\in\overline{\Omega}_1} g(x,\lambda).$$

Consider the equation,

$$\begin{cases} [(p(x)u_x)_x+g(x,\lambda)u]\,\chi_{\Omega_1}+ \\ \qquad\displaystyle\sum_{i=1}^{m}\left[\frac{1}{b_i-a_i}[p(b_i)u_x(b_i^+)-p(a_i)u_x(a_i^-)]+c_i(\lambda)u\right]\chi_{\Omega_{0,i}}=0 \\ \alpha u(0)-u_x(0)=0 \\ \beta u(1)+u_x(1)=0 \end{cases}\tag{2.2}$$

where $c_i(\lambda)$ is decreasing in $(-\infty, +\infty)$ for each $i = 1, \ldots, m$. Suppose also that

$(i)\alpha \neq 0$ *and* $\beta \neq 0$;
$(ii)\xi(\lambda) \to +\infty$ *as* $\lambda \to -\infty$;
$(iii)\eta(\lambda) \to -\infty$ *as* $\lambda \to +\infty$;
$(iv)c_i(\lambda) \to -\infty$ *as* $\lambda \to +\infty$, $i = 1, \ldots, m$.

Then there exist an infinite sequence of numbers $(\lambda_j)_{j\in\mathbb{N}}$ with

$$\lambda_1 > \lambda_2 > \ldots > \lambda_j > \ldots$$

such that if φ_j is the solution of (2.2) when $\lambda = \lambda_j$ then φ_j has exactly $(j-1)$ zeros in the interval $(0,1)$ for each $j = 1, 2, \ldots$

To prove this theorem we establish several lemmas extending preliminaries results of Sturm-Liouville theory to equations of type (P_0). We state these lemmas below and point out to the differences in the proofs when compared with the usual Sturm-Liouville theory.

Consider the following equations

$$\begin{cases} [(p_1(x)u_x)_x - g_1(x)u]\,\chi_{\Omega_1}(x) \\ \qquad\qquad + \displaystyle\sum_{i=1}^{m}\left[\frac{1}{b_i - a_i}[p_1(b_i)u_x(b_i^+) - p_1(a_i)u_x(a_i^-)] + c_i u\right]\chi_{\Omega_{0,i}}(x) = 0 \\ u(0) = \alpha_1,\ u_x(0) = \beta_1 \end{cases} \tag{2.3}$$

and

$$\begin{cases} [(p_2(x)v_x)_x - g_2(x)v)]\,\chi_{\Omega_1}(x) \\ \qquad\qquad + \displaystyle\sum_{i=1}^{m}\left[\frac{1}{b_i - a_i}[p_2(b_i)v_x(b_i^+) - p_2(a_i)v_x(a_i^-)] + d_i v\right]\chi_{\Omega_{0,i}}(x) = 0 \\ v(0) = \alpha_2,\ v_x(0) = \beta_2. \end{cases} \tag{2.4}$$

Let u and v be solutions of (2.3) and (2.4) respectively. The following lemma shows that the zeros of u and v alternate with each other.

LEMMA 2.1. *Let u and v be solutions respectively of (2.3) and (2.4) with $p_1 = p_2 = p$ where p, g_1, g_2 are real functions with $g_1(x) \geq g_2(x)$ continuous in $(0,1)$, $p > 0$ in $(0,1)$ is differentiable in Ω_1 and $c_i \leq d_i$ are constants for $i = 1, \ldots, m$. Then in each component of $Q = \{x \in (0,1) : u(x) \neq 0\}$ there exists one zero of v.*

Proof. Let (z_1, z_2) be a connected component of Q, that is $u(z_1) = u(z_2) = 0$ and $u(z) \neq 0$ for $x \in (z_1, z_2)$ and suppose that v does not have zeros in (z_1, z_2). Without loss of generality, we may assume that u and v are positive in the interval (z_1, z_2). Multiplying

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(2.3) by v, (2.4) by u and using the fact that p is differentiable in Ω_1, we obtain

$$\left[\frac{d}{dx}\left[p(x)(u_x v - u v_x)\right] + (g_2(x) - g_1(x))uv\right]\chi_{\Omega_1}(x)$$
$$+ \sum_{i=1}^{m}\Big[\frac{1}{b_i - a_i}\big[\big[p(b_i)u_x(b_i^+) - p(a_i)u_x(a_i^-)\big]\, v \qquad (2.5)$$
$$- \big[p(b_i)v_x(b_i^+) - p(a_i)v_x(a_i^-)\big]u\big] + (c_i - d_i)uv\Big]\chi_{\Omega_{0,i}}(x) = 0.$$

for $x \in (0,1)$.

We have two cases where there is some difference with the usual proof.

Case 1: $z_1 = [a_i, b_i]$ and $z_2 = [a_j, b_j]$ for some $i < j$, i, $j = 1, \ldots, m$.

Integrating (2.5) on (z_1, z_2) and using the fact that u and v are constant in each $\Omega_{0,k}$ for $k = i+1, \ldots, j-1$ we have

$$p(z_2)u_x(z_2)v(z_2) - p(z_1)u_x(z_1)v(z_1) = \int_{z_1}^{z_2}(g_1 - g_2)uv\chi_{\Omega_1}(x)dx$$
$$+ \int_{z_1}^{z_2}\sum_{k=i+1}^{j-1}(d_k - c_k)uv\chi_{\Omega_{0,k}}(x)dx.$$

Using the hypotheses on g_1, g_2, c_k and d_k it follows that the right hand side is positive, which is a contradiction.

Case 2: $z_1 = [a_i, b_i]$ for some $i = 1, \ldots, m$ and $z_2 \neq [a_j, b_j]$ for any $j = 1, \ldots, m$. This case can be treated in the same way as Case 1. ▮

The proof of the following lemma is very simple.

Lemma 2.2. *Let p_1, p_2, g_1, g_2 be real functions with g_1 and g_2 continuous and p_1 and p_2 positive in $(0,1)$. If u and v are solutions of (2.3) and (2.4), respectively, then for those $x \in (0,1)$ where $v(x) \neq 0$ we have*

$$\frac{d}{dx}\left(\frac{u}{v}(p_1 u_x v - p_2 u v_x)\right)\chi_{\Omega_1}(x)$$
$$+ \sum_{i=1}^{m}\left[(c_i - d_i)u^2 + \frac{1}{b_i - a_i}[[p_1(b_i)u_x(b_i^+) - p_1(a_i)u_x(a_i^-)]u\right.$$
$$\left. - [p_2(b_i)v_x(b_i^+) - p_2(a_i)v_x(a_i^-)]\frac{u^2}{v}]\right]\chi_{\Omega_{0,i}}(x)$$
$$= \left[(g_1 - g_2)u^2 + (p_1 - p_2)u_x^2 + p_2\left(u_x - \frac{uv_x}{v}\right)^2\right]\chi_{\Omega_1}(x). \qquad (2.6)$$

We now compare the distribution of zeros of solutions of equations (2.3) and (2.4).

Lemma 2.3. *Let p_1, p_2, g_1, g_2 be real valued functions with g_1 and g_2 continuous, p_1 and p_2 positive and satisfying in $(0,1)$, $p_1(x) \geq p_2(x) > 0$, $g_1(x) \geq g_2(x)$. Let $d_i \geq c_i$, $i = 1, \ldots, m$, be constants. Suppose*

$(i)\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$;
(ii)if $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$ then $\frac{p_1(0)\beta_1}{\alpha_1} \geq \frac{p_2(0)\beta_2}{\alpha_2}$;
$(iii)$the identity $g_1 \equiv g_2 \equiv 0$ does not hold in any subinterval of $(0,1)$.

Let u and v be solutions of (2.3) and (2.4) respectively. If u has n zeros in the interval $0 < x \leq 1$ then $v(x)$ has at lest n zeros in the same interval and the i^{th} zero of $v(x)$ is smaller then the i^{th} zero of $u(x)$.

Proof. Let $z_1, z_2, \ldots, z_n$ be the zeros of $u(x)$ in $(0,1]$ and suppose that $0 < z_1 < z_2 < \ldots < z_n \leq 1$. By Lemma 2.1 is sufficient to prove there exists a zero of v between 0 and z_1. If $u(0) = 0$ then by Lemma 2.1 v has a zero in $(0, z_1)$, and the result is proved. Suppose that $u(0) \neq 0$. If z_1 is not an interval the proof is the usual one.

If $z_1 = [a_i, b_i]$ for some $i = 1, \ldots, m$, integrating (2.6) and using the fact that u and v are constant in each $\Omega_{0,i}$, $i = 1, \ldots, m$, we have

$$- \alpha_1^2 \left[p_1(0)\frac{\beta_1}{\alpha_1} - p_2(0)\frac{\beta_2}{\alpha_2} \right] = \sum_{i=1}^{m} \int_0^{a_i} (d_i - c_i) u^2 \chi_{\Omega_{0,i}}(x) dx + \int_0^{a_i} \left[(g_1 - g_2)u^2 + (p_1 - p_2)u_x^2 + p_2 \left(u_x - u\frac{v_x}{v} \right)^2 \right] \chi_{\Omega_1}(x) dx.$$

But this contradicts (ii) since the left hand side is ≤ 0. ▌

Lemma 2.4. *Let p_1, p_2, g_1, g_2 and (i)-(iii) as in the Lemma 2.3, u and v solutions of (2.3) and (2.4), respectively. Let d be a point of $(0,1]$, $d \notin \Omega_0$ such that $u(d) \neq 0$ and $v(d) \neq 0$. If $u(x)$ and $v(x)$ have the same number of zeros in the interval $(0,d)$ then*

$$p_1(d)\frac{u_x(d)}{u(d)} > p_2(d)\frac{v_x(d)}{v(d)}.$$

Proof. Let l be the number of zeros of $u(x)$ and $v(x)$ in $(0,d)$. Suppose that $l > 1$. If z_l is the largest zero of $u(x)$ which is less then d then v has exactly l zeros in the interval $(0, z_l)$ and no zeros in (z_l, d). Since $d \in \Omega_1$ then $d \in (b_j, a_{j+1})$ for some $j = 0, \ldots, m$. We consider the following possibilities: (i) $z_l \in (b_j, d)$; (ii) $z_l \in (b_i, a_{i+1}) \subset \Omega_1$ for some $i = 0, 1, \ldots, j-1$ and (iii) $z_l = [a_i, b_i]$ for some $i < j$. In all cases we integrate (2.6) obtained in Lemma 2.2, and use in (ii) and (iii) that the solutions u and v are constant in each $\Omega_{0,k}$, $k = i + 1, \ldots, j$. ▌

To obtain the decreasing of the numbers of zeros of the solution $u(\cdot, t)$ of (P_0) as time increases we need a minimum principle for equations of the type (P_0) and for this we need the following lemmas.

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LEMMA 2.5. *Let E be a region of $[0,1]\times(0,+\infty)$. Suppose that u is constant in each interval (a_i,b_i), $i=1,\dots,m$ and $Su\le 0$ where S is given by*

$$\begin{aligned}Su(x,t) = {} & [p(x)u_{xx}+r(x)u_x+q(x,t)u]\chi_{\Omega_1}(x)\\ & +\sum_{i=1}^{m}[\frac{1}{b_i-a_i}[p(b_i)u_x(b_i^+,t)-p(a_i)u_x(a_i^-,t)]+h_i(t)u]\chi_{\Omega_{0,i}}(x)-u_t(x,t)\end{aligned}$$

for (x,t) in E, $q(x,t)<0$ and $h_i(t)<0$, for $i=1,\dots,m$. Let K be a disk such that it and its boundary ∂K are contained in E. Suppose that the minimum of u in E is $M\le 0$, $u>M$ in the interior of K, and $u=M$ at some point P on the boundary of K. Then the tangent to K at P is parallel to the x-axis.

Proof. Let $K=K_R(\bar{x},\bar{t})$, where $(\bar{x},\bar{t})$ denotes the center and R the radius of K. Suppose that the tangent to K at P is not parallel to the x-axis. We assume that P is the only boundary point where $u=M$. Otherwise, we may replace K by K', with $K'\subset K$ and $\partial K\cap\partial K'=\{P\}$. Let $P=(x_*,t_*)$. We have two cases to consider: $x_*\in\Omega_1$ or $x_*\in\Omega_0$.

Case 1: $x_*\in\Omega_1$

We take K such that if $(x,t)\in\partial K$ then x and x_* are in the same interval of the Ω_1. Construct a disk K_1 with center in P and radius R_1 sufficiently small, such that $R_1<|x_*-\bar{x}|$. Decrease R_1 sufficiently, such that if $(x,t)\in\partial K_1$ then x is in the same interval of Ω_1 that contains x_*.

The boundary of K_1 consists of the two arcs, $C'=\partial K_1\cap(K\cup\partial K)$ and $C''=\partial K_1\setminus C'$.

There exists a positive constant η such that $u\ge M+\eta$ on C' and $u\ge M$ on C''. In $(0,1)\times(0,\infty)$ consider the function

$$v(x,t)=e^{-\alpha[(x-\bar{x})^2+(t-\bar{t})^2]}-e^{-\alpha R^2},$$

where $\alpha>0$ is sufficiently large such that

$$\begin{aligned}Sv(x,t) = {} & [4\alpha^2p(x)(x-\bar{x})^2-2\alpha p(x)-2\alpha r(x)(x-\bar{x})\\ & +q(x,t)[1-e^{-\alpha(R^2-(x-\bar{x})^2-(t-\bar{t})^2)}]+2\alpha(t-\bar{t})]e^{-\alpha[(x-\bar{x})^2+(t-\bar{t})^2]}>0\end{aligned}$$

for $(x,t)\in K_1\cup\partial K_1$. This is possible since $|x-\bar{x}|\ge|x_*-\bar{x}|-R_1>0$.

Consider $w=u-\varepsilon v$, where ε is a positive constant. Then $Sw(x,t)=Su(x,t)-\varepsilon Sv(x,t)<0$, for (x,t) in $K_1\cup\partial K_1$. Choose ε sufficiently small such that $w=u-\varepsilon v\ge M+\eta-\varepsilon v>M$ on C' and $w=u-\varepsilon v\ge M-\varepsilon v>M$ on C''. Then $w>M$ on $\partial K_1=C'\cup C''$.

Since $w(x_*,t_*)=u(x_*,t_*)=M$, the minimum of w in K_1 occurs in a interior point of K_1. Let (x',t') be this point; then $w_x(x',t')=w_t(x',t')=0$ and $Sw(x',t')\ge 0$, which is a contradiction.

Case 2: $x_*\in\Omega_0$, that is $x_*\in(a_j,b_j)$ for some $j=1,\dots,m$.

If the tangent to K at P is not parallel to the x-axis since u is constant in (a_j,b_j), we have $u(x',t_*)=M$ where (x',t_*) is a point in interior of K.

Note that if $x_* = a_j$ or $x_* = b_j$, for some $j = 1, \ldots, m$ then, as before, we can replace K by $\tilde{K}$ such that $\partial\tilde{K} \subset (a_j, b_j)$ with $\partial K \cap \partial\tilde{K} = \{P\}$ and we can proceed accordingly. ▮

The proof of the following lemma can be found in [6].

Lemma 2.6. *Let E, S, u and M as in Lemma 2.5. Suppose that $u > M$ in some interior point (x_0, t_0) of E. If l is any horizontal segment in the interior of E which contains (x_0, t_0) then $u > M$ in l.*

Lemma 2.7. *Consider E and S as in the Lemma 2.5. Let u be constant in each interval (a_i, b_i), $i = 1, \ldots, m$, and $Su \le 0$. Suppose that $u > M$ in the horizontal strip in E, $\{(x,t) \in E : t_0 < t < t_1\}$ with t_0 and t_1 fixed. Then $u > M$ on all line segment $\{(x,t) \in E : t = t_1\}$.*

Proof. Suppose $P = (x_1, t_1)$ is on the line $t = t_1$ where $u = M$. Construct a disk K with center at P and radius sufficiently small such that the lower half of K is entirely in the part of E where $t > t_0$. We have two cases to consider: $x_1 \in \Omega_1$ or $x_1 \in \Omega_0$.

Case 1: $x_1 \in \Omega_1$

Take K such that, if $(x,t) \in \partial K$ then x is in same interval of Ω_1 that contains x_1. In $(0,1) \times (0,\infty)$, define the function $v(x,t) = e^{-[(x-x_1)^2 + \alpha(t-t_1)]} - 1$, where α is a positive constant chosen sufficiently large so that

$$Sv(x,t) = \Big[4p(x)(x-x_1)^2 - 2p(x) - 2r(x)(x-x_1) + q(x,t)[1 - e^{(x-x_1)^2 + \alpha(t-t_1)}] + \alpha\Big] e^{-[(x-x_1)^2 + \alpha(t-t_1)]} > 0$$

when $(x,t) \in K$ and $t \le t_1$.

The parabola

$$(x - x_1)^2 + \alpha(t - t_1) = 0 \qquad (2.7)$$

is tangent to the line $t = t_1$ at $P = (x_1, t_1)$. Denote by C' the arc in ∂K, including the endpoints, which is below the parabola (2.7) and by C'' the arc of the parabola inside the disk K. Let D be the region enclosed by C' and C''. There exists a positive constant $\eta > 0$ such that $u \ge M + \eta$ on C'. Notice that $v = 0$ on C''.

Consider now the function $w = u - \varepsilon v$, where ε is a positive constant sufficiently small, such that w has the following properties: $(i) Sw = Su - \varepsilon Sv < 0$ in D; $(ii) w = u - \varepsilon v \ge M + \eta - \varepsilon v > M$, on C' and $(iii) w \ge M$, in $\bar{D}$. Remark that $w = u \ge M$ on C''.

Condition (i) shows that w can not attain its minimum on D. From conditions (ii) and (iii) it follows that (x_1, t_1) is a minimum point of w in $\bar{D}$. Therefore $w_t(x_1, t_1) \le 0$. As $v_t(x_1, t_1) = -\alpha < 0$, we obtain $u_t(x_1, t_1) = w_t(x_1, t_1) - \alpha\varepsilon < 0$.

Since $x = x_1$ is a minimizing point of $u(\cdot, t_1)$ we have $u_x(x_1, t_1) = 0$ and $u_{xx}(x_1, t_1) \ge 0$. Then $Su(x_1, t_1) > 0$, which contradicts the hypothesis $Su \le 0$.

Case 2: $x_1 \in \Omega_0$, that is, $x_1 \in (a_j, b_j)$ for some $j = 1, \dots, m$.

Take K so that if $(x,t) \in \partial K$ then $x \in (a_j, b_j)$ and let $t_0 < t_2 < t_1$ such that $C' = \{(x,t) \in \partial K : t \le t_2\}$ is in the boundary of K, $C'' = \{(x,t) \in K \cup \partial K; t = t_1\} \cup \{(x,t) \in \partial K; t_2 < t < t_1\}$ and D is the region between C' and C''.

There exists a positive constant η such that $u \ge M + \eta$ on C'. In $(0,1) \times (0,\infty)$ define the function $v(x,t) = e^{-\alpha(t-t_1)} - 1$, where the positive constant α is taken sufficiently large, so that

$$Sv(x,t) = \Big(h_j(t)(1 - e^{\alpha(t-t_1)}) + \alpha\Big)\, e^{-\alpha(t-t_1)} > 0$$

for (x,t) in $K \cup \partial K$ and $t \le t_1$.

Consider $w = u - \varepsilon v$, where we choose $\varepsilon > 0$ sufficiently small to have $(i) Sw = Su - \varepsilon Sv < 0$ in $K \cup \partial K$ and $(ii) w = u - \varepsilon v \ge M + \eta - \varepsilon v > M$ in C'.

Let (x', t') be the minimum point of w in $D \cup C' \cup C''$; then (x',t') is in $I = \{(x,t) \in K \cup \partial K : t = t_1\}$ or in $I' = \{(x,t) \in K \cup \partial K : t_2 < t < t_1\}$.

Since w is constant in (a_j, b_j) we have $w_x(a_j^-, t') \le 0$, $w_x(b_j^+, t') \ge 0$, $w_t(x', t') \le 0$ and $w(x',t') \le M \le 0$. Then $Sw(x',t') \ge 0$, contradicting (i).

If $x_1 = a_j$ or $x_1 = b_j$, for some $j = 1, \dots, m$ we have $u(x, t_1) = M$ for all $x \in [a_j, b_j]$. In this case we consider $\tilde{x} \in (a_j, b_j)$ and repeat the argument above for $P = (\tilde{x}, t_1)$. ▮

Theorem 2.2. *Consider the region E and the operator S as in Lemma 2.5. If the minimum M of u is attained in an interior point (x_*, t_*) of E and $M \le 0$ then $u \equiv M$ on each line segment $\{(x,t) \in E : t = \bar{t}\}$ for every $\bar{t} < t_*$.*

Proof. Suppose there exist $\bar{t} < t_*$ and $\bar{x} \in (0,1)$ such that $u(\bar{x}, \bar{t}) > M$. Then by Lemma 2.6, $u(x_*, \bar{t}) > M$. Let $\tau = \sup\{t < t_* : u(x_*, t) > M\}$. By continuity $u(x_*, \tau) = M$. Let τ_1 such that $u(x_*, t) > M$ for some interval $\tau_1 < t < \tau$.

As $u(x_*, t) > M$ for $\tau_1 < t < \tau$, using Lemma 2.6 we get $u(x,t) > M$ for $\tau_1 < t < \tau$. From Lemma 2.7, we have $u(x, \tau) > M$ in E and in particular we obtain the contradiction $u(x_*, \tau) > M$. ▮

Our next theorem shows that the number of zeros of the solutions of the linearizations of equation (P_0) decreases with time.

Theorem 2.3. *Let $r, p : (0,1) \to \mathbb{R}$ be continuous in Ω_1; $p > 0$ and differentiable in Ω_1; $h_i : [t_0, t_1] \to \mathbb{R}$, $i = 1, \dots, m$; $q, w : [0,1] \times [t_0, t_1] \to \mathbb{R}$, bounded, w continuous, $w \not\equiv 0$, w_x continuous in $[0,1] \times (t_0, t_1]$, w_t, w_{xx} continuous in $(0,1) \times (t_0, t_1]$ and $w(\cdot, t)$ constant in each $\Omega_{0,i}$ $i = 1, \dots, m$ and satisfying*

$$\begin{aligned} w_t = {} & [p(x) w_{xx} + r(x) w_x + q(x,t) w]\chi_{\Omega_1} \\ & + \sum_{i=1}^{m} [\frac{1}{b_i - a_i}[p(b_i) w_x(b_i^+, t) - p(a_i) w_x(a_i^-, t)] + h_i(t) w]\chi_{\Omega_{0,i}} \end{aligned} \tag{2.8}$$

in $(0,1)\times(t_0,t_1]$. *Suppose also that* $\beta_0(t)$ *and* $\beta_1(t)$ *are continuous in* $[t_0,t_1]$ *and*

$$\begin{aligned} &w(0,t)(p(0)w_x(0,t)+\beta_0(t)w(0,t))\geq 0\\ &w(1,t)(p(1)w_x(1,t)+\beta_1(t)w(1,t))\leq 0. \end{aligned} \tag{2.9}$$

Then the number of components of the set $\{x, 0<x<1; w(x,t)\neq 0\}$ *decreases in* $t_0\leq t\leq t_1$.

Proof. We can assume, without loss of generality that $q(x,t)<0$, $h_i(t)<0$, for $i=1,\dots,m$, $\beta_0(t)=-1$ and $\beta_1(t)=1$. If not we can consider $\tilde{w}(x,t)=w(x,t)e^{\varphi(x,t)-\lambda t}$, where $\lambda>0$ is a constant to be chosen, $\varphi(x,t)$ is a smooth function satisfying $\varphi_x(0,t)\geq\frac{1}{p(0)}[\beta_0(t)+1]$ and $\varphi_x(1,t)\leq\frac{1}{p(1)}[\beta_1(t)-1]$. Then $\tilde{w}$ satisfy

$$\begin{aligned} \tilde{w}_t = {} & [p(x)\tilde{w}_{xx}+\tilde{r}(x)\tilde{w}_x+\tilde{q}(x,t)\tilde{w}]\chi_{\Omega_1}\\ & +\sum_{i=1}^{m}\left[\frac{1}{b_i-a_i}[p(b_i)\tilde{w}_x(b_i^+,t)-p(a_i)\tilde{w}_x(a_i^-,t)]+\tilde{h}_i(t)]\tilde{w}\right]\chi_{\Omega_{0,i}} \end{aligned}$$

and

$$\begin{aligned} \tilde{w}(0,t)(p(0)\tilde{w}_x(0,t)-\tilde{w}(0,t))\geq 0\\ \tilde{w}(1,t)(p(1)\tilde{w}_x(1,t)+\tilde{w}(1,t))\leq 0, \end{aligned}$$

where $\tilde{r}(x)=r(x)-2p(x)\varphi_x$, $\tilde{q}(x,t)=q(x,t)-p(x)\varphi_{xx}+p(x)\varphi_x^2-r(x)\varphi_x+\varphi_t-\lambda$ and $\tilde{h}_i(t)=h_i(t)-\frac{1}{b_i-a_i}[p(b_i)\varphi_x(b_i,t)-p(a_i)\varphi_x(a_i,t)]+\varphi_t-\lambda$, $i=1,\dots,m$. Take λ sufficiently large such that $\tilde{q}(x,t)<0$ and $\tilde{h}_i(t)<0$, $i=1,\dots,m$.

Let $Q(t)=\{x\in(0,1);\ w(x,t)\neq 0\}$ for $t_0\leq t\leq t_1$. We will show that $Q(t_0)$ has at least as many components as $Q(t_1)$.

Let σ be a component of $Q(t_1)$ in $(0,1)$ and S_σ the component of

$$((0,1)\times[t_0,t_1])\cap\{(x,t):w(x,t)\neq 0\}$$

containing σ. We will prove that (i) $S_\sigma\cap\{(x,t_0):0<x<1\}\neq\emptyset$ and (ii) if σ and σ_1 are disjoint components of $Q(t_1)$ then $S_\sigma\cap S_{\sigma_1}=\emptyset$.

(i) Suppose that $S_\sigma\cap\{(x,t_0):0<x<1\}=\emptyset$. We may assume $w(x,t)>0$ on S_σ, so

$$M=\max\{w(x,t):(x,t)\in\bar{S_\sigma}\}>0.$$

Suppose the maximum occurs at $(x',t')\in\bar{S_\sigma}$. Let

$$N_\varepsilon(x',t')=D_\varepsilon(x',t')\cap((0,1)\times[t_0,t_1]),$$

where $D_\varepsilon(x',t')$ is a disk with center in (x',t') and radius ε. Then N_ε is convex, $N_\varepsilon\cap S_\sigma\neq\emptyset$ and if ε is small $w>0$ in N_ε. Thus $N_\varepsilon\subset S_\sigma$, for ε sufficiently small.

Since $S_\sigma \cap \{(x,t_0) : 0 < x < 1\} = \emptyset$ we have $t' > t_0$. If $t_0 < t' \leq t_1$ and $0 < x' < 1$ then $w_x(x',t') = 0$, $w_{xx}(x',t') \leq 0$, $w_t(x',t') \geq 0$ and $w(x',t') = M > 0$.

It follows from (2.8) that

(i) If $x' \in \Omega_1$ then $w_t(x',t') = p(x')w_{xx}(x',t') + q(x',t')M$ which is a contradiction, since the first member is greater then zero and the second member is strictly negative.

(ii) If $x' \in \Omega_0$ then $x' \in \Omega_{0,j}$ for some $j = 1, \ldots, m$ and

$$w_t(x',t') = \frac{1}{b_j - a_j}[p(b_j)w_x(b_j^+,t') - p(a_j)w_x(a_j^-,t')] + h_j(t')M$$

since (x',t') is the maximum point. Then $w_x(a_j^-,t') \geq 0$ and $w_x(b_j,t') \leq 0$ and as in (i) we get a contradiction.

If $t_0 < t' \leq t_1$, $x' = 0$ then $(x,t') \in S_\sigma$ for $x > 0$ small, and $w(0,t') = M > 0$, from (2.9) we get $w_x(0,t') \geq \frac{M}{p(0)} > 0$ and M is not the maximum. Similarly, if $x' = 1$ we have $(x,t') \in S_\sigma$ for small $1 - x > 0$ and $w(1,t') = M > 0$ and again from (2.9) we obtain $w_x(1,t') \leq \frac{-M}{p(1)} < 0$ and M is not the maximum. Therefore $S_\sigma \cap \{(x, t = t_0) : 0 < x < 1\} \neq \emptyset$.

To prove (ii) let σ and σ_1 be disjoint components of $Q(t_1)$. We shall prove that $S_\sigma \cap S_{\sigma_1} = \emptyset$. Suppose $S_\sigma \cap S_{\sigma_1} \neq \emptyset$. Then $S_\sigma = S_{\sigma_1}$ and we assume, as before, $w > 0$ in S_σ. The set

$$S_\sigma \cap \{(x,t); 0 < x < 1, t_0 < t < t_1\}$$

is open and connected, hence path connected. There is a simple path γ in $S_\sigma \cap \{(x,t) : 0 < x < 1,\ t_0 < t < t_1\}$ with one initial point in $\sigma \cap \{t = t_1\}$ and end point in $\sigma_1 \cap \{t = t_1\}$. Adding a line segment in $\{(x, t = t_1) : 0 < x < 1\}$ joining these points, we obtain simple closed curve bounding a region E.

Note that in any interval in $\{(x,t_1) : 0 < x < 1\}$ that intersects σ and σ_1 we must have $w(x,t_1) = 0$ at some point $x \in (0,1)$. On $\partial E \cap \{(x,t) : 0 < x < 1, t < t_1\}$ and at the end points of the $\partial E \cap \{(x, t_1 : 0 < x < 1)\}$ we have $w > 0$.

It follows from Theorem 2.2 that $w > 0$ in $\bar{E}$. In particular $w(x,t_1) > 0$ in any interval joining σ and σ_1, and this is a contradiction. ▌

3. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

Let $p(x)$, $q(x,t)$, $h_i(t)$, $i = 1, \ldots, m$, $\gamma_0(t)$ and $\gamma_1(t)$ be real valued functions. Suppose $p(x) > 0$ and differentiable in Ω_1; $h_i(t)$, $i = 1, \ldots, m$, $q(x,t)$, $\gamma_0(t)$ and $\gamma_1(t)$ and their first t-derivatives are bounded and continuous on $-\infty < t < \infty$, $0 \leq x \leq 1$ and $\dot{\gamma_0}(t)$ and $\dot{\gamma_1}(t)$ are locally Hölder continuous. Let

$$\mathcal{D}(L(t)) = \{u \in H^1_{\Omega_0}(0,1) : (p(x)u_x)_x \in L^2_{\Omega_0}(0,1), p(0)u_x(0) = \gamma_0(t)u(0), \\ p(1)u_x(1) = \gamma_1(t)u(1)\},$$

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and define $L(t) : \mathcal{D}(L(t)) \to L^2_{\Omega_0}(0,1)$ by

$$(L(t)v)(x) = [(p(x)v_x(x))_x + q(x,t)v(x)]\chi_{\Omega_1}(x) \\ + \sum_{i=1}^{m}[\frac{1}{b_i - a_i}[p(b_i)v_x(b_i^+) - p(a_i)v_x(a_i^-)] + h_i(t)v(x)]\chi_{\Omega_{0,i}}(x). \quad (3.10)$$

Let

$$\mathcal{D}(L_\pm) = \{u \in H^1_{\Omega_0}(0,1) : (p(x)u_x)_x \in L^2_{\Omega_0}(0,1), p(0)u_x(0) = \gamma_0^\pm u(0), \\ p(1)u_x(1) = \gamma_1^\pm u(1)\},$$

and define $L_\pm : \mathcal{D}(L_\pm) \to L^2_{\Omega_0}(0,1)$ by

$$(L_\pm v)(x) = [(p(x)v_x(x))_x + q_\pm(x)v(x)]\chi_{\Omega_1}(x) \\ + \sum_{i=1}^{m}[\frac{1}{b_i - a_i}[p(b_i)v_x(b_i^+) - p(a_i)v_x(a_i^-)] + h_{i_\pm}v(x)]\chi_{\Omega_{0,i}}(x), \quad (3.11)$$

where $q_\pm(x) = \lim_{t\to\pm\infty} q(x,t)$, $h_{i_\pm} = \lim_{t\to\pm\infty} h_i(t)$, $\gamma_0^\pm = \lim_{t\to\pm\infty}\gamma_0(t)$ and $\gamma_1^\pm = \lim_{t\to\pm\infty}\gamma_1(t)$.

The operators $-L(t)$, for each fixed t, and $-L_\pm$ are sectorial operators. Let $\lambda_j(t)$ be the j^{th} eigenvalue of $L(t)$, with $\lambda_j(t) > \lambda_{j+1}(t)$ and $\lambda_j^\pm$ be the j^{th} eigenvalue of the $L_\pm$. Notice that $\lambda_j(t) \to \lambda_j^\pm$ as $t \to \pm\infty$. Let $\psi_j^\pm$ be the j^{th} eigenfunction of the $L_\pm$.

THEOREM 3.4. *Suppose the hypotheses above about $p(x)$, $q(x,t)$, $h_i(t)$, $i = 1, \dots, m$, $\gamma_0(t)$ and $\gamma_1(t)$ are all satisfied. Suppose also that*

$$\int_{-\infty}^{\infty}\left[[\sup_{0\le x\le 1}(|q_t(x,t)|)]^2 + \sum_{i=1}^{m}|\dot{h}_i(t)|^2 d + \sup_{t\le s\le t+1}(|\dot{\gamma}_1(s)| + |\dot{\gamma}_0(s)|)\right] dt < \infty$$

and

$$\sum_{n=0}^{\infty}\sup_{n\le s\le n+1}(|\dot{\gamma}_1(s)| + |\dot{\gamma}_0(s)|) < \infty.$$

Then there are classical solutions $v_j^\pm(x,t)$ of the problem

$$\begin{cases} v_t = (p(x)v_x)_x + q(x,t)v, \ x \in \Omega_1, \ t \in (-\infty,\infty) \\ \dot{v}_{\Omega_{0,i}}(t) = \frac{1}{b_i - a_i}[p(b_i)v_x(b_i^+,t) - p(a_i)v_x(a_i^-,t)] + h_i(t)v_{\Omega_{0,i}}(t), \ i = 1,\dots,m \\ p(0)v_x(0,t) = \gamma_0(t)v(0,t) \\ p(1)v_x(1,t) = \gamma_1(t)v(1,t) \end{cases} \quad (3.12)$$

with v_j^- *defined on* $-\infty < t < \infty$ *and* v_j^+ *defined for* $t < t_j$, *for some* t_j *such that*

$$v_j^+(x,t) = \exp\left(\int_0^t \lambda_j(s)\,ds\right)\left[\psi_j^+(x) + \mathrm{o}(1)\right], \text{ as } t \to +\infty \tag{3.13}$$

and

$$v_j^-(x,t) = \exp\left(-\int_t^0 \lambda_j(s)\,ds\right)\left[\psi_j^-(x) + \mathrm{o}(1)\right], \text{ as } t \to -\infty \tag{3.14}$$

with convergence in $\mathrm{C}^1(\Omega_1)$.

Proof. We consider the case $t \to +\infty$; the case $t \to -\infty$ can be treated in a similar way. We change variables to have the boundary conditions independent of t. Consider $\phi(\cdot,t) \in \mathrm{H}^1_{\Omega_0}(0,1)$,

$$\phi(x,t) = \phi_1(x)(\gamma_1(t) - \gamma_1^+) + \phi_0(x)(\gamma_0(t) - \gamma_0^+)$$

where $\phi_1, \phi_0 \in \mathrm{H}^1_{\Omega_0}(0,1)$ are such that $\phi_1'(0) = 0, \phi_1'(1) = 1$, $\phi_0'(0) = 1$ and $\phi_0'(1) = 0$.

Let v be a solution of (3.12) ans let $v(x,t) = e^{\phi(x,t)}w(x,t)$. Then w satisfies

$$w_t = e^{-\phi(\cdot,t)}(L(t)(e^{\phi(\cdot,t)}w)) - \phi_t w,$$

with boundary conditions $p(0)w_x(0) = \gamma_0^+ w(0)$ and $p(1)w_x(1) = \gamma_1^+(1)$. Defining $\Lambda(t)w = e^{-\phi(\cdot,t)}(L(t)(e^{\phi(\cdot,t)}w))$ we get

$$w_t = (\Lambda(t) - \phi_t)w, \tag{3.15}$$

and $\Lambda(t)$ has the same spectrum as $L(t)$. Note that $\phi(x,t) \to 0$ and $\Lambda(t)u \to L_+u$ as $t \to +\infty$ for each $u \in \mathrm{H}^1_{\Omega_0}(0,1)$.

Let $\{T(t,s), t \geq s\}$ be the evolution operators for equation (3.15), in the sense of Theorem 7.1.3 in Henry [4], so that $w(t,\cdot) = T(t,s)w(s,\cdot)$ when w solves the equation in the interval $[s,t]$.

Consider $-L_+$ as the principal operator and define $u \in \mathcal{D}(L_+) \to \mathrm{P}u \in \mathrm{H}^2(\Omega_1)$ by $\mathrm{P}u = u|_{\Omega_1}$. Then $\mathcal{D}(L_+) \hookrightarrow \mathrm{H}^2(\Omega_1)$, $\mathrm{L}^2_{\Omega_0}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega_1)$ and by interpolation $X^\alpha = \mathcal{D}((L_+)^\alpha) \hookrightarrow \mathrm{H}^{2\alpha}(\Omega_1)$ for $0 \leq \alpha < 1$, where $\|\psi\|_\alpha = \|(-L_+)^\alpha\psi\|_{L^2_{\Omega_0}}$. So convergence in $\|\cdot\|_\alpha$ implies convergence in $\mathrm{C}^1(\Omega_1)$ for $\alpha > 3/4$.

Furthermore, $\|\cdot\|_{1/2}$ is equivalent to the $\mathrm{H}^1_{\Omega_0}(0,1)$ norm. Indeed,

$$\begin{aligned}\|\psi\|^2_{1/2} &= \|(-L_+)^{1/2}\psi\|^2_{L^2_{\Omega_0}} = \langle(-L_+)\psi,\psi\rangle_{L^2_{\Omega_0}} \\ &= \int_{\Omega_1} p(x)\psi_x^2\,dx + \int_0^1 \left[-q_+(x)\chi_{\Omega_1}(x) - \sum_{i=1}^m h_{i_+}\chi_{\Omega_{0,i}}(x)\right]\psi^2(x)\,dx \\ &\qquad + \gamma_0\psi^2(0) - \gamma_1\psi^2(1).\end{aligned}$$

Since $\mathrm{H}^1(0,1) \hookrightarrow \mathrm{C}^0([0,1])$, it follows that $\|\cdot\|_{1/2}$ is equivalent to the $\mathrm{H}^1_{\Omega_0}(0,1)$ norm.

Using Theorem 7.1.3 (c) in Henry [4] and from the equivalence of the norms $\|\cdot\|_{1/2}$ and $\|\cdot\|_{\mathrm{H}^1_{\Omega_0}}$ there exists $\bar{c}$, depending on $-L_+$ and $(\sup_{0\le x\le 1}|q(x,t)|)^2 + \sum_{i=1}^m |h_i(t)|^2$, such that

$$\|T(t,s)\psi\|_{L^2_{\Omega_0}} \le \bar{c}\|\psi\|_{L^2_{\Omega_0}} \tag{3.16}$$

$$\|T(t,s)\psi\|_{H^1_{\Omega_0}} \le \bar{c}(t-s)^{-1/2}\|\psi\|_{L^2_{\Omega_0}} \tag{3.17}$$

for $0 < t - s \le 1$.

Fix τ and consider t so that $0 < t - \tau \le 1$. Write (3.15) as

$$\dot{w} = [\Lambda(\tau) + (\Lambda(t) - \Lambda(\tau) - \phi_t)]w.$$

From the variation of constants formula we get

$$T(t,\tau)\psi = e^{\Lambda(\tau)(t-\tau)}\psi + \int_\tau^t e^{\Lambda(\tau)(t-s)}\Big(\int_\tau^s \dot{\Lambda}(r)\,dr - \phi_t(s,\cdot)\Big)T(s,\tau)\psi\,ds,$$

where

$$\dot{\Lambda}(r)u = [\tilde{q}_r(x,r)u(x) + \eta_r(x,r)u_x(x)]\chi_{\Omega_1}(x) + \sum_{i=1}^m \dot{\tilde{h}}_i(r)u(x)\chi_{\Omega_{0,i}}(x)$$

with,

$$\begin{aligned}
\tilde{q}(x,t) &= q(x,t) + (p(x)\phi_x(x,t))_x + p(x)\phi_x^2(x,t)\\
\eta(x,t) &= 2p(x)\phi_x(x,t)\\
\tilde{h}_i(t) &= h_i(t) + \frac{1}{b_i - a_i}[p(b_i)\phi_x(b_i^+,t) - p(a_i)\phi_x(a_i^-,t)].
\end{aligned}$$

If $0 \le \alpha < 1$,

$$\begin{aligned}
\|T(t,\tau)\psi - e^{\Lambda(\tau)(t-\tau)}\psi\|_\alpha \le \int_\tau^t \|(c-L_+)^\alpha e^{\Lambda(\tau)(t-s)}\|_{\mathcal{L}(L^2_{\Omega_0})}\times&\\
\int_\tau^s \|\dot{\Lambda}(r)\,dr\|_{\mathcal{L}(H^1_{\Omega_0},L^2_{\Omega_0})}\|T(s,\tau)\psi\|_{H^1_{\Omega_0}}\,ds&\\
+\int_\tau^t \|(c-L_+)e^{\Lambda(\tau)(t-s)}\|_{\mathcal{L}(L^2_{\Omega_0})}\,\|\phi_t(s,\cdot)T(s,\tau)\psi\|_{L^2_{\Omega_0}}\,ds.&
\end{aligned}$$

Also,

$$\begin{aligned}
\|\dot{\Lambda}(r)u\|^2_{L^2_{\Omega_0}} &\le (2m)^2\left[\sup_{0\le x\le 1}|\tilde{q}_r(x,r)| + \sup_{0\le x\le 1}|\eta_r(x,r)| + \sum_{i=1}^m |\dot{\tilde{h}}_i(r)|\right]^2 \|u\|^2_{\mathrm{H}^1_{\Omega_0}}\\
&\le K_1^2\varepsilon_1(r)^2\|u\|^2_{\mathrm{H}^1_{\Omega_0}}
\end{aligned}$$

where

$$
\begin{aligned}
K_1 = 2m \max\Bigg\{ & 1, \sup_{0\le x\le 1} |(p(x)\phi_1'(x))'| + \sup_{0\le x\le 1} |(p(x)\phi_0'(x))'| \\
& + 2\Bigg(\sup_{0\le x\le 1} |p(x)\phi_1'(x)| + \sup_{0\le x\le 1} |p(x)\phi_0'(x)| \Bigg) \Bigg[(\sup_{0\le x\le 1} |\phi_0'(x)| \\
& + \sup_{0\le x\le 1} |\phi_1'(x)|)(\sup_{\tau\le r\le \tau+1} |\gamma_1(r)-\gamma_1^+| + \sup_{\tau\le r\le\tau+1} |\gamma_0(r)-\gamma_0^+|) + 1 \Bigg] \\
& + \sum_{i=1}^m \frac{1}{b_i-a_i} |p(b_i)\phi_1'(b_i^+) - p(a_i)\phi_1'(a_i)| \\
& + \sum_{i=1}^m \frac{1}{b_i-a_i} |p(b_i)\phi_0'(b_i^+) - p(a_i)\phi_0'(a_i)| \Bigg\}
\end{aligned}
$$

and,

$$
\varepsilon_1(r) = \sup_{0\le x\le 1} |q_r(x,r)| + \sum_{i=1}^m |\dot h_i(r)| + \sup_{r\le s\le r+1} (|\dot\gamma_1(s)| + |\dot\gamma_0(s)|).
$$

Then

$$
\| \int_\tau^s \dot\Lambda(r)dr \|_{\mathcal{L}(\mathrm{H}^1_{\Omega_0},\mathrm{L}^2_{\Omega_0})} \le \int_\tau^t \|\dot\Lambda(r)\|_{\mathcal{L}(\mathrm{H}^1_{\Omega_0},\mathrm{L}^2_{\Omega_0})} \le K_1 \int_\tau^t \varepsilon_1(r)dr.
$$

From the above estimate, together with (3.16) and (3.17), it follows that

$$
\begin{aligned}
\|T(t,\tau)\psi - e^{\Lambda(\tau)(t-\tau)}\psi\|_\alpha \le \bar c c_\alpha K_1 \int_\tau^t \varepsilon_1(r)dr \int_\tau^t (t-s)^{-\alpha}(s-\tau)^{-1/2} ds \|\psi\|_{\mathrm{L}^2_{\Omega_0}} \\
+ \bar c c_\alpha \int_\tau^t (t-s)^{-\alpha} \sup_{x\in[0,1]} |\phi_t(x,s)| ds \|\psi\|_{\mathrm{L}^2_{\Omega_0}} \\
\le \bar c c_\alpha K_1 \mathbf{B}(1/2, 1-\alpha) \int_\tau^t \varepsilon_1(r)dr (t-\tau)^{1/2-\alpha} \|\psi\|_{\mathrm{L}^2_{\Omega_0}} \\
+ \bar c c_\alpha K_2 \frac{1}{1-\alpha} \sup_{\tau\le s\le\tau+1} (|\dot\gamma_1(s)| + |\dot\gamma_0(s)|)(t-\tau)^{1/2-\alpha} \|\psi\|_{\mathrm{L}^2_{\Omega_0}}
\end{aligned}
$$

where $\mathbf{B}(\cdot,\cdot)$ is the beta function and $K_2 = \sup_{0\le x\le 1} |\phi_1(x)| + \sup_{0\le x\le 1} |\phi_0(x)|$. If

$$
c_\alpha' = \bar c c_\alpha \max\left\{ K_1 \mathbf{B}(1/2, 1-\alpha), K_2 \frac{1}{1-\alpha} \right\},
$$

then

$$\|T(t,\tau)\psi - e^{\Lambda(\tau)(t-s)}\psi\|_\alpha \le \\ c'_\alpha\Big[\int_\tau^t \varepsilon_1(r)dr + \sup_{\tau\le s\le\tau+1}(|\dot\gamma_1(s)|+|\dot\gamma_0(s)|)\Big](t-\tau)^{1/2-\alpha}\|\psi\|_{\mathrm{L}^2_{\Omega_0}}. \quad (3.18)$$

Consider the continuous linear operators $A_n = e^{\Lambda(n)}$ and $B_n = T(n+1,n) - A_n$ for $n \in \mathbb{Z}^+$. Inequality (3.18) implies

$$\sum_{n=0}^{\infty}\|B_n\| \le c'_\alpha \int_0^\infty \varepsilon_1(r)dr + \sum_{n=0}^{\infty}\sup_{n\le s\le n+1}(|\dot\gamma_1(s)|+|\dot\gamma_0(s)|) < \infty.$$

Also,

$$\sum_{n=0}^{\infty}\|A_{n+1}-A_n\|_{\mathcal{L}(L^2_{\Omega_0})} = \sum_{n=0}^{\infty}\|\int_0^1 e^{(1-s)\Lambda(n)}(\int_n^{n+1}\dot\Lambda(r)\,dr)e^{s\Lambda(n+1)}\,ds\| \\ \le c'\sum_{n=0}^{\infty}\int_n^{n+1}\varepsilon_1(r)\,dr \le c'\int_0^\infty \varepsilon_1(r)\,dr < \infty.$$

Letting $e^{L_+} = \lim_{n\to+\infty} A_n$, the hypotheses of Theorem 1 in [5] are satisfied. Let $\{w_n; n \ge n_0\}$ be the sequence provided by Theorem 1, with $\frac{w_n}{\|w_n\|} \to \psi_j^+$ as $n \to +\infty$ and $w_{n+1} = (A_n + B_n)w_n = T(n+1,n)w_n$, for $n \ge n_0$.

Define, $w(t) = T(t,n_0)w_{n_0}$. For $n \ge n_0$, we have $w(n) = w_n$. If $0 \le s \le 1$ and $t \ge n_0$, using (3.18) with $\alpha = 0$, we obtain

$$\|w(t+s) - e^{s\Lambda(t)}w(t)\|_{L^2_{\Omega_0}} \\ \le c'_0\left[\int_t^{t+s}\varepsilon_1(r)dr + \sup_{t\le r\le t+1}(|\dot\gamma_1(r)|+|\dot\gamma_0(r)|)\right]\|w(t)\|_{\mathrm{L}^2_{\Omega_0}}.$$

Thus $\|w(t+s) - e^{s\Lambda(t)}w(t)\|_{L^2_{\Omega_0}} = \mathrm{o}(\|w(t)\|_{L^2_{\Omega_0}})$ as $t \to +\infty$ uniformly for $0 \le s \le 1$ and $t \ge n_0$. By Corollary 1 in [5] for any constant $c \ne 0$, we have

$$w(t) = c\exp(\int_0^t \lambda_j(s)\,ds)[\psi_j^+ + \mathrm{o}(1)] \ \text{ as } \ t \to +\infty$$

where the convergence is in $L^2_{\Omega_0}(0,1)$.

We now show the convergence is in fact in the $\|\cdot\|_\alpha$ norm for any $\alpha < 1$, and that ensures the $\mathrm{C}^1(\Omega_1)$-convergence. To show this let

$$z(t) = \frac{1}{c}\exp(-\int_0^t \lambda_j(s)\,ds)w(t),$$

so $z(t) \to \psi_j^+$ in $L^2_{\Omega_0}(0,1)$ as $t \to +\infty$.

There is an eigenfunction $\psi_j(t)$ of $\Lambda(t)$ such that $\psi_j(t) \to \psi_j^+$ as $t \to +\infty$ in $\mathrm{H}^1_{\Omega_0}(0,1)$. Since, $w(t+1) = T(t+1, n_0)w_{n_0} = T(t+1,t)T(t,n_0)w_{n_0} = T(t+1,t)w(t)$ it follows that

$$\begin{aligned}\|\exp(\int_t^{t+1} \lambda_j(s)\,ds)z(t+1) - e^{\Lambda(t)}z(t)\|_\alpha &= \|T(t+1,t)z(t) - e^{\Lambda(t)}z(t)\|_\alpha \\ &\le c'_\alpha \left[\int_t^{t+1} \varepsilon_1(r)dr + \sup_{t\le r\le t+1}(|\dot\gamma_1(r)| + |\dot\gamma_0(r)|)\right] \|z(t)\|_{\mathrm{L}^2_{\Omega_0}} \to 0,\end{aligned}$$

as $t \to +\infty$. From $\lambda_j(t) \to \lambda_j^+$ as $t \to +\infty$, we get $\int_t^{t+1}\lambda_j(s)\,ds \to \lambda_j^+$ as $t \to +\infty$, and hence,

$$\begin{aligned}&\exp(\int_t^{t+1} \lambda_j(s)\,ds)\|z(t+1) - \psi_j(t)\|_\alpha \\ &\quad\le \|\exp(\int_t^{t+1}\lambda_j(s)\,ds)z(t+1) - e^{\Lambda(t)}z(t)\|_\alpha + \|e^{\Lambda(t)}z(t) - e^{\Lambda(t)}\psi_j(t)\|_\alpha \\ &\qquad\qquad + \|e^{\Lambda(t)}\psi_j(t) - \exp(\int_t^{t+1}\lambda_j(s)\,ds)\psi_j(t)\|_\alpha \\ &\quad= \mathrm{o}(1) + \|e^{\Lambda(t)}(z(t) - \psi_j(t))\|_\alpha.\end{aligned}$$

Since, $\|e^{\Lambda(t)}(z(t) - \psi_j(t))\|_\alpha \le d_\alpha\|e^{c-L_+ + \Lambda(t)}\|\,\|z(t) - \psi_j(t)\|_{L^2_{\Omega_0}} \to 0$ as $t \to +\infty$, then $\|z(t+1) - \psi_j(t)\|_\alpha \to 0$ as $t \to +\infty$ and thus $\|z(t) - \psi_j^+\|_\alpha \to 0$, as $t \to +\infty$. ∎

Remark 3.1. *Since $L(t)\psi_j^\pm = \lambda_j^\pm\psi_j^\pm$ Theorem 2.1 implies that $\psi_j^\pm$ has only simple zeros and vanishes exactly $(j-1)$ times in $(0,1)$; due to the $\mathrm{C}^1(\Omega_1)$-convergence obtained in Theorem 3.4 and since $v_j^\pm$ are solutions of (3.12) it follows that $v_j^\pm(t,\cdot)$ has exactly $(j-1)$ simple zeros in $(0,1)$ when $\pm t$ is sufficiently large.*

Corollary 3.1. *With the same assumptions and notations of Theorem 3.4 the adjoint equation*

$$\begin{aligned}\eta_t &+ [(p(x)\eta_x)_x + q(x,t)\eta]\chi_{\Omega_1} \\ &+ \sum_{i=1}^m \left[\frac{1}{b_i - a_i}[p(b_i)\eta_x(b_i^+,t) - p(a_i)\eta_x(a_i^-,t)] + h_i(t)\eta\right]\chi_{\Omega_{0,i}} = 0\end{aligned} \qquad (3.19)$$

with boundary conditions

$$\begin{aligned}p(0)\eta_x(0,t) &= \gamma_0(t)\eta(0,t)\\ p(1)\eta_x(1,t) &= \gamma_1(t)\eta(1,t),\end{aligned}$$

has classical solutions $\eta_j^{\pm}(x,t)$ (η_j^- *defined for* $t < t_j^*$ *for some* t_j^*) *such that*

$$\eta_j^+(x,t) = \exp(-\int_0^t \lambda_j(r)dr)[\psi_j^+(x) + \mathrm{o}(1)] \text{ as } t \to +\infty$$
$$\eta_j^-(x,t) = \exp(\int_t^0 \lambda_j(r)dr)[\psi_j^-(x) + \mathrm{o}(1)] \text{ as } t \to -\infty$$

with convergence in $\mathrm{C}^1(\Omega_1)$.

Proof. The result follows by applying Theorem 3.4 with $S(t) = -L(t)$. ▮

REMARK 3.2. *The application* $t \longmapsto \int_0^1 \eta(t,x)v(t,x)dx$, *where* $\eta(t,x)$ *solves* (3.19) *and* $v(t,x)$ *solves* (3.12), *is constant for* $t_0 < t < t_1$.

THEOREM 3.5. *With the same assumptions and notations of Theorem 3.4, let* $v(x,t)$ *be any solution, not identically zero, of* (3.12) *for* $t > t_0$.
Suppose that

$$t \in (0,\infty) \mapsto \left[(\sup_{0 \le x \le 1} |q(x,\cdot)|)^2 + \sum_{i=1}^m |h_i(\cdot)|^2 \right]^{1/2} \in \mathrm{L}^\infty(0,\infty) \cap \mathrm{L}^1_{\Omega_0}(0,\infty).$$

Then there exists an integer $j \ge 1$ *and a constant* $c \ne 0$, *such that*

$$v(x,t) = \exp(\int_0^t \lambda_j(s)\, ds)[c\psi_j^+(x) + \mathrm{o}(1)] \text{ as } t \to +\infty$$

with convergence in the sense of $\mathrm{C}^1(\Omega_1)$. *In fact, given any constant* κ, *if we define* m *by* $\lambda_{m+1}^+ < -\kappa \le \lambda_m^+$ *then there exist constants* $c_1, c_2, \ldots, c_m$ *such that* $v(x,t) = \sum_{i=1}^m c_i v_i^+(x,t) + \mathrm{o}(\exp(-\kappa t))$ *as* $t \to +\infty$.

Proof. We know that $\|v(t,\cdot)\| = \mathrm{O}(e^{\omega t})$ as $t \to +\infty$, for some constant ω. Thus

$$\mu = \limsup_{t \to +\infty} \frac{1}{t} \log(\|v(t,\cdot)\|_{\mathrm{L}^2_{\Omega_0}} \le \omega < +\infty.$$

We prove that $\mu > -\infty$ as $v \ne 0$. Write (3.12) as

$$v_t = B(t)v + C(t)v$$

with $\mathcal{D}(B(t)) = \mathcal{D}(C(t))$ where

$$\mathcal{D}(C(t)) = \{\varphi \in \mathrm{H}^1_{\Omega_0}(0,1), (p(x)\varphi_x)_x \in \mathrm{L}^2_{\Omega_0}(0,1),$$
$$p(0)u_x(0) = \gamma_0(t)u(0),\ p(1)u_x(1) = \gamma_1(t)u(1)\},$$

$$B(t)v(x) = [(p(x)v_x)_x]\chi_{\Omega_1} + \sum_{i=1}^{m} \frac{1}{b_i - a_i}[p(b_i)v_x(b_i^+) - p(a_i)v_x(a_i^-)]\chi_{\Omega_{0,i}}$$

and

$$C(t)v(x) = q(x,t)v\chi_{\Omega_1} + \sum_{i=1}^{m} h_i(t)v\chi_{\Omega_{0,i}}.$$

Then

$$\|\frac{dv}{dt} - B(t)v\|^2_{\mathrm{L}^2_{\Omega_0}} \leq \left(\left(\sup_{0\leq x\leq 1} |q(x,t)| \right)^2 + \sum_{i=1}^{m} |h_i(t)|^2 \right) \|v\|^2_{\mathrm{L}^2_{\Omega_0}},$$

where $B(t)$ is symmetric. The hypotheses of Proposition 2.1 in Agmon [1] are verified. Hence there exists a constant $\sigma > 0$ (depending on $v(t_0)$) such that $\|v(t)\|_{\mathrm{L}^2_{\Omega_0}} \geq \exp(-\sigma(t-t_0))\|v(t_0)\|_{\mathrm{L}^2_{\Omega_0}}$ for $t \geq t_0$ and

$$\mu = \limsup_{t\to+\infty}(\frac{1}{t}\log(\|v(t,\cdot)\|_{\mathrm{L}^2_{\Omega_0}})) \geq -\sigma > -\infty.$$

Set $\lambda_0^+ = +\infty$ and choose $j \geq 1$ such that $\lambda_j^+ \leq \mu < \lambda_{j-1}^+$. From (3.18) we have

$$\begin{aligned}
&\|v(t+s) - e^{L_+ s}v(t)\|_{\mathrm{L}^2_{\Omega_0}} \\
&\leq \|T(t+s,t)v(\cdot,t) - e^{L(t)s}v(\cdot,t)\|_{\mathrm{L}^2_{\Omega_0}} + \|e^{L(t)s} - e^{L_+ s}\|\|v(\cdot,t)\|_{\mathrm{L}^2_{\Omega_0}} \\
&\leq c_0' \left[\int_t^{t+s} \varepsilon_1(r)dr + \sup_{t\leq r\leq t+1} (|\dot\gamma_1(r)| + |\dot\gamma_0(r)|) + \|e^{L(t)s} - e^{L_+ s}\| \right] \|v(\cdot,t)\|_{\mathrm{L}^2_{\Omega_0}}.
\end{aligned}$$

Since

$$\begin{aligned}
\|(L(t) - L_+)v\|^2_{\mathrm{L}^2_{\Omega_0}} &= \int_{\Omega_1} |(q(x,t) - q_+(x))v|^2 dx + \sum_{i=1}^{m} \int_{\Omega_{0,i}} |(h_i(t) - h_{i_+})v|^2 dx \\
&\leq \left[\left(\sup_{0\leq x\leq 1} |q(x,t) - q_+(x)| \right)^2 + \sum_{i=1}^{m} |h_i(t) - h_{i_+}|^2 \right] \int_0^1 |v|^2 dx.
\end{aligned}$$

and $\|e^{L_+ s}\| \leq Ne^{\omega s}$, we have $\|e^{L(t)s} - e^{L_+ s}\| \leq Ne^{\omega}(e^{N\|L(t)-L_+\|} - 1) \to 0$ as $t \to +\infty$, uniformly in $0 \leq s \leq 1$. Thus

$$\|v(t+s) - e^{L_+ s}v(t)\|_{\mathrm{L}^2_{\Omega_0}} = \mathrm{o}(\|v(t)\|_{\mathrm{L}^2_{\Omega_0}}), \text{ as } t \to +\infty.$$

Choose α and β such that $\lambda_{j+1}^+ < \alpha < \beta < \lambda_j^+ \leq \mu$; then $\sigma(e^{L_+}) \cap \{z \in \mathbb{C} : e^\alpha \leq |z| \leq e^\beta\} = \emptyset$. We can then decompose $\mathrm{L}^2_{\Omega_0}(0,1) = X^1 \oplus X^2$ and the operator $L_+ = L_+^1 \oplus L_+^2$

where $X^1 = [\psi_1^+, \dots, \psi_j^+]$, is the subspace generated by $\psi_1^+, \dots, \psi_j^+$ and $X^2 = (X^1)^\perp$. The $\sigma(e^{L_+^1 t}) = \{z \in \sigma(e^{L_+ t}) : |z| > e^{\beta t}\}$ and $\sigma(e^{L_+^2 t}) = \{z \in \sigma(e^{L_+ t}) : |z| < e^{\alpha t}\}$.

Since $\alpha < \mu$ and $\mu = \limsup_{t\to+\infty} \frac{1}{t}\log(\|v(t)\|_{\mathrm{L}^2_{\Omega_0}})$ it follows that $\|v(t)\|_{L^2_{\Omega_0}}$ is not $\mathrm{o}(e^{\alpha t})$ as $t \to +\infty$. By Corollary 2 in [5] we have $\|v^2(t)\| > \|v^1(t)\|$ for t sufficiently large. But $\|v(t)\| = \max(\|v^1(t)\|, \|v^2(t)\|)$ and hence the component of the $\|v(t)\|_{\mathrm{L}^2_{\Omega_0}}$ in $[\psi_k^+, 1 \le k \le j]$ is small compared to $\|v(t)\|_{\mathrm{L}^2_{\Omega_0}}$.

Now choose α and β such that $\lambda_j^+ \le \mu < \alpha < \beta < \lambda_{j-1}^+$ and $\sigma(e^{L_+}) \cap \{z \in \mathbb{C} : e^\alpha \le |z| \le e^\beta\} = \emptyset$. Decompose $\mathrm{L}^2_{\Omega_0}(0,1) = X^1 \oplus X^2$ and $L_+ = L_+^1 \oplus L_+^2$, where $X^1 = [\psi_1^+, \dots, \psi_{j-1}^+]$ and $X^2 = [\psi_j^+ \psi_{j+1}^+, \dots]$. Then $\sigma(e^{L_+^1 t}) = \{z \in \sigma(e^{L_+ t}) : |z| > e^{\beta t}\}$ and $\sigma(e^{L_+^2 t}) = \{z \in \sigma(e^{L_+ t}) : |z| < e^{\alpha t}\}$.

Since $\mu = \limsup_{t\to+\infty} \frac{1}{t}\log(\|v(t,\cdot)\|_{\mathrm{L}^2_{\Omega_0}})$ and $\mu < \beta$ it follows that $\|v(t)\|_{\mathrm{L}^2_{\Omega_0}} = \mathrm{o}(e^{\beta t})$ as $t \to +\infty$. By Corollary 2 in [5] the component of $\|v(t)\|_{\mathrm{L}^2_{\Omega_0}}$ which is independent of $[\psi_k^+ : 1 \le k \le j-1]$ is small compared to $\|v(t)\|_{\mathrm{L}^2_{\Omega_0}}$. Then there exist real functions $\sigma(t)$ and $\alpha_k(t)$ such that

$$\frac{\|v(t) - \sigma(t)\psi_j^+\|_{\mathrm{L}^2_{\Omega_0}}}{\|v(t)\|_{\mathrm{L}^2_{\Omega_0}}} \to 0 \quad \text{and} \quad \frac{|\sigma(t)|}{\|v(t)\|_{\mathrm{L}^2_{\Omega_0}}} \to 1, \tag{3.20}$$

as $t \to +\infty$. Let $z(t) = \exp(-\int_0^t \lambda_j(s)ds)v(t)$ and $c(t) = \exp(-\int_0^t \lambda_j(s)ds)\sigma(t)$. Using (3.20) we get

$$\frac{\|z(t) - c(t)\psi_j^+\|_{\mathrm{L}^2_{\Omega_0}}}{\|z(t)\|_{\mathrm{L}^2_{\Omega_0}}} \to 0 \quad \text{and} \quad \frac{|c(t)|}{\|z(t)\|_{\mathrm{L}^2_{\Omega_0}}} \to 1, \tag{3.21}$$

as $t \to +\infty$. We now show that $c(t)$ converges to a nonzero limit as $t \to +\infty$. We have

$$\begin{aligned} \|z(t+s)\exp(\int_t^{t+s} \lambda_j(r)dr) - e^{sL(t)}z(t)\|_{\mathrm{L}^2_{\Omega_0}} &= \|T(t+s,t)z(t) - e^{sL(t)}z(t)\|_{\mathrm{L}^2_{\Omega_0}} \\ \le c_0' \left[\int_t^{t+s} \varepsilon_1(r)dr + \sup_{t\le r\le t+1}(|\dot\gamma_1(r)| + |\dot\gamma_0(r)|)\right] \|z(t)\|_{\mathrm{L}^2_{\Omega_0}} & \end{aligned} \tag{3.22}$$

for $0 \le s \le 1$ and $\int_0^{+\infty} \varepsilon_1(r)dr + \sum_{n=0}^\infty \sup_{n\le r\le n+1}(|\dot\gamma_1(r)| + |\dot\gamma_0(r)|) < \infty$.

As in Henry [5], this implies that $|c(n+1) - c(n)| \le \beta_n |c(n)|$ for $n = 1, \dots$ with $\sum_{n=1}^{+\infty} \beta_n < \infty$. Thus

$$\frac{|c(n+1)|}{|c(n)|} \le 1 + \beta_n$$

Since $\sum_{n=1}^\infty \beta_n < \infty$ it follows that $\prod_{n\ge n_0}(1+\beta_n) < \infty$, and by comparison $\prod_{n\ge n_0} \frac{c(n+1)}{c(n)}$ converges, so $c(n) \to c^* \ne 0$ as $n \to +\infty$. The estimative above for z shows that $c(t) \to c^*$ as $t \to +\infty$ and therefore $z(t) \to c^*\psi_j^+$, proving the first part of the theorem.

Let $\hat{v} = v - c^* v_j^+$. This solution is such that $\|\hat{v}(\cdot,t)\|_{L^2_{\Omega_0}} = o(\exp(\int_0^t \lambda_j))$ as $t \to +\infty$ so that $\|\hat{v}(\cdot,t)\|_{L^2_{\Omega_0}} = o(\exp(t(\lambda_{j+1}^+ + \varepsilon)))$ as $t \to +\infty$, for any $\varepsilon > 0$, and by iteration we obtain the expansion $v(\cdot,t) = \sum_{j=1}^m c_j v_j^+(\cdot,t) + o(e^{-kt})$, for $k \in (\lambda_{m+1}^+, \lambda_m^+]$.

Note that we proved only convergence in $L^2(0,1)$, but using the same procedure used in the proof of Theorem 3.4, we can show that this implies convergence in $\|\cdot\|_\alpha$ for any $\alpha < 1$ ensuring the convergence in $C^1(\Omega_1)$. ▮

4. TRANSVERSALITY OF STABLE AND UNSTABLE MANIFOLDS

Let $\varphi_\pm$ be hyperbolic equilibrium points of (P_0) and $\bar{u} = \bar{u}(x,t)$ a solution of (P_0), in $-\infty < t < +\infty$ such that

$$\lim_{t\to\pm\infty} \bar{u}(x,t) = \varphi_\pm(x).$$

If φ is an equilibrium of (P_0) the stable and unstable manifolds of φ are defined by

$$W^s(\varphi) = \{u_0 \in H^1_{\Omega_0}(0,1); \text{the solution } u(t;u_0) \text{ of } (P_0) \text{ exists for all } t \geq 0 \text{ and } u(t;u_0) \to \varphi \text{ as } t \to +\infty\}$$

and

$$W^u(\varphi) = \{u_0 \in H^1_{\Omega_0}(0,1) : \text{there exists a solution } u(t,\cdot) \text{ of } (P_0) \text{ for } t \leq 0, \; u(0,\cdot) = u_0 \text{ and } u(\cdot,0) \to \varphi \text{ as } t \to -\infty\}.$$

Let $q(x,t) = f'(\bar{u}(x,t)) + c(x)$, $h_i(t) = f'(\bar{u}_{\Omega_{0,i}}(t)) + \hat{c}_i$ for $i = 1,\dots,m$, where $\hat{c}_i = \frac{1}{b_i - a_i}\int_{a_i}^{b_i} c(x)dx$, $\gamma_0(t) = g'(\bar{u}(0,t)) - b(0)$ and $\gamma_1(t) = g'(\bar{u}(1,t)) - b(1)$, and consider the linearized equation

$$\begin{cases} v_t = (p(x)v_x)_x + q(x,t)v, \;\; x \in \Omega_1, \;\; t \in (-\infty,\infty) \\ \dot{v}_{\Omega_{0,i}}(t) = \frac{1}{b_i - a_i}[p(b_i)v_x(b_i^+,t) - p(a_i)v_x(a_i^-,t)] + h_i(t)v_{\Omega_{0,i}}(t), \;\; i = 1,\dots,m \\ p(0)v_x(0,t) = \gamma_0(t)v(0,t) \\ p(1)v_x(1,t) = \gamma_1(t)v(1,t). \end{cases} \tag{4.23}$$

Suppose the application

$$t \in (0,\infty) \mapsto \left[(\sup_{0\leq x\leq 1} |q(x,t)|)^2 + \sum_{i=1}^m |h_i(t)|^2 \right]^{1/2}$$

is in $L^\infty(0,\infty) \cap L^1_{\Omega_0}(0,\infty)$.

At the point $\bar{u}(t_0,\cdot) \in W^s(\varphi_+) \cap W^u(\varphi_-)$ the tangent spaces to the manifolds $W^s(\varphi_+)$ and $W^u(\varphi_-)$ can then be described by

$$T_{\bar{u}(t_0,\cdot)}W^s(\varphi_+) = \{v(t_0,\cdot) \in \mathrm{H}^1_{\Omega_0}(0,1);\ v(x,t) \text{ solves (4.23) in } t_0 < t < +\infty \\ \text{and } \|v(t,\cdot)\|_{\mathrm{L}^2_{\Omega_0}} = \mathrm{O}(e^{-\varepsilon t}) \text{ as } t \to +\infty \text{ for some } \varepsilon > 0\}$$

and

$$T_{\bar{u}(t_0,\cdot)}W^u(\varphi_-) = \{v(t_0,\cdot) \in \mathrm{H}^1_{\Omega_0}(0,1):\ v(x,t) \text{ solves (4.23) in } -\infty < t < t_0 \\ \text{and } \|v(t,\cdot)\|_{\mathrm{L}^2_{\Omega_0}} = \mathrm{O}(e^{\varepsilon t}) \text{ as } t \to -\infty \text{ for some } \varepsilon > 0\}.$$

Using the solutions $\{\eta^+_k(x,t),\ k \geq 1\}$ obtained in Corollary 3.1 and the solutions $\{v^-_j(x,t),\ j \geq 1\}$ obtained in Theorem 3.4 we can write $T_{\bar{u}(t_0,\cdot)}W^s(\varphi_+)$ and $T_{\bar{u}(t_0,\cdot)}W^s(\varphi_-)$.

Suppose that λ^+_m is the m$^{\text{th}}$ eigenvalue of the linearization of (P_0) at φ_+ with $\lambda^+_m > 0 > \lambda^+_{m+1}$ and λ^-_n is the n$^{\text{th}}$ eigenvalue of the linearization of (P_0) at φ_- with $\lambda^-_n > 0 > \lambda^-_{n+1}$. Then

$$T_{\bar{u}(t_0,\cdot)}W^s(\varphi_+) = \{\psi \in \mathrm{H}^1_{\Omega_0}(0,1) : \int_0^1 \psi\eta^+_k(t_0,\cdot) = 0 \text{ for } 1 \leq k \leq m\}$$

and

$$T_{\bar{u}(t_0,\cdot)}W^u(\varphi_-) = \text{subspace generate by } \{v^-_j(t_0,\cdot) : 1 \leq j \leq n\}.$$

Theorem 4.6 (Transversality Theorem). *Suppose the linearization of* (P_0) *at* φ_- *has* n_- *positive eigenvalues and the linearization of* (P_0) *at* φ_+ *has* m_+ *positives eigenvalues. If* $n_- \geq m_+ \geq 0$ *then*

$$W^u(\varphi_-) \pitchfork_{\bar{u}(\cdot,t_0)} W^s(\varphi_+), \tag{4.24}$$

that is, the sum of the subspaces $\mathrm{T}_{\bar{u}(\cdot,t_0)}W^u(\varphi_-)$ *and* $\mathrm{T}_{\bar{u}(\cdot,t_0)}W^s(\varphi_+)$ *is the whole space* $\mathrm{H}^1_{\Omega_0}(0,1)$. *Conversely,* $n_- \geq m_+$ *is a necessary condition for transversality.*

Proof. Suppose that (4.24) does not occur, that is, the sum of subspace generated by $\{v^-_j(t_0,\cdot) : 1 \leq j \leq n_-\}$ and $\{\psi \in \mathrm{H}^1_{\Omega_0}(0,1) : \int_0^1 \eta^+_k(t_0,\cdot)\psi = 0,\ k = 1,\ldots,m_+\}$ is not contained in $\mathrm{H}^1_{\Omega_0}(0,1)$. Then there are constants c_j, $j = 1,\ldots,m$, not all zero, such that

$$\eta(x,t) = \sum_{j=1}^{m_+} c_j\eta^+_j(x,t) \quad \text{and} \quad \int_0^t \eta(x,t_0)v^-_i(x,t_0)dx = 0 \tag{4.25}$$

for $1 \leq i \leq n_-$. Let $1 \leq k \leq m_+$, be such that $c_j = 0$ for $j > k$ and $c_k \neq 0$. By Corollary 3.19 we have $\eta(x,t) = \exp(-\int_0^t \lambda_k(r)dr)[c_k\psi^+_k(x) + \mathrm{o}(1)]$, as $t \to +\infty$.

Since this convergence is in $C^1(\Omega_1)$ the solution $\eta(\cdot,t)$ vanishes $(k-1)$ times in $(0,1)$ for large positive t. By Theorem 2.3, applied to the adjoint equation with the sign of t reversed, $\eta(\cdot,t)$ vanishes no more than $(k-1)$ times in $(0,1)$, for every t.

By Theorem 3.5, reversing the sign of t, there exist $c \neq 0$ and $j \geq 1$ so that

$$\eta(x,t) = \exp(\int_t^0 \lambda_j)[c\psi_j^-(x) + o(1)]$$

as $t \to -\infty$. Suppose $j \leq n_-$. Then from (4.25), we have

$$\begin{aligned} 0 &= \langle \eta(x,t_0), v_j^-(x,t_0)\rangle_{L^2_{\Omega_0}} \\ &= \int_0^1 \eta(x,t)v_j^-(x,t)dx \\ &= \int_0^1 [c\psi_j^-(x)\psi_j^-(x) + o(1)] \\ &= c + o(1), \text{ as } t \to -\infty \end{aligned}$$

which is false. Thus $j > n_-$ and $\eta(\cdot,t)$ vanishes $(j-1)$ times in $(0,1)$ for large negative t. We must have $j \leq k$ as noted before, so $n_- < j \leq k \leq m_+$, contrary to our hypothesis. ∎

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