On the orbit structure of $\mathbb{R}^n$-actions on $n$-manifolds

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We begin by proving that a locally free $C^2$-action of $\mathbb{R}^{n-1}$ on $T^{n-1} \times [0,1]$ tangent to the boundary and without compact orbits in the interior has all non-compact orbits of the same topological type. Then, we consider the set $A^2(\mathbb{R}^n, N)$ of $C^2$-actions of $\mathbb{R}^n$ on a closed connected orientable real analytic $n$-manifold $N$. We define a subset $A^\alpha_n \subset A^2(\mathbb{R}^n, N)$ and prove that if $\varphi \in A^\alpha_n$ has a $T^{n-1} \times \mathbb{R}$-orbit, then every $n$-dimensional orbit is also a $T^{n-1} \times \mathbb{R}$-orbit.

The subset $A^\alpha_n$ is big enough to contain all real analytic actions that have at least one $n$-dimensional orbit. We also obtain information on the topology of $N$.

Key Words: Action of $\mathbb{R}^n$, orbit structure.

1. INTRODUCTION

In this paper we begin by proving the following result that is a generalization of Corollary 2.6 in [4].

Theorem A. Let $\psi$ be a locally free $C^2$-action of $\mathbb{R}^{n-1}$ on $T^{n-1} \times [0,1]$, $n \geq 2$, tangent to the boundary. If there are no compact orbits in the interior, then all non-compact orbits have the same topological type.

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Due to Theorem 2.1, there is no restriction in assuming that the manifold with boundary, in Theorem A, is $T^{n-1} \times [0, 1]$.

$N$ will denote a closed connected orientable real analytic $n$-manifold with $n \geq 2$. Let $\mathcal{H}_n$ be the family of orientable $n$-manifolds obtained by gluing two copies of $T^{n-2} \times D^2$, $\mathcal{H}_2$ contains only $S^2$ and $\mathcal{H}_3$ consists of 3-manifolds that admit a Heegaard splitting of genus one. Denote by $A^r(\mathbb{R}^n, N)$ the set of $C^r$-actions of $\mathbb{R}^n$ on $N$, $2 \leq r \leq \omega$, with $C^r$ infinitesimal generators. It was proved in [1] that if $\varphi \in A^r(\mathbb{R}^n, N)$, then all $n$-dimensional orbits of $\varphi$ have the same topological type, i.e., are $T^k \times \mathbb{R}^{n-k}$-orbits for some fixed $k$, $0 \leq k \leq n$. Moreover, if the type is $T^{n-1} \times \mathbb{R}$, then $N$ is either homeomorphic to $T^n$ or $N \in \mathcal{H}_n$. It is not difficult to construct counterexamples of this results when $r = \infty$.

In this paper we define a subset $\mathcal{S}_n \subset A^2(\mathbb{R}^n, N)$, see Definition 2.2, which contains all actions $\varphi \in A^2(\mathbb{R}^n, N)$ that have at least one $n$-dimensional orbit. Then, we prove:

**Theorem B.** If $\varphi \in \mathcal{S}_n$ has one $T^{n-1} \times \mathbb{R}$-orbit, then every $n$-dimensional orbit is also a $T^{n-1} \times \mathbb{R}$-orbit. Moreover,

1. if $\text{Sing}_{n-2}(\varphi) = \emptyset$, then $N$ is a $T^{n-1}$ bundle over $S^1$;
2. if $\text{Sing}_{n-2}(\varphi) \neq \emptyset$, then $\text{Sing}_{n-2}(\varphi)$ is the union of two $T^{n-2}$-orbits and $N \in \mathcal{H}_n$.

The connection between the two results is that the first is used in the proof of second. It would be interesting to obtain analogous results for actions in $A^2(\mathbb{R}^n, N)$ that have one $T^k \times \mathbb{R}^{n-k}$-orbit with $0 \leq k < n - 1$.

### 2. PRELIMINARIES AND PROOF OF RESULTS

$M (N)$ will denote a closed connected and orientable real analytic $m$-manifold ($n$-manifold). A $C^r$-action of Lie group $G$ on $M$ is a $C^r$-map $\varphi : G \times M \rightarrow M$, $1 \leq r \leq \omega$, such that $\varphi(e, p) = p$ and $\varphi(gh, p) = \varphi(g, \varphi(h, p))$, for each $g, h \in G$ and $p \in M$, where $e$ is the identity in $G$. $O_p = \{\varphi(g, p); g \in G\}$ is called the $\varphi$-orbit of $p$. $G_p = \{g \in G; \varphi(g, p) = p\}$ is called the isotropy group of $p$. For each $p \in M$ the map $g \mapsto \varphi(g, p)$ induces an injective immersion of the homogeneous space $G/G_p$ in $M$ with image $O_p$. When $G = \mathbb{R}^n$, the possible $\varphi$-orbits are injective immersions of $T^k \times \mathbb{R}^l$, $0 \leq k + l \leq n$, where $T^k = S^1 \times \cdots \times S^1, k$ times.

For each $0 \leq i \leq n - 1$ let $\text{Sing}_i(\varphi) = \{p \in M; \dim O_p = i\}$ and $\text{Sing}(\varphi) = \cup_{i=0}^{n-1} \text{Sing}_i(\varphi)$. If $p \in \text{Sing}(\varphi)$, $O_p$ is called a singular orbit and when $p \in \text{Sing}_0(\varphi)$, $O_p$ is also called a point orbit and $p$ a fixed point by $\varphi$. We also write $p \in \text{Sing}_i^j(\varphi)$, $i = 1, \ldots, n - 1$, when $O_p$ is a $T^i$-orbit. If $\text{Sing}(\varphi) = M$, we call $\varphi$ a singular action.

For each $w \in \mathbb{R}^n \setminus \{0\}$ $\varphi$ induces a $C^r$-flow $(\varphi^w_t)_{t \in \mathbb{R}}$ given by $\varphi^w_t(p) = \varphi(tw, p)$ and its corresponding $C^{r-1}$-vector field $X_w$ defined by $X_w(p) = D_1\varphi(0, p) \cdot w$. If $\{w_1, \ldots, w_n\}$ is a base of $\mathbb{R}^n$ the associated vector fields $X_{w_1}, \ldots, X_{w_n}$ determine completely the action $\varphi$ and are called a set of infinitesimal generators of $\varphi$. Note that $[X_{w_j}, X_{w_j}] = 0$ for any two of them.

**Definition 2.1.** Let $\varphi \in A^r(\mathbb{R}^n, N)$ and $p \in N$.

a) $\varphi$ is of type $j$ at $p$, $0 \leq j \leq n$, if there exists a neighborhood $V$ of $p$ such that the union of the $j$-dimensional orbits of $\varphi|_V$ form an open and dense subset of $V$. 

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b) \( \varphi \) is \( j \)-finite at \( p \), if there exists a neighborhood \( V \) of \( p \) that intersects only a finite number of \( j \)-dimensional orbits.

**Definition 2.2.** We say that \( \varphi \in \mathscr{A}_n \subset A^2(\mathbb{R}^n, N) \) if \( \varphi \) is of type \( n \) and \( n \)-finite at each \( p \in \text{Sing}_i(\varphi) \) with \( 0 \leq i \leq n-3 \), \( \text{Sing}_{n-2}(\varphi) = \text{Sing}_{n-2}^c(\varphi) \) and for each \( p \in \text{Sing}_{n-2}^c(\varphi) \) there exists a neighborhood \( V_p \) of \( \mathcal{O}_p \) in \( N \) that satisfies one of the following two properties:

1. \( V_p \) is \( \varphi \)-invariant, homeomorphic to \( T^{n-2} \times D^2 \), where \( D^2 \) is an open disk, \( V_p \cap (\bigcup_{i=1}^{n-2} \text{Sing}_i(\varphi)) = \mathcal{O}_p \) and \( \text{Front}(V_p) \) is either a \( T^{n-1} \)-orbit or a \( T^{n-2} \)-orbit.

2. \( V_p \) contains at most a finite number of \( i \)-dimensional orbits with \( i = n - 1, n \).

**Infinitesimal generators adapted to a \( T^{n-1} \)-orbit.** Assume that \( \mathcal{O}_p \) is a \( T^{n-1} \)-orbit of \( \varphi \in A^r(\mathbb{R}^n, N) \) and let \( G_p \) be its isotropy group. Call \( G^n_p \) the connected component of \( G_p \) that contains the origin and let \( H \) be a \( (n-1) \)-dimensional subspace of \( \mathbb{R}^n \) such that \( \mathbb{R}^n = H \oplus G^n_p \). Note that \( G_p \cap H \) is isomorphic to \( \mathbb{Z}^{n-1} \). Let \( \{w_1, \ldots, w_{n-1}\} \) be a base of \( \mathbb{R}^n \) such that \( \{w_1, \ldots, w_{n-1}\} \) is a set of generators of \( G_p \cap H \), \( w_n \in G^n_p \) and write \( X_i = X_{w_i} ; i = 1, \ldots, n \). Note that if \( q \in \mathcal{O}_p \), then for every \( k \in \{1, \ldots, n-1\} \) the orbit of \( X_k \) by \( p \) is periodic of period one and also \( X_n(q) = 0 \). We shall say that \( X_1, \ldots, X_n \) is a set of infinitesimal generators adapted to \( \mathcal{O}_p \). The action \( \psi_p \in A^r(\mathbb{R}^{n-1}, N) \), \( r \geq 2 \), with infinitesimal generators \( X_1, \ldots, X_{n-1} \) will be called the action induced by \( \varphi \) and \( \mathcal{O}_p \). The understanding of the holonomy of \( \mathcal{O}_p \) as an orbit of \( \psi_p \) will bring light on the orbit structure of \( \varphi \) in the neighborhood of \( \mathcal{O}_p \).

Let \( \mathcal{O}_p \) be a \( T^{n-1} \)-orbit of \( \psi \in A^r(\mathbb{R}^{n-1}, N) \), \( \{w_1, \ldots, w_{n-1}\} \) be a set of generators of its isotropy group \( G_p \) and \( X_1 = X_{w_1} , \ldots, X_{n-1} = X_{w_{n-1}} \). For each \( k \in \{1, \ldots, n-1\} \), let \( \psi_k \in A^r(\mathbb{R}^{n-2}, N) \) be the action defined by \( X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_{n-1} \). Put a Riemannian metric on \( N \) and let \( \xi \) be the norm one vector field defined in a neighborhood of \( \mathcal{O}_p \) that is orthogonal to the orbits of \( \psi \). Let \( S_k \) be the circle orbit of \( X_k \) through \( p, k = 1, \ldots, n-1 \), and consider the ring \( A = S^1 \times (-1, 1) \) with coordinates \( (\theta, x) \). Define \( f_k : A \to N \) by \( f_k(\theta, x) = \xi^x \circ X^k_\theta(p) \) and note that \( f_k(S^1 \times \{0\}) = S_k \) and \( f_k(0, 0) = p \). Fix \( k \in \{1, \ldots, n-1\} \). Since \( S_k \) is a submanifold of \( \mathcal{O}_p \), it is transversal to the orbits of \( \psi_k \), there exists \( \varepsilon > 0 \) such that \( f_k \) restricted to \( A_{\varepsilon} = S^1 \times (-\varepsilon, \varepsilon) \) is an embedding transversal to the orbits of \( \psi_k \). Let \( D^{n-2}_k(\delta) = \{ t = (t_1, \ldots, t_k-1, t_{k+1}, \ldots, t_{n-1}); t_j \in (-\delta, \delta) \} \) and consider the \( C^r \)-map \( h_k : A_{\varepsilon} \times D^{n-2}_k(\delta) \to N \) defined by \( h_k(\theta, x, t) = \psi_k(t, f_k(\theta, x)) \). There exists \( \delta > 0 \) such that \( h_k \) restricted to \( A_{\varepsilon} \times D^{n-2}_k(\delta) \) is a diffeomorphism onto its image \( V_k \). Moreover, in these coordinates the infinitesimal generators of \( \psi \) take the form:

\[
\begin{align*}
X_i(\theta, x, t) &= \frac{\partial}{\partial t_i}, \quad i = 1, \ldots, k-1, k+1, \ldots, n-1 \\
X_k(\theta, x, t) &= \sum_{k \neq j=1}^{n-1} a_{jk}(\theta, x) \frac{\partial}{\partial t_j} + b_k(\theta, x) \frac{\partial}{\partial \theta} + c_k(\theta, x) \frac{\partial}{\partial x}.
\end{align*}
\]
A map like \( h_k \) will be called a **cylindrical coordinate system adapted to \( \mathcal{O}_p \) at \( S_k \). The vector field
\[
\tilde{X}_k = b_k(\theta, x) \frac{\partial}{\partial \theta} + c_k(\theta, x) \frac{\partial}{\partial x}
\]
defines a local flow on \( A_k \) having \( S^1 \times \{0\} \subset A_k \) as an orbit. When \( \psi = \psi_\varphi \) for some \( \varphi \in A^r(\mathbb{R}^n, N) \), then we also have
\[
X_n(\theta, x, t) = \sum_{k \neq j = 1}^{n-1} a_{jk}(\theta, x) \frac{\partial}{\partial t_j} + d_k(\theta, x) \frac{\partial}{\partial \theta} + e_k(\theta, x) \frac{\partial}{\partial x}.
\]
The vector fields \( \tilde{X}_k \) and \( \tilde{X}_n = d_k \partial/\partial \theta + e_k \partial/\partial x \) define a local \( C^r \)-action \( \tilde{\varphi}_k \) of \( \mathbb{R}^2 \) on \( A \) having \( S^1 \times \{0\} \) as a singular orbit.

The ring \( \Sigma_k = f_k(A_k) \) is transversal to the orbits of \( \psi \) and so is \( J = \cap_{k=1}^{n-1} \Sigma_k \). Note that \( p \in J \). The vector fields \( \tilde{Y}_k = (f_k)_* \tilde{X}_k \) and \( \tilde{Y}_n = (f_k)_* \tilde{X}_n \) are tangent to \( \Sigma_k \) and define a local \( C^r \)-action of \( \mathbb{R}^2 \) on \( \Sigma_k \). The map \( \alpha_k : [0, 1] \rightarrow \Sigma_k \) given by \( \alpha_k(t) = \tilde{Y}_k^t(p) \) is a parametrization of \( S_k \). Let \( \omega_k : (J, p) \rightarrow (J, p) \) be the Poincaré map of \( \alpha_k \) and
\[
\text{Hol} : \pi_1(\mathcal{O}_p, p) \cong \mathbb{Z}^k \rightarrow \text{Diff}^r(J, p)
\]
the holonomy of \( \mathcal{O}_p \) as a leaf of the foliation defined by the orbits of \( \psi \). Then, \( \omega_k = \text{Hol}(\alpha_k) \). Write \( J \) as the union of two intervals \( J^+ \cup J^- \) with \( J^+ \cap J^- = \{p\} \). Since \( \mathcal{O}_p \) is two-sided in \( N \), each \( \omega_i \) leaves \( J^+ \) \( (J^-) \) invariant.

**Remark 2.1.** Note that \( \{X_1, \ldots, X_{k-1}, \tilde{X}_k, X_{k+1}, \ldots, X_{n-1}, \tilde{X}_n\} \) define a local \( \mathbb{R}^n \)-action \( \tilde{\varphi} \) on \( A \times D_k^{n-2}(\epsilon) \) and that \( \mathcal{O}_{(\theta, x,t)}(\tilde{\varphi}) = \mathcal{O}_{(\theta, x,t)}(h_k \circ \varphi \circ h_k^{-1}) \) for each \( (\theta, x, t) \in A \times D_k^{n-2}(\epsilon) \).

The following lemma is essencial in the proof of Theorem A.

**Lemma 2.1.** Let \( \mathcal{O}_p \) be a \( T^{n-1} \)-orbit of \( \psi \in A^1(\mathbb{R}^{n-1}, N) \) and assume that \( \psi \) has no \( T^{n-1} \)-orbits, aside \( \mathcal{O}_p \), in a neighborhood \( V \) of \( \mathcal{O}_p \). Then there exists a neighborhood \( I^+ \) of \( p \) in \( J^+ \) such that for each \( k \in \{1, \ldots, n-1\} \) one of the following statements is verified: 

1. \( \omega_k|_{I^+} = \text{id} \); i.e., every \( \tilde{Y}_k \)-orbit near \( S_k \) is periodic.
2. Either \( \omega_k|_{I^+} \) or \((\omega_k|_{I^+})^{-1}\) is a topological contraction, i.e., every \( \tilde{Y}_k \)-orbit near \( S_k \) spirals towards \( S_k \).

**Proof.** We give the proof for \( k = n-1 \); the other cases are similar. Assume that \( \omega_{n-1} \) does not satisfy neither (1) nor (2). Then, there is a sequence \( \{q_l \in J^+; l \in \mathbb{N}\} \) such that \( \omega_{n-1}(q_l) = q_l \) and \( \lim_{l \to \infty} q_l = p \). We claim that \( p \) is an isolated fixed point of \( \omega_j \) for at least one \( j \in \{1, \ldots, n-2\} \). Otherwise, for each \( 1 \leq j \leq n-2 \) there exists a sequence \( \{q_{jk} \in J^+; k \in \mathbb{N}\} \) such that \( \omega_j(q_{jk}) = q_{jk} \) and \( \lim_{k \to \infty} q_{jk} = p \). If \( q_{jk} \in V \) and \( \omega_i(q_{jk}) = q_{jk} \) for each \( i \in \{1, \ldots, n-1\} \), then the \( \psi \)-orbit of \( q_{jk} \) is a
Lemma 2.2. If \( \varphi \in A^1(\mathbb{R}^n, N) \) has an orbit \( \mathcal{O} \) diffeomorphic to \( T^{n-1} \times \mathbb{R} \), then \( \text{Front}(\mathcal{O}) \) is the union of at most two \( T^k \)-orbits with \( k \in \{n-2, n-1\} \).

The proof of this lemma is given in [1].

Corollary 2.1. Let \( \mathcal{O} \) be an \( T^{n-1} \)-orbit of \( \varphi \in A^1(\mathbb{R}^n, N) \) and \( V \) be a neighborhood of \( \mathcal{O} \) such that \( V \setminus \mathcal{O} \) has exactly two connected components \( V_1 \) and \( V_2 \). Assume that there exist points \( p, q \in V_1 \) such that \( \dim \mathcal{O}_p = n \), \( \mathcal{O}_p \neq \mathcal{O}_q \) and \( \text{cl}(\mathcal{O}_p) \supset \mathcal{O} \subset \text{cl}(\mathcal{O}_q) \). Then, \( \mathcal{O}_p \) is not a \( T^{n-1} \)-orbit.

Proposition 2.1. Assume that \( \varphi \in A^2(\mathbb{R}^n, N) \) with \( n \geq 2 \). If \( \mathcal{O} \) is a \( T^k \times \mathbb{R}^{n-k} \)-orbit with \( k \leq n-2 \), then \( \text{cl}(\mathcal{O}) \) cannot contain a \( T^{n-1} \)-orbit.

If \( X \in \mathcal{C}(M^2) \), let \( \mathcal{C}(X) \) be the set of diffeomorphisms \( f \in \text{Diff}^\infty(M^2) \) that preserve orbits of \( X \). Assume that the orbit \( \gamma_p \) of \( X \) by \( p \) is periodic of period \( \tau \). Let \( \Sigma \) be a cross section to \( X \) at \( p \) and \( P_X : (\Sigma, p) \to (\Sigma, p) \) be Poincaré map.

Lemma 2.3. There exists a cross section \( \Sigma_p \subset \Sigma \) and a neighborhood \( V \subset \text{Diff}^\infty(M^2) \) of the identity map in the \( C^0 \) topology such that every \( f \in \mathcal{C}(X) \cap V \) induces a local diffeomorphism \( f_X : (\Sigma_p, p) \to (\Sigma_p, p) \) of class \( C^\tau \), with \( f_X \circ P_X = P_X \circ f_X \).

Proof. Let \( \tau > 0 \) be the period of \( \gamma_p \). We can assume that \( \Sigma = h^{-1}(\{0\} \times (-1,1)) \), where \( h : V \to (-1,1)^2 \) is a flow box for \( X \) at \( p \). Let \( \pi : V \to \Sigma \) be the projection along of the orbits of \( X \) and recall that \( P_X = \pi \circ X^\tau \). Put \( U = h^{-1}((-1/2,1/2)^2) \) and

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There exists a neighborhood $\mathcal{U}$ of $id \in \text{Diff}^s(M^2)$, in the $C^0$ topology, and $\epsilon > 0$ such that $f(U) \subset V$ and $f(\Sigma_\epsilon) \subset U$ for each $f \in \mathcal{U}$ and also $X^r(\Sigma_\epsilon) \subset U$. Choose $\Sigma_p = \Sigma_\epsilon$. For each $f \in \mathcal{U}$ there is defined a map $f_X : \Sigma_p \to \Sigma$ by $f_X(q) = \pi(f(q))$. We are going to show that $f_X \circ P_X = P_X \circ f_X$. Let $[x,y] \subset \Sigma$ be the arc with extremes $x$ and $y$ and define $x \leq y$ if $[p,x] \subset [p,y]$. Assume that $P_X(q) \leq q$ for every $q \in \Sigma_p$. For a fixed $f \in \mathcal{U}$ there are two possibilities $f_X(q) \geq q$ or $f_X(q) \leq q$. Let us consider the case $P_X(q) \leq q$ and $f_X(q) \geq q$. If $P_X(q) = q$, i.e., the orbit of $q$ by $X$ is periodic, then the orbit by $f_X(q)$ is also closed, therefore $f_X \circ P_X(q) = f_X(q)$. If $P_X(q) < q$, then $f_X(P_X(q))$ belongs to the orbit of $X$ by $f_X(q)$ and $P_X(q) \leq f_X(P_X(q)) < f_X(q)$. Thus, $f_X \circ P_X(q) = P_X \circ f_X(q)$. The other cases are analogous.

Proof (of Proposition 2.1). We begin proving this proposition for $n = 2$. Let $\varphi \in A^2(\mathbb{R}^2, N)$ and assume that there exist an $\mathbb{R}^2$-orbit $P$ such that its closure, $\text{cl}(P)$, contains an $S^1$-orbit $O_0$, see Figure 2. Then, there exists at least one $\mathbb{R}$-orbit $O$ contained in $\text{Front}(P)$ such that $\text{cl}(O) \supset O_0$. If $G_0$ and $G$ are the isotropy groups of $O_0$ and $O$, respectively, then $G = G_0^0$. Let $X_i = X_{w_i}$, $i = 1, 2$, where $\{w_1, w_2\}$ is a set of generators of $G_0$ such that $w_2 \in G_0^0$. Note that $X_i|_{O \cap O_0} \equiv 0$ and $O_0$ is a periodic orbit of $X_1$ of period 1. Let $\Sigma$ be a transversal section to $X_1$ at $p \in O_0$ and $P_{X_1} : (\Sigma, p) \to (\Sigma, p)$ be the Poincaré map. Without loss of generality we can assume that $P_{X_1}$ is a topological contraction, otherwise we consider the Poincaré map of $-X_1$. Let $\mathcal{U}$ and $\Sigma_p$ be as in the Lemma 2.3 and $\delta > 0$ such that $X_1^t \in \mathcal{U}$ for all $t \in (-\delta, \delta)$. Fix a $t \neq 0$ in the interval $(-\delta, \delta)$ and put $f = X_1^t$. It follows from the commutativity of $X_1$ with $X_2$ that $f \in \mathcal{C}(X_1)$. Note that $fX_1(q) = q$ if $q \in O \cap \Sigma_p$ and that $fX_1(q) \neq q$ if $q \in P$. By Lemma 2.3, $fX_1 \circ P_{X_1} = P_{X_1} \circ fX_1$, and by N. Kopell Lemma $fX_1 = id$. This contradiction completes the proof in the case $n = 2$.

Assume now that $n \geq 3$ and that there exist a $T^k \times \mathbb{R}^{n-k}$-orbit $P$, with $k < n-1$, such that $\text{cl}(P)$ contains a $T^{n-1}$-orbit $O_0$. If $G_0$ and $G_P$ are the isotropy groups of $O_0$ and $P$, respectively, then $G_P \subset G_0$. Let $X_1, \ldots, X_n$ the infinitesimal generators adapted to $O_0$ such that the linear $(n-2)$-subspace $H$ of $\mathbb{R}^n$ generated by $\{w_1, \ldots, w_{n-2}\}$ contains $G_P$. There exists a neighborhood $V$ of $O_0$ such that the action $\psi_\varphi \in A^2(\mathbb{R}^{n-1}, N)$ induced by

![FIG. 1.](image-url)
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$\varphi$ and $O_0$, i.e., generated by $X_1, \ldots, X_{n-1}$, has not $T^{n-1}$-orbits inside $V$. Otherwise, by Corollary 2.1, either $P$ is an $T^{n-1} \times \mathbb{R}$-orbit or $O_0 \not\subset \text{cl}(P)$. If $O$ is a $T^s \times \mathbb{R}^{n-s-1}$-orbit of $\varphi$ such that $O \subset \text{cl}(P)$, then $O_0 \subset \text{cl}(O)$ and $s < n - 1$. Consequently, if $G_O$ is the isotropy group of $O$, then $G_O \subset G_0$ and $G_O^0 = G_0^0$. Since $k, s < n - 1$, we can assume that $H \cap G_O$ is isomorphic to $\mathbb{Z}^s$ and $w_{n-1}, w_n \notin H$. Let $p \in O_0$ and consider $\omega_1, \ldots, \omega_{n-1}$ as in the proof of Lemma 2.1. By Lemma 2.1 $\omega_{n-1}$ (or $\omega_{n-1}^1$) is a topological contraction. Therefore, if $q \in \Sigma_{n-1} \cap P$, then $O_{h_{n-1}^1(q)}(\widehat{\varphi}_{n-1})$ is a $\mathbb{R}^2$-orbit in $A$ that contains the $\widehat{\varphi}_{n-1}$-orbit $S^1 \times \{0\}$ in its closure. By the first part of the proof this is a contradiction.

\[\text{FIG. 2.}\]

Remark 2.2. Let $\varphi \in \mathcal{A}_n$, $p \in \text{Sing}_{n-2}(\varphi)$ and $V_p$ a neighborhood of $O_p$. (a) Assume that $V_p$ satisfies (1) and $\text{Front}(V_p)$ is a $T^{n-1}$-orbit. Since $r \geq 2$, it follows by Proposition 2.1, that there is no $T^s \times \mathbb{R}^{n-s}$-orbit with $s \neq n - 1$ inside $V_p$. Thus, we can say that one of the following possibilities is satisfied:

(a1) $V_p \setminus O_p$ is a $T^{n-1} \times \mathbb{R}$-orbit;
(a2) $V_p$ contains infinitely many $T^{n-1} \times \mathbb{R}$-orbits;
(a3) $V_p \setminus O_p$ contains only $(n - 1)$-dimensional orbits.

(b) Assume now that $V_p$ satisfies (2), then:

(b1) if there is one $T^s \times \mathbb{R}^{n-s-1}$-orbit, $s \neq n - 1$, such that its closure contains $O_p$, then every $n$-orbit in $V_p \setminus O_p$ is not homeomorphic to $T^{n-1} \times \mathbb{R}$;
(b2) if there are no $T^s \times \mathbb{R}^{n-s-1}$-orbits, $s \neq n - 1$, which contain $O_p$ in its closure, then there is only one $n$-orbit and it is homeomorphic to $T^{n-1} \times \mathbb{R}$.

Proposition 2.2. Let $O_0$ be a $T^{n-1}$-orbit of $\varphi \in A^2(\mathbb{R}^n, N)$. Then, there exists a $\varphi$-invariant neighborhood $V_0$ of $O_0$, such that every connected component $U$ of $V_0 \setminus O_0$ satisfies one of the following properties:

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(1) \( U \) is an \( T^{n-1} \times \mathbb{R} \)-orbit;

(2) \( U \) contains infinitely many \( T^{n-1} \)-orbits approaching \( O_0 \) and every \( n \)-dimensional orbit inside \( U \) is a \( T^{n-1} \times \mathbb{R} \)-orbit;

(3) There exist \( s \in \{0, 1, \ldots, n-2\} \) such that \( U \) is the union of \( T^s \times \mathbb{R}^{n-s-1} \)-orbits.

Proof. By the continuity of the infinitesimal generators of \( \varphi \) there is a neighborhood \( V \) of \( O_0 \) such that every orbit by points in \( V \) has dimension at least \( n-1 \). Since \( r \geq 2 \), it follows from Proposition 2.1 that there are no \( T^s \times \mathbb{R}^{n-s} \)-orbits, \( s \neq n-1 \) approaching \( O_0 \). Therefore, if \( U \) one connected component of \( V \setminus O_0 \), then there are two possibilities:

(i) There are infinitely many \( n \)-orbits in \( U \) approaching \( F \). In this case we will show that (2) is verified. The other possibility is that there exists a sequence \( \{O_i\}_{i \in \mathbb{N}} \) of \( n \)-dimensional orbits inside \( U \), which are not homeomorphic to \( T^{n-1} \times \mathbb{R} \) and that approach \( O_0 \). Since \( \text{cl}(O_i) \cap \text{Sing}(\varphi) = \emptyset \) for each \( j \in \{0, \ldots, n-2\} \), then the compact set \( \text{cl}(O_i) \setminus O_i \) is the union of \((n-1)\)-dimensional orbits, of which at least one is compact, this contradicts the Proposition 2.1.

(ii) There are only a finite number of \( n \)-orbits in \( U \). If it happens to exist a \( T^{n-1} \times \mathbb{R} \)-orbit \( O \) such that \( \text{Front}(O) \supset O_0 \), then we can assume that \( U = O \), therefore (1) is verified. If there is no such \( T^{n-1} \times \mathbb{R} \)-orbit, then we can assume that the orbit of each \( p \in U \) is \((n-1)\)-dimensional. We also can assume that \( \varphi \) has no \( T^{n-1} \)-orbits inside \( U \), otherwise (2) is verified. Reducing the size of \( V \) it follows, from Lemma 2.1 that there exist \( s \in \{0, \ldots, n-2\} \) such that \( O_p \) is a \( T^s \times \mathbb{R}^{n-s-1} \)-orbit for each \( p \in U \). Thus, (3) is verified and this completes the proof.

Theorem 2.1 (Chatelet-Rosenberg, [3]). Let \( N \) be a compact orientable \( n \)-manifold with non-empty boundary. Suppose that \( \psi \) is a \( C^2 \) locally free action of \( \mathbb{R}^{n-1} \) on \( N \), then \( N \) is diffeomorphic to \( T^{n-1} \times [0, 1] \).

Figure 3 illustrates Theorem B for some \( \varphi \in \mathcal{A}_2 \).

FIG. 3.

Proof (of Theorem B). Let \( \mathcal{O} \) be a \( T^{n-1} \times \mathbb{R} \)-orbit and \( \mathcal{U} \) the family of all \( \varphi \)-invariant neighborhoods \( U \supset \mathcal{O} \) homeomorphic to \( T^{n-1} \times \mathbb{R} \) that do not contain a \( T^s \times \mathbb{R}^{n-s} \)-orbit.
with $s < n - 1$. The inclusion relation defines a parcial order in $\mathcal{W}$ and by Zorn’s Lemma there exists a maximal element $U_M$ in $\mathcal{W}$. We are going to show that $N \setminus U_M$ is either an $T^{n-1}$-orbit or the union of two $T^{n-2}$-orbits. Assume that $N \setminus U_M$ has non-empty interior, then $\text{cl}(U_M) \setminus U_M$ has two connected components. By Definition 2.2 at each $p \in \text{Sing}_i(\varphi)$, $i = 1, \ldots, n - 3$, $\varphi$ is of type $n$ and $n$-finite. This fact and Lemma 2.2 implies that $(\text{cl}(U_M) \setminus U_M) \cap \text{Sing}_i(\varphi) = \emptyset$, $i = 1, \ldots, n - 3$. Moreover, there exists one connected component $F$ of $\text{cl}(U_M) \setminus U_M$ that is not a $T^{n-2}$-orbit. We know that $F$ is $\varphi$-invariant and will show that $F \cap \text{Sing}_{n-2}(\varphi) = \emptyset$. In fact, if there exists $p \in F$ such that $O_p$ is a $T^{n-2}$-orbit and $V_p$ is a neighborhood of $O_p$ that satisfies Definition 2.2 (1), then $U_M \cup V_p$ would be a member of $\mathcal{W}$ containing $U_M$ properly. If $V_p$ satisfies condition (2), then, since $F \neq F_p$, there are $T^s \times \mathbb{R}^{n-s-1}$-orbits, $s \neq n - 1$, arriving at $O_p$ and by Remark 2.2 we would have $T^l \times \mathbb{R}^{n-l}$-orbits, $l \neq n - 1$, inside $U_M$. Therefore $F$ is an $T^{n-1}$-orbit. If (1) or (2) of Proposition 2.2 is verified, then there exists an open $\varphi$-invariant set $V$ homeomorphic to $T^{n-1} \times \mathbb{R}$, which does not contain $T^s \times \mathbb{R}^{n-s}$-orbits with $s \neq n - 1$ and such that $\text{Front}(V) \subset F$. If (3) of Proposition 2.2 is verified, then by Theorem 2.1 there exists an open $\varphi$-invariant set $V$ homeomorphic to $T^{n-1} \times \mathbb{R}$ and such that $\text{Front}(V) \subset F$. The open set $U_M \cup V \in \mathcal{W}$ and contains properly $U_M$, but this contradicts the fact that $U_M$ is maximal. Thus, $\text{Front}(U_M) = N \setminus U_M$.

Assume that $\text{Sing}_{n-2}(\varphi) = \emptyset$, then $\text{Front}(U_M)$ is homeomorphic to $T^{n-1}$. Therefore, $N$ is $T^{n-1}$ bundle over $S^1$. If $\text{Sing}_{n-2}(\varphi) \neq \emptyset$, then $\text{Front}(U_M)$ is the union of two $T^{n-2}$-orbits consequently, $N \in \mathcal{H}_n$.

REFERENCES