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# Polar multiplicities, equisingularity and Euler obstruction of map germs from $\mathbb{C}^n$ to $\mathbb{C}^n$

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We study how to minimize the number of invariants that is sufficient for the Whitney equisingularity of a one parameter deformation of corank one finitely determined holomorphic germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$. According to a result of Gaffney, these are the 0-stable invariants and all polar multiplicities which appear in the stable types of a stable deformation of the germ. First we describe all stable types, then we show how the invariants in the source and the target are related and reduce the number using these relations. We also investigate the relationship between the local Euler obstruction and the polar multiplicities of the stable types. We show an algebraic formula for the local Euler obstruction in terms of the polar multiplicities and show that the Euler obstruction is an invariant for the Whitney equisingularity. October, 2003 ICMC-USP

## 1. INTRODUCTION

Gaffney describes in [3] the following problem: "Given a 1-parameter family of map germs $F\colon \mathbb{C} \times \mathbb{C}^n, (0,0) \to \mathbb{C} \times \mathbb{C}^p, (0,0)$, find analytic invariants whose constancy in the family implies the family is Whitney equisingular." He shows that for the class of finitely determined map germs of discrete stable type, the Whitney equisingularity (hence the topological triviality) of such a family is guaranteed by the invariance of the zero stable types and the polar multiplicities associated to all stable types.

The number of invariants depends on the dimensions $(n, p)$ and it can be very big according to $n$ and $p$ are big. Then a natural question arises: "For a fixed pair of dimensions

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$(n, p)$, what is the minimum number of invariants in Gaffney's theorem that are necessary to guarantee the Whitney equisingularity of the family?"

When $p = 1$, Teissier in [18] showed that the constancy of the sequence $\mu^* = (\mu^0, \ldots, \mu^{n+1})$, where $\mu^j$ denotes the Milnor number of the intersection of $F^{-1}(0)$ with a generic $j$-plane, $j = 0, \ldots, n+1$, is enough to guarantee the Whitney equisingularity of the family.

Other cases where an answer to this question is known are $n = p = 2$ and $n = 2$, $p = 3$. They were studied by Gaffney in [3]. More recently, the first named author dealt with the cases $n = p = 3$ in [20] and $n = 3$, $p = 4$ in [21]. Also Vohra in [22] used Gaffney's approach to study map germs from n-space to the plane ($n \geq 3$), both for map germs germs of corank one.

In particular, the number of invariants required to guarantee the Whitney equisingularity in each of these situations was shown to be smaller than the a priori number given by the general result of Gaffney.

In this paper we deal with the case $n = p$ and consider germs of corank one. We reduce the number of invariants needed by finding relations among them and using the fact that they are upper semi-continuous.

Another invariant that is associated to the polar varieties is their local Euler obstruction. This local invariant for singular varieties was introduced by R. MacPherson in [16] in a purely obstructional way. Then, Gonzales-Sprinberg gave in [7] an algebraic interpretation of the local Euler obstruction and Lê and Teissier used this interpretation to show that the local Euler obstruction is an alternate sum of the polar multiplicities of the local polar variety (see [11]).

Here we apply these results to obtain explicit algebraic formulae for the Euler obstruction of the stable types of map germs from $\mathbb{C}^n$ to $\mathbb{C}^n$.

In section 2 we recall the basic definitions and results needed to prove our main result, which is shown in section 3. In section 4 we present the formulae for the local Euler obstruction and in section 5 we give an algorithm to compute the ideals that define the stable types in the source.

## 2. NOTATION AND PRELIMINARIES

We follow the notation used by Gaffney in [3] and denote by $\mathcal{O}(n, p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$. Let $\mathcal{O}_e(n, p)$ denotes the set of germs at the origin but not necessarily origin preserving.

For a germ $f \in \mathcal{O}_e(n, p)$, $J(f)$ denotes the ideal generated by the set of $p \times p$ minors of the derivative of $f$. The critical set of $f$, denoted by $\Sigma(f)$, is the set of points $x \in \mathbb{C}^n$ such that $J(f)(x) = 0$. The discriminant of $f$ is the image of the critical set by $f$,that is, $\Delta(f) = f(\Sigma(f))$. The determinant of the derivative of $f \in \mathcal{O}_e(n, n)$ is denoted by $J[f]$.

Our interest is in $\mathcal{A}$-finitely determined map-germs, where $\mathcal{A}$ denotes the usual Mather group of germs of holomorphic difeomorphisms in the source and in the target. We denote by $F$ a versal unfolding of such an $f$.

Definition 2.1. We say that a stable type $\mathcal{Q}$ appears in $F$ if for any representative $F = (id, f_u(x))$ of $F$, there exists a point $(s, y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the germ $f_u : \mathbb{C}^n, S \to$

$\mathbb{C}^p, y$ is a stable germ of type $\mathcal{Q}$ where $S = f^{-1}(y) \cap \Sigma(f_u)$. The points $(s, y)$ and $(s, x)$ with $x \in S$ are called points of stable type $\mathcal{Q}$ in the target and in the source, respectively.

If $f$ is stable, we denote the set of points in $\mathbb{C}^s \times \mathbb{C}^p$ of type $\mathcal{Q}$ by $\mathcal{Q}(f)$ and the set $\mathcal{Q}_S(f) = f^{-1}(\mathcal{Q}(f)) - \mathcal{Q}_\Sigma(f)$, where $\mathcal{Q}_\Sigma(f)$ denotes $f^{-1}(\mathcal{Q}(f)) \cap \Sigma(f)$.

If $f$ is finitely determined, we denote by $\overline{\mathcal{Q}(f)} = (\{0\} \times \mathbb{C}^p) \cap \overline{\mathcal{Q}(F)}$ and $\overline{\mathcal{Q}_S(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_S(F)}$, $\overline{\mathcal{Q}_\Sigma(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_\Sigma(F)}$, where the bar over a set means the closure of this set.

Definition 2.2. We say that $\mathcal{Q}$ is a zero-dimensional stable type for the pair $(n, p)$ if $\mathcal{Q}(f)$ has dimension 0 where $f$ is a representative of the stable type $\mathcal{Q}$.

We observe that the set $\overline{\mathcal{Q}(F)} = \cap F(j^{(p+1)} F^{-1}(\overline{\mathcal{A}z_i}))$ is closed and analytic, where $z_i$ is the $p+1$ jet of the stable type $\mathcal{Q}$ and $\mathcal{A}z_i$ is the $\mathcal{A}$-orbit of $z_i$.

A finitely determined germ $f$ has discrete stable type if there exist a versal unfolding $F$ of $f$ in which appears only a finite number of stable types. If $(n, p)$ is in the nice range of dimensions, then any finitely determined germ $f$ has a discrete stable type.

We say that an unfolding $F$ of $f$ is Whitney equisingular if there exists a regular stratification of the source and the target, i.e. satisfy the Whitney conditions **a** and **b**. In this case we say that the family $F$ is Whitney equisingular.

It is shown in [3] p. 208, that if $f$ has discrete stable type, and $F$ is a versal unfolding which only a finite number of stable types, then the family $F$ is Whitney equisingular.

Suppose that $\mathcal{Q}(F) = \{p_1, \cdots, p_r\}$ is the set of points of zero-dimensional type, where $F$ is a versal unfolding of $f$. The 0-stable invariant of type $\mathcal{Q}$ of $f$, denoted by $m(f; \mathcal{Q})$ is the multiplicity of the ideal $m_s \mathcal{O}_{\overline{\mathcal{Q}(F)},(0,0)}$ in $\mathcal{O}_{\overline{\mathcal{Q}(F)},(0,0)}$, where $m_s$ denotes the ideal generated by the coordinates of the space of parameters $\mathbb{C}^s$.

Let $F : \mathbb{C} \times \mathbb{C}^n, (0, 0) \to \mathbb{C} \times \mathbb{C}^p, (0, 0)$, $F = (u, \overline{f}(u, x))$, be a 1-parameter unfolding of a finitely determined germ $f$, such that $\overline{f}(u, -)$ preserves the origin for all $u$. Let $T := \mathbb{C} \times \{0\}$.

We say that $F$ is Whitney equisingular along the parameter space $T$ if the pair $(F^{-1}(0) - T, T)$ satisfies the Whitney conditions **a** and **b**.

The unfolding $F$ is a *good unfolding* of $f$ if there exist neighborhoods $U$, $W$ of the origin in $\mathbb{C} \times \mathbb{C}^n$ and $\mathbb{C} \times \mathbb{C}^p$ respectively such that $F^{-1}(W) = U$, $F$ maps $U \cap \Sigma(F) - T$ to $W - T$ and if $(t_0, y_0) \in W - T$, then the germ $f_{t_0} : \mathbb{C}^n, S \to \mathbb{C}^p, y_0$ is stable, where $S = F^{-1}(t_0, y_0) \cap \Sigma(F)$.

A good unfolding is said to be *excellent* if all the 0-stable invariants are constant in the unfolding and $f$ is of discrete type. In the equidimensional case $(n = p)$, it is also assumed that the degree of $f$, $\delta(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{f^*(m_n)\mathcal{O}_n}$, is constant in the unfolding. Using the polar multiplicities of the polar varieties of the stable types (defined by Teissier in [19]) and Thom's isotopy lemmas, Gaffney showed the following principal result.

THEOREM 2.1. ([3], p. 207) *Suppose that* $F : \mathbb{C} \times \mathbb{C}^n, (0,0) \to \mathbb{C} \times \mathbb{C}^p, (0,0)$ *is an excellent unfolding of a finitely determined germ* $f \in \mathcal{O}(n,p)$. *Also suppose that the polar invariants of all stable types defined in:*

*1.the discriminant* $\Delta(f_t) = f_t(\Sigma(f_t))$,
*2.the singular set* $\Sigma(f_t)$ *and also in the set*
*3.*$X(f_t) = \overline{\left(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t)\right)}$,

*are constant at the origin for all* $t$. *Then the unfolding is Whitney equisingular.*

*Remark 2. 1.* 1. The theorem also implies that such unfolding is topologically trivial; for the proof of this result Gaffney uses Thom's second isotopy lemma for complex analytic mappings, see [3] page 204.

2. The theorem remains valid if we replace the term "an excellent unfolding" in the hypothesis by "a 1-parameter unfolding which, when stratified by stable types and by the parameter axis $T$, has only the parameter axis $T$ as 1-dimensional stratum at the origin" ([22]). We shall apply this version of the Theorem in order to give sufficient conditions for the Whitney equisingularity of 1-parameter unfolding.

In the next section we shall need the following results.

THEOREM 2.2. (Lê-Greuel, [10], page 47) *Let* $X_1$ *be an I.C.I.S., with a singularity at* $0 \in \mathbb{C}^n$. *Let* $X$ *be an I.C.I.S. defined in* $X_1$ *by* $f_k = 0$, *and let* $f_1, ..., f_{k-1}$ *be the generators of the ideal that defines* $X_1$ *at* $0$ *in* $\mathbb{C}^n$. *Then*

$$\mu(X_1,0) + \mu(X,0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(f_1,...,f_{k-1},J(f_1,...,f_k))}.$$

In the case of a zero-dimensional I.C.I.S. we can use the following simpler formula. Let $f : \mathbb{C}^k, 0 \to \mathbb{C}^k, 0$ be a germ such that $X = f^{-1}(0)$ is an I.C.I.S. Then $\mu(X,0) = \delta(f) - 1$ (see [12] p. 78).

Another elementary result that we use here is the following. Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ be a finitely determined germ. Then $f : \Sigma(f) \subset \mathbb{C}^n, 0 \to \Delta(f) \subset \mathbb{C}^n, 0$ is bimeromorphic (see [2] p.154, or [12]).

## 3. EQUISINGULARITY OF MAP GERMS IN $\mathcal{O}(N,N)$

### 3.1. The stable types in $\mathcal{O}(n,n)$

As highlighted in the introduction, our aim is to minimize the number of invariants defined in the stable types of $f$ whose constancy in the family $f_t$ implies the family is Whitney equisingular.

The strategy is to apply Theorem 2.1 and the techniques used by Gaffney in [3], that is, stratify the source and the target by the stable types and establish relations among the

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invariants on the strata. As these invariants are upper semi-continuous, the relations will allow us to reduce the number of invariants required in Gaffney's theorem.

We present here an explicit description of all stable types in the source and target for corank one map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^n, 0)$. This description is done in terms of subschemes of multiple points of a germ $f$. The description of the 0-stable types is shown in [14]. We generalize this description for all $r$-stable types, with $0 \leq r \leq n-1$.

For this we first give the following preliminary definition. Given a continuous mapping $f : X \to Y$ on analytic spaces, we define the $k^{tk}$ multiple point space of $f$ as

$$D^k(f) := \text{closure}\left\{(x_1, x_2, ..., x_k) \in X^k : f(x_1) = ... = f(x_k) \ \text{for}\, x_i \neq x_j, i \neq j\right\}.$$

Let $f \in \mathcal{O}(n, n)$ be a finitely determined map germ of corank 1. Choosing linearly adapted coordinates, we write $f(x, z) = (x_1, \ldots, x_{n-1}, g(x, z))$, whith $x = (x_1, \ldots, x_{n-1}) \in \mathbb{C}^{n-1}$, $z \in \mathbb{C}$ and $g : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ is a polynomial. For each partition $\mathcal{P} = (r_1, \cdots, r_\ell)$ of an integer $m \leq n$, i.e, $r_1 + \cdots + r_\ell = m$, we consider the subset $D^\ell(f, \mathcal{P})$ of $\mathbb{C}^{n-1} \times \mathbb{C}^\ell$, with $\ell := \text{length}(\mathcal{P})$, defined by

$$D^\ell(f,\mathcal{P}) = closure\left\{ (x, z_1, \cdots, z_\ell) \in \mathbb{C}^{n-1} \times \mathbb{C}^\ell : \begin{array}{l} z_i \neq z_j, \\ f(x, z_i) = f(x, z_j) \text{ and} \\ f \text{ has a singularity of type} \\ A_{r_j} \text{ at } (x, z_j) \end{array} \right\}$$

where 'closure ' means the analytic closure in $\mathbb{C}^{n-1} \times \mathbb{C}^\ell$. We remark that when $m = n$, the subsets $D^n(f, \mathcal{P})$ are called zero-schemes and are related to the 0-stable types [13]. We will soon give a structure of subschemes to the sets $D^\ell(f, \mathcal{P})$ as well.

Nearby the $(A_{r_1} + \cdots + A_{r_\ell}) = A_{r_1, \cdots, r_\ell}$ multi-germs, there are points in the target with $(r_1+1)+(r_2+1)+\cdots+(r_\ell+1) = m+\ell$ pre-images. We define an $(m+\ell)-$tuple scheme in $\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$, which, on the appropriate diagonal, specializes to the ideal defining $A_{r_1, \cdots, r_\ell}$ multi-germs. (See Lemma 3.1).

We denote the coordinates of $\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$ by

$$(x, \mathbf{z}) = (x, z_0^1, \cdots, z_{r_1}^1, z_0^2, \cdots, z_{r_2}^2, \cdots, z_0^\ell, \cdots, z_{r_\ell}^\ell)$$

and define the sheaf of ideals $\mathcal{J}^\ell(f, \mathcal{P}) = \langle h_1, h_2, \cdots, h_{m+\ell-1}\rangle \subset \mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}}$ by

$$h_i := V^{-1}. \begin{vmatrix} 1 & z_0^1 & \cdots & (z_0^1)^{i-1} & g_0^1 & (z_0^1)^{i+1} & \cdots & (z_0^1)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_1}^1 & \cdots & (z_{r_1}^1)^{i-1} & g_{r_1}^1 & (z_{r_1}^1)^{i+1} & \cdots & (z_{r_1}^1)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_0^\ell & \cdots & (z_0^\ell)^{i-1} & g_{r_0}^\ell & (z_0^\ell)^{i+1} & \cdots & (z_0^\ell)^{m+\ell-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{r_\ell}^\ell & \cdots & (z_{r_\ell}^\ell)^{i-1} & g_{r_\ell}^\ell & (z_{r_\ell}^\ell)^{i+1} & \cdots & (z_{r_\ell}^\ell)^{m+\ell-1} \end{vmatrix}$$

where $V = V(z_0^1, \cdots, z_{r_1}^1, \cdots, z_0^\ell, \cdots, z_{r_\ell}^\ell)$ is the Vandermonde determinant and $g_i^k := g(x, z_i^k)$. In $\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$ there is a diagonal of particular interest, namely,

$$\Delta(\mathcal{P}) = \{(x, \mathbf{z}) \in \mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell} | z_i^k = z_j^k,\ \forall i, j = 0, \cdots, r_k,\ \forall k = 1, \cdots, \ell\}$$

which can be parametrized by $(x, z^1, \cdots, z^\ell)$:

$$(x, \mathbf{z}) = (x, z^1, \cdots, z^1, z^2, \cdots, z^2, \cdots, z^\ell, \cdots, z^\ell) \tag{3.1}$$

with $z^i$ repeated $r_i + 1$ times. This corresponds to an embedding $j_\ell : \mathbb{C}^{n-1} \times \mathbb{C}^\ell \to \mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}$. We denote by $j_\ell^* : \mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}} \to \mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^\ell}$ the induced surjection.

Let $\mathcal{I}_{\Delta(\mathcal{P})} = \langle z_i^k - z_0^k, i = 1, \cdots, r_k; k = 1, \cdots, \ell \rangle$ be the ideal defining $\Delta(\mathcal{P})$. Then $j_\ell^*$ induces an isomorphism $\frac{\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}}}{\mathcal{I}_{\Delta(\mathcal{P})}} \cong \mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^\ell}$. Note that a generic point of $V(\mathcal{I}_{\Delta(\mathcal{P})})$ is one of the form (3.1) with $z^i \neq z^j$, for $i \neq j$.

Let $\mathcal{J}_\Delta^\ell(f, \mathcal{P})$ be the sheaf of ideals in $\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}}$ defined by

$$\mathcal{J}_\Delta^\ell(f, \mathcal{P}) := \mathcal{J}^\ell(f, \mathcal{P}) + \mathcal{I}_{\Delta(\mathcal{P})}$$

and consider the ideal $\mathcal{I}^\ell(f, \mathcal{P}) := j_\ell^*(\mathcal{J}_\Delta^\ell(f, \mathcal{P}))$ of $\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^\ell}$. Then $j_\ell^*$ also induces an isomorphism

$$j_\ell^* : \frac{\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^{m+\ell}}}{\mathcal{J}_\Delta^\ell(f, \mathcal{P})} \to \frac{\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^\ell}}{\mathcal{I}^\ell(f, \mathcal{P})}. \tag{3.2}$$

Next lemma shows us that at a generic point of $\Delta(\mathcal{P})$ we have $D^\ell(f, \mathcal{P}) = V(\mathcal{J}_\Delta(f, \mathcal{P}))$, that is, $f$ has a singularity of type $A_{r_j}$ at $(x, z^j)$ and $f(x, z^1) = \cdots = f(x, z^\ell)$. Therefore, by an abuse of notation, we will call the subscheme $V(\mathcal{I}^\ell(f, \mathcal{P})) \subset \mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^\ell}$ equipped with the structural sheaf $\frac{\mathcal{O}_{\mathbb{C}^{n-1} \times \mathbb{C}^\ell}}{\mathcal{I}^\ell(f, \mathcal{P})}$ by $D^\ell(f, \mathcal{P})$ as well.

Lemma 3.1. *([13], lemma 2.7) At a generic point of $\Delta(\mathcal{P})$ we have,*

$$\begin{aligned} \mathcal{J}_\Delta(f, \mathcal{P}) \ &= \ \left\langle \frac{\partial g}{\partial z}(x, z^1), \cdots, \frac{\partial^{r_1} g}{\partial z^{r_1}}(x, z^1), \cdots, \frac{\partial g}{\partial z}(x, z^\ell), \cdots, \frac{\partial^{r_\ell} g}{\partial z^{r_\ell}}(x, z^\ell) \right\rangle \\ &\quad + \left\langle g(x, z^i) - g(x, z^1);\ 2 \leq i \leq \ell \right\rangle + \mathcal{I}_{\Delta(\mathcal{P})}. \end{aligned}$$

We remark that for the partition $\mathcal{P} = (1, \cdots, 1)$ of $m$ with $1 + \cdots + 1 = m = \ell$, we have $D^\ell(f, \mathcal{P}) = D^\ell(f)$, where $D^\ell(f)$ denotes the set of $\ell$-multiple points of $f$.

For the partition $\mathcal{P} = (r_i)$, then $r_i = m$, $\ell = 1$, and $D^1(f, \mathcal{P}) = \Sigma^{1, \cdots, 1}(f)$, where $\Sigma^{1, \cdots, 1}(f)$ is the set of singularities of type $\Sigma^{1, \ldots, 1}$ of $f$ with $1, \cdots, 1$ repeated $r_i$-times, i.e, $f$ has an singularity of type $A_{r_i}$.

PROPOSITION 3.1. *Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ be a finitely determined map germ of corank 1. Then, for any partition $\mathcal{P} = (r_1, \ldots, r_\ell)$ of $m \leq n$ we have:*

*1. $f$ is stable if and only if $V\left(\mathcal{J}^\ell_\Delta(f, \mathcal{P})\right)$ is smooth of dimension $n - m$, or empty;*
*2. The ideal $\mathcal{J}^\ell_\Delta(f, \mathcal{P})$ at 0 is either an ICIS of dimension $n - m$, or is empty;*

*Proof.* We denote by $H^\ell$ the map germ defined by the generators $h_i$ of the ideal $\mathcal{J}^\ell(f, \mathcal{P})$ and $E(\mathcal{P})$ the map germ defined by the generators of the ideal $\mathcal{I}_{\Delta,\mathcal{P}}$.

To prove the item 1., we have from 2.13 of [13] that $f$ is stable of type $A_{r_i}$ at 0 if and only if the map germ $\left(H^\ell, E(\mathcal{P})\right)$ is a submersion, and this is equivalent to say that $D^\ell(f, \mathcal{P})$ is smooth of dimension $n - m$.

2. Suppose that $f$ is finitely determined and choose a representative $f : U \to V$ as in the Geometric Theorem of Mather-Gaffney (see [23]), we shall show that for any partition $\mathcal{P}$ of an $m \leq n$ and at any point $(x, \mathbf{z}) \neq (0, 0)$ the mapping $(H^\ell, E(\mathcal{P}))$ is a submersion. If necessary we restrict $U$ such that $f$ is a singularity of type $A_{r_1,\ldots,r_\ell}$, after reordering if necessary, we can suppose that $(x, \mathbf{z})$ is a generic point of $\mathcal{I}_{\Delta(\mathcal{P})}$ for some partition $\mathcal{P} = (r_1, \ldots, r_\ell)$, hence $(x, \mathbf{z}) = (x, z^1, \cdots, z^1, z^2, \cdots, z^2, \cdots, z^\ell, \cdots, z^\ell)$. Now we suppose that $f$ is a singularity of type $A_{r_i}$ in $(x, z^i)$. It follows by the Geometric Theorem that the multi-germ of $f$ at $\left\{(x, z^1), \ldots, (x, z^\ell)\right\}$ is stable, then by the theorem 2.13 of [13] the mapping $(H^\ell, E(\mathcal{P}))$ is a submersion, thus for any point distinct from 0, the $2m + \ell - 1$ functions generating $\mathcal{J}^\ell_\Delta(f, \mathcal{P})$ define a submersion, therefore the variety $V\left(\mathcal{J}^\ell_\Delta(f, \mathcal{P})\right)$ is an ICIS with isolated singularity at 0 of dimension $n - m$. ∎

As a consequence of this Proposition and the isomorphism given in (3.2) we obtain the following

COROLLARY 3.1. *Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ be a map germ of corank 1. Then, for any partition $\mathcal{P} = (r_1, \ldots, r_\ell)$ of $m \leq n$, the germ of $D^\ell(f, \mathcal{P})$ at 0 is an ICIS of dimension $n - m$.*

We will give now an explicit description of the stable types in the source and the target of any finitely determined map germ $f \in \mathcal{O}(n, n)$ with corank one.

For each partition $\mathcal{P} = (r_1, \ldots, r_\ell)$ of $m \leq n$, we denote by $D^\ell_1(F, \mathcal{P})$ the projection of $D^\ell(F, \mathcal{P})$ to the $(x, z)$-space. We remember that each of the sets $D^\ell_1(F, \mathcal{P})$ is a subset of $\Sigma(f)$.

EXAMPLE 3.1. For finitely determined map germs $f \in \mathcal{O}(2, 2)$ of corank 1, the possible partitions of 1 and 2 are (1), (1, 1) and (2). Then $D^1_1(f, (1)) = \Sigma(f)$, $D^2_1(f, (1, 1)) = D^2(f)$ is the set of double points of $f$ and $D^1_1(f, (2)) = \Sigma^{1,1}(f)$ is the set of cusps of $f$.

For finitely determined map germs $f \in \mathcal{O}(3, 3)$ of corank 1, the possible partitions of 1, 2 and 3 are (1), (1, 1) (2), (2, 1), (1, 1, 1) and (3). Then, $D^1_1(f, (1)) = \Sigma(f)$ is the singular set of $f$, $D^2_1(f, (1, 1)) = D^2(f)$ is the curve of double points of $f$, $D^1_1(f, (2)) = \Sigma^{1,1}(f)$ is a cuspidal curve, $D^3(f,(1,1,1)) = D^3(f)$ is the set of triple points and $D^1_1(f,(3)) = \Sigma^{1,1,1}(f)$ is the set of Swallowtails of $f$.

In the next theorem we will use the following notation.

Let $X(f) := \overline{(f^{-1}(\Delta(f)) - \Sigma(f))}$. For each partition $\mathcal{P}$ of $m \leq n$, we define the sets $X^\ell_1(f,\mathcal{P})$ by,

$$X^\ell_1(f,\mathcal{P}) := f^{-1}(f(D^\ell_1(f,\mathcal{P}))) - \left(D^\ell_1(f,\mathcal{P}) \cap \Sigma(f)\right).$$

Theorem 3.1. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined map germ of corank 1. Then,*

*1. The stables types in the* **source** *are $D^\ell_1(f,\mathcal{P}) \subset \Sigma(f)$ and $X^\ell_1(f,\mathcal{P}) \subset X(f)$, for all partitions $\mathcal{P} = (r_1,\dots,r_\ell)$ of all $m \leq n$.*

*2. The stables types in the* **target** *are $f(D^\ell_1(f,\mathcal{P})) \subset \Delta(f)$, for all partitions $\mathcal{P} = (r_1,\dots,r_\ell)$ of each $m \leq n$.*

*3. The dimensions of $X^\ell_1(f,\mathcal{P})$ and of $f(D^\ell_1(f,\mathcal{P}))$ are both $n-m$.*

*Proof.* 1. A stable map germ $f \in \mathcal{O}(n,n)$ has an $A_k$ singularity if it is $\mathcal{A}$-equivalent to the germ

$$(x_1,....,x_{n-1},z) \to (x_1,....,x_{n-1},z^{k+1} + x_1 z^{n-1} + ... + x_{k-1}z).$$

Moreover, any stable corank 1 map germ is an $A_k$ singularity for some natural number $k$, hence the set of points in $\mathbb{C}^n$ where a stable map has an $A_k$ singularity is a smooth sub-manifold of codimension $k$. The image of this set by $f$ is also a smooth sub-manifold of codimension $k$. Since $f$ is finitely determined, it follows by the Geometric criterion of Mather-Gaffney (see [23]), that there exist neighborhoods $U$ and $V$ of 0 in $\mathbb{C}^n$ such that $f^{-1}(0) \cap U \cap \Sigma(f) = 0$ and for each $y \in V$, $y \neq 0$, the germ $f : (\mathbb{C}^n, S) \to \mathbb{C}^n, y$ is stable ( $S = f^{-1}(y) \cap U \cap \Sigma(f)$), hence for each $x \in S$, the germ $f : \mathbb{C}^n, x \to \mathbb{C}^n, y$ is an $A_k$ for some $k$ and these sub-manifolds in the discriminant are in general position. But this occurs if and only if $r_1 + r_2 + ..... + r_j = m \leq n$. We call such multi germ and $A_{\mathcal{P}}$-singularity and the result follows from the Lemma 3.1.

2. and 3. From the corollary given in the page 19 of [6], we know that there exist neighborhoods of the origin $U_1$ in $\mathbb{C}^{n-1} \times \mathbb{C}^\ell$ and $U_2$ in $\mathbb{C}^n$ such that the map $p : D^m(f,\mathcal{P}) \to U_2$ induced by the projection $U_1 \to U_2$ is proper and finite. Since $f$ is proper and finite, the map $f \circ p$ is also proper and finite, then $V = (f \circ p)(D^m(f,\mathcal{P}))$ is an analytic subvariety $n-m$-dimensional, in particular, $f(D^\ell_1(f,\mathcal{P}))$ is $n-m$-dimensional. Since $D^\ell_1(f,\mathcal{P})$ is $n-m$-dimensional and $f$ is proper and finite, we also obtain that $X^\ell_1(f,\mathcal{P})$ is an analytic space of dimension $n-m$. ▮

The following proposition describes the structure of the sets $D^\ell_1(f,\mathcal{P}) \subset \Sigma(f)$.

Proposition 3.2. *Let $p_\ell : \mathbb{C}^{n-1} \times \mathbb{C}^\ell \to \mathbb{C}^n$ be the projection that forgets the last $\ell - 1$ coordinates and $p^*_\ell$ the associated surjection. Let $\mathcal{I}^\ell_1(f,\mathcal{P}) := (p^*_\ell)^{-1}(\mathcal{I}^\ell(f,\mathcal{P}))$. Then $p^*_\ell$ induces an isomorphism*

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$$p_\ell^* : \frac{\mathcal{O}_n}{\mathcal{I}_1^\ell(f,\mathcal{P})} \to \frac{\mathcal{O}_{n-1+\ell}}{\mathcal{I}^\ell(f,\mathcal{P})}.$$

*For each partition $\mathcal{P} = (r_1, \dots r_\ell)$ of $m \leq n$, the set $D_1^\ell(f,\mathcal{P})$ is an ICIS.*

In order to obtain the set $D_1^\ell(f,\mathcal{P})$ we consider the following diagram, which shows how the sets $D^\ell(f,\mathcal{P})$ are related with the set $D^\ell(f)$ for all partitions $\mathcal{P}$ of $m \neq n$.

$$\begin{array}{ccccccc}
 & & & & & & D^\ell(f,\underbrace{(1,\dots,1)}) \\
 & & & & & & \downarrow\ (\ell-times) \\
 & & \vdots & & & & D^{\ell-1}(f,(1,\dots,1)) \\
 & & \downarrow & & & & \downarrow \\
\vdots & & D^2(f,(2,2)) & & & & \vdots \\
\downarrow & \nearrow & & \searrow & & & \downarrow \\
D^1(f,(4)) & \to & D^2(f,(3,1)) & \to & D^3(f,(2,1,1)) & \to & D^4(f,(1,1,1,1)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & D^1(f,(3)) & \to & D^2(f,(2,1)) & \to & D^3(f,(1,1,1)) \\
 & & & & \downarrow & & \downarrow \\
 & & & & D^1(f,(2)) & \to & D^2(f,(1,1)) \\
 & & & & & & \downarrow \\
 & & & & & & U \cap \Sigma(f) \xrightarrow{f} V \subset \mathbb{C}^n
\end{array}$$

In the above diagram, all horizontal and diagonal arrows are inclusions, vertical arrows are projections, and $D^k(f,1,\dots,1))$ denotes the set $D^k(f)$ of $k$ multiple points of $f$, with $1 \leq k \leq \ell$.

*Proof.* Since $p_\ell^*$ is surjective, we show that it is injective. Suppose that it is not injective, there exists a non zero element $g \in \dfrac{\mathcal{O}_{n-1+\ell}}{\mathcal{J}_\Delta(f,\mathcal{P})}$ such that $p_\ell^*(g) = 0$, therefore $g_{|p(D^\ell(f,\mathcal{P}))} = 0$ and $p(D^\ell(f,\mathcal{P}))$ is contained in the sub-manifold of the set $V(\mathcal{I}_\Delta(f,\mathcal{P}))$ defined by the zero set of $g$, hence $p_\ell(D_1^\ell(f,\mathcal{P})) \subset V(\mathcal{I}_\Delta(f,\mathcal{P}))$, and this implies that $p_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P})) \subset \mathcal{I}_\Delta^1(f,\mathcal{P})$.

From above we have that

$$p_\ell^* : \frac{\mathcal{O}_{n-1+\ell}}{\mathcal{I}_\Delta(f,\mathcal{P})} \to \frac{\mathcal{O}_n}{p_\ell^*(\mathcal{I}(f,\mathcal{P}))}$$

is an isomorphism. Then, from the lemma 1.8 of [12], we conclude that if

$$d(\mathcal{I}_\Delta(f,\mathcal{P})) \text{ and } d(j_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P})))$$

are the minimal number of generators of $\mathcal{I}_\Delta(f,\mathcal{P})$ and $p_\ell^*\mathcal{I}_\Delta(f,\mathcal{P})$ respectively, hence

$$n-1-\ell-d(\mathcal{I}_\Delta(f,\mathcal{P}))=n-d(p_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P}))),$$

and $d(p_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P})))=m$.

We choose a projection $p$ that is finite (this is possible from [6]). Then,

$$p(V((\mathcal{I}_\Delta(f,\mathcal{P})))=V(p_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P})))=D_1^\ell(f,\mathcal{P})$$

is of dimension $n-m$. Since the ideal $p_\ell^*(\mathcal{I}_\Delta(f,\mathcal{P}))=\mathcal{I}_\Delta^1(f,\mathcal{P})$ that defines $D_1^\ell(f,\mathcal{P})$ has $m$ generators, we obtain that $D_1^\ell(f,\mathcal{P})$ is an ICIS. ▮

In the following we shall show how the polar multiplicities in the target and in the source are related. We first give this relation for the polar invariants in the target.

### 3.2. The polar multiplicities in the discriminant $\Delta(f)$

To show how the polar multiplicities of the stable types in the target are related we describe the polar varieties of the strata $f(D_1^\ell(f,\mathcal{P}))$, for each partition $\mathcal{P}$ of $m\leq n$.

For a map germ $f\in\mathcal{O}(n,n)$ with corank 1, each set $f(D_1^\ell(f,\mathcal{P}))$ is of dimension $n-m$ and the polar varieties with codimension $j$ ( $0\leq j\leq n-m$), of this stratum are obtained in the following way: Choose generic projections $p:\mathbb{C}^{n-1}\times\mathbb{C}^\ell\to\mathbb{C}^n$ and $p_{n-m-j+1}:\mathbb{C}^n\to\mathbb{C}^{n-m-j+1}$. Then, each polar variety is defined by:

$$P_j(f\circ p(D^\ell(f,\mathcal{P})))=\overline{\Sigma(p_{n-m-j+1}|f\circ p(D^\ell(f,\mathcal{P})^0)}$$

for $0\leq j\leq n-m$, where $D^\ell(f,\mathcal{P})^0$ denotes the regular part of $D^\ell(f,\mathcal{P})$.

We see in [19] that the codimension of the polar variety $P_j(f\circ p(D^\ell(f,\mathcal{P})))$ in $\mathbb{C}^n$ is $j$. The polar invariants associate to each polar variety are the multiplicities $m(P_j(f\circ p(D^\ell(f,\mathcal{P}))))$, $0\leq j\leq n-m$. We will denote them simply by $m_j(f\circ p(D^\ell(f,\mathcal{P})))$.

To compute the multiplicities $m_j((f\circ p(D^\ell(f,\mathcal{P}))))$, we need, by definition, to consider the sets

$$\overline{\Sigma(p_{n-m-j+1}|f\circ p(D^\ell(f,\mathcal{P})^0)}.$$

However, it is better to work with the sets

$$X_j(\mathcal{P}):=\overline{\Sigma(p_{n-m-j+1}\circ f\circ p|D^\ell(f,\mathcal{P}))},$$

which can also be described by

$$X_j(\mathcal{P})=V\left(\mathcal{I}^\ell(f,\mathcal{P}),\ J(p_{n-m-j+1}\circ f\circ p,\ \mathcal{I}^\ell(f,\mathcal{P})\right).$$

The advantage to work with these sets are that they are in the source and the equations that define the associate polar varieties are computable. Our strategy is to compute the polar invariants, choosing generic projections for these sets. Besides, considering that $f$ is bimeromorphic, we can compute the polar invariants in the target.

Let $\mathcal{P} = (r_1, \cdots, r_\ell)$ be a partition of $m \leq n$ with $r_1 \geq r_2 \geq \cdots \geq r_\ell \geq 1$. Define $N(\mathcal{P})$ to be the order of the subgroup of $S_\ell$ which fixes $\mathcal{P}$. Here $S_\ell$ acts on $\mathbb{R}^\ell$ by permuting the coordinates. For example, if $\mathcal{P} = (4,4,4,2,2,2,1,1)$, we have $N(\mathcal{P}) = (3!)^2 2!$. We remark that if $\mathcal{P} \neq (r_i)$, then $N(\mathcal{P}) \neq 1$.

Proposition 3.3. *Let $\mathcal{P}$ be a partition of $m \leq n$. Then,*

$$m_j((f \circ p(D^\ell(f,\mathcal{P}))) = \frac{1}{N(\mathcal{P})} deg((p_{n-m-j} \circ f \circ p)|X_j(\mathcal{P}))).$$

*Proof.* For each $j = 0, ..., n-m$, we have $X_j(\mathcal{P}) \subset D^\ell(f,\mathcal{P})$. If $\mathbf{y} = (x, z_1, z_2, \ldots, z_\ell) \in X_j(\mathcal{P})$ and $\sigma \in S_\ell$, then

$$\mathbf{y}_\sigma = (x, z_{\sigma(1)}, z_{\sigma(2)}, ..., z_{\sigma(\ell)}) \in X_j(\mathcal{P})$$

if and only if $r_{\sigma(k)} = r_i$ for all $k, i = 1, ..., \ell$. Note that there exist $N(\mathcal{P})$ of these $\sigma$. The points $\mathbf{y}$, $\mathbf{y}_\sigma$ are different, but the corresponding sets $\{z_1, z_2, ..., z_\ell\}$ are equal, i.e., the germ $f$ in $\{(x,z_1),(x,z_2),...,(x,z_\ell)\}$ has an ordinary $\ell$-multiple point in $X_j(\mathcal{P})$. We observe that each of these points point gives $N(\mathcal{P})$ points in $(p_{n-m-j} \circ f \circ p)^{-1}(z) \subset X_j(\mathcal{P})$. ∎

*Remark 3. 1.* Since the projections $p_{n-m-j}$ and $p$ are generic and the germ $f$ is finitely determined, the variety $X_j(\mathcal{P})$ is an ICIS. Therefore it is Cohen Macaulay and

$$deg(K_{n-m-j}|X_j(\mathcal{P})) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(K_{n-m-j}, \mathcal{I}^\ell(f,\mathcal{P}), J(K_{n-m-j+1}, \mathcal{I}^\ell(f,\mathcal{P}))}$$

where $K_{n-m-j} = p_{n-m-j} \circ f \circ p$.

Next theorem is the main theorem of this section.

Theorem 3.2. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined map germ of corank 1. Then,*

$$\begin{aligned}\Sigma_{i=0}^{n-m-1}(-1)^i N(\mathcal{P}) m_i(f \circ p(D^\ell(f,\mathcal{P}))) \;&=\; (-1)^{n-m}\mu(D^\ell(f,\mathcal{P})) + 1 + \\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})))}.\end{aligned}$$

*Proof.* We choose a projection $p_{n-m} : \mathbb{C}^n \to \mathbb{C}^{n-m}$ such that the degree of $p_{n-m}|f \circ p(D^\ell(f,\mathcal{P}))$ is equal to the multiplicity of $f \circ p(D^\ell(f,\mathcal{P}))$ at the origin. Analogously, we choose a projection $p_{n-m-1} : \mathbb{C}^n \to \mathbb{C}^{n-m-1}$ such that its degree is the multiplicity $m_1(f \circ p(D^\ell(f,\mathcal{P})))$ of the polar variety $P_1(f \circ p(D^\ell(f,\mathcal{P})))$ at the origin.

Let

$$X_1 = V(p_{n-m-1} \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})),$$

$$X = V(p_{n-m} \circ f \circ p, \mathcal{I}^{\ell}(f, \mathcal{P})).$$

As these varieties are I.C.I.S., we can apply theorem 2.2 to obtain that

$$\mu(X_1) + \mu(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(p_{n-m-1} \circ f \circ p, \mathcal{I}^{\ell}(f, \mathcal{P}), J[p_{n-m} \circ f \circ p, \mathcal{I}^{\ell}(f, \mathcal{P})])}$$

Since the germ $f$ is bimeromorphic and the projections are generic, we have

$$deg((p_{n-m-1} \circ f \circ p)|X_1(\mathcal{P})) = \deg(p_{n-m-1}|P_1(f \circ p(D^{\ell}(f, \mathcal{P})))).$$

From Proposition 3.3 and Remark 3.1, it follows that

$$\mu(X_1) + \mu(X) = N(\mathcal{P}) m_1(f \circ p(D^{\ell}(f, \mathcal{P}))). \tag{3.3}$$

In the same way, choosing a projection $p_{n-m-2} : \mathbb{C}^n \to \mathbb{C}^{n-m-2}$ such that the degree of $p_{n-m-2}|P_2(f \circ p(D^{\ell}(f, \mathcal{P})))$ is $m_2(f \circ p(D^{\ell}(f, \mathcal{P})))$, and letting $X_2 = V(p_{n-m-2} \circ f, \mathcal{I}^{\ell}(f, \mathcal{P}))$ we obtain that

$$\mu(X_1) + \mu(X_2) = dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(p_{n-m-2} \circ f \circ p, \mathcal{I}^{\ell}(f, \mathcal{P}), J(p_{n-m-1} \circ f \circ, \mathcal{I}^{\ell}(f, \mathcal{P})))}. \tag{3.4}$$

Again, as $f$ is bimeromorphic and the projection $p_{n-m-2}$ is generic, we have that

$$deg((p_{n-m-2} \circ f)|X_2(\mathcal{P})) = \deg(p_{n-m-2}|P_2(f \circ p(D^{\ell}(f, \mathcal{P})))).$$

From Proposition 3.3 and Remark 3.1 it follows that

$$\mu(X_1) + \mu(X_2) = N(\mathcal{P}) m_2(f \circ p(D^{\ell}(f, \mathcal{P}))).$$

Then,

$$N(\mathcal{P}) m_1(f \circ p(D^{\ell}(f, \mathcal{P}))) - \mu(X) + \mu(X_2) = N(\mathcal{P}) m_2(f \circ p(D^{\ell}(f, \mathcal{P}))).$$

Repeating this argument we will obtain the sets

$$X_{n-m-1} = V(p_1 \circ f \circ p, \mathcal{I}^{\ell}(f, \mathcal{P}))$$

$$X_{n-m} = D^{\ell}(f, \mathcal{P})$$

and the equality

$$\Sigma_{i=1}^{n-m-1} (-1)^i N(\mathcal{P}) m_i(f \circ p(D^{\ell}(f, \mathcal{P}))) - \mu(X_{n-m}) + \mu(X) =$$

$$= dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^{\ell}(f,\mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}^{\ell}(f,\mathcal{P})))}.$$

Therefore we have $\mu(X) = deg(p_{n-m} \circ f \circ p, \mathcal{I}^{\ell}(f,\mathcal{P})) - 1$. Since $f \circ p : D^{\ell}(f,\mathcal{P}) \to f(D^{\ell}_1(f,\mathcal{P}))$ is finite and bimeromorphic,

$$deg(p_{n-m}|f(D^{\ell}_1(f,\mathcal{P}))) = deg(p_{n-m} \circ f, \mathcal{I}^{\ell}(f,\mathcal{P}))$$

and $deg(p_{n-m} \circ f \circ p, \mathcal{I}^{\ell}(f,\mathcal{P})) = N(\mathcal{P}) m_0(f(D^{\ell}_1(f,\mathcal{P})))$. ▮

In the next theorem we show how the number of points of type $A_{\mathcal{P}}$ are related to the polar multiplicities of a stable unfolding.

THEOREM 3.3. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ of corank 1. Then*

$$dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}^1_{\Delta}(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}^1_{\Delta}(f,\mathcal{P})))} = N(\mathcal{P}) m_{n-m}(f(D^{\ell}_1(f,\mathcal{P}))) + \Sigma_{\mathcal{P}} c_{\mathcal{P}} \sharp A_{\mathcal{P}}.$$

*Proof.* Choose a versal unfolding $F$ of $f$ and consider $F : D^{\ell}_1(f,\mathcal{P}) \subset \mathbb{C}^n \times \mathbb{C}^s \to f(D^{\ell}_1(f,\mathcal{P})) \subset \mathbb{C}^n \times \mathbb{C}^s$. We know that $D^{\ell}_1(f,\mathcal{P})$ is an ICIS in $\mathbb{C}^n \times \mathbb{C}^s$. As $(p_1, \pi_s) : \mathbb{C}^n \times \mathbb{C}^s \to \mathbb{C} \times \mathbb{C}^s$ is a generic linear projection, we have $\Sigma((p_1,\pi_s) \circ F)|D^{\ell}_1(f,\mathcal{P})) = V(\mathcal{I}^1_{\Delta}(f,\mathcal{P}), J((p_1,\pi_s) \circ F, \mathcal{I}^1_{\Delta}(f,\mathcal{P})))$.

The invariant $m_{n-m}(f(D^{\ell}_1(f,\mathcal{P})))$ is controlled by the degree of the projection

$$\pi_s | V(\mathcal{I}^1_{\Delta}(f,\mathcal{P}), J((p_1,\pi_s) \circ F, \mathcal{I}^1_{\Delta}(f,\mathcal{P}))),$$

that is, by the colength $e_J(f)$ of the maximal ideal $m_s$ in $\mathcal{O}_n$ of the source. This is given by

$$e_J(f) = dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}^1_{\Delta}(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}^1_{\Delta}(f,\mathcal{P})))}.$$

The possible components of $V(\mathcal{I}^1_{\Delta}(f,\mathcal{P}), J((p_1,\pi_s) \circ F, \mathcal{I}^1_{\Delta}(f,\mathcal{P})))$ are the closure of the following sets $F^{-1}(P_{n-m}(D^{\ell}_1(f,\mathcal{P}),\pi_s)), \cup_{\mathcal{P}} F^{-1}(A_{\mathcal{P}})$.

To count the contribution of $\pi_s$ restricted to these components, we use the normal forms of the stable types. We choose a generic parameter $s$ and neighborhoods $U_2 \subset \mathbb{C}^n \times \mathbb{C}^s$ and $U_1 \subset \mathbb{C}^s$ such that for every point in $U_1$ we have $e_J(f)$ pre-images in $V \cap U_2$ counting multiplicity. Then

$$\begin{aligned} e_J(f) &= \textstyle\sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{s+n,x}}{(m_s, \mathcal{I}^1_{\Delta}(f,\mathcal{P}), J((p_1,\pi_s) \circ F, \mathcal{I}^1_{\Delta}(f,\mathcal{P})))} \\ &= \textstyle\sum_{x \in S} dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{(\mathcal{I}^k_1(f_s,\mathcal{P}), J(p_1 \circ f_s, \mathcal{I}^k_1(f_s,\mathcal{P})))}. \end{aligned}$$

where $S = {\pi_s}^{-1}(0) \cap V$. As the parameter $s$ is generic, we can suppose that $f_s$ is a stable multi germ of type $A_{\mathcal{P}}$, with $\mathcal{P}$ being a partition of $n$ and we count its contribution in the variety $V$.

Therefore the contribution of the points $A_{\mathcal{P}}$ in $e_J(f)$ is a constant $C_{\mathcal{P}} \in \mathbb{Z}^+$, this means that these points can or not appear in $V$.

Since $F$ restricted to each component is finite and bimeromorphic, we have

$$deg(\pi_s|V) = N(\mathcal{P})m_{n-m}(f(D_1^2(f))) + \Sigma_{\mathcal{P}} C_{\mathcal{P}} \sharp A_{\mathcal{P}}.$$

The theorem follows now by joining all above equalities and remark 3.1. ▮

### 3.3. The polar multiplicities in $\Sigma(f)$

We know from [18] that the absolute polar multiplicities of a hypersurface $X$ with isolated singularity are related to the Milnor numbers of the plane sections $\mu^k$, by the following equalities

$$m_k(X) = \mu^{k+1}(X) + \mu^k(X)$$

for $0 \leq k \leq d-1$, where $d = dim(X)$. This result is also valid for an I.C.I.S. (see [10],[4]). The absolute polar multiplicities are defined when the dimension of $X$ is $\geq 1$. The multiplicity $m_d(X)$ cannot be defined directly like the other $m_k$, $0 \leq k \leq d-1$, because the singularities of $p_1|X$ are isolated points. However, Gaffney [4] defines this multiplicity for spaces that are I.C.I.S as follows.

Definition 3.1. The $d-$th polar multiplicity at 0, denoted by $m_d(X^d)$, of a $d$-dimensional I.C.I.S. $X^d$ is defined as

$$m_d(X^d) = dim_{\mathbb{C}} \frac{\mathcal{O}_X}{J(p_1, f)}$$

where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0)$, $f^{-1}(0) = X^d$ and $p_1 : \mathbb{C}^n \to \mathbb{C}$ is a generic linear projection.

*Remark 3. 2.* As $V(p_1, f)$ is I.C.I.S., then by Theorem 2.2 we have

$$m_d(X^d) = \mu(X^d) + \mu(X^d \cap p_1^{-1}(0)).$$

When $f \in \mathcal{O}(n,n)$ is a finitely determined map germ, $\Sigma(f)$ is a hypersurface with an isolated singularity. If $f$ is of corank 1, we apply Definition 3.1 and all properties above to obtain the following proposition.

Proposition 3.4. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ of corank 1. Then, for each partition $\mathcal{P}$ of $m \leq n$*

$$\begin{aligned}\Sigma_{i=0}^{n-m-1}(-1)^i m_i(D^\ell(f,\mathcal{P})) &= (-1)^{n-m}\mu(D^\ell(f,\mathcal{P})) + 1 + \\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \tfrac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1\circ f\circ p, \mathcal{I}^\ell(f,\mathcal{P})))}.\end{aligned}$$

$$\begin{aligned}\Sigma_{i=0}^{n-m-1}(-1)^i m_i(D_1^\ell(f,\mathcal{P})) &= (-1)^{n-m}\mu(D_1^\ell(f,\mathcal{P})) + 1 + \\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \tfrac{\mathcal{O}_n}{(\mathcal{I}_\Delta^1(f,\mathcal{P}), J(p_1\circ f, \mathcal{I}_\Delta^1(f,\mathcal{P})))}.\end{aligned}$$

### 3.4. The Lê numbers of $X(f_t)$

We consider now the set
$X(f_t) = \overline{\left(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t)\right)}$. According to Theorem 2.1, we need the constancy of the polar invariants of all stable types defined in $X(f_t)$. From theorem 3.1, we see that these stable types are the sets $X_1^\ell(f_t,\mathcal{P}) = f_t^{-1}(f_t(D_1^\ell(f,\mathcal{P}))) - \left(D_1^\ell(f_t,\mathcal{P}) \cap \Sigma(f)\right)$, for each partition $\mathcal{P}$ of $m \leq n$.

Unfortunately, we do not know how the polar invariants of all stable types $X_1^\ell(f_t,\mathcal{P})$ are related, since these sets are not hypersurfaces.

On the other side, the set $X(f_t) = \overline{\left(f_t^{-1}(\Delta(f_t)) - \Sigma(f_t)\right)}$, is an hypersurface in $\mathbb{C}^n \times \mathbb{C}$, possibly with non isolated singularities. In this case, we can apply the following result of Lê and Teissier given in [11], which shows that the hypersurface $X(f_t) \subset \mathbb{C} \times \mathbb{C}^n$ is Whitney equisingular along the parameter space if the associate Lê numbers are constant, see [15]. These numbers are the generalization of the Milnor numbers for hypersurfaces with isolated singularities.

We denote by $h_t$ the function that defines the set $X(f_t)$ and call

$$X_0(f_t) = V(f_t(x, h_t(x,z))/J[f_t]).$$

THEOREM 3.4. *( [11], p. 95.) The pair $(X(f_t) - \Sigma(X_0(f)), T)$ is Whitney equisingular if, and only if the Lê numbers of $X(f_t)$ and the Lê numbers of all generic planar sections of $X(f_t)$ are constant on $T$, i.e., $\lambda^i(X_0(f_t))$ and $\lambda^i(X_0(f_t)/H^k)$ are constant on $T$, for all $i = 0, ..., k-1$, $k = 1, ..., n-2$, where $H^k$ is a generic plane of $\{t\} \times \mathbb{C}^n$.*

Using the relations given by Gaffney and Gassler in [5], p. 710 we obtain that:

$$\lambda^k(X_0(f_t)/H^{n-k}) = \lambda^k(X_0(f_t)) + m_k(X_0(f_t)),$$

and

$$\lambda^i(X_0(f_t)/H^{n-k}) = \lambda^i(X_0(f_t))$$

for $i = 1, \ldots, k-1$.

Since $\lambda^i(X_0(f_t))$ is lexicographically upper semi continuous and $m_i(X_0(f_t))$ is also upper semi continuous, we obtain the following

PROPOSITION 3.5. *The set $(X(f_t) - \Sigma(X_0(f))$ is Whitney equisingular along the parameter space $T = \mathbb{C} \times \{0\}$ if, and only if, $\lambda^i(X_0(f_t)/H^{n-k})$ and $\lambda^k(X_0(f_t))$ are constants on $T$ for all $i = 1, \ldots, k-1$ and $k = 1, ..., n-2$.*

From the results shown here, we obtain the main result of this paper:

THEOREM 3.5. *Let $F : \mathbb{C} \times \mathbb{C}^n, (0,0) \to \mathbb{C} \times \mathbb{C}^n, (0,0)$ be an unfolding of a finite germ $f \in \mathcal{O}(n,n)$ with corank 1. Then $F$ is Whitney equisingular if, and only if, for all partitions*

*$\mathcal{P}$ of each $m \leq n$, the polar multiplicities $m_{2i+1}(f_t \circ p(D^\ell(f_t, \mathcal{P})))$, $m_{2i+1}(D_1^\ell(f_t, \mathcal{P}))$, the Lê numbers $\lambda^i(X_0(f_t)/H^{n-k})$, $\lambda^k(X_0(f_t))$, and the polar multiplicities of $X_1^\ell(f_t, \mathcal{P})$ are constants at the origin for $f_t$.*

## 4. THE EULER OBSTRUCTION

The local Euler obstruction for nonsingular varieties, introduced in [16] by R. MacPherson, in a purely obstructional way is an invariant that is also associated to the polar invariants. The local Euler obstruction plays an important role in his affirmative response to a conjecture of Deligne and Grothendieck on the existence of Chern classes for singular complex algebraic varieties (see [8],[16],[11].)

In [11], Lê and Teissier proved a formula for the multiplicity of the local polar varieties and, with the aid of Gonzales-Sprinberg's purely algebraic interpretation of the local Euler obstruction, they showed that the local Euler obstruction is an alternate sum of the multiplicity of the local polar varieties.

Here we apply these results to obtain explicit and algebraic formulae for the Euler obstruction of the stable type of mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$.

**The local Euler obstruction** (see [16] or [7] for the definition and more details). Suppose that $X \subset \mathbb{C}^n$ is an analytic space of dimension $d$, $\nu$ the transformation of Nash of $X$. Let $p \in X$ and $z = (z_1, \cdots, z_n)$ be local coordinates in $\mathbb{C}^n$ such that $z_i(p) = 0$.

Let $\| z \|^2 = \Sigma z_i \overline{z_i}$. Since $\| z \|^2$ is a real-valued function, $d\| z \|^2$ may be considered as a section of $(T\mathbb{C}^n)^*$ where $*$ denotes the real dual bundle retaining only its orientation from the complex structure. We can also consider $d\| z \|^2$ as a restriction to a section $r$ of $(TX)^*$. In [1] it is showed that for small $\epsilon$, the section $r$ is non zero over $\nu^{-1}$ where $0 \leq \| z \| \leq \epsilon$. Therefore let $B_\epsilon = \{z / \| z \| \leq \epsilon\}$ and $S_\epsilon = \{z / \| z \| = \epsilon\}$. The obstruction to extending $r$ as a non zero section of $TX^*$ from $\nu^{-1}(S_\epsilon)$ to $\nu^{-1}(B_\epsilon)$, which we denote by $Eu(TX^*, r)$, lies in $H^d(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon); \mathbb{Z})$. If $O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))}$ denotes the orientation class in $H_d(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon); \mathbb{Z})$, then we define the local Euler obstruction of $X$ at $p$ to be $Eu(TX^*, r)$ evaluated on $O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))}$ or symbolically

$$Eu_p(X) = \left\langle Eu(TX^*, r), O_{(\nu^{-1}(B_\epsilon), \nu^{-1}(S_\epsilon))} \right\rangle$$

to $\nu^{-1}(B_\epsilon)$

The following result shows how the local Euler obstruction and the polar multiplicities are related.

THEOREM 4.1. (Lê Dũng Trang et B. Teissier, [11]) *Let $X$ be a reduced analytic space at $0 \in \mathbb{C}^{n+1}$ of dimension $d$. Then*

$$Eu_0(X) = \Sigma_{i=0}^{d-1} (-1)^{d-i-1} m_i(X)$$

*where $m_i(X)$ is the polar multiplicity of the polar variety $P_i(X)$.*

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From this Theorem, we can now deduce the formulae for the Euler obstructions of the stable types at $\Delta(f)$ from Theorem 3.2, and at $\Sigma(f)$ from proposition 3.4.

COROLLARY 4.1. *Let $f \in \mathcal{O}(n,n)$ be a finitely determined germ of corank 1. Then for each partition $\mathcal{P}$ of $m \leq n$ and a generic projection $p : \mathbb{C}^{n-1} \times \mathbb{C}^{\ell} \to \mathbb{C}^n$,*

$$\begin{aligned} N(\mathcal{P}) Eu_0(f \circ p(D^\ell(f,\mathcal{P}))) &= (-1)^{n-m}\mu(D^\ell(f,\mathcal{P})) + 1 \\ &\quad +(-1)^{n-m+1} dim_{\mathbb{C}} \tfrac{\mathcal{O}_{n+\ell-1}}{(\mathcal{I}_\Delta(f,\mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}_\Delta(f,\mathcal{P})))} \end{aligned}$$

$$\begin{aligned} Eu_0(D^\ell(f,\mathcal{P})) &= (-1)^{n-m}\mu(D^\ell(f,\mathcal{P})) + 1 + \\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \tfrac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})))}. \end{aligned}$$

$$\begin{aligned} Eu_0(D_1^\ell(f,\mathcal{P})) &= (-1)^{n-m}\mu(D_1^\ell(f,\mathcal{P})) + 1 + \\ &\quad (-1)^{n-m+1} dim_{\mathbb{C}} \tfrac{\mathcal{O}_{n}}{(\mathcal{I}_\Delta^1(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}_\Delta^1(f,\mathcal{P})))}. \end{aligned}$$

From this formula we can deduce the following special cases:

1. Case $\mathcal{P} = (1)$, here $Eu_0(\Sigma(f)) = (-1)^n m_i(\Sigma(f)) + (-1)^{n+1}\mu(\Sigma(f)) + 1$,

and
$$Eu_0(\Delta(f)) = \mu(\Sigma(f)) + (-1)^n dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(J[f], J(p_1 \circ f, J[f]))} + 1$$

2. Case $\mathcal{P} = (r_i)$. Here we obtain the germs of stable type 1, i.e, all stable types contained in the source of $f$. Let $\mathcal{I}^I(f)$ be the defining ideal of $\Sigma^I(f)$, where $I = (1, \cdots, 1)$, and the number 1 is repeated $r_i$ times, then

$$Eu_0(f(\Sigma^I(f))) - (-1)^{d+1}\mu(\Sigma^I(f)) - 1 = (-1)^{d+1} dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}^I(f), J(p_1 \circ f, \mathcal{I}^I(f)))}$$

As a consequence of these results and the Theorems 2.1 and 4.1 we show that the Euler obstruction is an invariant for the Whitney equisingularity.

THEOREM 4.2. *Suppose that $f \in \mathcal{O}(n,n)$ is a finitely determined germ of corank 1 and $F = (t, f_t)$ is a good 1−parameter unfolding. If $F$ is Whitney equisingular along $T = \mathbb{C} \times \{0\}$, then $Eu_0(D_1^\ell(f_t,\mathcal{P}))$ and $Eu_0(f_t \circ p(D^\ell(f_t,\mathcal{P})))$, are constant for $t$ close to the origin, for all partition $\mathcal{P}$ of $m \leq n$.*

## 5. ALGORITHM TO COMPUTE THE IDEALS OF THE STABLE TYPES IN THE SOURCE

We show here a generalization, for all $m \leq n$, of the algorithm given in [17] for the case $m = n$. This algorithm computes the ideals of the stable types in the source for all partition $\mathcal{P}$ of $m < n$. We implemented this algorithm using the software Maple.

>restart;
>with(linalg);
Define the germ $f$ and a partition $\mathcal{P}$ of $m \leq n$:
>$f_1(\mathbf{x}, z) = (\mathbf{x}, f(\mathbf{x}, z))$
Find the dimension of the "space" and the length of the partition
>
>test:=proc($f, L, j$)
local $nL, A, V, i, dv, da$;
$nL := nops(L)$;
with(linalg):
$V$ :=vandermonde($L$);
$dv$ :=factor($det(V)$);
for $i$ from 1 to $nL$ do
$V[i, j]$ :=subs($y = V[i, 2], f$);
od;
$A$ :=convert($V, matrix$); $da$ :=factor($det(A)$);
print($V$);
print(factor(da)/factor(dv));
(here it is chosen the partition of $m$ and its the length, therefore we are choosing the variables z[i,j], for instance, if $\mathcal{P} = (r_1, ..., r_\ell)$ we choose the variables

$$z[1, 0], ...z[1, r_1], z[2, 0], ...z[2, r_2], ..., z[\ell, 0], ...z[\ell, r_\ell])$$

RETURN(subs($z[1, 1] = z[1, 0], z[2, 1] = z[2, 0], da/dv$));
(here we exchange the variables $z[1, r_1] = z[1, 0]$)
> end:
> $r := test(y^5 + x[1] * y + x[2]^2 * y^2 + x[2] * y^3, [z[1, 0], z[1, 1], z[2, 0], z[2, 1]], 2)$;

EXAMPLE 5.1. Using this algorithm we find the ideals $I^\ell(f, \mathcal{P})$ for all partition $\mathcal{P}$ of 3 for the following germs

$$\begin{array}{ll} I & (x, y, z^4 + (xy + 1y^3 + y^4)z + xz^2) \\ II & (x, y, z^4 + (xy + y^3)z + xz^2) \\ III & (x, y, z^4 + (x^2 + xy^2 + y^4)z + xz^2) \end{array}$$

**Case I**:

$$I^2(f,(1,1)) = (2z_1z^2 + 2zz_1^2 + xy + y^3 + y^4, -z^2 - 4z_1z - z_1^2 + x, 2z + 2z_1)$$

$$I^2(f,(1,1,1)) = (y(y^3 + y^2 + x), x, 0, 1, 0)$$

$$I^2(f,(2,1)) = (y(y^3 + y^2 + x), x, 0, 1)$$

$$I^2(f,2)) = (-8z^3 + xy + y^3 + y^4, 6z^2 + x)$$

Analogously we compute for the cases II and III.

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