CADERNOS DE MATEMÁTICA **4**, 319–330 October (2003)
ARTIGO NÚMERO SMA#179

Parameter Dependent Quasi-Linear Parabolic Equations

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Abstract. In this work we are concerned with non-linear reaction diffusion equations with principal part given by p-laplacian and positive diffusion coefficient $a_\varepsilon \in L^\infty$, $0 < \delta < a_\varepsilon$, which has global attractors $\mathcal{A}_\varepsilon$. We obtain some properties about the convergence of the solutions of these equations and show the upper semicontinuity of the attractors. October, 2003 ICMC-USP

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma := \partial\Omega$. Consider the second order partial differential equation subjected to homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{d}{dt}u^\varepsilon(t) - \mathrm{div}(a^\varepsilon|\nabla u^\varepsilon(t)|^{p-2}\nabla u^\varepsilon(t)) = Bu^\varepsilon(t), & t > 0 \\ u^\varepsilon(0) = u_0 \in W_0^{1,p}(\Omega), \end{cases} \tag{1}$$

where $p > \max\{2, n/2\}$, $a^\varepsilon \in L^\infty(\Omega)$, $a^\varepsilon(x) \geq \delta > 0$ almost everywhere in Ω for every $\varepsilon \in [0, \varepsilon_0]$ and a^ε converges to a in $L^\infty(\Omega)$ when $\varepsilon \to 0$. According with [4], by supposing that B satisfies conditions **H-1-H-5** enumerated below, for each $\varepsilon \in [0, \varepsilon_0]$, the problem (1) has a global attractor $\mathcal{A}_\varepsilon$ in $W_0^{1,p}(\Omega)$ and $\mathcal{A}_\varepsilon$ is embedded in $L^\infty(\Omega)$. In this work, using some uniform estimates u^ε, with respect to $\varepsilon \in [0, \varepsilon_0]$, we obtain convergence of the solutions u^ε to u^0, as ε goes zero and as a consequence of this fact we get the upper semicontinuity of the global attractors $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$, on $W_0^{1,p}(\Omega)$ topology.

To properly state ours results we need to introduce some notation and some conditions that we will assume through this work. Let $H = L^2(\Omega)$, and for every $\varepsilon \in [0, \varepsilon_0]$, consider the operator A^ε, defined as follow:

* Partially Supported by PROCAD-CAPES - 0092-01-0, Brazil

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$$\begin{cases} D(A^\varepsilon) = \{u \in W_0^{1,p}(\Omega);\ \mathrm{div}(a^\varepsilon |\nabla u|^{p-2}\nabla u) \in H\}, \\ A^\varepsilon u = -\mathrm{div}(a^\varepsilon |\nabla u|^{p-2}\nabla u),\ \text{if } u \in D(A^\varepsilon), \end{cases} \tag{2}$$

with p and a^ε as we have described above.

Next we will state a set of hypothesis concerning the perturbation B. They are taken from [4] and [5] in order to show global existence and the existence of global attractors. Let us suppose that B is a Nemitskiĭ operator associated to some real function $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ and denote B_1 and B_2 the Nemitskiĭ operators associated to $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. Actually, we suppose that

H-1 $-\mathbf{b}_1$ *is increasing,* $\mathbf{b}_1(0) = 0$,*in such way that* $A^\varepsilon - B_1$ *is the subdifferential* $\partial\phi^\varepsilon$ *of the convex lower semicontinuous map*

$$\phi^\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{p}\, a^\varepsilon\, |\nabla u(x)|^p - \int_0^u \mathbf{b}_1(v)dv \right) dx, & u \in W_0^{1,p}(\Omega), \\ +\infty\ , & \text{otherwise} \end{cases}$$

(we also impose conditions in $\mathbf{b}_1$ *in such a way that* $D(\phi^\varepsilon) = W_0^{1,p}(\Omega)$, $\varepsilon \in [0, \varepsilon_0]$*);*

H-2 $\mathbf{b}_2(r) \geq k_1 r + k_2$ *for all* $r \in \mathbb{R}$*, and there exists a constant* $c_1 > 0$ *such that for all fixed* $\delta \in \mathbb{R}$ *for some choice of* $c_2 = c_2(\delta)$, $\mathbf{b}_2(r) \leq c_1 r + c_2(\delta)$, *if* $r \geq -\delta$;

H-3 $\mathbf{b}_2$ *satisfies an* L^∞ *local Lipschitz condition, meaning that if* $\mathcal{K} := \{u \in H,\ \|u\|_{L^\infty(\Omega)} \leq \eta\}$ *for some* $\eta > 0$*, then there exists* $\omega = \omega(\eta) > 0$*, such that* $\|B_2(u) - B_2(v)\|_H \leq \omega\|u - v\|_H$*, for every* $u, v \in \mathcal{K}$;

H-4 *There exists a real increasing positive function* $\mathcal{L}_0(\cdot)$, $\gamma \in (0,1)$ *and* $c \in \mathbb{R}$ *such that*

$$\|B_2 u\|_H^2 \leq \gamma \|\partial\phi^\varepsilon u\|_H^2 + \mathcal{L}_0(\|u\|_H)[(\phi^\varepsilon(u))^2 + c], \quad \forall u \in D(\partial\phi^\varepsilon) \text{ and } \varepsilon \in [0, \varepsilon_0];$$

H-5 *There exist positive constants* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\langle B_2 u, u\rangle + \alpha\phi^\varepsilon(u) \leq \beta(\|u\|_H^2 + 1) \quad \forall u \in D(\partial\phi^\varepsilon) \text{ and } \varepsilon \in [0, \varepsilon_0].$$

REMARK 1.1. *We observe that* **H-2** *does not imply linear growth. Also, if* k_1 *is negative we may incorporate* $k_1 r$ *to* b_1 *and assume that* k_1 *is zero. Therefore we may always assume that* k_1 *is non-negative.*

Following [5], the assumptions **H-1** and **H-4** imply that the problem (1) has a solution in $W_0^{1,p}(\Omega)$, for every $\varepsilon \in [0, \varepsilon_0]$, while the assumption **H-5** implies the global existence of this solutions. As we are not assuming that B is a globally Lipschtz operator, we can not assure uniqueness of solutions for (1). In order to obtain existence of global attractors we have to ensure some strong smoothing and attraction properties, and the main tool we employ to overcome this difficulty is a comparison result developed in [3]. The assumption **H-2** is used to compare the solutions of problem (1) with solutions of two auxiliary problems, both with linear perturbations. By this way we obtain $L^\infty(\Omega)$ bounds, uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$, for solutions of problem (1). With those uniforms bounds and **H-3** we can assume, without loss of generality, that the perturbation operator B satisfies a global Lipschitz condition.

After we have come to this point, we can state that the problem

$$\begin{cases} \dfrac{d}{dt}u^\varepsilon(t) + \partial\phi^\varepsilon u^\varepsilon(t) = B_2 u^\varepsilon(t), \quad t > 0, \varepsilon \in [0, \varepsilon_0], \\ u^\varepsilon(0) = u_0 \in W_0^{1,p}(\Omega), \end{cases} \tag{3}$$

has a global attractors $\mathcal{A}_\varepsilon$ in $W_0^{1,p}(\Omega)$, embedded in $L^\infty(\Omega)$, [4], for each $\varepsilon \in [0, \varepsilon_0]$, and the L^∞ bounds for the attractors are uniform on ε. Note that the problem (3) is obtained of the problem (1) by incorporate B_1 to the operator ϕ_ϵ defined in **H-1**.

Using estimates in $W_0^{1,p}(\Omega)$ of the solutions of (3), uniforms with respect to $\varepsilon \in [0, \varepsilon_0]$ and uniforms with respect to initial data in bounded sets of $W_0^{1,p}(\Omega)$, we finally show that the family of global attractors $\{\mathcal{A}_\varepsilon;\ \varepsilon \in [0, \varepsilon_0]\}$ is upper semicontinuous at $\varepsilon = 0$, on $W_0^{1,p}(\Omega)$ topology.

This work is organized as follow: In Section 2 we obtain some uniforms bounds, with respect to $\varepsilon \in [0, \varepsilon_0]$, of the solutions of (3) and using the arguments contained in [4] we get existence of global attractors of this problem. In Section 3 we obtain some properties about the convergence of the solutions of the problem (3), as ε goes to zero and obtain the upper semicontinuity of the family of global attractors $\{\mathcal{A}_\varepsilon;\ \varepsilon \in [0, \varepsilon_0]\}$ in $\varepsilon = 0$.

2. UNIFORM BOUNDS. EXISTENCE OF GLOBAL ATTRACTORS

In this section, with the assumptions **H-1**–**H-5**, we will obtain some estimates for solutions u^ε of problem (3), uniformly on $\varepsilon \in [0, \varepsilon_0]$. Those uniform bounds will ensure the existence of global attractors in $W_0^{1,p}(\Omega)$. Also they will be helpful to obtain the upper semicontinuity of this attractors.

LEMMA 2.1. *If u^ε is a solution of* (3)*, then exist a positive constant $K_1 > 0$ such that $\|u^\varepsilon(t)\|_H < K_1$ for every $t \geq 0$ and $\varepsilon \in [0, \varepsilon_0]$.*

Proof: In fact, as we have

$$\delta \int_\Omega |\nabla u(x)|^p dx \leq \int_\Omega a^\varepsilon(x) |\nabla u(x)|^p dx \leq \sup \|a^\varepsilon\|_{L^\infty(\Omega)} \int_\Omega |\nabla u(x)|^p dx, \tag{4}$$

for every $u \in W_0^{1,p}(\Omega)$, from [**H-1**], [**H-5**] (4) and Young's inequality we obtain that

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon(t)\|_H^2 \leq -C(\delta, p, \alpha, \Omega)\|u^\varepsilon(t)\|_H^p + \beta(\|u^\varepsilon(t)\|_H^2 + 1) \leq -C_1\|u^\varepsilon(t)\|_H^p + C_2,$$

where $C_1 = C_1(\delta, p, \alpha, \beta, \Omega)$, and $C_2 = C_2(\delta, p, \alpha, \beta, \Omega)$ are positive constants independent of ε. Then $y(t) := \|u^\varepsilon(t)\|_H^2$ satisfies the inequality

$$\frac{d}{dt} y(t) \leq -2C_1 (y(t))^{\frac{p}{2}} + 2C_2.$$

It follows from Lemma III.5.1 in [7] that there exists $T_0 > 0$ such that if $t > T_0$,

$$\|u^\varepsilon(t)\|_H^2 \leq \left(\frac{2C_2}{C_1}\right)^{\frac{2}{p}} + 1.$$

On the other hand, if $t \leq T_0$ we have that

$$\frac{1}{2}\|u^\varepsilon(t)\|_H^2 \leq \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle B_2 u^\varepsilon(s), u^\varepsilon(s)\rangle ds \leq \frac{1}{2}\|u_0\|_H^2 + \int_0^{T_0} \beta(\|u^\varepsilon(s)\|_H^2 + 1) ds.$$

¿From Gronwall Inequality we obtain that

$$\|u^\varepsilon(t)\|_H \leq C(u_0, \beta, T_0), \quad t \leq T_0.$$

Taking

$$K_1 = \max\left\{\left(\frac{2C_2}{C_1}\right)^{\frac{2}{p}} + 1,\ C(u_0, \beta, T_0)\right\},$$

it follows that

$$\|u^\varepsilon(t)\|_H \leq K_1$$

for every $t \geq 0$ and $\varepsilon > 0$. ∎

Lemma 2.2. *If u^ε is a solution of (3), then there exist a positive constant $K_2 > 0$, independent of ε, such that*

$$\|u^\varepsilon(t)\|_{W_0^{1,p}(\Omega)} < K_2,$$

for every $t \geq 0$ and $\varepsilon \in [0, \varepsilon_0]$.

Proof: Since

$$\frac{d}{dt}\phi^\varepsilon(u^\varepsilon(t)) = \langle B_2 u^\varepsilon(t) - \frac{d}{dt}u^\varepsilon(t), \frac{d}{dt}u^\varepsilon(t) - B_2 u^\varepsilon(t)\rangle + \langle B_2 u^\varepsilon(t) - \frac{d}{dt}u^\varepsilon(t), B_2 u^\varepsilon(t)\rangle,$$

then

$$\frac{1}{2}\|B_2 u^\varepsilon(t) - \frac{d}{dt}u^\varepsilon(t)\|_H^2 + \frac{d}{dt}\phi^\varepsilon(u^\varepsilon(t)) \leq \frac{1}{2}\|B_2 u^\varepsilon(t)\|_H^2,$$

for every $t \geq 0$. From [**H-4**] and Lemma 2.1 we get that

$$\frac{(1-\gamma)}{2}\|B_2 u^\varepsilon(t) - \frac{d}{dt}u^\varepsilon(t)\|_H^2 + \frac{d}{dt}\phi^\varepsilon(u^\varepsilon(t)) \leq \frac{1}{2}\mathcal{L}_0(K_1)[(\phi^\varepsilon(u^\varepsilon(t)))^2 + c], \qquad (5)$$

for $t \geq 0$.

On the other hand, since $\phi^\varepsilon(0) = 0$, we have that

$$\frac{1}{2}\frac{d}{dt}\|u^\varepsilon(t)\|_H^2 + \phi^\varepsilon(u^\varepsilon(t)) \leq \frac{1}{2}\frac{d}{dt}\|u^\varepsilon(t)\|_H^2 + \langle \partial\phi^\varepsilon(u^\varepsilon(t)), u^\varepsilon(t)\rangle \leq \langle B_2 u^\varepsilon(t), u^\varepsilon(t)\rangle.$$

Then, [**H-5**], imply that

$$\frac{1}{2}\|u^\varepsilon(t+r)\|_H^2 + \int_t^{t+r}(1+\alpha)\phi^\varepsilon(u^\varepsilon(s))ds \leq \frac{1}{2}\|u^\varepsilon(t)\|_H^2 + \int_t^{t+r}\beta(\|u^\varepsilon(s)\|_H^2 + 1)ds. \quad (6)$$

¿From Lemma 2.1 and the Uniform Gronwall Lemma ([7], Lemma III.1.1), (5) and (6), we get that there exists $K_2 > 0$, independent of ε, such that

$$\phi^\varepsilon(u^\varepsilon(t+r)) \leq K_2,$$

for every $t \geq 0$. The equivalence given by (4) shows this lemma. ∎

The next result ensure uniforms L^∞ estimates on $\varepsilon \in [0, \varepsilon_0]$ for solutions of auxiliary problems. Those estimates together with results in [3] will guarantee uniforms L^∞ estimates for solutions of problem (3).

Lemma 2.3. *If u^ε is the solution of*

$$\begin{cases} \frac{d}{dt}u^\varepsilon(t) + \partial\phi^\varepsilon u^\varepsilon(t) = c_1 u^\varepsilon(t) + c_2, \quad t > 0, \\ u^\varepsilon(t,x) = 0, \quad t > 0, \; x \in \partial\Omega, \\ u^\varepsilon(0) = u_0 \in W_0^{1,p}(\Omega), \end{cases} \qquad (7)$$

$\varepsilon \in [0, \varepsilon_0]$, *there exists a positive constant* $K_3 > 0$, *independent of* ε, *such that*

$$\|u^\varepsilon(t)\|_{L^\infty(\Omega)} < K_3,$$

for every $\varepsilon \in [0, \varepsilon_0]$ *and for* $t > T_1$, *with* T_1 *large enough.*

Proof: We know that $u^\varepsilon(t) \in D(A^\varepsilon)$, for $t > 0$. Then, according to [4], we have that $u^\varepsilon(t) \in L^\infty(\Omega)$. Thus, there is a positive constant $C = C(c_1, c_2, \delta, \|u^\varepsilon(t)\|_H, \|u_t^\varepsilon(t)\|_H)$ such that $\|u^\varepsilon(t)\|_\infty \leq C$. Thus we have to verify only that $\|u_t^\varepsilon(t)\|_H$ is uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$ for large enough values of t. In fact, if $0 < s < t$ and $h > 0$, we obtain that

$$\frac{1}{2}\|u^\varepsilon(t+h) - u^\varepsilon(t)\|_H^2 \leq \frac{1}{2}\|u^\varepsilon(s+h) - u^\varepsilon(s)\|_H^2 + \int_s^t c_1\|u^\varepsilon(\tau+h) - u^\varepsilon(\tau)\|_H^2 d\tau.$$

Multiplying by $1/h^2$ and letting h goes to zero, we have that

$$\|\frac{d}{dt}u^\varepsilon(t)\|_H^2 \leq \|\frac{d}{dt}u^\varepsilon(s)\|_H^2 + 2c_1\int_s^t \|\frac{d}{dt}u^\varepsilon(\tau)\|_H^2 d\tau. \tag{8}$$

Also,

$$\begin{aligned}\frac{d}{dt}\phi^\varepsilon(u^\varepsilon(t)) &= \langle \partial\phi^\varepsilon(u^\varepsilon(t)), \frac{d}{dt}u^\varepsilon(t)\rangle = \langle c_1 u^\varepsilon(t) + c_2 - \frac{d}{dt}u^\varepsilon(t), \frac{d}{dt}u^\varepsilon(t)\rangle \\ &= \frac{1}{2c_1}\frac{d}{dt}\|c_1 u^\varepsilon(t) + c_2\|_H^2 - \|\frac{d}{dt}u^\varepsilon(t)\|_H^2.\end{aligned}$$

By integration, from 0 to t,

$$\int_0^t \|\frac{d}{dt}u^\varepsilon(\tau)\|_H^2 d\tau \leq \phi^\varepsilon(u_0) + \frac{1}{2c_1}\left(c_1 K_1 + c_2\right)^2, \tag{9}$$

where K_1 is given by Lemma 2.1. From (8) and (9) we get that

$$\|\frac{d}{dt}u^\varepsilon(t)\|_H^2 \leq \|\frac{d}{dt}u^\varepsilon(s)\|_H^2 + C, \tag{10}$$

for $0 < s < t$, where by (4),

$$C = 2c_1\phi^\varepsilon(u_0) + \left(c_1 K_1 + c_2\right)^2$$

is independent of $\varepsilon \in [0, \varepsilon_0]$. Integrating (10) from 0 to t, (9) implies that

$$\|\frac{d}{dt}u^\varepsilon(t)\|_H^2 \leq \frac{1}{t}\int_0^t \|\frac{d}{dt}u^\varepsilon(s)\|_H^2 ds + C \leq \frac{1}{t}\frac{C}{2c_1} + C,$$

for $t > 0$. Hence, if T_1 is large enough,

$$\|\frac{d}{dt}u^\varepsilon(t)\|_H^2 \leq 1 + C,$$

for $t \geq T_1$ and for every $\varepsilon \in [0, \varepsilon_0]$. ∎

At this point we can proof the following result:

THEOREM 2.1. *Suppose that the operator B satisfies the assumptions* **H-1 – H-5***, then the problem* (3) *has a global attractor $\mathcal{A}_\varepsilon$ in $W_0^{1,p}(\Omega)$, for every $\varepsilon \in [0, \varepsilon_0]$. Furthermore, $\mathcal{A}_\varepsilon$ are uniformly embedded in $L^\infty(\Omega)$.*

Proof: From Lemma 2.3 and results contained in [3] we obtain L^∞ estimates for solutions of the problem (3), uniformly on $\varepsilon \in [0, \varepsilon_0]$. With the same arguments used in [4], we get the theorem. ∎

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3. CONVERGENCE PROPERTIES

In this section we show that the solutions of the problem (3) tends to the solutions of a limit problem, as ε goes to zero. In particular, we also obtain that the family of global attractors $\{\mathcal{A}_\varepsilon \subset W_0^{1,p}(\Omega);\ 0 \leq \varepsilon \leq \varepsilon_0\}$ of the problem (3) is upper semicontinuity at $\varepsilon = 0$, on $W_0^{1,p}(\Omega)$ topology.

We start with two technical lemmas.

Lemma 3.4. *Let $\{S^\varepsilon(t) : H \to H, t \geq 0\}$ be the semigroup generated by $\partial\phi^\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$ and u^ε the solution of the problem* (3). *Then*

$$\|S^\varepsilon(h)u^\varepsilon(t) - u^\varepsilon(t)\|_H \to 0,$$

as $h \to 0$, uniformly in $[0, \varepsilon_0]$, for each $t > 0$. Furthermore, if $T > 0$ we obtain that

$$\|u^\varepsilon(T-h) - S^\varepsilon(h)u^\varepsilon(T-h)\|_H \to 0,$$

as $h \to 0$, uniformly in $[0, \varepsilon_0]$.

Proof: For every $\varepsilon \in [0, \varepsilon_0]$, following [6], we obtain that

$$\|S^\varepsilon(h)u^\varepsilon(t) - u^\varepsilon(t)\|_H \leq 3\|u^\varepsilon(t) - J_h^{\phi^\varepsilon} u^\varepsilon(t)\|_H,$$

where $J_h^{\phi^\varepsilon} = (I + h\partial\phi^\varepsilon)^{-1}$.

Now, since for every $\lambda > 0$ and $u \in H$, Proposition 2.11 in [2] shows that

$$\frac{1}{2\lambda}\|J_\lambda^{\phi^\varepsilon} u - u\|_H^2 + \phi^\varepsilon(J_\lambda^{\phi^\varepsilon} u) = \min_{v \in H}\left[\frac{1}{2\lambda}\|v - u\|_H^2 + \phi^\varepsilon(v)\right],$$

we obtain by Lemma 2.2,

$$\frac{1}{2h}\|u^\varepsilon(t) - J_h^{\phi^\varepsilon} u^\varepsilon(t)\|_H^2 + \phi^\varepsilon(J_h^{\phi^\varepsilon} u^\varepsilon(t)) \leq \phi^\varepsilon(u^\varepsilon(t)) \leq K_2,$$

with K_2 uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$.

With the same arguments above, we obtain that

$$\|u^\varepsilon(T-h) - S^\varepsilon(h)u^\varepsilon(T-h)\|_H \leq \sqrt{2hK_2},$$

with K_2 uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$. ∎

Lemma 3.5. *Let u^ε be the solution of the problem* (3) *and u the solution of the problem*

$$\begin{aligned} &\frac{d}{dt}u(t) + Au(t) - B_1u(t) = B_2u(t), \qquad t > 0\\ &u(0) = u_0 \in W_0^{1,p}(\Omega), \end{aligned} \tag{11}$$

where A is the operator A^0 given by (2). For every $\varepsilon \in [0, \varepsilon_0]$, define the function $b^\varepsilon(x) = a^\varepsilon(x) - a(x)$, almost everywhere in Ω. Then

$$\int_0^t \left| \langle b^\varepsilon |\nabla u^\varepsilon(s)|^{p-2} \nabla u^\varepsilon(s), \nabla u(s) - \nabla u^\varepsilon(s) \rangle \right| ds \to 0,$$

as $\varepsilon \to 0$, for every $t > 0$.

Proof: Firstly note that

$$\|b^\varepsilon\|_{L^\infty(\Omega)} = \|a^\varepsilon - a\|_{L^\infty(\Omega)} \to 0,$$

as $\varepsilon \to 0$. Now, for every $s \geq 0$, Hölder inequality yelds that

$$\begin{aligned} &\left| \langle b^\varepsilon |\nabla u^\varepsilon(s)|^{p-2} \nabla u^\varepsilon(s), \nabla u(s) - \nabla u^\varepsilon(s) \rangle \right| \\ &\leq \|b^\varepsilon\|_{L^\infty(\Omega)} \int_\Omega |\nabla u^\varepsilon(s)|^{p-2} |\nabla u^\varepsilon(s)| |\nabla u(s) - \nabla u^\varepsilon(s) \rangle| dx \\ &\leq \|b^\varepsilon\|_{L^\infty(\Omega)} \int_\Omega \left[\tfrac{1}{p'} |\nabla u^\varepsilon(s)|^{(p-1)p'} + \tfrac{1}{p} |\nabla u(s) - \nabla u^\varepsilon(s)|^p \right] dx, \end{aligned}$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Since, by Lemma 2.2, the integral on second member of the inequality above is uniformly bounded for $\varepsilon \in [0, \varepsilon_0]$ and $s \geq 0$, we get the result. ∎

The next result shows that the solutions of the problem (3) goes to the solution of the problem (11), as ε goes to zero, in H.

THEOREM 3.2. *Let u^ε be the solution of the problem* (3) *and u the solution of the problem* (11). *Then,*

$$\|u^\varepsilon(t) - u(t)\|_H \to 0,$$

as $\varepsilon \to 0$, uniformly in bounded sets of $[T_1, +\infty)$, with T_1 given by Lemma 2.3.

Proof: First observe that there exist a positive constant $C = C(p, n, \delta)$ such that

$$\begin{aligned} \langle Au - A^\varepsilon u^\varepsilon, u - u^\varepsilon \rangle &= \langle a \left(|\nabla u|^{p-2} \nabla u - |\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon \right), \nabla u - \nabla u^\varepsilon \rangle \\ &\quad - \langle b^\varepsilon |\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon, \nabla u - \nabla u^\varepsilon \rangle \\ &\geq C \|\nabla u - \nabla u^\varepsilon\|^p_{L^p(\Omega)} - \langle b^\varepsilon |\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon, \nabla u - \nabla u^\varepsilon \rangle. \end{aligned}$$

Additionally, from Lemma 2.3, assumption **H-2** and results contained in [5], there exists a positive constant K_4 such that

$$\|u^\varepsilon(t)\|_{L^\infty(\Omega)} \leq K_4,$$

for t large enough. Again, Lemma 2.3 implies that K_4 is uniform with respect to $\varepsilon \in [0,\varepsilon_0]$. Also, the operator B_2 satisfies a local Lipschitz condition given by assumption **H-3**. Therefore, if T_1 is like at Lemma 2.3, u^ε e u are the unique solutions of the problems (3) and (11) respectively. Moreover, there exists $\omega > 0$ such that

$$\begin{aligned}\frac{1}{2}\|u(t)-u^\varepsilon(t)\|_H^2 \;\leq\; & \int_0^t \left|\langle b^\varepsilon |\nabla u^\varepsilon(s)|^{p-2}\nabla u^\varepsilon(s), \nabla u(s)-\nabla u^\varepsilon(s)\rangle\right| ds \\ & +\omega\int_0^t \|u(s)-u^\varepsilon(s)\|_H^2 ds.\end{aligned}$$

The Gronwall Inequality and Lemma 3.5 imply that

$$\|u(t)-u^\varepsilon(t)\|_H^2 \to 0,$$

as $\varepsilon \to 0$, uniformly in bounded sets of $[T_1,+\infty)$. ∎

Finally we have the main result of this work.

THEOREM 3.3. *Let u^ε be the solution of problem* (3) *and u the solution of the problem* (11). *Then,*

$$\|u^\varepsilon(t)-u(t)\|_{W_0^{1,p}(\Omega)} \to 0,$$

as $\varepsilon \to 0$, on bounded sets of $[T_1,+\infty)$, with T_1 given by Lemma 2.3.

Proof: For $T_1 < t < T$, since that

$$\|u^\varepsilon(t)\|_{W_0^{1,p}(\Omega)} \leq K_2,$$

uniformly in $[0,\varepsilon_0]$, every sequence in $\{u^\varepsilon(t);\, 0\leq\varepsilon\leq\varepsilon_0\}$, has a subsequence $\{u^{\varepsilon_n}(t)\}$ that converge weakly to $u(t)$, as $\varepsilon_n \to 0$, on $W_0^{1,p}(\Omega)$ topology. We will show that

$$\|u^{\varepsilon_n}(t)\|_{W_0^{1,p}(\Omega)} \to \|u(t)\|_{W_0^{1,p}(\Omega)},$$

as $\varepsilon \to 0$. To do this, we first note, from Lemma 3.5 and Theorem 3.2, that

$$\begin{aligned}&\int_{T_1}^T \|u(s)-u^{\varepsilon_n}(s)\|_{W_0^{1,p}(\Omega)}^p ds \leq \frac{1}{C}\int_{T_1}^T \langle Au(s)-A^{\varepsilon_n}u^{\varepsilon_n}(s), u(s)-u^{\varepsilon_n}(s)\rangle ds \\ +\; & \frac{1}{C}\int_{T_1}^T \left|\langle b^{\varepsilon_n}|\nabla u^{\varepsilon_n}(s)|^{p-2}\nabla u^{\varepsilon_n}(s), \nabla u(s)-\nabla u^{\varepsilon_n}(s)\rangle\right| ds \\ \leq\; & \frac{\omega}{C}\int_{T_1}^T \|u(s)-u^{\varepsilon_n}(s)\|_H^2 ds \\ +\; & \frac{1}{C}\int_{T_1}^T \left|\langle b^{\varepsilon_n}|\nabla u^{\varepsilon_n}(s)|^{p-2}\nabla u^{\varepsilon_n}(s), \nabla u(s)-\nabla u^{\varepsilon_n}(s)\rangle\right| ds \to 0,\end{aligned}$$

as $\varepsilon_n \to 0$, where ω and C are taken from proof of Theorem 3.2.

Therefore, there is a subsequence, $\{u^{\varepsilon_{nk}}(t)\}$, such that

$$\|u^{\varepsilon_{nk}}(t) - u(t)\|_{W_0^{1,p}(\Omega)} \to 0, \quad \varepsilon_{n_k} \to 0,$$

almost everywhere in $[T_1, T]$.

Now, fixing $t \in [T_1, T]$,

$$\begin{aligned} |\phi^{\varepsilon_{nk}}(u^{\varepsilon_{nk}}(t)) - \phi(u(t))| &\leq |\phi^{\varepsilon_{nk}}(u^{\varepsilon_{nk}}(t)) - \phi^{\varepsilon_{nk}}(u^{\varepsilon_{nk}}(\theta))| \\ + \quad |\phi^{\varepsilon_{nk}}(u^{\varepsilon_{nk}}(\theta)) - \phi(u(\theta))| &+ |\phi(u(\theta)) - \phi(u(t))|. \end{aligned} \tag{12}$$

Choosing $\theta \in [T_1, T]$, close enough of t, in such way that

$$\|u^{\varepsilon_{nk}}(\theta)\|_{W_0^{1,p}(\Omega)} \to \|u(\theta)\|_{W_0^{1,p}(\Omega)}, \quad \varepsilon_{n_k} \to 0,$$

we can conclude that

$$\phi^{\varepsilon_{nk}}(u^{\varepsilon_{nk}}(t)) \to \phi(u(t)), \quad \varepsilon_{n_k} \to 0,$$

since that the right side of the inequality

$$\begin{aligned} |\phi^{\varepsilon_{nk}}(u^{\varepsilon_{nk}}(t)) - \phi^{\varepsilon_{nk}}(u^{\varepsilon_{nk}}(\theta))| &\leq \int_\theta^t \left| \left\langle \partial\phi^{\varepsilon_{nk}} u^{\varepsilon_{nk}}(s), \frac{d}{dt} u^{\varepsilon_{nk}}(s) \right\rangle \right| ds \\ &\leq \frac{1}{2} \int_\theta^t |B_2 u^{\varepsilon_{nk}}(s)|_H^2 ds, \end{aligned}$$

is uniformly bounded in ε_{nk} for $s \in [T_1, T]$ and, by similar arguments, we estimate the third term in the right side on (12).

Thus,

$$\|u^{\varepsilon_{nk}}(t)\|_{W_0^{1,p}(\Omega)} \to \|u(t)\|_{W_0^{1,p}(\Omega)}, \quad \varepsilon_{n_k} \to 0,$$

for every $t \in [T_1, T]$. The result follows since that, for every $t \in [T_1, T]$,

$$u^{\varepsilon_{nk}}(t) \overset{W_0^{1,p}(\Omega)}{\rightharpoonup} u(t) \quad \text{and} \quad \|u^{\varepsilon_{nk}}(t)\|_{W_0^{1,p}(\Omega)} \to \|u(t)\|_{W_0^{1,p}(\Omega)},$$

as $\varepsilon_{n_k} \to 0$, showing that

$$u^{\varepsilon_{nk}}(t) \to u(t), \quad \varepsilon_{n_k} \to 0$$

on $W_0^{1,p}(\Omega)$, everywhere in $[T_1, T]$. ∎

As an application of Theorem 3.3 we have the following result.

Corollary 3.1. *The family of global attractors*

$$\{\mathcal{A}_\varepsilon \subset W_0^{1,p}(\Omega);\ 0 \le \varepsilon \le \varepsilon_0\}$$

of the problem (3) *is upper semicontinuity in* $\varepsilon = 0$, *on* $W_0^{1,p}(\Omega)$ *topology.*

Proof: Consider the initial value problems (3) and (11) with initial data $\psi_\varepsilon \in \mathcal{A}_\varepsilon$, for every $\varepsilon \in [0, \varepsilon_0]$. Theorem 3.3 implies that there exists $\delta > 0$ such that

$$\|u^\varepsilon(t) - u^0(t)\|_{W_0^{1,p}(\Omega)} < \frac{\delta}{2},$$

for every $\varepsilon \in [0, \varepsilon_0]$ and $t \ge T_1$.

On the other hand, $\mathcal{A}_0$ attracts bounded sets of $W_0^{1,p}(\Omega)$ and the results contained in Section 2 imply that there exists a bounded set $\mathcal{B} \subset W_0^{1,p}(\Omega)$ such that

$$\mathcal{A}_\varepsilon \subset \mathcal{B},$$

for every $\varepsilon \in [0, \varepsilon_0]$. Therefore, there is $T_2 > T_1$ in such way that

$$\sup_{\psi_\varepsilon \in \mathcal{A}_\varepsilon \ \ \varepsilon \in [0,\varepsilon_0]} \operatorname{dist}_{W_0^{1,p}(\Omega)}(u^0(T_2), \mathcal{A}_0) \le \frac{\delta}{2}.$$

Thus, for every $\varepsilon \in [0, \varepsilon_0]$, we obtain that

$$\operatorname{dist}_{W_0^{1,p}(\Omega)}(u^\varepsilon(T_2), \mathcal{A}_0) \le \operatorname{dist}_{W_0^{1,p}(\Omega)}(u^\varepsilon(T_2), u^0(T_2)) + \operatorname{dist}_{W_0^{1,p}(\Omega)}(u^0(T_2), \mathcal{A}_0) \le \delta.$$

On the other hand, for every $\varepsilon \in [0, \varepsilon_0]$, we have that

$$u^\varepsilon(T_2)\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon,$$

then we conclude that

$$\operatorname{dist}_{W_0^{1,p}(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \le \delta,$$

for every $\varepsilon \in [0, \varepsilon_0]$, completing the proof of the corollary. ∎

REFERENCES

1. BARBU, V., *Nonlinear Semigroups and Differential Equations in Banach Space,* Noordhoff International Publishing (1976).
2. BRÉZIS, H., *Operateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, North-Holland Publishing Company, Amsterdam (1973).
3. CARVALHO, A. N. and GENTILE C. B., Comparison Results for Nonlinear Parabolic Problems with Monotone Principal Part, *Journal of Mathematical Analysis and Applications* **259**, 01 (2001), pp. 319-337.

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4. CARVALHO, A. N. and GENTILE C. B., Asymptotic Behavior of Parabolic Equations with Subdifferential Principal Part. *Journal of Mathematical Analysis and Applications* **280**, 2 (2003), pp 252-272.

5. ÔTANI M., Nonmonotone Perturbations for Nonlinear Parabolic Equations Associated Whit Subdifferential Operators, Cauchy Problems, *Journal of Differential Equations,* 46 (1982) pp. 268-299.

6. PLANT, A. T., Four Inequalities for Monotone Gradient Vector Fields, *Arch. Rational Mech. Anal.* **82**, 4 (1983), pp. 377-389.

7. TEMAN R., *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York (1988).

8. VRABIE, I.I., *Compactness Methods for Nonlinear Evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics, London (1987).

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