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On transversality for quadratic spaces over rings with many units[1]

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In this note we deal with the existence of transversals in quadratic spaces over rings with many units. As applications we extend to these rings some classical results on quadratic Pfister spaces over semilocal rings. April, 2000
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1. INTRODUCTION

The purpose of this note is to develop a transversality theory for quadratic spaces over rings with many units.

Transversality for quadratic spaces over commutative rings was already considered by R. Baeza [1] in the semilocal case (see also [2]) and by B. McDonald and B. Kirkwood [6] and L. Walter [8] in the case of rings with many units with some restrictive conditions. In particular, their results on transversality are recovered in our study (see Section 2).

As applications of our results on transversality we extend to rings with many units basic facts on quadratic Pfister spaces which are well known in the case of (for instance) semilocal rings (see [1]). Specifically we show that if R is a ring with many units then quadratic Pfister spaces over R are round and isotropic quadratic Pfister spaces over R are hyperbolic (see Section 3).

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Following [7] a *ring with many units* is a commutative ring with identity R which satisfies the following local-global principle: given a polynomial $f \in R[X_1 \dots X_n]$ if f represents a unit over $R_{\mathfrak{M}}$, for every maximal ideal $\mathfrak{M}$ of R, then f represents a unit over R. Such a ring is also called in the literature a *local-global ring* (or an *LG-ring*, for short). Such kind of rings includes the semilocal rings, the zero-dimensional rings or, more generally, the rings which are von Neumann regular modulo their Jacobson radical.

Basic facts on quadratic spaces over rings with many units can be seen, for instance, in [5] and [3]. Basic facts and basic terminology on quadratic spaces over commutative rings are considered here as in [1].

Throughout R will denote a ring with many units. The multiplicative group of the units of R is denoted by R^*. By $\mathrm{Max}(R)$ we will denote the set of all the maximal ideals of R. Unadorned $\otimes$ means $\otimes_R$.

2. THE TRANSVERSALITY THEOREMS

¿From now on we will denote by (E, q) (or simply E, for short) a nonsingular quadratic module or, following [1], simply a quadratic space over R, that is, E is a finitely generated projective R-module and $\Phi_q : E \to \mathrm{Hom}_R(E, R)$ given by $\Phi_q(x) : y \mapsto \varphi_q(x, y)$, is an isomorphism of R-modules, where $\varphi_q : E \times E \to R$ given by $\varphi_q(x, y) = q(x+y)-q(x)-q(y)$ for $x, y \in E$, denotes the corresponding symmetric bilinear form associated to the quadratic form q. We also assume that E has constant rank. So, by [7] (see also [5]) E is a free R-module. Furthermore, according to the rank of E is either even or odd, E has orthogonal decompositions of the form either $E = \langle e_1, f_1\rangle \perp \dots \perp \langle e_n, f_n\rangle$ or $E = \langle e_1, f_1\rangle \perp \dots \perp \langle e_n, f_n\rangle \perp \langle g\rangle$, where $\varphi_q(e_i, f_i) = 1$, $1-4q(e_i)q(f_i) \in R^*$, for $1 \le i \le n$, and $q(g) \in R^*$ (see [5], Theorem 7.3 and [4], Section 2). Such a basis is called a *canonical basis* for E over R. If, in addition, $q(e_i)q(f_i) \in R^*$, for all $1 \le i \le n$ we call such a basis *strictly canonical*. By [4], Proposition 2.2, E has an strictly canonical basis, provided that $|R/\mathfrak{M}| \ge 4$, for every $\mathfrak{M} \in \mathrm{Max}(R)$. We also observe that if $\mathrm{rank}(E)$ is odd then necessarily $2 \in R^*$ and E admits an orthogonal decomposition of the type $E = \langle e_1\rangle \perp \dots \perp \langle e_n\rangle$ with $q(e_i) \in R^*$, $1 \le i \le n$, if and only if $2 \in R^*$ (see [4], Section 2).

An element $x \in E$ is called *primitive* (or *unimodular*) if $x \not\equiv 0 \,(\mathrm{mod}\ \mathfrak{M}E)$, for every $\mathfrak{M} \in \mathrm{Max}(R)$.

Given an orthogonal decomposition $E = E_1 \perp \dots \perp E_s$ of E we call the element $x \in E$ *transversal with respect to this decomposition* if $x = x_1 + \cdots + x_s$, with $x_i \in E_i$ and $q(x_i) \in R^*$, for $1 \le i \le s$. Correspondingly, if $\{v_1, \dots, v_n\}$ is a basis of E over R, an element $x \in E$ is called *transversal with respect to this basis* if in the representation $x = x_1 + \cdots + x_n$, with $x_i = \alpha_i v_i$, $\alpha_i \in R$, $1 \le i \le n$, all the α_i are units. Note that if $\mathrm{rank}(E) = 1$ then every primitive element $x \in E$ is transversal.

Our aim in this section is, given a primitive element $x \in E$, to present conditions under which there exists $\sigma \in \mathcal{S}(E)$ such that $\sigma(x)$ is transversal with respect to an orthogonal decomposition (resp. strictly canonical basis) of E. Here $\mathcal{S}(E)$ denotes the group generated by all the symmetries of E. By a symmetry of E we mean an R-linear map $\sigma_y : E \to E$, where $y \in E$ with $q(y) \in R^*$ and $\sigma_y : x \mapsto x - q(y)^{-1}\varphi_q(x, y)y$, for every $x \in E$. Obviously if a primitive element $x \in E$ is transversal then $\mathrm{id}_E \in \mathcal{S}(E)$ already satisfies the required. So

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we will exclude these elements from our considerations as well as we will assume throughout that the quadratic space E has rank ≥ 2. We will denote by $\mathrm{Prim}(E)$ the set of all the primitive elements of E which are not transversal.

Specifically we will prove the following theorems. Recall that a primitive element $x \in E$ is called isotropic (resp. strictly anisotropic) if $q(x) = 0$ (resp. $q(x) \in R^*$).

Theorem 2.1 *Let $E = E_1 \perp \ldots \perp E_s$, with $s \geq 2$, be an orthogonal decomposition of E and $x \in \mathrm{Prim}(E)$. Then there exists $\sigma \in \mathcal{S}(E)$ such that $\sigma(x)$ is transversal with respect to this decomposition in the following cases:*

(i) *$|R/\mathfrak{M}| > 5$ for all $\mathfrak{M} \in \mathrm{Max}(R)$ and $\mathrm{rank}(E_i) = 1$ for all $1 \leq i \leq s$.*

(ii) *$|R/\mathfrak{M}| \geq 4$ for all $\mathfrak{M} \in \mathrm{Max}(R)$ and at least one component of the orthogonal decomposition of E has rank ≥ 2.*

(iii) *$|R/\mathfrak{M}| \geq 3$ for all $\mathfrak{M} \in \mathrm{Max}(R)$ and at least two components of the orthogonal decomposition of E has rank ≥ 2.*

Theorem 2.2 *Let $E = E_1 \perp \ldots \perp E_s$ be a orthogonal decomposition of E having at least two components with rank ≥ 2 and $x \in \mathrm{Prim}(E)$. Assume that x is isotropic (resp. strictly anisotropic). If s is even (resp. odd) then there exists $\sigma \in \mathcal{S}(E)$ such that $\sigma(x)$ is transversal with respect to this decomposition.*

Theorem 2.3 *Let* B *be a strictly canonical basis of E and $x \in \mathrm{Prim}(E)$. Then there exists $\sigma \in \mathcal{S}(E)$ such that $\sigma(x)$ is transversal with respect to* B *in the following cases:*

(i) *$\mathrm{rank}(E) = 2$.*

(ii) *$\mathrm{rank}(E) = 3$ and $|R/\mathfrak{M}| \geq 4$ for all $\mathfrak{M} \in \mathrm{Max}(R)$.*

(iii) *$\mathrm{rank}(E) > 3$ and $|R/\mathfrak{M}| \geq 3$ for all $\mathfrak{M} \in \mathrm{Max}(R)$.*

In order to prove these theorems we will need the following lemma which will allow us to reduce our study on transversality to the case of quadratic spaces over a field.

We start by considering the polynomial ring $R[X]$ in an infinite set of indeterminates $X = \{ X_{i1}, \ldots, X_{in} \mid i \geq 1 \}$, where $n = \mathrm{rank}(E)$. Now take the $R[X]$-module $E[X] = \{ \sum_{1 \leq i \leq n} h_i v_i \mid h_i \in R[X] \} \simeq R[X] \otimes E$, where $\{ v_1, \ldots, v_n \}$ is a basis of E over R. We also denote by q (resp. φ_q) the extension of q (resp. φ_q) to $E[X]$.

Given $y \in E$ let $\tau_y : E \to E$ denote the R-linear map given by $\tau_y(x) = q(y)x - \varphi_q(x, y)y$, for every $x \in E$. Note that if $q(y) \in R^*$ then $\tau_y = q(y)\sigma_y$. Now take $x \in E$ and $y_j = \sum_i \lambda_{ji} v_i \in E$, $1 \leq j \leq k$ and put $h(x, y_1, \ldots, y_k) = \tau_{y_k} \cdots \tau_{y_1}(x)$. Let $h(x, y_1, \ldots, y_k) = h_1(x, y_1, \ldots, y_k) + \cdots + h_s(x, y_1, \ldots, y_k)$ be a representation of $h(x, y_1, \ldots, y_k)$ with respect to an orthogonal decomposition $E = E_1 \perp \ldots \perp E_s$, with $s \geq 2$, of E. Similarly define $\tau_z : E[X] \to E[X]$, for every $z \in E[X]$, and take the polynomial $h(x, z_1, \ldots, z_k) = \tau_{z_k} \cdots \tau_{z_1}(x)$ where $z_j = \sum_i X_{ji} v_i \in E[X]$, $1 \leq j \leq k$. Clearly $h(x, z_1, \ldots, z_k) = h_1(x, z_1 \ldots, z_k) + \cdots +$

$h_s(x, z_1 \ldots, z_k)$ is the corresponding representation of $h(x, z_1, \ldots, z_k)$ with respect to the orthogonal decomposition $E[X] = E_1[X] \perp \ldots \perp E_s[X]$ of $E[X]$.

Note that $h(x, y_1, \ldots, y_k)$ is transversal with respect to the orthogonal decomposition of E above considered if and only if $\prod_{1 \le i \le s} q(h_i(x, y_1, \ldots, y_k)) \in R^*$. If in addition $\prod_{1 \le j \le k} q(y_j) \in R^*$ then

$$\sigma_{y_k} \cdots \sigma_{y_1}(x) = \left(\prod_{1 \le j \le k} q(y_j) \right)^{-1} h(x, y_1, \ldots, y_k)$$

is also transversal with respect the same orthogonal decomposition of E.

Put $H(x, y_1, \ldots, y_k) := \left(\prod_{1 \le i \le s} q(h_i(x, y_1, \ldots, y_k)) \right) \left(\prod_{1 \le j \le k} q(y_j) \right)$.
Clearly $H(x, y_1, \ldots, y_k) \in R^*$ it means that the polynomial

$$H(x, z_1 \ldots, z_k) := \left(\prod_{1 \le i \le s} q(h_i(x, z_1 \ldots, z_k)) \right) \left(\prod_{1 \le j \le k} q(z_j) \right) \in R[X]$$

represents a unit over R.

Lemma 2.4 *Let $x \in Prim(E)$. Assume that for each $\mathfrak{M} \in Max(R)$ there exist $k \ge 1$ and $y_1, \ldots, y_k \in E$ such that $H(x, y_1, \ldots, y_k) \notin \mathfrak{M}$. Then there exists $m \ge 1$ such that $H(x, z_1 \ldots, z_m)$ represents a unit over R.*

Proof: For every $k \ge 1$, denote $\mathcal{H}_k$ the set of all $H(x, y_1, \ldots, y_k)$, with $y_1, \ldots, y_k \in E$. Let $\Gamma_k = \Gamma(\mathcal{H}_k) = \{\mathfrak{M} \in \mathrm{Max}(R) \mid \mathfrak{M} \not\supseteq \mathcal{H}_k\}$. By the assumption $\{ \Gamma_k \mid k \ge 1 \}$ is a open cover of $\mathrm{Max}(R)$, so there exists a finite subcover $\{ \Gamma_k \mid 1 \le k \le m \}$ of $\mathrm{Max}(R)$, since $\mathrm{Max}(R)$ with the Zariski topology is compact.
Finally, we observe that $\Gamma_k \subseteq \Gamma_{k+1}$, for every $k \ge 1$. In fact, for each $\mathfrak{M} \in \Gamma_k$ there exist $y_1, \ldots, y_k \in E$ such that $H(x, y_1, \ldots, y_k) \notin \mathfrak{M}$. Now, since $s \ge 2$ we can take $y \in E$ such that $q(y) \notin \mathfrak{M}$ and $y \in \{ h(x, y_1, \ldots, y_k) \}^\perp \bmod \mathfrak{M}E$. Thus we have $h(x, y_1, \ldots, y_k, y) \equiv q(y) h(x, y_1, \ldots, y_k) \bmod \mathfrak{M}E$ and consequently $H(x, y_1 \ldots, y_k, y) \equiv q(y)^3 H(x, y_1, \ldots, y_k) \notin \mathfrak{M}$. So $\mathfrak{M} \in \Gamma_{k+1}$ and therefore $\mathrm{Max}(R) = \Gamma_m$ which completes the proof, since R is a ring with many units. □

Now we are able to prove the transversality theorems.

For the following we remark that, for any R-submodule E' of E, every element of $\mathcal{S}(E')$ extends to an element of $\mathcal{S}(E)$ in an obvious way, since any element $y \in E' \subseteq E$ such that $q(y) \in R^*$ determines $\sigma_y \in \mathcal{S}(E')$ as well as $\sigma_y \in \mathcal{S}(E)$. By $\mathbb{F}_p$ we mean the finite field with p (p a prime integer) elements.

Proof of Theorem 2.1: Firstly we observe that it suffices to prove this result in the case that the given orthogonal decomposition of E has only two components. In fact, assume that the assertion holds for primitive elements in orthogonal sums of quadratic spaces over R with only two components. Then, there is $\sigma_1 \in \mathcal{S}(E)$ such that $\sigma_1(x) = x_1 + x_2$ with $x_1 \in E_1$, $x_2 \in E' = E_2 \perp \ldots \perp E_s$ and $q(x_1)q(x_2) \in R^*$. Thus x_2 is primitive in E' and by recurrence and by the assumption there is $\sigma_2 \in \mathcal{S}(E')$ such that $\sigma_2(x_2)$ is transversal with respect to the orthogonal decomposition $E_2 \perp \ldots \perp E_s$. Now, taking $\sigma = \sigma_2\sigma_1$ the result follows.

Therefore, in order to end our proof, it remains to verify the case $E = E_1 \perp E_2$. By Lemma 2.4 we can assume that R is a field. Now (ii) and (iii) follow by applying arguments similar to those used by Baeza in [1], Theorem III.5.2. In the case (i) the characteristic of R is different from two, $E = \langle e_1 \rangle \perp \langle e_2 \rangle$, with $q(e_1)q(e_2) \neq 0$, and $x = \alpha e_1 + \beta e_2$ with either $\alpha = 0$ or $\beta = 0$ not simultaneously. Suppose that $\beta = 0$. Since $|R| \geq 7$ then there exist elements $a, b \in R$ such that $ab(q(e_1)^2 a^4 - q(e_2)^2 b^4) \neq 0$ and σ_y, with $y = ae_1 + be_2$, satisfies the required. □

Proof of Theorem 2.2: We start by observing again that we can assume that R is a field, by Lemma 2.4. Also, if $|R| \geq 3$ then the result follows by Theorem 2.1 (iii). So, let $R = \mathbb{F}_2$ and recall that any quadratic space over $\mathbb{F}_2$ has even rank. We will divide the proof in several steps.

Step 1: $s = 2$ and $x = x_1 + x_2$, with $x_i \in E_i$, is isotropic.

Since x is not transversal then $q(x) = 0$ implies $q(x_i) = 0$, $i = 1, 2$.

Assume firstly that one of the components of x is zero, say $x_2 = 0$. In this case $x = x_1 \in E_1$ is isotropic and there exists $y_1 \in E_1$ such that $q(y_1) = 0$ and $\varphi_q(x_1, y_1) = 1$, i. e., E_1 contains a hyperbolic plane. Taking $g \in E_2$ such that $q(g) = 1$, and $y = y_1 + g$ we get $q(y) = 1$ and $\sigma_y(x) = (x_1 + y_1) + g$, which is clearly transversal.

Now assume $x_i \neq 0$, $i = 1, 2$. In this case there exist $y_i \in E_i$ such that $\langle x_i, y_i \rangle \subseteq E_i$, $i = 1, 2$, are hyperbolic planes. Taking $y = x_1 + y_1 + x_2$ we get $q(y) = \varphi_q(x_1, y_1) = 1$ and $\sigma_y(x) = x_1' + x_2'$ where $x_1' = y_1 \neq 0$ and $x_2' = 0$. Now the result follows as above.

Step 2: $s = 3$ and $x = x_1 + x_2 + x_3$, with $x_i \in E_i$, is anisotropic.

Then $1 = q(x) = q(x_1) + q(x_2) + q(x_3)$ and we can assume that $q(x_1 + x_2) = 0$ and $q(x_3) = 1$. If $x_1 + x_2 \neq 0$ then $x_1 + x_2$ is an isotropic element of $E_1 \perp E_2$ and the result follows by Step 1. If $x_1 + x_2 = 0$ then $x_1 = x_2 = 0$ and therefore $x = x_3 \in E_3$. Clearly there exists $y_3 \in E_3$ such that $\varphi_q(x_3, y_3) = 1$. If $q(y_3) = 1$, taking $y = y_1 + y_2 + y_3$, with $y_i \in E_i$, such that $q(y_i) = 1$, $i = 1, 2$, we have $q(y) = 1$ and $\sigma_y(x_3) = y_1 + y_2 + (x_3 + y_3)$ with $y_1 + y_2 \in E_1 \perp E_2$ isotropic and $x_3 + y_3 \in E_3$ anisotropic. Now the result follows by Step 1. If $q(y_3) = 0$, taking $y = y_1 + x_3 + y_3$, with $y_1 \in E_1$ such that $q(y_1) = 1$, we get $q(y) = 1$ and $\sigma_y(x_3) = y_1 + y_3$. Since y_1 is anisotropic and $y_3 \in E_2 \perp E_3$ is isotropic, the result again follows by Step 1.

Step 3: $s \geq 3$ is odd and the assertion holds for anisotropic elements in any orthogonal sum of s quadratic spaces over R.

In this case the assertion also holds for isotropic elements in any orthogonal sum of $s+1$ quadratic spaces over R. In fact, let $F = F_1 \perp \ldots \perp F_s \perp F_{s+1}$ be an orthogonal sum of $s+1$ quadratic spaces over R and $y \in F$, $y = y_1 + \cdots + y_s + y_{s+1}$ be primitive and isotropic. So we have $0 = q(y) = q(y_1) + \cdots + q(y_s) + q(y_{s+1})$. Put $y' = y_1 + \cdots + y_s$ and $F' = F_1 \perp \ldots \perp F_s$. Then we have $y = y' + y_{s+1} \in F = F' \perp F_{s+1}$. By Step 1 we can assume that y is transversal with respect to the orthogonal decomposition $F = F' \perp F_{s+1}$, so $q(y') = 1 = q(y_{s+1})$. Thus $y' = y_1 + \cdots + y_s \in F_1 \perp \ldots \perp F_s$ is anisotropic and the result follows by the assumption.

Step 4: $s \geq 4$ is even and the assertion holds for isotropic (resp. anisotropic) elements in any orthogonal sum of s (resp. $s-1$) quadratic spaces over R.

Is this case the assertion also holds for anisotropic elements in any orthogonal sum of $s+1$ quadratic spaces over R. In fact, let $F = F_1 \perp \ldots \perp F_s \perp F_{s+1}$ be an orthogonal sum of $s+1$ quadratic spaces over R and $y \in F$, $y = y_1 + \cdots + y_s + y_{s+1}$ be anisotropic. So we have $1 = q(y) = q(y_1) + \cdots + q(y_s) + q(y_{s+1})$ and we can assume that $q(y_{s+1}) = 1$ and, consequently, $q(y_1 + \cdots + y_s) = q(y_1) + \cdots + q(y_s) = 0$.

If $y_1 + \cdots + y_s \neq 0$ then $y_1 + \cdots + y_s$ is isotropic in $F_1 \perp \ldots \perp F_s$ and the result follows by the assumption.

If $y_1 + \cdots + y_s = 0$ then $y = y_{s+1}$ and by Step 2 there exists $\sigma \in \mathcal{S}(F_{s-1} \perp F_s \perp F_{s+1})$ such that $\sigma(y) = \sigma(y_{s+1}) = y'_{s-1} + y'_s + y'_{s+1}$ is transversal with respect to the decomposition $F_{s-1} \perp F_s \perp F_{s+1}$. Now $y'_{s-1} \in F_1 \perp \ldots \perp F_{s-1}$ is anisotropic and again by the assumption there exists $\tau \in \mathcal{S}(F_1 \perp \ldots \perp F_{s-1})$ such that $\tau(y'_{s-1}) = y'_1 + \ldots + y'_{s-2} + y''_{s-1}$ is transversal with respect to the decomposition $F_1 \perp \ldots \perp F_{s-1}$. Therefore we have $\tau\sigma(y) = \tau\sigma(y_{s+1}) = \tau(y'_{s-1} + y'_s + y'_{s+1}) = \tau(y'_{s-1}) + \tau(y'_s + y'_{s+1}) = \tau(y'_{s-1}) + y'_s + y'_{s+1} = y'_1 + \cdots + y'_{s-2} + y''_{s-1} + y'_s + y'_{s+1}$ and the result follows.

Now, by induction and Steps 1-4 the proof is complete. □

Proof of Theorem 2.3: We firstly prove that if there exists $\sigma \in \mathcal{S}(E)$ such that $\sigma(x)$ is transversal with respect to a given orthogonal decomposition of E then also there exists $\tau \in \mathcal{S}(E)$ such that $\tau(x)$ is transversal with respect to a given strictly canonical basis of E. We will prove this assertion only in the case that $\text{rank}(E)$ is odd. Similar arguments hold in the even case. By Lemma 2.4 we can assume that R is a field.

Let $\mathrm{B} = \{e_i, f_i, g \mid 1 \leq i \leq s\}$ be an strictly canonical basis of E, $E_i = \langle e_i, f_i \rangle$ and $F = \langle g \rangle$. So $E = E_1 \perp \ldots \perp E_s \perp F$ is an orthogonal decomposition of E. Take $x \in \text{Prim}(E)$. By the assumption there exists $\sigma_0 \in \mathcal{S}(E)$ such that $\sigma_0(x) = x_1 + \cdots + x_s + \gamma g$, with $x_i \in E_i$ and $q(x_i)\gamma \neq 0$, $1 \leq i \leq s$. Then each $x_i \in E_i$ is primitive. If x_i is transversal with respect to the strictly canonical basis $\{e_i, f_i\}$ of E_i we are done. If not, then $x_i = \alpha_i e_i + \beta_i f_i$, with either $\alpha_i = 0$ or $\beta_i = 0$. Suppose that $\alpha_i = 0$. Then $\sigma_{e_i}(x_i) = \sigma_{e_i}(\beta_i f_i) = -\beta_i q(e_i)^{-1} e_i + \beta_i f_i$ and the result follows. Note that this last argument also proves the assertion (i) of the theorem. The assertions (ii) and (iii) follow now from Theorem 2.1. □

Corollary 2.5 *Let* B *be a strictly canonical basis of* E *and* $x \in \mathit{Prim}(E)$. *Assume that* x *is isotropic (resp. strictly anisotropic). If* $\mathit{rank}(E) = 2k$ *(resp.* $\mathit{rank}(E) = 2k+1$*), for* $k \geq 2$, *then there exists* $\sigma \in \mathcal{S}(E)$ *such that* $\sigma(x)$ *is transversal with respect to* B.

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Proof: It is an immediate consequence of Theorem 2.2 and Theorem 2.3(i). □

Remarks 2.6 1) The converse of Theorem 2.2 also holds if there exists $\mathfrak{M} \in \mathrm{Max}(R)$ such that $R/\mathfrak{M} = \mathbb{F}_2$. In fact, let $x \in \mathrm{Prim}(E)$ and assume that there exists $\sigma \in \mathcal{S}(E)$ such that $\sigma(x)$ is transversal with respect to a given orthogonal decomposition $E = E_1 \perp \ldots \perp E_s$ of E. Thus $q(x) = q(\sigma(x)) = s \pmod{\mathfrak{M}}$ for every $\mathfrak{M} \in \mathrm{Max}(R)$ such that $R/\mathfrak{M} = \mathbb{F}_2$. Therefore x is either isotropic or anisotropic modulo $\mathfrak{M}$, for such maximal ideals $\mathfrak{M}$ of R, if and only if correspondingly s is either even or odd.

2) The following examples justify the necessity of the restrictions on the cardinality of the residual fields of R as well as on the rank of the components in the orthogonal decomposition of E as considered in Theorems 2.1 and 2.2.

(i) In the quadratic space $E = \langle e_1 \rangle \perp \langle e_2 \rangle$ over any commutative ring R, any isotropic element is transversal with respect to this orthogonal decomposition of E.

(ii) In the quadratic space $E = \langle e_1 \rangle \perp \langle e_2 \rangle$ over $R = \mathbb{F}_5$, with $q(e_1)^2 = q(e_2)^2$ any symmetry of E either fixes or permutes the lines Re_i, $i = 1, 2$.

(iii) In the quadratic space $\langle e_1 \rangle \perp \langle e_2 \rangle \perp \langle e_3 \rangle$ over $R = \mathbb{F}_3$, with $q(e_1) = q(e_2) = q(e_3)$, any transversal element with respect to this orthogonal decomposition of E is isotropic.

(iv) For the quadratic space $E = E_1 \perp \langle e_2 \rangle \perp \langle e_3 \rangle$ over $R = \mathbb{F}_3$, with $\mathrm{rank}(E_1) \geq 2$ and $q(e_2) = q(e_3)$, there is not $\sigma \in \mathcal{S}(E)$ such that $\sigma(e_2 + e_3)$ is transversal with respect to this orthogonal decomposition of E.

3) Theorem 2.1 is already known in the following cases:

(i) R is semilocal (see [1], Theorem III 5.2).

(ii) R is a ring with many units, $2 \in R^*$ and all residual fields of R are infinite (see [6]).

(iii) R is a ring with many units, $2 \in R^*$ and $|R/\mathfrak{M}| > 5$ for all $\mathfrak{M} \in \mathrm{Max}(R)$ (see [8], Proposition III 6.1).

4) Theorem 2.3 (ii) and (iii) is also known in the case of semilocal rings (see [1] Theorem III 5.1).

5) The restriction $k \geq 2$ in Corollary 2.5 is necessary. In fact, if $k = 1$ and $\mathrm{rank}(E) = 2$ then any primitive and isotropic element of E is already transversal with respect to any strictly canonical basis of E. Furthermore, in the quadratic space $E = \langle e, f \rangle \perp \langle g \rangle$ over $\mathbb{F}_3$, with $q(e) = q(f) = \varphi_q(e, f) = 1$ and $q(g) = 2$, the anisotropic element $x = e + 2f$ remains invariant under the action of any symmetry of E.

3. ROUND SPACES

In this section we maintain the notation (E, q) for a quadratic space over R. Given a quadratic space (E, q) over R, we will denote by $D(q)$ the set of all the represented values by (E, q), that is, $D(q) = \{q(x) \mid x \in E\}$. For the represented units we set $D(q)^* = D(q) \bigcap R^*$.

Every element $\lambda \in R^*$ together with a quadratic space (E, q) over R determine a new quadratic space $\langle \lambda \rangle \otimes (E, q) := (\langle e \rangle \otimes E, \langle \lambda \rangle \otimes q)$ over R where $\langle \lambda \rangle \otimes q(ae \otimes x) = a^2 \lambda q(x)$, for all $a \in R$ and $x \in E$. An element $\lambda \in R^*$ is called a *similarity norm* of a quadratic

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space (E, q) if $(E, q) \simeq \langle\lambda\rangle \otimes (E, q)$. Clearly the set of all the similarity norms of (E, q) is a subgroup of R^* which is denoted by $N(q)$.

A quadratic space (E, q) is called *round* if $N(q) = D(q)^*$.

Remarks 3.1 (see [1], Ch. IV, Section 1)

1) An element $\lambda \in R^*$ is a similarity norm of a quadratic space (E, q) if and only if there exists a R-linear isomorphism $\sigma : E \to E$ such that $q(\sigma(x)) = \lambda q(x)$, for all $x \in E$.

2) For every quadratic space (E, q) over R, $1 \in D(q)$ implies $N(q) \subseteq D(q)^*$ and $D(q)^* \subseteq N(q)$ if and only if (E, q) is round.

For every $a \in R^*$ and every quadratic space (E, q) over R we denote by $\langle 1,\, a\rangle \otimes (E, q)$ the quadratic space $\langle 1\rangle \otimes (E, q) \perp \langle a\rangle \otimes (E, q)$. Also, by $[1,\, b]$ we denote the quadratic space (E, q) over R where $E = \langle e, f\rangle$, $q(e) = 1 = \varphi_q(e, f)$ and $q(f) = b$ with $b \in R$ such that $1 - 4b \in R^*$.

Since R is a ring with many units it easily follows from ([4], Section 2) and Remarks 3.1 that the unique round quadratic spaces of rank 2 over R are of the type $[1,\, b]$.

The following theorem is a consequence of Theorem 2.1 (iii) and its proof is similar to the proof given in [1], Theorem IV.2.1.

Theorem 3.2 *Assume that $|R/\mathfrak{M}| \geq 3$ for all $\mathfrak{M} \in Max(R)$. Let (E, q) be a round quadratic space over R and $a \in R^*$. Then the quadratic space $\langle 1,\, a\rangle \otimes (E, q)$ is round.* □

Corollary 3.3 *For any $a_1, \ldots, a_n \in R^*$ and $b \in R$ such that $1 - 4b \in R^*$, the quadratic space*

$$\langle\langle a_1, \ldots, a_n, b]] := \langle 1,\, a_1\rangle \otimes \cdots \otimes \langle 1,\, a_n\rangle \otimes [1,\, b]$$

is round.

Proof: Considering that integral extensions of R also are rings with many units ([5], Corollary 2.3) and that Cancellation Witt Theorem also holds for quadratic spaces over rings with many units ([3], Ch. 2, Section 6), this proof easily follows as a consequence of Theorem 3.2 by arguments identical to those used in [1], Theorem IV.2.4. □

The following result is a consequence of Theorem 2.2 and its proof is similar to the proof given in [1], Theorem IV.3.1. Recall that a quadratic space is called *isotropic* if it contains a primitive isotropic element. Also we say that a quadratic space (E, q) is *hyperbolic* if there exists a direct summand P of E such that $(E, q) \simeq (\mathbb{H}(P), q_P)$, where $\mathbb{H}(P) = P \bigoplus \mathrm{Hom}_R(P, R)$ and $q_P(x, h) = h(x)$, for every $x \in P$ and $h \in \mathrm{Hom}_R(P, R)$.

Theorem 3.4 *Let (E, q) be a round quadratic space over R, $a \in R^*$ and $(F, q') = \langle 1,\, a\rangle \otimes (E, q)$. If (F, q') is isotropic then $-a \in D(q)^*$ and, in particular, (F, q') is hyperbolic.* □

A quadratic space of the type $\langle\langle a_1, \ldots, a_n, b]]$ as considered in Corollary 3.3 is called a *quadratic Pfister space*.

Application of Theorem 3.4 to quadratic Pfister spaces yields the following corollary.

Corollary 3.5 *Isotropic quadratic Pfister spaces over R are hyperbolic.* $\square$

Remark 3.6 Corollary 3.5 implies that the level $s(R) := \min\{n \in \mathbb{N} \mid -1 = a_1 + \cdots + a_n, a_i \in R\}$ of R (if it is finite) is a power of 2, provided that $2 \in R^*$ (see [1], Theorem A.3 for the semilocal case and [8], Corollary 3.5.6 for rings with many units).

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