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# Structural stability of singular actions of $\mathbb{R}^n$ having a first integral

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We begin by characterizing the family of real analytic closed orientable $n$-manifolds, $n \geq 2$, that admit an analytic action of $\mathbb{R}^n$ with at least one orbit diffeomorphic to $T^{n-1} \times \mathbb{R}$. Next, for each closed orientable $n$-manifold $N$ we define a subset $\mathscr{C}_n$ of singular actions in $A^\omega(\mathbb{R}^n, N)$ such that each action $\varphi \in \mathscr{C}_n$ has a non-constant first integral as well as some generic properties and then we prove that $\varphi$ is $C^1$ structurally stable. An action of $\mathbb{R}^n$ is called singular if every orbit has dimension less than $n$. October, 2003 ICMC-USP

*Key Words:* Singular action, structural stability, compact orbit.

## 1. INTRODUCTION

Let $M$ be a closed connected real analytic $m$-manifold and $A^r(\mathbb{R}^n, M)$ the set of $C^r$, $1 \leq r \leq \omega$, actions of $\mathbb{R}^n$ on $M$ whose infinitesimal generators are also of class $C^r$. We consider in $A^r(\mathbb{R}^n, M)$ the $C^1$-topology induced by the $C^1$-distance between infinitesimal generators. An action of $\mathbb{R}^n$ on a manifold is called singular if every orbit has dimension less than $n$.

In this paper we characterize the family of analytic closed orientable $n$-manifolds that admit an analytic action of $\mathbb{R}^n$ with at least one orbit diffeomorphic to $T^{n-1} \times \mathbb{R}$, see Theorem 2.1. This family consists of $T^n$ and $\mathcal{H}_n$, the family of manifolds obtained by glueing two copies of $T^{n-2} \times D^2$ by an orientation preserving diffeomorphism of $T^{n-1} =$

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$\partial(T^{n-2} \times D^2)$. An action $\varphi \in A^r(\mathbb{R}^n, M)$ is said to be of *type* $j$, $0 \leq j \leq n$, if the union of the $j$-dimensional orbits is an open dense subset of $M$. Denote by $A^r_j(\mathbb{R}^n, M)$ the subspace of actions of type $j$. For each analytic closed connected orientable $n$-manifold $N$ we define a subset $\mathscr{C}_n \subset A^\omega_{n-1}(\mathbb{R}^n, N)$ such that each action $\varphi \in \mathscr{C}_n$ has a non-constant first integral as well as some generic properties, see Definition 2.4. Our main result is to prove that each action in $\mathscr{C}_n$ is $C^1$ structurally stable, see Theorem 2.2. It is proved in Lemma 2.4 that if $\varphi \in \mathscr{C}_n$, then there exists $\psi_\varphi \in A^\omega(\mathbb{R}^{n-1}, N)$ with the same orbits than $\varphi$. C. Perelló proved in [8] that Morse-Smale $C^\omega$ vector fields on an analytic orientable closed surface form an open and dense set. It follows from his theorem that if $\varphi \in \mathscr{C}_2$, then $\psi_\varphi$ is not $C^1$-structurally stable.

The fact that there are $C^1$ structurally stable singular actions is not new. In fact, Kato and Morimoto proved in [5] the following theorem: Let $X \in \mathfrak{X}^r(N)$, $r \geq 1$ and $n \geq 3$, be an Anosov vector field. Then, $Y \in \mathfrak{X}^r(N)$ satisfies $[X, Y] = 0$ if and only if $Y = cX$, with $c \in \mathbb{R}$. As a consequence of this result, we can define an open set $\mathscr{B} \subset A^r(\mathbb{R}^n, N)$ of singular actions such that each element in $\mathscr{B}$ is $C^1$ structurally stable.

The fact that there are $C^1$ structurally stable actions with non-constant first integrals is also not new. Write $T^3 = S^1 \times T^2$ and let $\mathcal{F}$ be the foliation of $T^3$ with leaves $\{\theta\} \times T^2$, $\theta \in S^1$. Saldanha in [9], defined a subset of locally free actions $\mathscr{C} \subset A^\infty(\mathbb{R}^2, T^3)$ such that each $\varphi \in \mathscr{C}$ has $\mathcal{F}$ as underlying foliation and is $C^1$ structurally stable.

What is new in our result is that each $\varphi \in \mathscr{C}_n$, besides being $C^1$ structurally stable, is a singular action and at the same time has a non-constant first integral. Finally, we show that this phenomenum is typical of analytic actions. In [1], to be published, we consider actions $\varphi \in A^r_n(\mathbb{R}^n, N)$. We defined the concept of transversally hyperbolic compact orbit and proved for $n = 2$ and $n = 3$ that if $\varphi \in A^r_n(\mathbb{R}^n, N)$, $r \geq 1$, and every compact orbit is transversally hyperbolic, then $\varphi$ is structurally stable. The problem of characterizing the structurally stable actions in $A^\omega(\mathbb{R}^n, N)$ is still an open and interesting problem.

## 2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULTS

Through this paper $M$ $(N)$ will denote a closed connected and orientable real analytic $m$-manifold ( $n$-manifold) and we shall assume that $m \geq n$. A $C^r$-action of Lie group $G$ on $M$ is a $C^r$-map $\varphi : G \times M \to M$, $1 \leq r \leq \omega$, such that $\varphi(e, p) = p$ and $\varphi(gh, p) = \varphi(g, \varphi(h, p))$, for each $g, h \in G$ and $p \in M$, where $e$ is the identity in $G$. $\mathcal{O}_p = \{\varphi(g, p);\ g \in G\}$ is called the *$\varphi$-orbit* of $p$. $G_p = \{g \in G;\ \varphi(g, p) = p\}$ is called the *isotropy group* of $p$. For each $p \in M$ the map $g \mapsto \varphi(g, p)$ induces an injective immersion of the homogeneous space $G/G_p$ in $M$ with image $\mathcal{O}_p$. When $G = \mathbb{R}^n$, the possible $\varphi$-orbits are injective immersions of $T^k \times \mathbb{R}^\ell$, $0 \leq k + \ell \leq n$, where $T^k = S^1 \times \cdots \times S^1$, $k$ times.

For each $0 \leq i \leq n-1$ let $\mathrm{Sing}_i(\varphi) = \{p \in M; \dim \mathcal{O}_p = i\}$ and $\mathrm{Sing}(\varphi) = \cup_{i=0}^{n-1} \mathrm{Sing}_i(\varphi)$. If $p \in \mathrm{Sing}(\varphi)$, $\mathcal{O}_p$ is called a *singular orbit* and when $p \in \mathrm{Sing}_0(\varphi)$, $\mathcal{O}_p$ is also called a *point orbit* and $p$ a *fixed point* by $\varphi$. We also write $p \in \mathrm{Sing}^c_i(\varphi)$, $i = 1, \ldots, n-1$, when $\mathcal{O}_p$ is a $T^i$-orbit. If $\mathrm{Sing}(\varphi) = M$, we call $\varphi$ a *singular action*.

For each $w \in \mathbb{R}^n \setminus \{0\}$ $\varphi$ induces a $C^r$-flow $(\varphi_w^t)_{t\in\mathbb{R}}$ given by $\varphi_w^t(p) = \varphi(tw, p)$ and its corresponding $C^{r-1}$-vector field $X_w$ is defined by $X_w(p) = D_1\varphi(0,p)\cdot w$. If $\{w_1, \dots, w_n\}$ is a base of $\mathbb{R}^n$, the associated vector fields $X_{w_1}, \dots, X_{w_n}$ determine completely the action $\varphi$ and are called a set of *infinitesimal generators* of $\varphi$. Note that $[X_{w_i}, X_{w_j}] = 0$ for any two of them.

Denote by $A^r(\mathbb{R}^n, M)$ the set of $C^r$-actions, $r \geq 1$, of $\mathbb{R}^n$ on $M$ such that their canonical infinitesimal generators are also $C^r$ vector fields. Given two actions $\{\varphi; X_1, \dots, X_n\}$ and $\{\psi; Y_1, \dots, Y_n\}$ define $d_k(\varphi, \psi) = \max\limits_{1\leq i\leq n} \|X_i - Y_i\|_k$. With this distance $A^r(\mathbb{R}^n, M)$ is a metric space and the corresponding topology is the $C^k$-topology. We say that $\varphi$ is an action of *type* $j$ and write $\varphi \in A_j^r(\mathbb{R}^n, M)$ if the union of the $j$-dimensional orbits is an open dense subset of $M$. Note that for analytic actions $A^\omega(\mathbb{R}^n, M) = A_0^\omega(\mathbb{R}^n, M) \cup A_1^\omega(\mathbb{R}^n, M) \cup \cdots \cup A_n^\omega(\mathbb{R}^n, M)$, see Lemma 2.3. This is not the case for non-analytic actions since it is possible to have $\varphi \in A^\infty(\mathbb{R}^n, M)$ which does not belong to any $A_j^\infty(\mathbb{R}^n, M)$, see [3].

Let $\varphi \in A^r(\mathbb{R}^k, M)$ and $X_1, \dots, X_k$ a set of infinitesimal generators of $\varphi$. We shall denote by $\mathscr{G}(\varphi)$ the commutative Lie subalgebra of $\mathfrak{X}^r(M)$ generated by those vector fields.

Definition 2.1. An action $\psi \in A^r(\mathbb{R}^k, M)$ is said to immerse (immerse properly) in $A^r(\mathbb{R}^n, M)$, $k \leq n$, if there exist $\varphi \in A^r(\mathbb{R}^n, M)$ such that $\mathscr{G}(\psi)$ is a subalgebra (proper subalgebra) of $\mathscr{G}(\varphi)$. We shall write $\psi \mapsto \varphi$ to indicate that $\varphi$ realizes the immersion of $\psi$. If $\psi \in A_k^r(\mathbb{R}^k, M)$ immerses properly in $A_n^r(\mathbb{R}^n, M)$ one shall say that $\psi$ embedds in $A_n^r(\mathbb{R}^n, M)$.

Let $\psi \in A^r(\mathbb{R}^k, M)$ and $X_1, \dots, X_k$ a set of infinitesimal generators. $\psi$ can allways be immersed in $A^r(\mathbb{R}^n, M)$. In fact, put $X_{k+i} = \sum_{j=1}^k a_{ji}X_j$, $1 \leq i \leq n-k$, where each $a_{ji} : M \to \mathbb{R}$ is a first integral, perhaps constant, of each $X_j$, $j = 1, \dots, k$. The action $\varphi$ generated by $X_1, \dots, X_k, X_{k+1}, \dots, X_n$ realizes the immersion of $\psi$. Note that the immersion is proper if at least one $a_{ji}$ is non-constant. On the other hand $\psi \in A_k^r(\mathbb{R}^k, M)$, in general, does not embedds in $A_n^r(\mathbb{R}^n, M)$.

The notions of topological equivalence and $C^k$ structural stability that we use here for actions are the standard one's. The following two lemmas extend to actions of $\mathbb{R}^n$ classical lemmas in the theory of flows, see [1]. Let $D_\varepsilon^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m;\ |x_i| < \varepsilon\}$, $\varepsilon > 0$, and $\frac{\partial}{\partial x_i} = (0, \dots, 0, 1, 0, \dots, 0)$ the constant vector field.

Lemma 2.1 ($k$-flow box). *Let $\varphi \in A^r(\mathbb{R}^k, M)$ with infinitesimal generators $X_1, \dots$, $X_k$, and $\mathcal{O}_p$ a $k$-dimensional orbit. There exists a $C^r$- diffeomorphism $h : V_p \to D_\varepsilon^m$, where $V_p$ is a neighborhood of $p$, such that $h_*X_i = \frac{\partial}{\partial x_i}$ in $D_\varepsilon^m$, for each $i = 1, \dots, k$.*

*Remark 2. 1.* Note that the diffeomorphism $h = h(\varphi) : V_p \to D_\varepsilon^m$ depends continuously on $\varphi$ in the following sense: given $\eta > 0$, there exists $\delta > 0$ such that if $\widetilde{\varphi} \in A^r(\mathbb{R}^k, M)$ is $\delta$ $C^1$-close to $\varphi$, then $h(\widetilde{\varphi}) : \widetilde{V}_p \to D_\varepsilon^m$ is $\eta$ $C^1$-close to $h(\varphi)$ in $V_p \cap \widetilde{V}_p$.

A pair $(V_p, h)$ as in Lemma 2.1 will be called a *k-flow box* at $p$. If $q \in \mathcal{O}_p$ with $q \neq p$, then there exists $u \in \mathbb{R}^k$ such that $X_u^1(p) = q$. We shall call $\gamma = \{X_u^t(p); 0 \leq t \leq 1\}$ an arc of $\varphi$ in $\mathcal{O}_p$. By using Lemma 2.1 one can also prove:

LEMMA 2.2 (Long $k$-flow box). *Let $\varphi \in A^r(\mathbb{R}^k, M)$, $\mathcal{O}_p$ a $k$-dimensional orbit of $\varphi$ and $\gamma \subset \mathcal{O}_p$ an arc of $\varphi$ in $\mathcal{O}_p$. Then, there exists $k$-flow box $(V_\gamma, h)$, where $V_\gamma$ is a neighborhood of $\gamma$.*

**Infinitesimal generators adapted to a singular orbit.** Let $\mathcal{O}_p$ be a singular $k$-dimensional orbit of $\varphi \in A^r(\mathbb{R}^n, M)$ and $G_p$ its isotropy group. $\mathcal{O}_p$ is a $T^\ell \times \mathbb{R}^{k-\ell}$-orbit with $0 \leq \ell \leq k < n$. Call $G_p^0$ the connected component of $G_p$ that contains the origin and let $H$ be a $k$-dimensional subspace of $\mathbb{R}^n$ such that $\mathbb{R}^n = H \oplus G_p^0$. Let $\{w_1, \ldots, w_n\}$ be a base of $\mathbb{R}^n$ such that $\{w_1, \ldots, w_k\}$ is a base of $H$, $\{w_{k+1}, \ldots, w_n\}$ is a base of $G_p^0$ and $\{w_1, \ldots, w_\ell\}$ generate the subgroup $G_p \cap H$. $\{X_i = X_{w_i}; i = 1, \ldots, n\}$ is a set of infinitesimal generators of $\varphi$. Note that $X_{k+1}(q) = \cdots = X_n(q) = 0$ for every $q \in \mathcal{O}_p$. We shall say that $X_1, \ldots, X_n$ is a set of *infinitesimal generators adapted to* $\mathcal{O}_p$.

Applying Lemma 2.1 to the action $\varphi$ restricted to $H$, we obtain a chart $h : V_p \to D_\varepsilon^m$ of $M$ such that if $(\theta, x) \in D_\varepsilon^m = D_\varepsilon^k \times D_\varepsilon^{m-k}$, then the vector fields $X_i$ in this chart can be writen

$$\begin{aligned} X_i(\theta, x) &= \frac{\partial}{\partial \theta_i}, \quad i = 1, \ldots, k \\ X_{k+i}(\theta, x) &= \sum_{j=1}^{k} a_{ji}(x) \frac{\partial}{\partial \theta_j} + \sum_{j=k+1}^{m} a_{ji}(x) \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, n-k \end{aligned} \tag{1}$$

A chart like the one above is called *adapted to $\mathcal{O}_p$ at $p$*. The vector fields

$$\widehat{X}_i = \sum_{j=k+1}^{m} a_{ji}(x) \frac{\partial}{\partial x_j}, \quad i = 1, \ldots, n-k,$$

define a local action $\varphi_T$ of $\mathbb{R}^{n-k}$ on $D_\varepsilon^{m-k}$ having $0 \in D_\varepsilon^{m-k}$ as a fixed point.

When $p$ is a fixed point of $\varphi$, then a chart adapted to $\mathcal{O}_p$ at $p$ will be any chart of $M$ which contains $p$. In this case $\widehat{X}_i = X_i$, $i = 1, \ldots, n$.

*Remark 2. 2.* Note that $\{X_1, \ldots X_k, \widehat{X}_1, \ldots, \widehat{X}_{n-k}\}$ define a local $\mathbb{R}^n$-action $\widehat{\varphi}$ on $D_\varepsilon^m$ and that $\mathcal{O}_{(\theta,x)}(\widehat{\varphi}) = \mathcal{O}_{(\theta,x)}(h \circ \varphi \circ h^{-1})$ for each $(\theta, x) \in D_\varepsilon^m$.

LEMMA 2.3. *Assuming $n \leq m$ the following decomposition holds:*

$$A^\omega(\mathbb{R}^n, M) = A_0^\omega(\mathbb{R}^n, M) \cup A_1^\omega(\mathbb{R}^n, M) \cup \cdots \cup A_n^\omega(\mathbb{R}^n, M).$$

*Moreover, if $\varphi \in A_n^\omega(\mathbb{R}^n, N)$, then there is only a finite number of $n$-dimensional orbits, all of them homeomorphic.*

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*Proof.* Let $X_1, \ldots, X_n$ be a set of infinitesimal generators of $\varphi$ and $\mathcal{O}_p$ an orbit of maximal dimension $k$, $0 < k \leq n$. Fix a finite number of charts $(U_i, x_i)$, $0 \leq i \leq \ell$, of $M$ such that $M = \cup_{i=1}^{\ell} U_i$ and assume that $X_1, \ldots, X_k$ are linearly independent in $U_1$. This last property propagates along the charts to an open and dense subset of $M$ using the fact that a real analytic function defined on an open set $U \subset \mathbb{R}^m$ is zero either on $U$ or on the complement of an open dense subset of $U$, see [7]. Let $\mathcal{O}_1, \mathcal{O}_2$ be two different $n$-dimensional orbits of $\varphi \in A^{\omega}(\mathbb{R}^n, N)$ and $G_1, G_2$ their respective isotropy groups. If $u \in G_p$, then $X_u^1|_{\mathcal{O}_1} = id$. Since $X_u^1$ is an analytic diffeomorphism, it follows that $X_u^1 = id$ on $N$. Thus, $u \in G_2$ i.e., $G_1 \subset G_2$. By the same argument $G_2 \subset G_1$ and therefore $G_1 = G_2$. Finally, using charts adapted to the singular orbits one shows that every $\varphi \in A_n^{\omega}(\mathbb{R}^n, N)$ has a finite number of $n$-dimensional orbits. ∎

Let $\mathcal{O}_p$ be a $T^{n-2}$-orbit of $\psi \in A^r(\mathbb{R}^{n-1}, N)$. It can be verified that $\widehat{X}_{n-1}$ has the following two properties:

(1) Although $\widehat{X}_{n-1}$ depends on the chart $(V_p, h)$, which in turn depends on $H$, the fact that $0 \in D_{\varepsilon}^2$ be a center (saddle, node, focus) of $D\widehat{X}_{n-1}(0)$ does not depend on the chart.

(2) If $q \in \mathcal{O}_p$ and $q \neq p$, there exists a chart $(V_p, h)$ adapted to $\mathcal{O}_p$ such that $q \in V_p$.

It follows from the two properties above that the following concept is well defined.

Definition 2.2. Let $\mathcal{O}_p$ be a $T^{n-2}$-orbit of $\psi \in A^r(\mathbb{R}^{n-1}, N)$. $\mathcal{O}_p$ is said to be transversally simple if there exists a chart adapted to $\mathcal{O}_p$ at $p$ such that $0 \in D_{\varepsilon}^2$ is a simple singularity of $\widehat{X}_{n-1}$. When $0$ is a center (saddle, node, focus) of $D\widehat{X}_{n-1}(0)$ we will say that $\mathcal{O}_p$ is transversally a center (saddle, node, focus).

**Separatrices of a $T^{n-2}$-orbit that is transversally a saddle.** Let $\mathcal{O}_p$ be a $T^{n-2}$-orbit of $\psi \in A^r(\mathbb{R}^{n-1}, N)$ that is transversally a saddle, $X_1, \ldots, X_{n-2}, X_{n-1}$ infinitesimal generators adapted to $\mathcal{O}_p$ and $h : V_p \longrightarrow D_{\varepsilon}^{n-2} \times D_{\varepsilon}^2$ a chart adapted to $\mathcal{O}_p$ at $p$. Let $\xi^s$ $(\xi^u)$ be the stable (unstable) submanifold of $\widehat{X}_{n-1}$ at $0 \in D_{\varepsilon}^2$, $\Sigma = h^{-1}(D_{\varepsilon}^2)$ and $\eta^s = h^{-1}(\xi^s)$ $(\eta^s = h^{-1}(\xi^s))$. We know that $X_i^1(p) = p$, $i = 1, \ldots, n-2$. For each $i = 1, \ldots, n-2$, let $D_i^{n-1} = \{t = (t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) \in D_{\varepsilon}^n\}$, $\Sigma_i = h^{-1}(D_i^{n-1})$ and $P_i : (\Sigma_i, p) \to (\Sigma_i, p)$ the Poincaré map of $X_i$ at $p$. Let $\pi_i : V_p \to \Sigma$ be the projection along the orbits of the action generated by $X_1, \ldots, X_{i-1}, X_{i+1} \ldots, X_{n-2}$ and $\omega_i : U \subset \Sigma \to \Sigma$ be the local diffeomorphism defined by $\omega_i = \pi_i \circ P_i$. Since $[X_i, \widehat{X}_{n-1}] = 0$, $i = 1, \ldots, n-2$, it follows that $\omega_i$ takes orbits of $\widehat{X}_{n-1}$ in $U$ into orbits of $\widehat{X}_{n-1}$ in $\Sigma$ and in particular $\omega_i((\eta^s \cup \eta^u) \cap U) \subset \eta^s \cup \eta^u$. Therefore, the four connected components of $\eta^s \cup \eta^u \setminus \{p\}$ give rise to at most four $\varphi$-orbits and each of them is a $T^{n-2} \times \mathbb{R}$-orbit. This orbits will be called *separatrices* of $\mathcal{O}_p$.

Lemma 2.4. *Let $\mathcal{O}_p$ be a transversally simple $T^{n-2}$-orbit of $\varphi \in A_{n-1}^r(\mathbb{R}^n, N)$. There exists a neighborhood $V$ of $\mathcal{O}_p$ and $\psi_\varphi \in A^r(\mathbb{R}^{n-1}, N)$ such that $\mathcal{O}_q(\psi_\varphi) = \mathcal{O}_q(\varphi)$ for each $q \in V$. In particular, if $r = \omega$, then $\mathcal{O}_q(\psi_\varphi) = \mathcal{O}_q(\varphi)$ for each $q \in N$.*

*Proof.* Let $X_1, \ldots, X_{n-2}, X_{n-1}, X_n$ infinitesimal generators adapted to $\mathcal{O}_p$. We know that $X_1, \ldots, X_{n-2}$ are linearly independent in a neighborhood $V_1$ of $\mathcal{O}_p$. Let $h : V_p \to D^m_\varepsilon$ be a chart adapted to $\mathcal{O}_p$ at $p$. Since $\mathcal{O}_p$ is transversally simple, we can assume that $0 \in D^2_\varepsilon$ is a simple singularity of $\widehat{X}_{n-1}$. Thus, there is a neighborhood $U$ of $p$ such that if $q \in U \setminus \mathcal{O}_p$, then $X_{n-1}(q) \neq 0$. It is clear that $X_{n-1}$ has no singularities on the saturated set $sat(U \cap V_1)$ of $U \cap V_1$ under the action defined by $\{X_1, \ldots, X_{n-2}\}$ and also that $sat(U \cap V_1)$ contains a neighborhood $V$ of $\mathcal{O}_p$. Call $\psi_\varphi$ the action of $\mathbb{R}^{n-1}$ on $N$ whose infinitesimal generators are $\{X_1, \ldots, X_{n-2}, X_{n-1}\}$. Since $\varphi \in A^r_{n-1}(\mathbb{R}^n, N)$, it follows that $X_n = \sum_{i=1}^{n-1} f_i X_i$ in $V$, where $X_i f_j = 0$ for $i, j = 1, \ldots, n-1$. Note that when $r = \omega$, $V = N$. ∎

Definition 2.3. A $T^{n-2}$-orbit $\mathcal{O}_p$ of $\varphi \in A^r_{n-1}(\mathbb{R}^n, N)$ is transversally simple (center, saddle, node, focus) if and only if it has the same attribute as an orbit of $\psi_\varphi \in A^r(\mathbb{R}^{n-1}, N)$, where $\psi_\varphi$ is given by Lemma 2.4.

Proposition 2.1. *If $\varphi \in A^r(\mathbb{R}^n, N)$ has an orbit $\mathcal{O}$ homeomorphic to $T^{n-1} \times \mathbb{R}$, then* Front$(\mathcal{O})$ *is the union of at most two $T^k$-orbits with $k \in \{n-2, n-1\}$.*

For each $n \geq 2$ let $\mathcal{H}_n$ be the family of all analytic closed connected and orientable manifolds that can be obtained by glueing two copies of $T^{n-2} \times D^2$. Note that $\mathcal{H}_2$ has only one element, which is $S^2$. Note also that $\mathcal{H}_3$ consists of 3-manifolds that admit a Heegaard splitting of genus one.

Theorem 2.1. *Let $N$ be a real analytic closed connected and orientable $n$-manifold, $n \geq 2$. Assume that $\varphi \in A^\omega(\mathbb{R}^n, N)$ has at least one $T^{n-1} \times \mathbb{R}$-orbit, then every $n$-dimensional orbit is a $T^{n-1} \times \mathbb{R}$-orbit and $\varphi \in A^\omega_n(\mathbb{R}^n, N)$. Furthermore,*
(1) *if $\mathrm{Sing}^c_{n-2}(\varphi) = \emptyset$, then $N$ is homeomorphic to $T^n$,*
(2) *if $\mathrm{Sing}^c_{n-2}(\varphi) \neq \emptyset$, then $\mathrm{Sing}^c_{n-2}(\varphi)$ is the union of two $T^{n-2}$-orbits and $N \in \mathcal{H}_n$.*

Corollary 2.1. *Let $N$ be a real analytic closed connected and orientable $n$-manifold and $\psi \in A^\omega_{n-1}(\mathbb{R}^{n-1}, N)$ that has a $T^{n-1}$-orbit $\mathcal{O}(\psi)$. If $\psi$ embedds in $A^\omega_n(\mathbb{R}^n, N)$, then $N$ is either homeomorphic to $T^n$ or $N \in \mathcal{H}_n$. Moreover, in both cases $\mathcal{O}(\psi)$ has a neighborhood $V$ that is a union of $T^{n-1}$-orbits.*

Definition 2.4. Let $\mathscr{C}^0_n$ be the set of actions $\varphi \in A^\omega_{n-1}(\mathbb{R}^n, N)$ such that there is at least one $T^{n-2}$-orbit $\mathcal{O}_p$ which is transversally a center and one $T^{n-2}$-orbit which is transversally a saddle. Let $\psi_\varphi \in A^\omega(\mathbb{R}^{n-1}, N)$ be the action constructed in Lemma 2.4 from $\varphi$ and $\mathcal{O}_p$. Now define the subset $\mathscr{C}_n = \mathscr{C}_n(N) \subset \mathscr{C}^0_n$ by saying that $\varphi \in \mathscr{C}_n$ if and only if

1) $\mathrm{Sing}_i(\varphi) = \emptyset$, $i = 0, \ldots, n-3$, and $\mathrm{Sing}_{n-2}(\varphi) = \mathrm{Sing}^c_{n-2}(\varphi)$,

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2) every $T^{n-2}$-orbit is transversally simple,
3) $\psi_\varphi \mapsto \varphi$ is a proper immersion,
4) If $\mathcal{O}_p$ is a $T^{n-2}$-orbit of $\varphi$ that is transversally a saddle, then its separatrices are not separatrices of any other $T^{n-2}$-orbit that is also transversally a saddle.

Note that if $\mathcal{O}$ is a $T^{n-2}$-orbit of $\varphi \in \mathscr{C}_n$, then from condition 3) one obtains that $\mathcal{O}$ is transversally a center or a saddle. The following statement is a corollary of Proposition 2.1.

COROLLARY 2.2. *Let $\varphi \in \mathscr{C}_n^0(N)$ and $\psi_\varphi$ as in Definition 2.4. Then $\psi_\varphi$ can not be embedded in $A_n^\omega(\mathbb{R}^n, N)$.*

THEOREM 2.2. *Let $N$ be a real analytic closed orientable $n$-manifold and $\varphi \in \mathscr{C}_n \subset A_{n-1}^\omega(\mathbb{R}^n, N)$. Then $\varphi$ is structurally stable and consequently $\mathscr{C}_n$ is an open set in $A^\omega(\mathbb{R}^n, N)$.*

## 3. PROPERTIES OF ACTIONS IN $\mathscr{C}_N$

In this section we prove some properties of actions $\varphi \in \mathscr{C}_n$ that are needed for the proof of Theorem 2.2. For the sake of clarity some of them are given first for $n = 2$ and then for $n > 2$.

LEMMA 3.1 (Persistency of transversally simple $T^{n-2}$-orbits). *Let $\mathcal{O}_0$ be a transversally simple $T^{n-2}$-orbit of $\psi \in A^r(\mathbb{R}^{n-1}, N)$. Given a neighborhood $V$ of $\mathcal{O}_0$ there exist a neighborhood $\mathcal{V}_\psi$ of $\psi$ in $A^r(\mathbb{R}^{n-1}, N)$ such that each $\xi \in \mathcal{V}_\varphi$ has only one singular orbit $\mathcal{O}_0(\xi) \subset V$ and $\mathcal{O}_0(\xi)$ is also a transversally simple $T^{n-2}$-orbit.*

*Proof.* The proof is obtained using Remark 2.1. ▮

*Remark 3. 1.* Lemma 3.1 is also valid for actions $\varphi \in A_{n-1}^r(\mathbb{R}^n, N)$.

For each $\varphi \in \mathscr{C}_n$ let $K_\varphi$ be the set of points $p \in N$ such that $\mathcal{O}_p(\varphi)$ is not a $T^{n-1}$-orbit and $c_1, \dots, c_k$ $(s_1, \dots, s_\ell)$ be the $T^{n-2}$-orbits of $\varphi$ that are transversally a center (saddle).

PROPOSITION 3.1. *$\mathscr{C}_2 \subset A_1^\omega(\mathbb{R}^2, N)$ is an open set in $A^\omega(\mathbb{R}^2, N)$.*

*Proof.* Let $\varphi \in \mathscr{C}_2$. Denote by $c_1, \dots, c_k$ the centers and by $s_1, \dots, s_\ell$ the saddles of $\varphi$. Since $\varphi \in \mathscr{C}_2$, it follow that $k \geq 1$ and $\ell \geq 1$. Let $X_1, X_2$ be infinitesimal generators of $\varphi$ such that the flow $\psi_\varphi$ of $X_1$ has the same orbits as $\varphi$ and $X_2 = fX_1$, with $f$ non-constant. Given a neighborhood of each singular point of $X_1$, by Lemma

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3.1, there exists a neighborhood $\mathcal{U} = \mathcal{U}(X_1)$ such that any $\widetilde{X}_1 \in \mathcal{U}$ has only one simple singularity in each of these neighborhoods, where $\tilde{s}_i$ is also a saddle and $\tilde{c}_i$ is either a focus or a center. Assume now that $\mathcal{V}$ is a neighborhood of $\varphi$ such that if $\widetilde{\varphi} \in \mathcal{V}$, then $\widetilde{X}_1 \in \mathcal{U}$. $\widetilde{\varphi}$ can not belong to $A_2^{\omega}(\mathbb{R}^2, N)$. If this were the case, then $\widetilde{\varphi}$ would have an $S^1 \times \mathbb{R}$-orbit intersecting any neighborhood of $\widetilde{c}_1$ and also an $\mathbb{R}$-orbit intersecting any neighborhood of $\widetilde{s}_1$, but this contradicts Proposition 2.1. Thus $\widetilde{\varphi} \in A_1^{\omega}(\mathbb{R}^2, N)$ and $\widetilde{X}_2 = \widetilde{f}\widetilde{X}_1$. Moreover, if $\mathcal{V}$ is sufficiently small, then $\widetilde{f}$ can not be constant. ▌

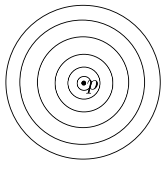
THEOREM 3.1. *If $\varphi \in \mathscr{C}_2$, then there exists a neighborhood $\mathcal{V}$ of $\varphi$ such that for each $\widetilde{\varphi} \in \mathcal{V}$*

$$K_{\widetilde{\varphi}} = \{\widetilde{c}_1, \ldots, \widetilde{c}_k\} \cup \{\widetilde{s}_1, \ldots, \widetilde{s}_\ell\} \cup \{\cup_{i=1}^{\ell} \widetilde{S}_{i1} \cup \widetilde{S}_{i2}\}$$

*where, for each $i = 1, \ldots, \ell$, $\widetilde{S}_{i1}$ and $\widetilde{S}_{i2}$ are $\mathbb{R}$-orbits satisfying* $\mathrm{Front}(\widetilde{S}_{i1}) = \widetilde{s}_i = \mathrm{Front}(\widetilde{S}_{i2})$.

*Proof.* By Proposition 3.1, it is enough to prove the theorem for a fixed $\varphi \in \mathscr{C}_2$. If $\mathcal{O}_q$ is a $\mathbb{R}$-orbit of $\psi_\varphi$, then $\alpha(q) = \omega(q) = s_i$ for some $i \in \{1, \ldots, k\}$. In fact, assume that $p \in \omega(q)$ is a regular point of $\psi_\varphi$. Take a transversal section $\Sigma$ to the flow $\psi_\varphi$ by $p$. There is a sequence $q_n \in \Sigma$, with $q_n = \psi_\varphi(t_n, q)$, converging to $p$. Let $f$ be the real analytic first integral of $\psi_\varphi$ given by Definition 2.4 condition 3). $f|_\Sigma$ is constant on $\{q_n\}$. This gives a contradiction and proves that $\omega(q) = s_i$ for some $i \in \{1, \ldots, k\}$. It follows from condition 4) that $\alpha(q) = s_i$, too. Since each saddle has two self-connections, the theorem is proved. ▌

To each fixed point $p$ of $\varphi \in A^r(\mathbb{R}^n, M)$, $1 \le r \le \omega$, there is associated a linear action $\varrho : \mathbb{R}^n \to \mathrm{Aut}(T_pM)$ given by $\varrho(v) = D\varphi_v(p)$ where $\varphi_v(\cdot) = \varphi(v, \cdot)$. Recall that if $\varphi \in A_1^r(\mathbb{R}^2, N^2)$, $1 \le r \le \omega$, has a fixed point $p \in N^2$ that is a center, then the phase portrait of the induced linear action $\varrho$ is like Figure 1, (i).



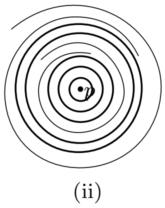




**FIG. 1.**

The following lemma gives a sufficient condition for the topological equivalence between $\varphi$ and $\varrho$ in the neighborhood of $p$.

LEMMA 3.2. *Assume that $\varphi \in A_1^1(\mathbb{R}^2, N^2)$ has a first integral that is not constant on any open set. If $p \in \mathrm{Fix}(\varphi)$ is a center, then $\varphi$ is topologically equivalent to $\varrho$ in a neighborhood of $p$.*

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*Proof.* Let $X_1, X_2$ be infinitesimal generators of $\varphi$ such that $p$ is a center of $X_1$ and $\psi_\varphi \in A^1(\mathbb{R}, N^2)$ is the flow of $X_1$. Let $h : V_p \to U_0$ be a chart of $N$ such that $\mathrm{Sing}(X_1) \cap V_p = \{p\}$, $h(p) = 0$ and $D(h_* X_1)(0) = \left(\begin{smallmatrix} 0 & -\beta \\ \beta & 0 \end{smallmatrix}\right)$, $\beta \neq 0$. There exists a neighborhood $0 \in U \subset U_0$ such that if $x \in U$, then $h_* X_1(x)$ is transversal to the ray $0x$. It follows that any orbit of $X_1$ by $q \in h^{-1}(U)$ is either closed or a spiral that accumulates in $p$ or in a closed orbit contained in $h^{-1}(U)$, see Figure 1 (ii). Let $C = \{q \in h^{-1}(U); \mathcal{O}_q(X_1) \text{ is closed}\}$. If $h^{-1}(U) \setminus C$ has non-empty interior, then the first integral would be constant in some open set. Thus $C = h^{-1}(U)$ and this proves the lemma. ∎

Proposition 3.2. *$\mathscr{C}_n \subset A^\omega_{n-1}(\mathbb{R}^n, N)$ is an open set in $A^\omega(\mathbb{R}^n, N)$.*

*Proof.* Denote by $c_1, \dots, c_k$ and by $s_1, \dots, s_\ell$ the $T^{n-2}$-orbits of $\varphi$ that are transversally a center and a saddle, respectively. Since $\varphi \in \mathscr{C}_n$, $k \geq 1$ and $\ell \geq 1$. Let $X_1, \dots, X_n$ be infinitesimal generators of $\varphi$ such that the action $\psi_\varphi$ generated by $X_1, \dots, X_{n-1}$ has the same orbits than $\varphi$ and $X_n = f_1 X_1 + \cdots + f_{n-1} X_{n-1}$, with $f_i$ non-constant for at least one $i = 1, \dots, n-1$. By Lemma 3.1, given a neighborhood of each $T^{n-2}$-orbit of $\psi_\varphi$ there exists a neighborhood $\mathcal{U}_i = \mathcal{U}_i(X_i)$ of $X_i$, $i = 1, \dots, n-1$, such that any action generated by $Y_1, \dots, Y_{n-1}$, $Y_i \in \mathcal{U}_i$ has only one transversally simple $T^{n-2}$-orbit in each of these neighborhoods, where $\tilde{s}_i$ is also transversally a saddle and $\tilde{c}_i$ is either transversally a focus or center. Now, assume that $\mathcal{V}$ is a neighborhood of $\varphi$ such that if $\phi \in \mathcal{V}$, then $Y_i \in \mathcal{U}_i$, $i = 1, \dots, n-1$. $\phi$ can not belong to $A^\omega_n(\mathbb{R}^n, N)$. If this were the case, then $\phi$ would have a $T^{n-1} \times \mathbb{R}$-orbit intersecting any neighborhood of $\widetilde{c}_1$ and also a $T^{n-2} \times \mathbb{R}$-orbit intersecting any neighborhood of $\widetilde{s}_1$, which contradicts Proposition 2.1. Thus, $\phi \in A^\omega_{n-1}(\mathbb{R}^n, N)$ and $Y_n = \widetilde{f}_1 Y_1 + \cdots + \widetilde{f}_{n-1} Y_{n-1}$. Finally if $\mathcal{V}$ is sufficiently small, then $\widetilde{f}_i$ can not be constant for every $i = 1, \dots, n-1$. ∎

Theorem 3.2. *If $\varphi \in \mathscr{C}_n$, then there exists a neighborhood $\mathcal{V} \subset \mathscr{C}_n$ of $\varphi$ such that $K_{\widetilde{\varphi}} = \{\widetilde{c}_1, \dots, \widetilde{c}_k\} \cup \{\widetilde{s}_1, \dots, \widetilde{s}_\ell\} \cup (\cup_{i=1}^{\ell} \widetilde{S}_i)$ for each $\widetilde{\varphi} \in \mathcal{V}$ and for each $i = 1, \dots, \ell$, $\widetilde{S}_i$ satisfies one of the following statements:*

*1. $\widetilde{S}_i$ is the union of two $T^{n-2} \times \mathbb{R}$-orbits $\widetilde{S}_{i1}, \widetilde{S}_{i2}$ such that $\widetilde{S}_{ij} \cup \widetilde{s}_i$ is homeomorphic to $T^{n-1}$, $j = 1, 2$.*
*2. $\widetilde{S}_i$ is a $T^{n-2} \times \mathbb{R}$-orbit such that $\mathrm{Front}(\widetilde{S}_i) = \widetilde{s}_i$.*

The structure of $\psi \in A^1(\mathbb{R}^{n-1}, N)$ in the neighborhood of a $T^{n-2}$-orbit that is transversally a center is well determined when $\psi$ has a non-constant first integral. More precisely:

Proposition 3.3. *Assume that $\psi \in A^1(\mathbb{R}^{n-1}, N)$ ($\varphi \in A^1_{n-1}(\mathbb{R}^n, N)$) has a $T^{n-2}$-orbit $\mathcal{O}_0$ that is transversally a center. If $\psi$ ($\varphi$) has a first integral that is non-constant on any open subset of $N$, then there exists a $\psi$-invariant ($\varphi$-invariant) neighborhood $V_0$ of $\mathcal{O}_0$ such that $V_0 \setminus \mathcal{O}_0$ is a union of $T^{n-1}$-orbits.*

*Proof.* Let $V$ be a neighborhood of $\mathcal{O}_0$ in $N$ such that $V \cap \mathrm{Sing}(\varphi) = \mathcal{O}_0$. Take a chart $h : V_p \to D^n_\varepsilon$ adapted to $\mathcal{O}_0$ in $p$ with $h(p) = 0$ and $V_p \subset V$. The infinitesimal generators of $\psi$ in these coordinates are like in (1). Since $0 \in D^2_\varepsilon$ is a center of $D\widehat{X}_{n-1}(0)$ and $\psi$ has a first integral that is not constant in $V$, so does $\widehat{X}_{n-1}$. By Lemma 3.2 there exists $\delta \in (0, \varepsilon)$ such that all orbits of $\widehat{X}_{n-1}$ within $D^2_\delta$ are closed. Let $\psi_0$ be the action of $\mathbb{R}^{n-2}$ on $N$ given by $X_1, \dots, X_{n-2}$. $\mathcal{O}_0$ is also an orbit of $\psi_0$ and $\Sigma = h^{-1}(D^2_\delta)$ is transversal at $p$ to the orbits of $\psi_0$. If $q \in \Sigma \setminus \{p\}$, then $C_q = h^{-1}(\mathcal{O}_{h(q)}(\widehat{X}_{n-1}))$ is homeomorphic to $S^1$ and is contained in $\mathcal{O}_q$, see Figure 2. Let's consider the holonomy of $\mathcal{O}_0$ as a leaf of the foliation defined by $\psi_0$

$$\mathrm{Hol} : \pi_1(\mathcal{O}_0) \cong \mathbb{Z}^k \to \mathrm{Diff}^1(\Sigma, p). \tag{2}$$

The orbit $\alpha_i$ of $X_i$ by $p$, $i = 1, \dots, n-2$, is closed and $\{[\alpha_1], \dots, [\alpha_{n-2}]\}$ is a set of generators of $\pi_1(\mathcal{O}_0)$. For each $i = 1, \dots, n-2$, let $D^{n-1}_i = \{t = (t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) \in D^n_\varepsilon\}$, $\Sigma_i = h^{-1}(D^{n-1}_i)$, $P_i : (\Sigma_i, p) \to (\Sigma_i, p)$ the Poincaré map of $X_i$ at $p$ and $\pi_i : V_p \to \Sigma$ the projection along the orbits of the action generated by $X_1, \dots, X_{i-1}, X_{i+1} \dots, X_{n-2}$. Then, $\omega_i = \mathrm{Hol}(\alpha_i) = \pi_i \circ P_i$. Note that $\omega_i(C_q) = C_{\omega_i(q)}$ for each $q \in \Sigma$. We state that $\omega_i(C_q) = C_q$ for each $q \in \Sigma \setminus \{p\}$ and each $i \in \{1, \dots, n-2\}$, which in turn implies that $\mathcal{O}_q$ is a $T^{n-1}$-orbit. In fact, if for some $q \in \Sigma \setminus \{p\}$ there exists $i \in \{1, \dots, n-2\}$ such that $\omega_i(q) \notin C_q$, then either $\omega_i(C_q)$ or $\omega_i^{-1}(C_q)$ would be a circle $C_{q'}$ in the interior of $C_q$. This would imply that all orbits of $\psi$ by points of the ring $R \subset \Gamma$ whose boundary is $C_q \cup C_{q'}$ would have a common orbit in its closure. Therefore, any first integral of $\psi$ would be constant on the saturated of $R$, which is an open set. Finally, when $\varphi \in A^1_{n-1}(\mathbb{R}^n, N)$ it is enough to consider $\psi_\varphi$. ∎

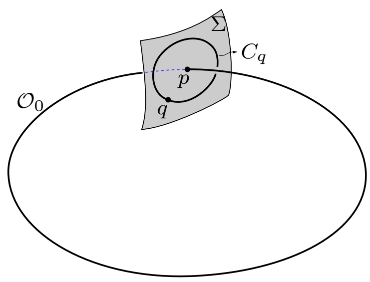


**FIG. 2.**

With the same notation used in Proposition 3.3 and similar arguments we obtain:

LEMMA 3.3. *Assume that $\psi \in A^1(\mathbb{R}^{n-1}, N)$ has a $T^{n-2}$-orbit $\mathcal{O}_0$ that is transversally a saddle. If $\psi$ has a first integral that is not constant on any open set, then for each $i = 1, \dots, n-2$, either $\omega_i = id$ or $\omega_i^2 = id$ in the space of orbits of $(h^* \widehat{X}_{n-1})|_\Sigma$.*

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Lemma 3.4. *If $\mathcal{O}$ is a $T^{n-2}\times\mathbb{R}$-orbit of $\varphi\in\mathscr{C}_n$, then $\mathcal{O}$ is a separatrix of some $T^{n-2}$-orbit $\mathcal{O}_0$ that is transversally a saddle.*

*Proof.* Let $C$ be a connected component of $\mathrm{cl}(\mathcal{O})\setminus\mathcal{O}$, where $\mathrm{cl}(\mathcal{O})$ denote the closure of $\mathcal{O}$. $C$ is a $\varphi$-invariant compact set. If $\mathrm{Sing}^c_{n-2}\cap C=\emptyset$, then $C$ is a $T^{n-1}$-orbit and consequently every first integral of $\varphi$ would be constant. If $\mathrm{Sing}^c_{n-2}\cap C\neq\emptyset$, call $\mathcal{O}_0$ the $T^{n-2}$-orbit contained in $C$. Since $\varphi\in\mathscr{C}_n$ any $T^{n-2}$-orbit is either transversally a center or a saddle. However, by Proposition 3.3 only the second case is possible. ▮

*Proof* (of Theorem 3.2). By Proposition 3.2 it is enough to prove the theorem for a fixed $\varphi\in\mathscr{C}_n$. Let $\mathcal{U}$ be the family of $\varphi$-invariant open sets $U\subset N\setminus K_\varphi$ such that every orbit in $U$ is a $T^{n-1}$-orbit. By Proposition 3.3, $\mathcal{U}$ is non-empty. The inclusion relation defines a parcial order in $\mathcal{U}$ and by Zorn's Lemma there exists a maximal element $U_M$ in $\mathcal{U}$. Clearly $N\setminus U_M=K_\varphi$. We are going to show that

$$N\setminus U_M=\{c_1,\dots,c_k\}\cup\{s_1,\dots,s_\ell\}\cup(\cup_{i=1}^{\ell}S_i),\tag{3}$$

where $S_i$ is the union of the separatrices of $s_i$, $i=1,\dots,\ell$. Let $F$ be a connected component of $\mathrm{Front}(U_M)$. Assume that $F\cap\mathrm{Sing}^c_2(\varphi)=\emptyset$, then $F$ is a $T^{n-1}$-orbit. Since $\varphi$ is analytic $F$ has trivial holonomy and there is a neighborhood $V_F$ such that each orbit inside $V_F$ is a $T^{n-1}$-orbit. The open set $U_M\cup V_F\in\mathcal{U}$ and contains $U_M$ properly albeit this is a contradiction. Assume now that $F\cap\mathrm{Sing}^c_{n-2}(\varphi)\neq\emptyset$, then there exists a $T^{n-2}$-orbit $\mathcal{O}\subset F$. If $\mathcal{O}=c_j$, then by Proposition 3.3 $c_j=F$. If $\mathcal{O}=s_i$, by Lemmas 3.3 and 3.4 there exists a neighborhood $V$ of $s_i$ such that every orbit in $V\setminus(S_i\cup s_i)$ is a $T^{n-1}$-orbit. Thus the relation (3) is proved. It remains to proved that each $S_i$ satisfies (1) or (2). With the notations introduced in the proof of the Proposição 3.3 put $\mathcal{O}_0=s_i$. $\widehat{\omega}_j=h\circ\omega_j\circ h^{-1}:(D^2_\varepsilon,0)\to(D^2_\varepsilon,0)$, $j=1,\dots,n-2$, is a local diffeomorphism that preserves orbits of $\widehat{X}_{n-1}$, then

$$D\widehat{\omega}_j(0)D\widehat{X}_{n-1}(0)=D\widehat{X}_{n-1}(0)D\widehat{\omega}_j(0).$$

This implies that there exists a neighborhood $\Sigma_0$ of $p$ in $\Sigma$ such that $\omega_j(\Sigma_0\cap\eta^k)\subset\eta^k$, $k=s,u$. Moreover, it follows from condition 4) of Definition 2.4 that $S_i$ satisfies (1) or (2). ▮

Proposition 3.4. *If $\varphi\in A^\omega_{n-1}(\mathbb{R}^n,N)$ with $N\notin\mathcal{H}_n$ and $\varphi$ satisfies conditions 1), 2) and 3) of Definition 2.4, then the existence of a $T^{n-2}$-orbit that is transversally a center implies the existence of another $T^{n-2}$-orbit that is transversally a saddle. In particular $\varphi\in\mathscr{C}^0_n(N)$.*

*Proof.* Let $\mathcal{O}$ be a $T^{n-2}$-orbit of $\varphi$ that is transversally a center and $\mathcal{U}$ be the family of $\varphi$-invariant open sets $U\supset\mathcal{O}$ and homeomorphic to $T^{n-2}\times D^2$ such that every orbit in $U \setminus \mathcal{O}$ is a $T^{n-1}$-orbit. By condition 3) and Proposition 3.3, $\mathcal{U}$ is non-empty. The inclusion relation defines a parcial order in $\mathcal{U}$ and by Zorn's Lemma there exists a maximal element $U_M$ in $\mathcal{U}$. As in the proof of Theorem 3.2, $\mathrm{Front}(U_M)$ can not be a $T^{n-1}$-orbit and by condition 1) $\mathrm{Front}(U_M)$ must contain a $T^{n-2}$-orbit $\mathcal{O}_1$. Now, if $\mathcal{O}_1$ is not transversally a saddle it has to be transversally a center and $N$ would be contained in $\mathcal{H}_n$. ▮

It follows from Corollary 2.2 that an action $\varphi$, as in Proposition 3.4, does not embedded in $A_n^\omega(\mathbb{R}^n, N)$.

Let $\mathcal{O}_p$ be a $T^k \times \mathbb{R}^{n-k-1}$-orbit, $1 \leq k \leq n-1$, of $\psi \in A^r(\mathbb{R}^{n-1}, N)$ and $\{w_1^0, \dots, w_{n-1}^0\}$ a base of $\mathbb{R}^{n-1}$, where $\{w_1^0, \dots, w_k^0\}$ is a set of generators of the isotropy group $G_p$. Let $X_i = X_{w_i^0}$, $i = 1, \dots, n-1$. Note that for each $q \in \mathcal{O}_p$, $\mathcal{O}_q(X_i)$, $i = 1, \dots, k$ is periodic of period one.

Lemma 3.5. *Assume that there exists a neighborhood $V$ of $p$ such that each orbit in $V \setminus \mathcal{O}_p$ is a $T^{n-1}$-orbit. Then, there exist a neighborhood $V_0$ of $p$ with $V_0 \subset V$ and $C^r$ functions $w_i : V_0 \to \mathbb{R}^{n-1}$, $i = 1, \dots, k$, such that $w_i(p) = w_i^0$ and for each $q \in V_0$, every orbit of $X_{w_i(q)}$, $i = 1, \dots, k$ is periodic of period one.*

*Proof.* Let $h : V_p \subset V \to D_\varepsilon^n$ with $h(p) = 0$, be a $(n-1)$-flow box at $p$. Let $D_i = D_i(\varepsilon) = \{(x_1, \dots, x_n) \in D_\varepsilon^n; x_i = 0\}$ and $\Sigma_i = \Sigma_i(\varepsilon) = h^{-1}(D_i)$. The functions $\tau_i : V_p \to (-\varepsilon, \varepsilon)$ given by $\tau_i(q) = -x_i(q)$, where $h(q) = (x_1(q), \dots, x_n(q))$, are such that $X_i^{\tau_i(q)}(q) \in \Sigma_i$, for $i = 1, \dots, n-1$. We know that $X_1^1(p) = p$, $i = 1, \dots, k$. Therefore, there exists $0 < \delta < \varepsilon$ such that $X_i^1(\Sigma_i(\delta)) \subset V_p$, $i = 1, \dots, k$. Let $\Sigma_p = \Sigma_p(\delta) = \cap_{i=1}^{n-1} \Sigma_i(\delta)$. $\Sigma_p$ is a transversal section to $\mathcal{O}_p$ at $p$. For each $i = 1, \dots, k$, consider the function $w_i : \Sigma_p \to \mathbb{R}^{n-1}$ given by

$$w_i(q) = \sum_{j=1}^{i-1} \tau_j(X_i^1(q)) w_j^0 + (1 + \tau_i(X_i^1(q))) w_i^0 + \sum_{j=i+1}^{n-1} \tau_j(X_i^1(q)) w_j^0. \tag{4}$$

It can be verified that every orbit of $X_{w_i(q)}$ inside $\mathcal{O}_q$, $q \in U$, is periodic of period one and $w_i(p) = w_i^0$, $i = 1, \dots, k$. We can extend the functions $w_i$ to the open set $V_0 = \cup_{q \in \Sigma_p} (\mathcal{O}_q \cap V)$ by defining $w_i(q) = w_i(\Sigma_p \cap \mathcal{O}_q)$ and this completes the proof. ▮

Lemma 3.6. *Assume that $\psi \in A^\omega(\mathbb{R}^{n-1}, N)$ immerses properly in $\mathscr{C}_n$. Let $S_1, S_2$ be the separatrices of a $T^{n-2}$-orbit $\mathcal{O}$ that is transversally a saddle and $G_0, G_1, G_2$ the isotropy subgroups of $\mathcal{O}$, $S_1, S_2$, respectively. Then, there exists a linear $(n-2)$-subspace $H$ of $\mathbb{R}^{n-1}$ transversal to $G_0$ such that $G_1 = G_0 \cap H = G_2$.*

*Proof.* For each $i = 1, 2$, $G_i$ is a subgroup of $G_0$ isomorphic to $\mathbb{Z}^{n-2}$. Let $H_i$ be the linear subspace of $\mathbb{R}^{n-1}$ generated by $G_i$. Then $\dim H_i = n-2$ and $H_i$ is transversal to $G_0^0$. We first show that $H_1 = H_2$. Assume that $H_1 \neq H_2$ and choose $u_1 \in G_1 \setminus G_2$, $u_2 \in G_2 \setminus G_1$ such that $w = u_1 - u_2 \in G_0^0$. Let $X_w, X_{u_1}, X_{u_2} \in \mathfrak{X}^r(N)$ be the associated

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vector fields. Then $X_w = X_{u_1} - X_{u_2}$ or equivalently $X_w^t = X_{u_1}^t \circ X_{u_2}^{-t}$. Take infinitesimal generators $X_1, \ldots, X_{n-1}$ of $\psi$ adapted to $\mathcal{O}$ so that $X_{n-1} = X_w$ and a chart $(V, h)$ adapted to $\mathcal{O}$ at $p \in \mathcal{O}$. We will show now that $DX_{u_i}^1(p) = id$, $i = 1, 2$. Consider the map $h \circ X_{u_i}^1 \circ h^{-1} : D_\varepsilon^n \to D_\varepsilon^n$. At $0 \in D_\varepsilon^n \subset \mathbb{R}^n$ take a base $\{\partial/\partial\theta_1, \ldots, \partial/\partial\theta_{n-2}, v_s, v_u\}$ such that $v_s$ ($v_u$) is tangent to $\xi^s$ ($\xi^u$). Since $S_i \cap \eta^s \neq \emptyset$ and $S_i \cap \eta^u \neq \emptyset$ it follows that $D(h \circ X_{u_i}^1 \circ h^{-1})(0)(v_s) = v_s$ and $D(h \circ X_{u_i}^1 \circ h^{-1})(0)(v_u) = v_u$. Since $u_i \in G_0$, it follows that $D(h \circ X_{u_i}^1 \circ h^{-1})(0)(\partial/\partial\theta_j) = \partial/\partial\theta_j$, $j = 1, \ldots, n-2$. We conclude that $DX_{u_i}(p) = id$. Thus, $DX_w^1(p) = id$, which is equivalent to $DX_w(p) = 0$. However, this contradicts the fact that $\mathcal{O}$ is transversally a saddle and proves that $H_1 = H_2 = H$. Finally the fact that $G_1 = G_0 \cap H = G_2$ follows applying Lemma 3.5 to the action $\psi|_H$ on $S_1 \cup S_2$. ▮

## 4. PROOF OF THE MAIN RESULTS

### 4.1. Proof of Theorem 2.1

The following proposition is an extension of a result used by Lima in [6] for $n = 3$. Let $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$.

Proposition 4.1. *Let $Y_1, \ldots, Y_{s-1}$, $s \geq 3$, be a set of commutative vector fields on $T^{s-2} \times D$ that are tangent to the boundary and also linearly independent there. Then, in every disk $\{\theta\} \times D \subset T^{s-2} \times D$ there exists an interior point where they are linearly dependent.*

*Proof.* Let $\varphi : \mathbb{R}^{s-1} \times (T^{s-2} \times D) \to T^{s-2} \times D$ be the action induced by $Y_1, \ldots, Y_{s-1}$. Since these vector fields are tangent to the boundary $\partial(T^{s-2} \times D) = T^{s-2} \times S^1$ and linearly independent there, we know that $\partial(T^{s-2} \times D)$ is an orbit of $\varphi$. Let $H$ be the isotropy group of this orbit. Choose infinitesimal generators $X_1, \ldots, X_{s-1}$ of $\varphi$ such that at each point $\theta = (\theta_1, \ldots, \theta_{s-1}) \in T^{s-1}$, $X_i(\theta) = \frac{\partial}{\partial\theta_i}$, $i = 1, \ldots, s-1$, and consider the map $\alpha : \partial D \to SO(s) \subset V_s(\mathbb{R}^s)$, where $V_s(\mathbb{R}^s)$ is the manifold of oriented bases of $\mathbb{R}^s$, given by $\alpha = (X_1, \ldots, X_s)$, where $X_s = X_1 \wedge \cdots \wedge X_{s-1}$. For each $s > 2$ $\pi_1(V_s(\mathbb{R}^s)) = \pi_1(SO(s)) = \mathbb{Z}_2$ and its generator is given by the inclusion of the circle $SO(2)$ into $SO(s)$. Since we can interpret $\alpha$ as been this inclusion, it follows that $\alpha$, as a map into $V_s(\mathbb{R}^s)$, can not be extended to $D$ and the proposition is proved. ▮

Corollary 4.1. *Let $\psi \in A^1(\mathbb{R}^{s-1}, M^s)$, $s \geq 3$, and $p \in \mathrm{Fix}(\psi)$ be an isolated singularity. Then, there exists a neighborhood $V$ of $p$ in $M^s$ which does not contain any $T^{s-1}$-orbit.*

*Proof.* Let $V$ be a neighborhood of $p$ such that $V \cap \mathrm{Sing}(\psi) = \{p\}$ and assume that there is a $T^{s-1}$-orbit $\mathcal{O} \subset V$. Then, $\mathcal{O}$ embedds in $\mathbb{R}^s$ as an orbit and we can assume that $\mathcal{O} = \partial(T^{s-2} \times D)$ with $T^{s-2} \times D \subset V$. By Proposition 4.1, there are infinitely many singular points of $\psi$ in the interior of $T^{s-2} \times D$ and this is a contradiction. ▮

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*Proof* (of Proposition 2.1). Front($\mathcal{O}$) has at most two connected components and each one of them contains at least a compact orbit. Let $C$ be a connected component and $\mathcal{O}_0 \subset C$ a $T^k$-orbit with isotropy group $G_0$. We will show that $C = \mathcal{O}_0$. Assume, for a moment, that $C$ contains another orbit $\mathcal{O}_1$ and take $p \in \mathcal{O}_0$ and $q \in \mathcal{O}_1$. Let $u_1, \ldots, u_{n-1}$ be a set of generators of $G$, the isotropy group of $\mathcal{O}$. Then, all $X_{u_i}$-orbits through $\mathcal{O}$ are periodic of period one. Even more, there are sequences $\{p_j \in \mathcal{O}; j \in \mathbb{N}\}$ and $\{t_{ij} \in [0,1]; i = 1, 2 \ldots, n-1 \text{ and } j \in \mathbb{N}\}$ such that

$$\lim_{j\to\infty} p_j = p \quad \text{and} \quad \lim_{j\to\infty} \varphi\Big(\sum_{i=1}^{n-1} t_{ij}u_i, p_j\Big) = q.$$

For each $i = 1, \ldots, n-1$, we can assume, extracting a subsequence if necessary, that $t_{ij} \to t_i \in [0,1]$. Then $\varphi(\sum_{i=1}^{n-1} t_i u_i, p) = q$, which contradicts the fact that $\sum_{i=1}^{n-1} t_i u_i \in G_0$.

Now, we will show that $k \in \{n-2, n-1\}$. Assume, for a moment, that $k < n-2$, then $s = n-k > 2$. Let $p \in \mathcal{O}_0$ and $h : V_p \to D_1^n$ a chart adapted to $\mathcal{O}_0$ at $p$ and $\varphi_T$ the induced local action of $\mathbb{R}^s$ on $D^s$. The image of $\mathcal{O} \cap V_p$ by $h$ intersects $D^s$ in a $T^{s-1} \times \mathbb{R}$-orbit $\widehat{\mathcal{O}}$ of $\varphi_T$ such that $0 \in D^s$ is a connected component of Front($\widehat{\mathcal{O}}$) and, therefore, an isolated singular point of $\varphi_T$. The generators $u_1, \ldots, u_{s-1}$ of the isotropy group of $\widehat{\mathcal{O}}$ define a local action of $\mathbb{R}^{s-1}$ on $D^s$ and this action has $T^{s-1}$-orbits arbitrarily close to 0. This contradicts Corollary 4.1 and proves that $k \in \{n-2, n-1\}$. ▮

*Proof* (of Theorem 2.1). Let $\mathcal{O}$ be a $T^{n-1} \times \mathbb{R}$-orbit and $\mathcal{U}$ the family of all $\varphi$-invariant neighborhoods $U \supset \mathcal{O}$, homeomorphic to $T^{n-1} \times \mathbb{R}$, that do not contain a $T^s \times \mathbb{R}^{n-s}$-orbit with $s \neq n-1$. The inclusion relation defines a parcial order in $\mathcal{U}$ and by Zorn's Lemma there exists a maximal element $U_M$ in $\mathcal{U}$. We are going to show that $\text{cl}(U_M) = N$. In fact, assume that $\text{cl}(U_M) \neq N$, and let $F$ be a connected component of $\text{cl}(U_M) \setminus U_M$. There are points of $F$ in the closure of at least one $T^{n-1} \times \mathbb{R}$-orbit $\mathcal{O}_1 \subset U_M$ and since Front($\mathcal{O}_1$) is $\varphi$-invariant, then $F \subset$ Front($\mathcal{O}_1$). By Proposition 2.1 and the fact that $U_M$ is maximal, we can assume that $F$ is a $T^{n-1}$-orbit. Since $N$ is orientable there exists another $T^{n-1} \times \mathbb{R}$-orbit $\mathcal{O}_2 \subset N \setminus U_M$ such that $F \subset$ Front($\mathcal{O}_2$). The open set $U_M \cup F \cup \mathcal{O}_2 \in \mathcal{U}$ and contains properly $U_M$. This contradicts the fact that $U_M$ is maximal and proves that every $n$-dimensional orbit is a $T^{n-1} \times \mathbb{R}$-orbit.

Let $\mathcal{O}_i$, $1 \leq i \leq \ell$, be the $n$-dimensional orbits of $\varphi$ and $G_i$ the isotropy group of $\mathcal{O}_i$. From the analyticity of $\varphi$ one obtains that $G_1 = \cdots = G_\ell$, which are isomorphic to $\mathbb{Z}^{n-1}$. Let $\{u_1, \ldots, u_{n-1}, u_n\}$ be a set of generators of $\mathbb{R}^n$, as a vector space, such that $\{u_1, \ldots, u_{n-1}\}$ is a set of generators of $G_1$ and $X_i = X_{u_i}$, $1 \leq i \leq n$, the associated vector fields. $\{X_1, \ldots, X_{n-1}\}$ are the infinitesimal generators of an action $\varphi_H$ of $\mathbb{R}^{n-1}$ on $N$ that is the restriction of $\varphi$ to the subspace $H$ generated by $\{u_1, \ldots, u_{n-1}\}$. Note that if $p \in \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_\ell$, then $\mathcal{O}_p(\varphi_H)$, the orbit of $\varphi_H$ by $p$, is a $T^{n-1}$-orbit and if $p \in N \setminus \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_\ell$, then it is, by Proposition 2.1, a $T^{n-1}$-orbit or a $T^{n-2}$-orbit. Even more, we can verify that the map $\overline{\varphi_H} : T^{n-1} \times N \to N$ given by $\overline{\varphi_H}(e^{2\pi i t_1}, \ldots, e^{2\pi i t_{n-1}}, p) = \varphi_H(t_1u_1, \ldots, t_{n-1}u_{n-1}, p)$ is in fact an analytic action of $T^{n-1}$ with the same underlying foliation than the one given by $\varphi_H$. Assume that $\text{Sing}^c_{n-2}(\varphi) = \emptyset$, then by Lemma 3.5 the

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isotropy group of every orbit of $\overline{\varphi_H}$ is the same; or, in other words, $\overline{\varphi_H}$ is a free action. Thus, $N$ is a principal $T^{n-1}$ bundle over $S^1$ and consequently $N$ is homeomorphic to $T^n$. Finally, assume $\mathrm{Sing}^c_{n-2}(\varphi) \neq \emptyset$. Since $U_M$ is homeomorphic to $T^{n-1} \times \mathbb{R}$ and $\mathrm{cl}(U_M) = N$, it follows that $\mathrm{Sing}^c_{n-2}(\varphi)$ is the union of two $T^{n-2}$-orbits. Thus $N \in \mathcal{H}_n$ . ▌

### 4.2. Proof of Theorem 2.2, $n = 2$

Before starting the proof of Theorem 2.2 we exhibit an example of an action $\varphi \in \mathscr{C}_2$ .

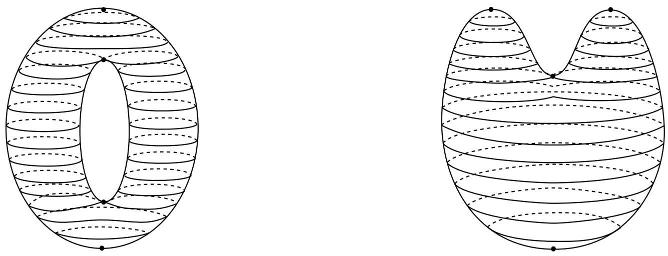

**FIG. 3.**

**Example:** Let $N$ be a closed orientable surface, $H : N \to \mathbb{R}$ an analytic Morse function with at least one singularity of index $-1$, $X_H \in \mathfrak{X}^\omega(N)$ the associated Hamiltonian. Assume that there are no connections between two different saddle points of $X_H$ . Then, $\{X_H, HX_H\}$ are infinitesimal generators of an action $\varphi \in \mathscr{C}_2$ (see Figure 3).

*Proof* (of Theorem 2.2, n=2). Without loss of generality we can assume that $\widetilde{c}_i = c_i$ , $i = 1, \ldots, k$ and $\widetilde{s}_j = s_j$ , $j = 1, \ldots, \ell$. Let $F_j : V_j \to \widetilde{V}_j$ be a topological equivalence between $X_1$ and $\widetilde{X}_1$ at $s_j$ and $U_j \subset V_j$ a cross shaped neighborhood of $s_j$ such that $\mathrm{Front}(U_j) = (\cup_{i=1}^4 B_{ji}) \cup (\cup_{i=1}^2 (A^1_{ji} \cup A^2_{ji}))$, where $B_{ji}$ is a piece of orbit of $X_1$ and $A^i_{j1}$ , $i = 1, 2$ intersects $S_{j1}$ and is transversal to $X_1$ , see Figure 4. Let $\widetilde{U}_j = F_j(U_j)$, $\widetilde{B}_{ji} = F_j(B_{ji})$ and $\widetilde{A}^l_{ji} = F_j(A^l_{ji})$. Reparametrizing the time $t$ of the flows $X^t_1$ and $\widetilde{X}^t_1$ , we can assume that $X^1_1(A^1_{ji}) = A^2_{ji}$ and $\widetilde{X}^1_1(\widetilde{A}^1_{ji}) = \widetilde{A}^2_{ji}$ for $i = 1, 2$. Let $W_j$ $(\widetilde{W}_j)$ be the saturated by $X_1$ $(\widetilde{X}_1)$ of closure of $U_j$ $(\widetilde{U}_j)$. Then, $\mathrm{Front}(W_j)$ $(\mathrm{Front}(\widetilde{W}_j))$ is the union of three closed orbits of $X_1$ $(\widetilde{X}_1)$. If $p \in W_j \setminus U_j$ , there exists $t \in (0, 1)$ such that $X^t_1(p) \in A^l_{ji}$ for some $i, l \in \{1, 2\}$. Extend $F_j$ to $W_j$ by defining $F_j(p) = \widetilde{X}^{-t}_1(F_j(X^t_1(p)))$. It is clear that $F_j : W_j \to \widetilde{W}_j$ is a topological equivalence between $X_1$ and $\widetilde{X}_1$ that preserves orientation. Note that it is possible to reduce the size of $W_j$ to guarantee that $W_{j_1} \cap W_{j_2} = \emptyset$ if $j_1 \neq j_2$ .

Next, we are going to extend the equivalences $F_j$ to a global topological equivalence $F$ between $X_1$ and $\widetilde{X}_1$ . Fix a Riemannian metric on $N$ and let $Z$ be the vector field obtained by rotating $X_1$ by a right angle. Clearly $\mathrm{Sing}(Z) = \mathrm{Sing}(X_1) = \mathrm{Sing}(\widetilde{X}_1)$ and taking $\mathcal{V}$, smaller, if necessary, $Z$ will be transversal to both $X_1$ and $\widetilde{X}_1$ at points in

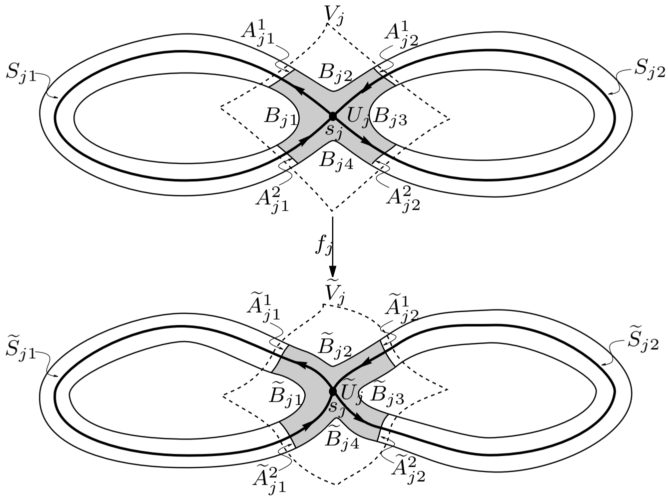


**FIG. 4.**

$N \setminus \mathrm{Sing}(Z)$. Each connected component $C$ of $N \setminus \cup_{j=1}^{\ell} W_j$ is homeomorphic either to an open disk or to $S^1 \times (0,1)$. In the first case, $C \cap \mathrm{Fix}(\varphi)$ is a center and in the second case, it is empty. To each $C$ corresponds a unique connected component $\widetilde{C}$ of $N \setminus \cup_{j=1}^{\ell} \widetilde{W}_j$ with the same properties than $C$. We shall define $F : C \to \widetilde{C}$ for each $C$. There are two cases:

$(i)$ *$C$ is an open disk.* There exists a unique $j \in \{1, \dots, \ell\}$, such that $\mathrm{Front}(C) \subset \mathrm{Front}(W_j)$, $\mathrm{Front}(\widetilde{C}) \subset \mathrm{Front}(\widetilde{W}_j)$ and $F_j(\mathrm{Front}(C)) = \mathrm{Front}(\widetilde{C})$. For each point $p \in \mathrm{Front}(C)$ let $L_p = \mathrm{cl}(\mathcal{O}_p(Z) \cap C)$ and $\widetilde{L}_{F_j(p)} = \mathrm{cl}(\mathcal{O}_{F_j(p)}(Z) \cap \widetilde{C})$. Fix a point $p_0 \in \mathrm{Front}(C)$ and extend $F_j : L_{p_0} \to \widetilde{L}_{F_j(p_0)}$ as a homeomorphism. If $q \in C$, there is a unique $L_p$ that contains $q$. Now, let $q_1 = \mathcal{O}_q(X_1) \cap L_{p_0}$. Define $F(q) = \mathcal{O}_{F_j(q_1)}(\widetilde{X}_1) \cap \widetilde{L}_{F_j(p)}$.

$(ii)$ *$C$ is homeomorphic to $S^1 \times (0,1)$.* $\mathrm{Front}(C) = A_0 \cup A_1$, where $A_0$ and $A_1$ are $S^1$-orbits. There exists $i, j \in \{1, \dots, \ell\}$, $i \neq j$ such that $A_0$ $(A_1)$ is a connected component of $\mathrm{Front}(W_i)$ $(\mathrm{Front}(W_j))$. For the unique $\widetilde{C}$ associated to $C$ we also have $\mathrm{Front}(\widetilde{C}) = \widetilde{A}_0 \cup \widetilde{A}_1$, where $\widetilde{A}_0$ $(\widetilde{A}_1)$ is a connected component of $\mathrm{Front}(\widetilde{W}_i)$ $(\mathrm{Front}(\widetilde{W}_j))$. Let $G : \mathrm{cl}(C) \to S^1 \times [0,1]$ and $\widetilde{G} : \mathrm{cl}(\widetilde{C}) \to S^1 \times [0,1]$ homeomorphisms such that the orbits of $X_1|_{\mathrm{cl}(C)}$ and $\widetilde{X}_1|_{\mathrm{cl}(\widetilde{C})}$ are $G^{-1}(S^1 \times \{t\})$ and $\widetilde{G}^{-1}(S^1 \times \{t\})$, $t \in [0,1]$. The two homeomorphisms $h_0 : S^1 \times \{0\} \to S^1 \times \{0\}$ and $h_1 : S^1 \times \{1\} \to S^1 \times \{1\}$ given by $h_0 = \widetilde{G} \circ F_i \circ G^{-1}$ and $h_1 = \widetilde{G} \circ F_j \circ G^{-1}$ are both isotopic to the identity and therefore isotopic among themselves. Let $h_t$ be the isotopy between $h_0$ and $h_1$ and define $H : S^1 \times [0,1] \to S^1 \times [0,1]$ by $H(\theta, t) = (h_t(\theta), t)$. $H$ is a homeomorphism preserving the foliation $S^1 \times \{t\}$ and the restriction of the topological equivalence $F$ to be closure of $C$ will be the homeomorphism $\widetilde{G}^{-1} \circ H \circ G : \mathrm{cl}(C) \to \mathrm{cl}(\widetilde{C})$. ▮

### 4.3. Proof of Theorem 2.2, general case

Fix $i \in \{1, \ldots, \ell\}$ and let $X_1, \ldots, X_{n-1}$ be infinitesimal generators of $\psi$ adapted to $\mathcal{O} = s_i$. Let $S_i$ be as in relation (3) and $G_i$ the isotropy group of $S_i$ (well defined because of Theorem 3.2 and Lemma 3.6). Call $H$ the $(n-2)$-linear subspace of $\mathbb{R}^{n-1}$ generated by $G_i$ and $\psi_0 = \psi|_H$ is defined by $X_1, \ldots, X_{n-2}$.

LEMMA 4.1. *Assume that $\psi \in A^\omega(\mathbb{R}^{n-1}, N)$ immerses properly in $\mathscr{C}_n$. Let $S$ be the union of the separatrices of a $T^{n-2}$-orbit $\mathcal{O}$ which is transversally a saddle and $p \in \mathcal{O}$. Then, there exists an eight shaped curve $\Gamma$ with $p \in \Gamma$ and $\Gamma \subset \mathrm{cl}(S)$, which is transversal to the orbits of $\psi_0$ within $\mathrm{cl}(S)$, such that $\Gamma \setminus \{p\} = \Gamma_1 \cup \Gamma_2 \subset S$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_i \cup \{p\}$ is homeomorphic to $S^1$, $i = 1, 2$.*

*Proof.* Let $h : V \longrightarrow D_\varepsilon^{n-2} \times D_\varepsilon^2$ be a chart adapted to $\mathcal{O}$ at $p$ and $\eta^s$ ($\eta^u$) be the stable (unstable) submanifold of $(h|_\Sigma)^* \widehat{X}_{n-1}$ at $p \in \Sigma = h^{-1}(D_\varepsilon^2)$. Put $\eta^i \setminus \{p\} = \eta_1^i \cup \eta_2^i$, $i = s, u$. Note that $\eta_j^s, \eta_j^u \subset S_j$, $j = 1, 2$. Let $q_j^i \in \eta_j^i$, $i = u, s$ be such that $\mathcal{O}_{q_j^s}(\psi_0) \neq \mathcal{O}_{q_j^u}(\psi_0)$ for $j = 1, 2$. Then, there exist $v_1, v_2 \in \mathbb{R}^{n-1} \setminus H$ such that $q_j^u \in \mathcal{O}_{q_j^s}(X_{v_j})$, $j = 1, 2$. Furthermore, since $v_1, v_2 \in \mathbb{R}^{n-1} \setminus H$ we know that $\mathcal{O}_{q_j^s}(X_{v_j})$, $j = 1, 2$ is transversal to the orbits of $\psi_0$. Consider the arc $\overline{q_1^i q_2^i} \subset \eta^i$ with extremes $q_1^i$ and $q_2^i$, $i = s, u$, and the arc $\overline{q_j^s q_j^u} \subset \mathcal{O}_{q_j^s}(X_{v_j})$ with extremes $q_j^s, q_j^u$, $j = 1, 2$. The curve

$$\Gamma = \overline{q_1^s q_2^s} \cup \overline{q_1^u q_2^u} \cup \overline{q_1^s q_1^u} \cup \overline{q_2^s q_2^u}$$

is transversal to the $\psi_0$-orbits. We can assume, without lost of generality that the connected components of $\Gamma \setminus \{p\}$ are $C^\infty$ curves. ▮

*Remark 4. 1.* The curve $\Gamma$, although self-intersecting at $p$, has a $C^\infty$ tangent vector field, which is linearly independent with $X_1, \ldots, X_{n-2}$ at each point $q \in \Gamma$ and also $\Gamma \setminus \{p\} \subset S$. Let $U \subset \Sigma$ be a cross shaped neighborhood of $p$ as in the proof of Theorem 2.2, $n = 2$. It is possible to extend $U$ to a 2-dimensional submanifold $\mu \supset \Gamma$ which is transversal to the orbits of $\psi_0$ and therefore also to those of $\psi$. The intersection of the orbits of $\psi$ with $\mu$ define a foliation, which is given by the orbits of a vector field $X$ that satisfies the following properties:

1. $X|_U = (h^* \widehat{X}_{n-1})|_U$,
2. $\mathrm{Sing}(X) = \{p\}$ and $p$ is a saddle singularity.
3. The connected components of $\Gamma \setminus \{p\}$ are the separatrices of $X$ at $p$.
4. Every orbit of $X$ by a points in $\mu \setminus \Gamma$ is closed.

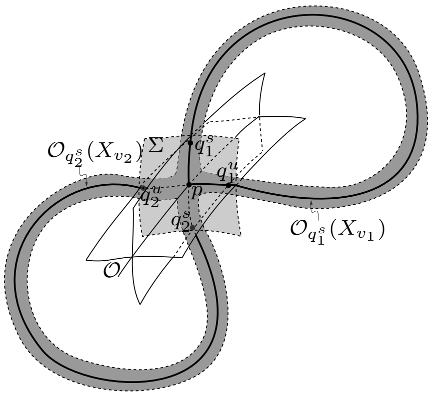


**FIG. 5.**

Let $\mathcal{V}$ be a neighborhood of $\varphi \in \mathscr{C}_n$ as in Proposition 3.2. By this proposition and Theorem 3.2 for each $\phi \in \mathcal{V}$ we have

$$K_\phi = \{\widetilde{c}_1, \ldots, \widetilde{c}_k\} \cup \{\widetilde{s}_1, \ldots, \widetilde{s}_\ell\} \cup (\cup_{i=1}^{\ell} \widetilde{S}_i)$$

where, for each $i = 1, \ldots, \ell$, $\widetilde{S}_i$ has the same properties as $S_i$ (Theorem 3.2, (1) or (2)).

Let $\{w_1^0, \ldots, w_{n-1}^0\}$ be a base of $\mathbb{R}^{n-1}$, where $\{w_1^0, \ldots, w_k^0\}$ is a set of generators of $G_p$, the isotropy group of a $T^k \times \mathbb{R}^{n-k-1}$-orbit $\mathcal{O}_p$, $n-2 \leq k \leq n-1$, of $\psi_\varphi$. Let $X_i = X_{w_i^0}$, $i = 1, \ldots, n-1$, and $w_i : V_0 \to \mathbb{R}^{n-1}$, $i = 1, \ldots, k$, the $C^r$ functions given by Lemma 3.5. if $\phi \in \mathcal{V}$ and $\{u_1, \ldots, u_{n-1}\}$ is a base of $\mathbb{R}^{n-1}$, we will denote by $\{Y_{u_1}, \ldots, Y_{u_{n-1}}\}$ the corresponding infinitesimal generators of $\psi_\phi$. Let $Y_i = Y_{w_i^0}$, $i = 1, \ldots, n-1$.

LEMMA 4.2. *Given $\varphi \in \mathscr{C}_n$ and $d > 0$ there exist a neighborhood $\mathcal{V}_0 \subset \mathcal{V}$ of $\varphi \in \mathscr{C}_n$ and a neighborhood $V_1 \subset V_0$ of $p$ that satisfy: if $\phi \in \mathcal{V}_0$, then there exist $C^\omega$ functions $\widetilde{w}_i : V_1 \to \mathbb{R}^{n-1}$, $i = 1, \ldots, k$, such that for each $q \in V_1$, $\|\widetilde{w}_i(q) - w_i(q)\| < d$ and every orbit of $Y_{\widetilde{w}_i(q)}$, $i = 1, \ldots, k$ inside $\mathcal{O}_q(\psi_\phi)$ is periodic of period one.*

*Proof.* We shall use the same notations as in the proof of Lemma 3.5. Let $\mathcal{V}_1 \subset \mathcal{V}$ be a neighborhood of $\varphi$ such that $Y_i^1(\Sigma_i(\delta)) \subset V_0$ for all $i = 1, \ldots, k$. There exists $\alpha > 0$ such that

$$\widetilde{\Sigma}_i = \cup_{|t_i|<\alpha} Y_1^{t_1} \circ \cdots \circ Y_{i-1}^{t_{i-1}} \circ Y_{i+1}^{t_{i+1}} \circ \cdots \circ Y_{n-1}^{t_{n-1}}(\Sigma_p(\delta)),\ i = 1, \ldots, n-1,$$

is $C^r$ close to $\Sigma_i$ and transversal to $Y_i$. Let $V_1 = \cup_{|t|<\alpha} Y_i^t(\widetilde{\Sigma}_i)$. For $\phi$ sufficiently $C^1$ close of $\varphi$ the $C^\omega$ functions $\widetilde{\tau}_i : V_1 \to (-\varepsilon, \varepsilon)$ such that $Y_i^{\widetilde{\tau}_i(q)}(q) \in \widetilde{\Sigma}_i$, $i = 1, \ldots, n-1$,

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are $C^1$ close of the corresponding $\tau_i|_{V_1}$ of $\psi_\varphi$. Note that $\Sigma_p = \Sigma_p(\delta) = \cap_{i=1}^{n-1}\widetilde{\Sigma}_i(\alpha)$. For each $i = 1, \ldots, k$, the functions $\widetilde{w}_i : \Sigma_p \to \mathbb{R}^{n-1}$ given by

$$\widetilde{w}_i(q) = \sum_{j=1}^{i-1} \widetilde{\tau}_j(Y_i^1(q))w_j^0 + (1 + \widetilde{\tau}_i(Y_i^1(q)))w_i^0 + \sum_{j=i+1}^{n-1} \widetilde{\tau}_j(Y_i^1(q))w_j^0$$

are such that every orbit of $Y_{\widetilde{w}_i(q)}$, $i = 1, \ldots, k$, inside $\mathcal{O}_q(\psi_\phi)$ is periodic of period one. Furthermore, there exist a neighborhood $\mathcal{V}_0 \subset \mathcal{V}_1$ of $\varphi$ such that $\phi \in \mathcal{V}_0$ implies that $\widetilde{w}_i$ is $d - C^1$–close of $w_i$. ▮

*Proof* (of Theorem 2.2, general case). Let $\psi_\varphi \in A^\omega(\mathbb{R}^{n-1}, N)$ be the action associated to $\varphi$ given by Lemma 2.4. Put a Riemannian metric on $N$ and let $\xi$ be the normal bundle to $\mathscr{F}(\psi_\varphi)$, the codimension 1 underlying foliation of $\psi_\varphi$. $\xi$ is orientable and integrable. Let $\mathcal{V}$ be a neighborhood of $\varphi \in \mathscr{C}_n$ as in Proposition 3.2. Without loss of generality, we can assume that if $\phi \in \mathcal{V}$, then there exists $p_j \in \widetilde{s}_j \cap s_j$, $j = 1, \ldots, \ell$. Let $\Gamma_j \ni p_j$ be the eight shaped curve given by Lemma 4.1 for the action $\psi_\varphi$ and $(\mu_j, X_\varphi^j)$ the 2-dimensional submanifold and the vector field, given by Remark 4.1. It is possible to construct each $\mu_j$ in such a way that $\xi$ be tangent to $\mu_j$ at each $p \in \mu_j \setminus \{p_j\}$. If $\mathcal{V}$ is sufficiently small, then $\mu_j$ is also transversal to the $\psi_\phi$-orbits for every $\phi \in \mathcal{V}$. The intersection of the $\psi_\phi$-orbits with $\mu_j$, $j = 1, \ldots, \ell$, are the orbits of a vector field $X_\phi^j$. By Theorem 3.2, $\widetilde{S}_j \cap \mu_j = \widetilde{\Gamma}_j$ is an eight shaped curve and the orbits of $X_\phi^j$ near of $\widetilde{\Gamma}_j$ are periodic. With similar arguments to those used in the case $n = 2$, we obtain a topological equivalence $F_j : \nu_j \to \widetilde{\nu}_j$ between $X_\varphi^j$ and $X_\phi^j$, where $\nu_j$ ($\widetilde{\nu}_j$) is a $X_\varphi^j$-invariant ( $X_\phi^j$-invariant) closed neighborhood of $\Gamma_j$ ($\widetilde{\Gamma}_j$) and $\nu_j \cup \widetilde{\nu}_j \subset \mu_j$. Given $\delta > 0$, we can reduce the size of $\mathcal{V}$ and also of $\nu_j$ in such a way that for every $p \in \nu_j$, $d(p, F_j(p)) < \delta$.

Let $W_j$ ($\widetilde{W}_j$) be the saturated by $\psi_\varphi$ ($\psi_\phi$) of $\nu_j$ ($\widetilde{\nu}_j$). Then Front($W_j$) (Front($\widetilde{W}_j$)) is the union of three $T^{n-1}$-orbits if $S_j$ ($\widetilde{S}_j$) satisfies Theorem 3.2, (1) and of two $T^{n-1}$-orbits if $S_j$ ($\widetilde{S}_j$) satisfies Theorem 3.2, (2). Fix $j \in \{1, \ldots, \ell\}$ and let $w_1^0, \ldots, w_{n-2}^0 \in \mathbb{R}^{n-1}$ be a generators of $G_j$, the isotropy group of $S_j$. Therefore, for each $p \in S_j$, $\mathcal{O}_p(X_{w_i^0})$ is periodic of period one $\forall i \in \{1, \ldots, n-2\}$. Fix $q_j \in \Gamma_j$ and let $w_i : V_j \to \mathbb{R}^{n-1}$, $i = 1, \ldots, n-2$, where $V_j$ is a neighborhood of $p_j$ in $\mu_j$, be the functions given by Lemma 3.5. Recall that $w_i(q_j) = w_i^0$. Note that for each $p \in V_j$, $w_1(p), \ldots, w_{n-2}(p)$ is a set of generators of the isotropy group of $\mathcal{O}_p((\psi_\varphi)_0)$. Thus, we can assume that each $w_i$ is defined on $\nu_j$ and also in $W_j$, if we wish. Let $\widetilde{w}_i : \widetilde{W}_j \to \mathbb{R}^{n-1}$, $i = 1, \ldots, n-2$, be the functions given by Lemma 4.2, for $k = n-2$. For each $p \in V_j^0$, $\widetilde{w}_1(p), \ldots, \widetilde{w}_{n-2}(p)$ is a set of generators of the isotropy group of $\mathcal{O}_p((\psi_\phi)_0)$ and thus, we can assume that $\widetilde{w}_i$ is defined on $\widetilde{\nu}_j$ or even in $\widetilde{W}_j$. We are going to extend $F_j$ to a map $W_j \to \widetilde{W}_j$. If $p \in W_j \setminus \nu_j$, there exists $t_1, \ldots, t_{n-2} \in (0, 1)$ such that $f(p) = X_{w_1(p)}^{t_1} \circ \cdots \circ X_{w_{n-2}(p)}^{t_{n-2}}(p) \in \nu_j$. Define

$$F_j(p) = Y_{\widetilde{w}_1(F_j(f(p)))}^{t_1} \circ \cdots \circ Y_{\widetilde{w}_{n-2}(F_j(f(p)))}^{t_{n-2}}(F_j(f(p))).$$

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$F_j : W_j \to \widetilde{W}_j$ is a topological equivalence that preserves orientation between $\psi_\varphi$ and $\psi_\phi$ and by reducing the size of $\mathcal{V}$ and $\nu_j$ we can assume that $d(p, F_j(p)) < \delta$ for each $p \in W_j$, where $\delta$ is given. Note that there is no problem in assuming that $W_j \cap W_k = \emptyset$ if $j \neq k$.

We are going to extend the equivalences $F_j$ to a global topological equivalence $F$ between $\psi_\varphi$ and $\psi_\phi$. Each connected component $C$ of $N \setminus \cup_{j=1}^{\ell} W_j$ is homeomorphic either to $T^{n-2} \times D$ or to $T^{n-1} \times (0,1)$, where $D$ is an open 2-disk. In the first case $C \cap \mathrm{Sing}^c_{n-2}(\psi_\varphi)$ is a $T^{n-2}$-orbit that is transversally a center and in the second case is empty. To each $C$ corresponds a unique connected component $\widetilde{C}$ of $N \setminus \cup_{j=1}^{\ell} \widetilde{W}_j$ with the same properties than $C$. We shall define $F : C \to \widetilde{C}$ for each $C$. There are two cases:

(*i*) *C is homeomorphic to* $T^{n-2} \times D$. There exists a unique $j \in \{1, \dots, \ell\}$ such that $\mathrm{Front}(C) \subset \mathrm{Front}(W_j)$, $\mathrm{Front}(\widetilde{C}) \subset \mathrm{Front}(\widetilde{W}_j)$ and $F_j(\mathrm{Front}(C)) = \mathrm{Front}(\widetilde{C})$. For each point $p \in \mathrm{Front}(C)$ let $L_p = \mathrm{cl}(\mathcal{O}_p(\xi) \cap C)$, where $\mathcal{O}_p(\xi)$ is the integral curve of $\xi$ by $p$ and $\widetilde{L}_{F_j(p)} = \mathrm{cl}(\mathcal{O}_{F_j(p)}(\xi) \cap \widetilde{C})$. Fix a point $p_0 \in \mathrm{Front}(C)$, and extend $F_j : L_{p_0} \to \widetilde{L}_{F_j(p_0)}$ as an homeomorphism. If $q \in C$, then there is a unique $p \in \mathrm{Front}(C)$ such that $q = \mathcal{O}_q(\psi_\varphi) \cap L_p$. Let $q_1 = \mathcal{O}_q(\psi_\varphi) \cap L_{p_0}$ and define $F(q) = \mathcal{O}_{F_j(q_1)}(\psi_\phi) \cap \widetilde{L}_{F_j(p)}$.

(*ii*) *C is homeomorphic to* $T^{n-1} \times (0,1)$. Then, $\mathrm{Front}(C) = A_0 \cup A_1$, where $A_0$ and $A_1$ are $T^{n-1}$-orbits of $\psi_\varphi$. There exist $i, j \in \{1, \dots, \ell\}$, $i \neq j$, such that $A_0$ ($A_1$) is a connected component of $\mathrm{Front}(W_i)$ ($\mathrm{Front}(W_j)$). For the corresponding $\widetilde{C}$, $\mathrm{Front}(\widetilde{C}) = \widetilde{A}_0 \cup \widetilde{A}_1$, where $\widetilde{A}_0$ ($\widetilde{A}_1$) is a connected component of $\mathrm{Front}(\widetilde{W}_i)$ ($\mathrm{Front}(\widetilde{W}_j)$). Let $\mathcal{O}(\xi)$ be an integral curve of $\xi$ that cuts every $T^{n-1}$-orbit in $\mathrm{cl}(C)$ and every $T^{n-1}$-orbit in $\mathrm{cl}(\widetilde{C})$ and put $p_0 = \mathcal{O}(\xi) \cap A_0$ $\widetilde{p}_0 = \mathcal{O}(\xi) \cap \widetilde{A}_0$, $p_1 = \mathcal{O}(\xi) \cap A_1$, $\widetilde{p}_1 = \mathcal{O}(\xi) \cap \widetilde{A}_1$, $[p_0, p_1]$ ($[\widetilde{p}_0, \widetilde{p}_1]$) the arc within $\mathcal{O}(\xi)$ with extremes $p_0$ and $p_1$ ($\widetilde{p}_0$ and $\widetilde{p}_1$). Define $g : [0,1] \to [p_0, p_1]$ and $\widetilde{g} : [0,1] \to [\widetilde{p}_0, \widetilde{p}_1]$ as homeomorphisms. For each $p \in [p_0, p_1]$ ($p \in [\widetilde{p}_0, \widetilde{p}_1]$) let $G_p$ ($\widetilde{G}_p$) the isotropy group of $p$ under $\psi_\varphi$ ($\psi_\phi$). $G_p$ ($\widetilde{G}_p$) is generated by $w_1(p), \dots, w_{n-2}(p), w_{n-1}(p)$ ($\widetilde{w}_1(p), \dots, \widetilde{w}_{n-2}(p), \widetilde{w}_{n-1}(p)$). Consider the map $\psi : \mathbb{R}^{n-1} \times [p_0, p_1] \to \mathrm{cl}(C)$ ($\widetilde{\psi} : \mathbb{R}^{n-1} \times [\widetilde{p}_0, \widetilde{p}_1] \to \mathrm{cl}(\widetilde{C})$) given by:

$$
\begin{aligned}
\psi(t_1, \dots, t_{n-1}, p) &= \psi_\varphi(t_1 w_1(p) + \cdots + t_{n-1} w_{n-1}(p), p) \\
\widetilde{\psi}(t_1, \dots, t_{n-1}, p) &= \psi_\phi(t_1 \widetilde{w}_1(p) + \cdots + t_{n-1} \widetilde{w}_{n-1}(p), p).
\end{aligned}
$$

Since $\psi(\mathbb{R}^{n-1} \times \{p\})$ ($\widetilde{\psi}(\mathbb{R}^{n-1} \times \{p\})$) is the $T^{n-1} - \psi_\varphi$-orbit ($T^{n-1} - \psi_\varphi$-orbit) by $p$, then $\psi$ ($\widetilde{\psi}$) induces a map $\psi_1 : T^{n-1} \times [p_0, p_1] \to C$ ($\widetilde{\psi}_1 : T^{n-1} \times [\widetilde{p}_0, \widetilde{p}_1] \to \widetilde{C}$). Define $H : T^{n-1} \times [0,1] \to \mathrm{cl}(C)$ and $\widetilde{H} : T^{n-1} \times [0,1] \to \mathrm{cl}(\widetilde{C})$ by $H(x,s) = \psi_1(x, g(s))$ and $\widetilde{H}(x,s) = \widetilde{\psi}_1(x, \widetilde{g}(s))$. Again, by reducing the size of $\mathcal{V}$, of $\nu_i$, $\nu_j$ and by the proximity of the functions $w_i$ and $\widetilde{w}_i$ in the neighborhood of a $T^{n-1}$-orbit given by Lemma 4.2, for $k = n-1$, we can assume that for every $x \in T^{n-1}$

$$
\begin{aligned}
&d((x,0), \widetilde{H}^{-1} \circ F_i \circ H(x,0)) < \delta \\
&d((x,1), \widetilde{H}^{-1} \circ F_j \circ H(x,1)) < \delta \, ,
\end{aligned}
$$

where $\delta > 0$ is given. Therefore $h_0 = \widetilde{H}^{-1} \circ F_i \circ H(\ ,0) : T^{n-1} \to T^{n-1}$ and $h_1 = \widetilde{H}^{-1} \circ F_j \circ H(\ ,1) : T^{n-1} \to T^{n-1}$ are both isotopic to identity and thus, isotopic between themselves. Let $h_s$ be the isotopy between $h_0$ and $h_1$ and define $F : \mathrm{cl}(C) \to \mathrm{cl}(\widetilde{C})$ by $F(q) = \widetilde{H}^{-1} \circ h \circ H^{-1}(q)$, where $h : T^{n-1} \times [0,1] \to T^{n-1} \times [0,1]$, $h(x,s) = (h_s(x), s)$. $F$ is the desired topological equivalence. ▌

### 4.4. The case $C^r$

The following lemma is proved in [2, page 74] and will be useful in this section.

Lemma 4.3. *Let $\mathscr{F}$ be a codimension one transversally orientable $C^r$ foliation, $r \geq 1$. Assume that $F \in \mathscr{F}$ is a compact leaf with trivial holonomy, then there exists an open neighborhood $V$ of $F$ saturated by $\mathscr{F}$ and a $C^r$-diffeomorphism $h : F \times (-1,1) \to V$ such that the leaves of $\mathscr{F}$ inside $V$ are the submanifolds $h(F \times \{t\})$, $t \in (-1,1)$ and $F = h(F \times \{0\})$. In particular, $V \setminus F$ has two connected components.*

Let $\mathcal{O}_0$ be a $T^{n-1}$-orbit of $\varphi \in A^r(\mathbb{R}^n, N)$ and $\{w_1, \ldots, w_{n-1}, w_n\}$ a base of $\mathbb{R}^n$ such that $\{w_1, \ldots, w_{n-1}\}$ is a set of generators of its isotropy group $G_0$. Put $X_i = X_{w_i}$, $i = 1, \ldots, n$.

Proposition 4.2. *Let $\varphi \in A^r(\mathbb{R}^n, N)$, $\mathcal{O}_0$ a $T^{n-1}$-orbit and assume that there exists a neighborhood $V_0$ of $\mathcal{O}_0$ such that every orbit in $V_0$ is a $T^{n-1}$-orbit. Then there exists a diffeomorphism $f : V \to T^{n-1} \times (-1,1)$, where $V \subset V_0$ is a neighborhood of $\mathcal{O}_0$, such that*

$$f_* X_i(\theta, x) = a_{1i}(x) \frac{\partial}{\partial \theta_1} + \cdots + a_{(n-1)i}(x) \frac{\partial}{\partial \theta_{n-1}}, \ i = 1, \ldots, n, \tag{5}$$

*where $\theta = (\theta_1, ..., \theta_{n-1}) \in T^{n-1}$ and the functions $a_{ij} : T^{n-1} \times (-1,1) \to \mathbb{R}$ are of class $C^r$, $j = 1, \ldots, n-1$.*

*Proof.* By Lemma 4.3 we can assume that $\mathcal{O}_0 = T^{n-1} \times \{0\}$, $V = T^{n-1} \times (-1,1)$ and that $\mathcal{O}_x = T^{n-1} \times \{x\}$ is the orbit of $\varphi$ by the point $(\theta, x) \in V$. Let $H$ be the vector subspace of $\mathbb{R}^n$ generated by $G_0$. By Lemma 3.5, there exist $C^r$ functions $w_i : (-1,1) \to H$, $i = 1, \ldots, n-1$, such that $G_x$, the isotropy group of $\mathcal{O}_x$, is generated by $\{w_1(x), \ldots, w_{n-1}(x)\}$. For each point $x \in (-1,1)$ there exists a diffeomorphism $f_x : \mathcal{O}_x \to \mathcal{O}_x$ such that $(f_x)_* X_{w_i(x)} = \frac{\partial}{\partial \theta_i}$, $i = 1, \ldots, n-1$. Since $X_j$ commutes with $X_{w_i(x)}$, for every $j = 1, \ldots, n$ and $i = 1, \ldots, n-1$, it follows that

$$(f_x)_* X_j(\theta, x) = \sum_{i=1}^{n-1} a_{ij}(x) \frac{\partial}{\partial \theta_i}, \quad j = 1, \ldots, n.$$

Define $f : T^{n-1}\times(-1,1) \to T^{n-1}\times(-1,1)$ by $f(\theta,x) = f_x(\theta)$. Since $w_i$ is a $C^r$ function, we conclude that $f$ is a $C^r$ diffeomorphism such that

$$f_*X_j(\theta,x) = \sum_{i=1}^{n-1} a_{ij}(x)\frac{\partial}{\partial\theta_i}, \quad j = 1,\ldots,n.$$

∎

PROPOSITION 4.3. *Let $\varphi \in \mathrm{Act}^r(\mathbb{R}^n, N)$. Assume that there exists a $\varphi$-invariant neighborhood $V_0$ of $\mathcal{O}_0$ such that every orbit inside $V_0$ is a $T^{n-1}$-orbit. Then, $\varphi$ can not be locally structurally stable at $\mathcal{O}_0$.*

*Proof.* By Proposition 4.2 we can assume that $\mathcal{O}_0 = T^{n-1}\times\{0\}$, $V_0 = T^{n-1}\times(-1,1)$ and that the infinitesimal generators have the form

$$X_i(\theta,x) = a_{1i}(x)\frac{\partial}{\partial\theta_1} + \cdots + a_{(n-1)i}(x)\frac{\partial}{\partial\theta_{n-1}}, \ i = 1,\ldots,n.$$

Given $\varepsilon > 0$ there exist $\delta > 0$ and a function $\widetilde{a}_{nn} : (-1,1) \to \mathbb{R}$ such that $\widetilde{a}'_{nn}(0) \neq 0$, $\widetilde{a}_{nn}(x) = 0$ if $x \geq |2\delta|$ and $\|\widetilde{a}_{nn}\|_1 < \varepsilon$. Let $A_{nn}(x) = (a_{ij}(x))$, $i,j = 1,\ldots,n-1$, and define for each $k = 1,\ldots,n-1$,

$$\widetilde{a}_{kn}(\theta,x) = \widetilde{a}_{nn}(x)\big\langle [A_{nn}(x)]^{-1}\cdot \alpha'_k(x),\, \theta\big\rangle + a_{kn}(x),$$

where $\alpha_k(x)$ is the column vector $\big(a_{k1}(x),\ldots,a_{k(n-1)}(x)\big)^t$. The vector fields $X_1,\ldots,$ $X_{n-1}$ together with

$$\widetilde{X}_n = \begin{cases} \sum_{k=1}^{n-1}\widetilde{a}_{kn}(\theta,x)\frac{\partial}{\partial\theta_k} + \widetilde{a}_{nn}(x)\frac{\partial}{\partial x}, & \text{in } T^{n-1}\times(-1,1) \\ X_n, & \text{otherwise} \end{cases}$$

define an action $\widetilde{\varphi} \in A^r(\mathbb{R}^n, N)$ that can be taken arbitrarily $C^1$-close of $\varphi$ by choosing appropriatly $\varepsilon$. Note that $\widetilde{\varphi}$ has a $T^{n-1}\times\mathbb{R}$-orbit inside $V_0$ and thus can not be topologically equivalent to $\varphi$. ∎

In Proposition 4.3, the fact that $\varphi \in A^r(\mathbb{R}^n, N)$ is crucial. In fact Saldanha in [9] showed that there are structurally stable actions of $\mathbb{R}^2$ on $T^3$ with all orbits being $T^2$-orbits.

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