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Codimension 2 singularities and their bifurcations in order 2 implicit differential equations

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We study in this paper singularities of codimension 2 in implicit differential equations $F(x,y,p)=0$ ($p=\frac{dy}{dx}$ or $\frac{dx}{dy}$), where F is a germ of a smooth function with $F_p=0$ and $F_{pp}\neq 0$ at the singular point. We also deal with the bifurcations of these singularities in generic 2-parameter families of equations. October, 2003 ICMC-USP

1. INTRODUCTION

Let $F(x,y,[dx:dy])=0$ be an implicit differential equation, where F is a germ of a smooth function in $(x,y,[dx:dy])\in\mathbb{R}^2\times\mathbb{R}P^1$. At a point where the vertical (resp. horizontal) line is not a solution, we can take an affine chart $p=\frac{dy}{dx}$ (resp. $q=\frac{dx}{dy}$), assume without loss of generality that the point of interest is the origin and suppose that $p=0$ by a rotation in the (x,y)-plane. The IDE can then be considered as a germ of a smooth function $\mathbb{R}^3,0\to\mathbb{R},0$. If the $F_p(0)\neq 0$ (when not indicated otherwise, subscript denote partial differentiation), the above equation can be written locally in the form $p=g(x,y)$ and studied using the methods from the theory of ordinary differential equations. When $F_p(0)=0$ the equation may define locally more than one direction in the plane. The cases that have been most intensively studied, with applications to differential geometry (see for example Section 4) and control theory (see [16] and [23]), are the IDE's that define at most two directions in the plane. This can happen locally when: (i) $F(0)=F_p(0)=0$ but $F_{pp}(0)\neq 0$, or (ii) if the equation is simply given in a quadratic form $a(x,y)dy^2+2b(x,y)dxdy+c(x,y)dx^2=0$. We label IDE's (i) *IDE's of order 2* and the aim of this paper is to study their singularities of codimension 2 and their generic bifurcations. We deal in [24] with codimension 2 singularities of case (ii) when all the coefficients vanish at a given point.

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We recall that the local configurations of all codimension ≤ 1 phenomena of IDE's of order 2 are known (see [13], [14], [15], [23] for the stable phenomena and [10], [17], [23] for the codimension 1 cases).

We obtain here topological models of the configurations of the solutions curves of IDE's of order 2 in a neighbourhood of a singularity of codimension 2. We also obtain models for generic families of such IDE's. We follow a similar strategy adopted in [20] for single vector fields (see section 2) and also use tools from singularity theory. We prove a general topological determinacy results for IDE's of order 2 (section 2 and Appendix for proof). In Section 3 we study all the codimension 2 cases and give some applications to differential geometry in Section 4. For background in singularity theory see [28].

2. TRANSVERSALITY, VERSAL UNFOLDINGS AND BIFURCATION SETS

We consider here topological equivalence between IDE's and say that two IDE's F and G are equivalent if there is a local homeomorphism taking the integral curves of F to those of λG where λ is a germ of a non-zero function. (We are considering direction fields so we can multiply by non-zero functions.)

When considering families of IDE's, we shall adopt, as in [20], the notion of fibre topological equivalence for families of multi-valued fields. Two families F_t and G_s are *fibre topologically equivalent* if there exist a local homeomorphism $s = \psi(t)$ between the parameter space and a family of local homeomorphisms of $\mathbb{R}^2$ depending on the parameter t, say h_t, such that for all t, h_t is a topological equivalence between F_t and $G_{\psi(t)}$. The map h_t is not required to be continuous in t. This approach may in general exclude bifurcation if the neighbourhood of the phase portrait shrinks to a point as the parameter tends to zero. We can avoid this situation by fixing the diameter of the neighbourhood for all values of the parameter near zero.

A t-parameter family F_t is said to be *topologically versal* if for any other s-parameter deformation G_s, there is a map ϕ such that G_s is fibre topologically equivalent to $F_{\phi(s)}$.

An approach for investigating IDE's that define at most two directions (IDE's of order 2) is given in [26], [14], and [27]. It consists of lifting the bi-valued direction field defined in the plane to a single field ξ on the surface $M = F^{-1}(0)$ in $\mathbb{R}^3$. This field is given by

$$\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p},$$

(see for example [1], [2]), and is determined by the restriction of the standard contact form $dy - pdx$ in $\mathbb{R}^3$ to the surface.

If 0 is a regular value of F then M is smooth and the projection $\pi : M \to \mathbb{R}^2$, given by $\pi(x, y, p) = (x, y)$, is a fold at points where $F_p = 0$ and $F_{pp} \neq 0$ (that is, π can be written in an appropriate system of coordinates in the source and target in the form $(u, v) \mapsto (u, v^2)$). The critical set of this projection is called the *criminant* and its image is the *discriminant* of the equation. The configuration of the solution curves of F at a point on the discriminant is determined by the pair (ξ, σ), where σ is the involution on M that interchanges points with the same image under the projection to $\mathbb{R}^2$.

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It is shown in [13], [14] that if ξ does not vanish at the point in question then the IDE can locally be reduced by smooth changes of coordinates in the plane to $dy^2 - xdx^2 = 0$. The integral curves in this case is a family of cusps transverse to the discriminant, which is a smooth curve.

If ξ has an elementary singularity (saddle/node/focus) with separatrices transverse to the criminant and not killed under projection, then it is called a *well-folded singularity* ([15]). Note that a singularity of ξ occurs at points where $F = F_p = F_x + pF_y = 0$. These are also the points where the unique direction determined by the IDE on the discriminant is tangent to the discriminant. At well-folded singularities, the equation is locally smoothly equivalent to $dy^2 + (-y + \lambda x^2)dx^2 = 0$, with $\lambda \neq 0, \frac{1}{16}$ ([15]). There are three topological models, a well-folded saddle if $\lambda < 0$, a well-folded node if $0 < \lambda < \frac{1}{16}$ and a well-folded focus if $\frac{1}{16} < \lambda$; Figure 1. (See also [16] for applications to control theory.)

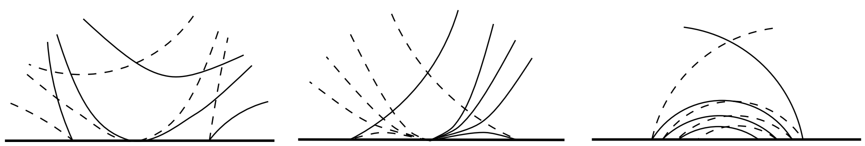

FIG. 1. Well-folded saddle (left), node (centre) and focus (right).

The family of cusps and the well-folded singularities are the only locally structurally stable configurations of singular IDE's with $F_{pp}(0) \neq 0$. The bifurcations in generic 1-parameter familes of IDE's of order 2 have also been classified. One of these is the *well-folded saddle node bifurcations* ($\lambda = 0$ above) and occurs when the discriminant is smooth and the lifted field ξ has a saddle-node singularity. Then the equation is locally smoothly equivalent to $dy^2 + (-y + x^3 + \mu x^4)dx^2 = 0$ ([17]).

When $\lambda = \frac{1}{16}$ the formula above yields the transition from well-folded node to well-folded focus ([18]).

It is not difficult to show that a generic 1-parameter family of IDE's with a well-folded saddle node (resp. well-folded node-focus change) singularity at $t = 0$ is (fibre) topologically equivalent to $dy^2 + (-y + x^3 + tx)dx^2 = 0$ (resp. $dy^2 + (-y + (\frac{1}{16} + t)x^2)dx^2 = 0$).

Bifurcations can also occur when the discriminant has a Morse singularity. These equations are labelled *Morse Type 1* in [4]. Generic Morse Type 1 singularities are locally topologically equivalent to $dy^2 + (\pm x^2 \pm y^2) = 0$ ([4]; see also [23]). A generic 1-parameter family of IDE's with a Morse Type 1 singularity at $t = 0$ is (fibre) topologically equivalent to $dy^2 + (\pm x^2 \pm y^2 + t) = 0$ ([4]). As t passes through 0, two well-folded saddles or foci singularities appear on one side of the transition and none on the other. The saddle or focus type are distinguished by the sign of x^2 in the normal form (+ for focus and − for saddle).

We complete the study of codimension 1 phenomena with the following observation.

LEMMA 2.1. *There are no Hopf bifurcations on the lifted field ξ of an IDE of order 2 at a regular point on the criminant (i.e a point on the surface M corresponding to a smooth point on the discriminant).*

Proof At a singularity of the IDE on a smooth discriminant we can write the equation, after a smooth change of coordinates, the IDE of the form $p^2 + (-y + g(x,y)) = 0$ with $j^1 g \equiv 0$. Therefore the surface of the equation is given locally by $y = h(x,p)$ for some germ of a smooth function h. Then the linear part of the projection of the lifted field ξ to the (x,p)-plane is given by

$$\begin{pmatrix} 0 & 2 \\ -g_{xx}(0,0) & 1 \end{pmatrix}$$

and the above matrix does not have pure imaginary eigenvalues. □

We associate to a family of IDE's $F(x,y,t,p) = 0$ a map

$$\begin{array}{rrcl} \Phi: & \mathbb{R}^3 \times \mathbb{R}^r, 0 & \to & J^k(3) \\ & ((x,y,p),t) & \mapsto & j^k F_t|_{(x,y,p)} \end{array}$$

where $J^k(3)$ denotes the k-jet space of functions of three variables, and $j^k F_t|_{(x,y,p)}$ is the k-jet of F_t at (x,y,p) (which is simply the Taylor expansion of order k of F_t at the point (x,y,p)). A singularity in the family is of codimension m if the conditions that define it yield a semi-algebraic set V of codimension $m+3$ in $J^k(3)$. The family F is said to be *generic* if the map Φ is transverse to V in $J^k(3)$. Observe that a necessary condition for genericity is that $r \geq m$.

Our interest here is when $m = 2$. The *bifurcation set* of a generic 2-parameter family consists of the closure of set of parameters where the associated germ F_t has a singularity of codimension 1. As we have observed before, a singularity of the IDE F (for t fixed) at a smooth point of the discriminant is given by $F = F_p = F_x + pF_y = 0$. Suppose, without loss of generality, that $F_y \neq 0$ at the point in consideration, so that the surface of the equation is given by $y = h(x,p)$ for some germ of a smooth function h. Then the linear part of the projection of the lifted field ξ to the (x,p)-plane is given by

$$(A_{ij}) = \begin{pmatrix} F_{xp} - \frac{F_x}{F_y} F_{yp} & F_{pp} - \frac{F_p}{F_y} F_{yp} \\ -F_{xx} + \frac{F_x}{F_y} F_{xy} - p(F_{xy} - \frac{F_x}{F_y} F_{yy}) & -F_y - F_{xp} + \frac{F_p}{F_y} F_{xy} - p(F_{yp} - \frac{F_p}{F_y} F_{yy}) \end{pmatrix}$$

and the components of the bifurcation set are as follows (recall that Hopf-bifurcations are excluded by Lemma 2.1):

(D) *Morse singularity on the discriminant*: this is the set of parameters t such that

$$F = F_p = F_x = F_y = 0.$$

(SN) *Saddle-node bifurcations*: this is the set of parameters t such that

$$F = F_p = F_x + pF_y = \det(A) = 0.$$

(NF) *Node-focus change*: this is the set of parameters t such that

$$F = F_p = F_x + pF_y = (A_{11} + A_{22})^2 - 4\det(A) = 0.$$

We obtain a stratification S of a neighbourhood U of the origin in the parameter space $\mathbb{R}^2$ given by $\{0\}$, D, SN, NF, $U - \{0\} \cup D \cup SN \cup NF$. (Some of the codimension 1 strata may of course be empty.)

In practice, the calculation can be simplified considerably using the fact that any IDE of order 2 can be written of the form

$$p^2 + f(x, y) = 0 \qquad (1)$$

(see [9]). It is easy to show that their deformations can also be written as $p^2 + f(x, y, t) = 0$.

Our aim is to show that any generic 2-parameter family of a codimension 2 singularity is (fibre) topologically versal. For this we take the following steps (compare with [20]):

– Obtain a model for the IDE at $t = 0$.
– Reduce when possible the N-jet of the family to a normal form.
– Obtain a condition for the family to be generic.
– Show that the bifurcation sets of generic families are homeomorphic.
– Obtain the configuration of the discriminant in each strata of $S - \{0\}$.
– Show that the number of singularities, their type and their position on the discriminant are constant in each stratum of $S - \{0\}$.
– Show that the configurations of the integral curves have a constant topological type in each stratum of $S - \{0\}$.

We need to determine the topological configuration of IDE's of order 2 with a codimension 2 singularity at the origin. We prove the general result below (see Appendix for the proof), where a germ of an IDE F is said to be finitely topologically determined if it is topologically equivalent to the IDE j^kF for some integer k.

THEOREM 2.1. *Let F be a smooth germ of an IDE of order 2. Suppose that its discriminant has a $\mathcal{K}$-finitely determined singularity at the origin and a finite order of contact with the unique direction determined by the equation. Then F is finitely topologically determined.*

3. SINGULARITIES OF CODIMENSION 2

The singularities of codimension 2 in IDE's of order 2 are the following:

1. *Bifurcations on a smooth discriminant*: the discriminant is smooth and the lifted field has a codimension 2 singularity.

2. *The non-transverse Morse case*: the discriminant has a Morse singularity and the unique direction determined by the IDE at the origin has an ordinary tangency with the discriminant.

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3. *The cusp case*: the discriminant has a cusp singularity with a limiting tangent transverse to the unique direction determined by the IDE.

The above conditions define semi-algebraic sets of codimension 5 in $J^k(3)$ (k appropriately chosen), and are denoted V_1, V_2, V_3 respectively.

3.1. Bifurcations on a smooth discriminant

It is shown in [17] that when the discriminant is smooth and the singularity is degenerate, the equation is smoothly equivalent to

$$(p + \epsilon x^r + a x^{2r-1})^2 = y$$

with $\epsilon = \pm 1$, a a modulus, and $r > 2$. It is not hard to show that we can rewrite the above equation in the following form

$$p^2 + (-y + \epsilon x^r + a x^{2r-1}) = 0.$$

The case of interest here is when $r = 4$. If we take the IDE of order 2 in an equivalent form $p^2 + f(x, y) = 0$, then its 4-jet is generically ($\frac{\partial^4 f}{\partial x^4}(0) \neq 0$) in V_1 if and only if

$$f(0) = f_x(0) = f_{xx}(0) = f_{xxx}(0) = 0.$$

PROPOSITION 3.1. *(i). A 2-parameter family of IDE's $p^2 + f(x, y, u, v)$ with $p^2 + j^4 f_{(0,0)}(x, y)$ in V_1 is generic if and only if*

$$\begin{pmatrix} f_{ux} & f_{uxx} \\ f_{vx} & f_{vxx} \end{pmatrix}$$

has maximal rank at the origin.
(ii). Any N-jet of a generic family, with $N > 7$, is smoothly equivalent to

$$p^2 + (-y + \epsilon x^4 + a x^7 + u x^2 + v x) = 0.$$

Proof The tangent space of the image of the map $\Phi : \mathbb{R}^3 \times \mathbb{R}^2, 0 \to J^4(3)$ at the image of the origin is generated by $j^4 f_x = 4\epsilon x^3$, $j^4 f_y = -1$, $j^4 F_p = 2p$, $j^4 f_u$ and $j^4 f_v$. Statement (i) then follows by a straightforward calculation. Statement (ii) follows by the usual formal reduction calculations (see for example the proof of Proposition 3.2). □

THEOREM 3.1. *(i). An IDE of order 2 with a smooth discriminant and with the associated lifted field ξ having one vanishing eigenvalue and an additional degeneracy is topologically equivalent to*

$$p^2 + (-y + \epsilon x^4) = 0, \quad \epsilon = \pm 1.$$

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(ii). Any generic 2-parameter family of the IDE (i) is fibre topologically equivalent to

$$p^2 + (-y + \epsilon x^4 + ux^2 + vx) = 0.$$

See Figure 3.

Proof We can take the generic family, using Proposition 3.1, in the form $p^2+f(x,y,u,v) = 0$ with

$$f(x,y,u,v) = -y + \epsilon x^4 + ax^7 + ux^2 + vx + g(x,y,u,v),$$

where g is a germ of a smooth function with a zero N-jet (N-large). As the discriminant is smooth, the stratum (D) of the bifurcation set is empty. A node-focus change occurs when $f = f_x = f_{xx} - \frac{1}{8}(f_y)^2 = 0$. However $f_y = 1$ at the origin, so the stratum (NF) is also empty.

A saddle-node bifurcation occurs when $f = f_x = f_{xx}$. These equations yield a curve in the (u,v)-plane given by $v^2 + \epsilon \frac{2}{27} u^3 + O(||(u,v)||^4) = 0$. This curve is diffeomorphic to a cusp $v^2 + u^3 = 0$.

We observe that the family of the lifted fields ξ associated to $p^2 + f(x,y,u,v) = 0$ is a topologically versal deformation of the singularity of ξ_0, and the bifurcation set of ξ coincides with that of the IDE. So the number and type of singularities that occur on the discriminant can easily be determined. These are as in Figure 3. For example, in the case $\epsilon = +1$ (resp. $\epsilon = -1$) and when there is only one singularity on the discriminant, it is of type well folded saddle (resp. node).

The configuration of the integral curves of $p^2 + f_{u,v}(x,y) = 0$, for (u,v) fixed, in a given component of the bifurcation set are obtained by projecting those of the lifted field $\xi_{u,v}$ to the (x,y)-plane. More precisely, the solutions of the IDE are the union of the solutions of the vector field α associated to $p = \sqrt{-f_{u,v}(x,y)}$ with those of β associated to $p = -\sqrt{-f_{u,v}(x,y)}$. The solution of α are the projections of the integral curves of $\xi_{u,v}$ in one region of M delimited by the criminant and those of β the projections of the integral curves in the remaining region.

To show that the configurations of the integral curves of the IDE are equivalent to those of the model and are constant in each stratum of S we exhibit homeomorphisms that do the job. For example, when $\epsilon = -1$ and (u,v) are in the open component where there are 3 singularities (two well-folded saddles and one well-folded node) we choose a neighbourhood U of the discriminant $f_{(u,v)} = 0$ as shown in Figure 2, where all the curves, except the thick ones, are solutions of the IDE. (We give a positive orientation to the boundary ∂U.) The segments BC and IJ are transverse to the integral curves of the IDE with the vertices belonging to the separatrices of the well-folded saddles.

We claim now that if we choose a neighbourhood U' of the discriminant of the model $-y - x^4 + ux + vx^2 = 0$ in the same way as above, then any increasing homeomorphism h from the curve $BCDEFGHIJ$ to the curve $B'C'D'E'F'G'H'I'J'$, sending vertices to vertices, induces a homeomorphism H from U to U'. (We denote by α' and β' the vector fields associated to the equation of the model.)

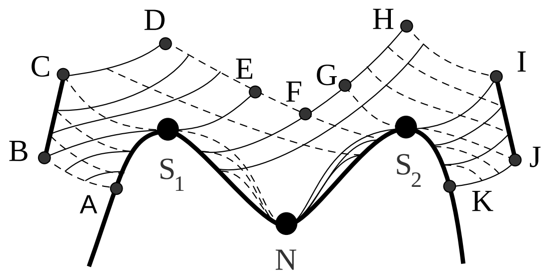

FIG. 2. A neighbourhood of the discriminant.

Suppose that the integral curves of α (resp. β) are the continuous (resp. discontinuous) ones in Figure 2. A point q in the triangular sector ABS_1 is the intersection of an integral curve of β starting from a point q_1 on BC and an integral curve of α from a point $\bar{q}$ on the discriminant. The integral curve of β from $\bar{q}$ intersect the curve BC at a point q_2 that satisfies $q_2 < q_1$. We consider the points $h(q_1)$ and $h(q_2)$ on $B'C'$. Because we have chosen our neighbourhoods properly and h is increasing, the points $h(q_1)$ and $h(q_2)$ determine a unique point $H(q)$ in the sector $A'B'S'_1$ which is the intersection of the integral curves of α' and β' by reversing the above process (here we may interchange α' and β' if necessary). We follow the same technique for the remaining sectors. It is clear that the resulting map H is a homeomorphism that sends integral curves to integral curves.

□

3.2. Non-transverse Morse singularity

An IDE (1) has a non transverse Morse singularity (i.e its 4-jet is in V_2) if and only if

$$f(0) = f_x(0) = f_y(0) = f_{xx}(0) = 0,\ f_{xxx} \neq (0).$$

In this case we can write the 2-jet of the IDE in the form $p^2 + xy$.

Proposition 3.2. *(i). The N-jet of an IDE of order 2 with a non-transverse Morse singularity is smoothly equivalent to*

$$p^2 + (xy + x^3 + g(x)) = 0$$

where g is a polynomial function with a zero 3-jet.
(ii). A 2-parameter family of IDE's $p^2 + f(x, y, u, v)$ with $p^2 + j^3 f_{(0,0)}(x, y)$ in V_2 is generic if and only if

$$\begin{pmatrix} 0 & 1 & 1 \\ f_u & f_{uy} & f_{uxx} \\ f_v & f_{vy} & f_{vxx} \end{pmatrix}$$

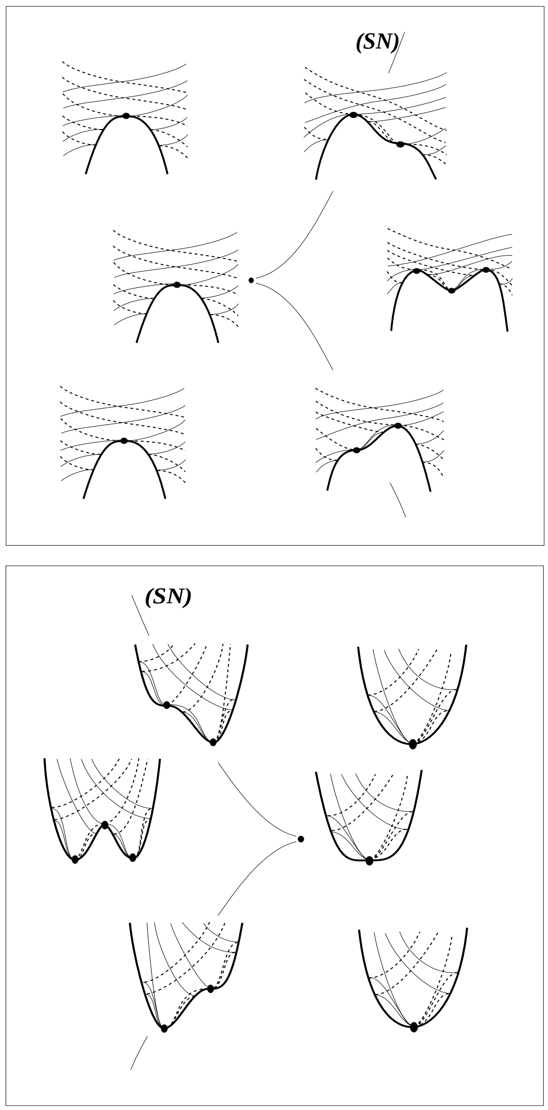

FIG. 3. Bifurcations of $p^2 + (-y + \epsilon x^4 + ux + vx^2) = 0$, $\epsilon = -1$ left and $\epsilon = +1$ right.

has maximal rank at the origin. When this is the case the N-jet of the family is smoothly equivalent to

$$p^2 + (xy + x^3 + ux^2 + v + g(x, u, v))$$

where g is a polynomial with a zero 3-jet.

Proof (i) This follows the usual formal reduction technique. Suppose we reduced the $(l-1)$-jet to the normal form and write the l-jet as $p^2 + b_l(x, y)p + (xy + c_l(x, y))$, where a_l, b_l are homogeneous polynomials of degree l. Then a change of coordinates $x = X + p(X, Y), y = Y + q(X, Y)$ yields an IDE $A_l p^2 + B_l p + C_l = 0$ where we still write x for X and y for Y and

$$\begin{aligned} A_l &= (1 + q_y)^2 + \overline{b}_l p_y(1 + q_y) + p_y^2((x + p)(y + q) + \overline{c}_l) \\ B_l &= 2q_x(1 + q_y) + \overline{b}_l((1 + p_x)(1 + q_y) + p_y q_x) \\ &\qquad + 2p_y(1 + p_x)((x + p)(y + q) + \overline{c}_l) \\ C_l &= q_x^2 + \overline{b}_l q_x(1 + p_x) + (1 + p_x)^2((x + p)(y + q) + \overline{c}_l) \end{aligned}$$

where $\overline{b}_l(x, y) = b_l(x + p, y + q)$ and $\overline{c}_l(x, y) = c_l(x + p, y + q)$.

If q is a homogeneous polynomial of degree $l+1$ and p of degree $l-1$, then $j^l B_l$ is given by $2q_x + b_l$. We can therefore reduce $j^l B_l$ to zero by choosing an appropriate polynomial q. We now divide the l-jet of the equation by A_l, so that the coefficient of p^2 is 1. Then the homogeneous part of degree l in C_l is given by $2xyp_x + yp + a_l$. It is clear that we can get rid of all the monomials divisible by y in a_l by choosing an appropriate p, so that the l-jet of the IDE becomes $p^2 + xy + h(x)$. Suppose now that $h(x) = \lambda x^k + h.o.t$ with $\lambda \neq 0$. Then a change of scale reduces the l-jet of the equation to $p^2 + xy \pm x^k + g(x)$, where g is a polynomial with a zero k-jet.

(ii) Follows by a straightforward calculation. □

THEOREM 3.2. *(i). An IDE of order 2 with a non-transverse Morse singularity is topologically equivalent to*

$$p^2 + (xy + x^3) = 0.$$

(ii) Any generic 2-parameter family of the IDE (i) is (fibre) topologically equivalent to

$$p^2 + (xy + x^3 + ux^2 + v) = 0.$$

See Figure 4.

Proof The first statement follows from Theorem 2.1 and its proof. For the second statement we proceed as in Theorem 3.1. We take the family of the form $p^2 + f(x, y, u, v) = 0$, with $f(x, y, u, v) = xy + x^3 + ux^2 + v + g(x, y, u, v)$, $j^3 g \equiv 0$ and $j^5 g(0, y, 0, 0) \equiv 0$. The components of the bifurcation set are then as follows.

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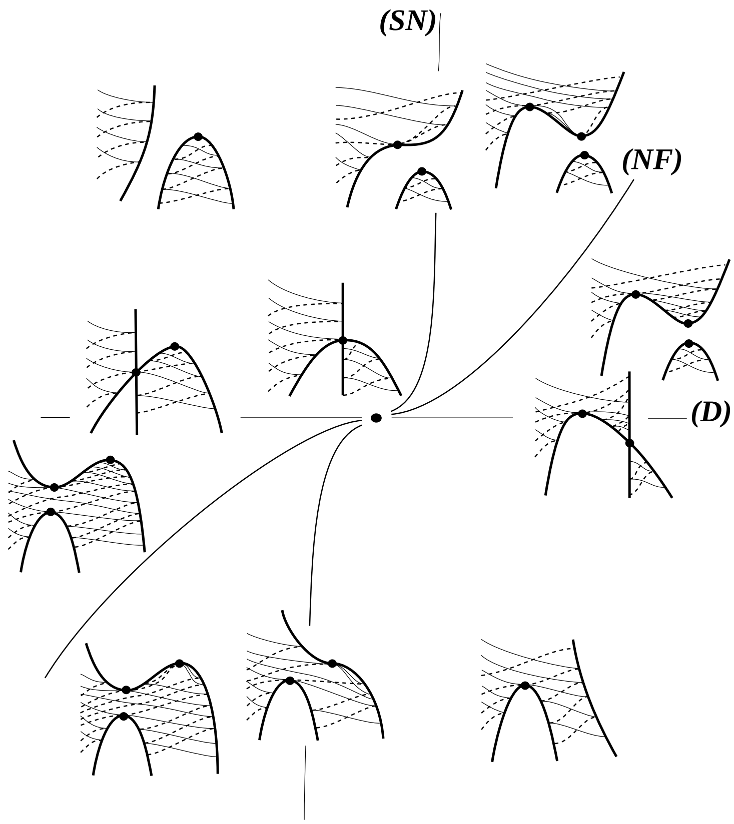

FIG. 4. Bifurcations of $p^2 + (xy + x^3 + ux^2 + v) = 0$.

(D) Singular discriminant: this is the set of parameters (u, v) such that $f = f_x = f_y = 0$ at some point in a neighbourhood of the origin. So we have

$$\begin{aligned} v + ux^2 + xy + x^3 + g(x, y, u, v) &= 0, \\ 2ux + y + 3x^2 + g_x(x, y, u, v) &= 0, \\ x + g_y(x, y, u, v) &= 0. \end{aligned}$$

By the implicit function theorem, we can write locally $(x, y, v) = h_1(u)$, so the stratum (D) is a smooth curve given by $v = k_1(u)$ for some germ of a smooth function k_1 with $j^5 k_1 \equiv 0$.

(SN) Saddle-node bifurcations: this is the set of parameters (u, v) such that $f = f_x = f_{xx} = 0$ at some point in a neighbourhood of the origin. Therefore

$$\begin{aligned} v + ux^2 + xy + x^3 + g(x, y, u, v) &= 0, \\ 2ux + y + 3x^2 + g_x(x, y, u, v) &= 0, \\ 2u + 6x + g_{xx}(x, y, u, v) &= 0. \end{aligned}$$

Again by the implicit function theorem, we can write locally $(x, y, v) = h_2(u)$, so the stratum (SN) is a smooth curve given by $v = k_2(u)$ for some germ of a smooth function k_2. A calculation shows that $j^5 k_2 = \frac{1}{27}u^3 + au^4 + bu^5$, where a and b are constants that depend on the coefficients of the monomials of degree 4 and 5 in g.

(NF) Node-focus change: this is the set of parameters (u, v) such that $f = f_x = f_{xx} - \frac{1}{8}(f_y)^2 = 0$ at some point in a neighbourhood of the origin. Therefore

$$\begin{aligned} v + ux^2 + xy + x^3 + g(x, y, u, v) &= 0, \\ 2ux + y + 3x^2 + g_x(x, y, u, v) &= 0, \\ 2u + 6x + g_{xx}(x, y, u, v) - \tfrac{1}{8}(x + g_y(x, y, u, v))^2 &= 0. \end{aligned}$$

so the stratum (NF) is also a smooth curve given by $v = k_3(u)$ for some germ of a smooth function k_3. A calculation shows that $j^5 k_3(u) = \frac{1}{27}u^3 + au^4 + (b - \frac{1}{186624})u^5$, with a and b are as in the SN stratum calculation.

The above calculations show that the bifurcation set of generic families of a non-transverse Morse singularity are all homeomorphic (and consists of 3 tangential smooth curves, Figure 4).

We need now to determine the number and type of singularities that appear in the members of a generic family. A singularity occurs when $f = f_x = 0$, which means that the discriminant is either singular or has a horizontal tangent. We can classify singularities of germs of functions up to a subgroup $\mathcal{K}^*$ of $\mathcal{K}$, where the action in the source is by germs of diffeomorphisms that preserve horizontal lines ([25]). The germ $xy + x^3$ is 3-$\mathcal{K}^*$-determined and a versal deformation is given by $xy + x^3 + ux^2 + v$. Therefore a generic family of a non-transeverse Morse type IDE induces a $\mathcal{K}^*$-versal deformation of the discriminant. It follows that the number of singularities and their position (both $\mathcal{K}^*$-invariant) on $f_{(u,v)} = 0$ can be deduced from the $\mathcal{K}^*$-versal deformation $xy + x^3 + ux^2 + v$. The $\mathcal{K}^*$-bifurcation set of f has two components. One consists of the set of parameters where the zero fibre of the associated function is singular, and is precisely the stratum (D). The other is the set

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of parameters where zero fibres of the corresponding functions have higher contact with the horizontal lines, and this is the stratum (SN). The number of singularities can now be computed in each stratum of S (see Figure 4). The type of theses singularities are determined by a direct computation and are as in Figure 4.

To show that the configurations in each stratum are topologically equivalent and equivalent to those of the model, we proceed as in the proof of Theorem 3.1. For example the homeomorphism from a neighbourhood of the discriminant in Figure 5 to that of the model is completely determined by an increasing homeomorphism h from the curve $NABCD$ to the curve $N'A'B'C'D'$, preserving vertices. (Observe that, in Figure 5, one of the the separatrices of the well folded-saddle S_1 intersects one of the separatrices of the well folded-saddle S_2.)

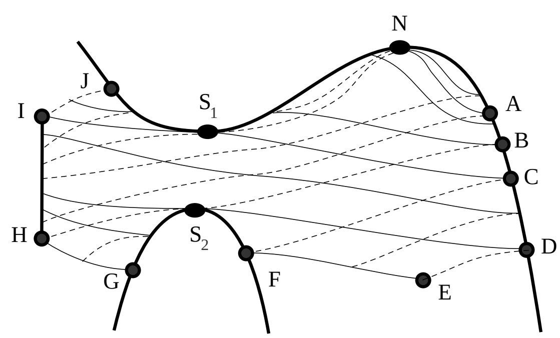

FIG. 5. A neighbourhood of the discriminant.

□

3.3. The cusp singularity

An IDE (1) has generically a cusp singularity if and only if

$$f(0) = f_x(0) = f_y(0) = ((f_{xy})^2 - f_{xx}f_{yy})(0) = 0.$$

We have generically $f_{xx}(0) \neq 0$ (that is the unique direction determined by the IDE is transverse to the limiting tangent direction to the discriminant). In this case we can set $j^2 f = \epsilon x^2$, $\epsilon = \pm 1$. Then the discriminant has a genuine cusp singularity when $f_{yyy}(0) \neq 0$.

Proposition 3.3. *(i). The N-jet of an IDE of order 2 with a discriminant that has a cusp singularity with a limiting tangent transverse to the unique direction determined by the IDE is smoothly equivalent to*

$$p^2 + (\epsilon x^2 + y^3 + g(y)) = 0, \ \epsilon = \pm 1,$$

where g is a polynomial with a zero 3-jet.

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(ii). A 2-parameter family of IDE's $p^2 + f(x, y, u, v) = 0$ *with* $p^2 + j^3 f_{(0,0)}(x, y)$ *in* V_3 *is generic if and only if the matrix*

$$\begin{pmatrix} f_u & f_{uy} \\ f_v & f_{vy} \end{pmatrix}$$

has maximal rank at the origin. When this is the case N*-jet of the family is smoothly equivalent to*

$$p^2 + (\epsilon x^2 + y^3 + uy + v + g(y, u, v))$$

where g *is a polynomial with a zero 3-jet.*

Proof The proof follows in the same way that of Proposition 3.2. □

THEOREM 3.3. *(i). A germ of an IDE of order 2 with a discriminant having a cusp singularity with limiting tangent transverse to the unique direction defined by the IDE is topologically equivalent to*

$$p^2 + (\epsilon x^2 + y^3) = 0, \ \epsilon = \pm 1.$$

(ii) Any generic 2-parameter family of the IDE (i) is (fibre) topologically equivalent to

$$p^2 + (\epsilon x^2 + y^3 + uy + v) = 0.$$

See Figure 6.

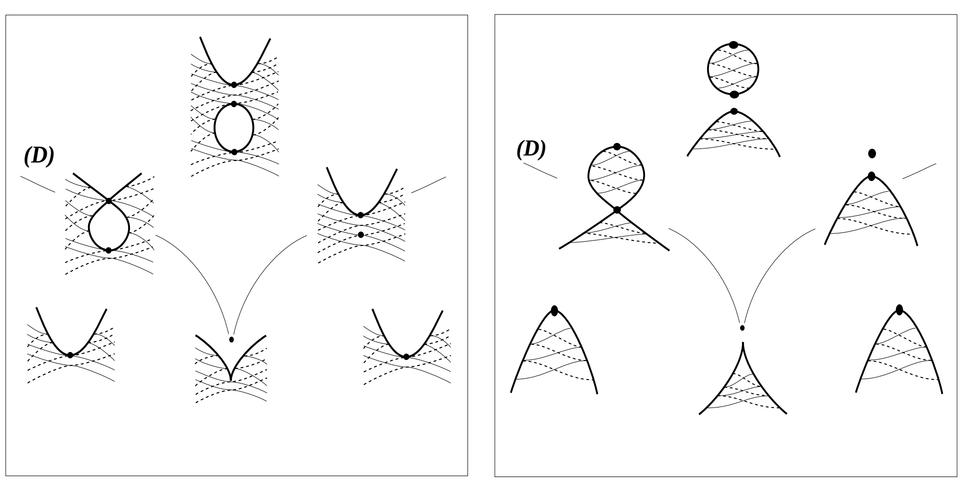

FIG. 6. Bifurcations of $p^2 + (\epsilon x^2 + y^3 + uy + v) = 0$, $\epsilon = -1$ left and $\epsilon = +1$ right.

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Proof The proof of the first statement follows from Theorem 2.1 and its proof (see also [23]).

For (ii) we proceed as in the previous cases. As $f_{xx}(0) \neq 0$, it follows that the component (SN) and (NF) of the bifurcation set are empty. The component (D) is diffeomorphic to a cusp. On one branch of this cusp the discriminant of the IDE's have a Morse singularity of type isolated point and on the other of type node.

To determine the number of singularities we consider the $\mathcal{K}^*$-singularity of the discriminant. The discriminant $f_{(0,0)}$ is $\mathcal{K}^*$-equivalent to x^2+y^3 and a $\mathcal{K}^*$-versal deformation of this singularity is given by x^2+y^3+uy+v. In particular a generic deformation of the IDI induces a $\mathcal{K}^*$-versal deformation of the discriminant. It follows that the number and position of the singularities on $f_{(u,v)}=0$ can be deduced from the versal deformation x^2+y^3+vy+u. These are as in Figure 6.

A calculation shows that all the simple singularities that appear in the deformation are of type well-folded saddles if $\epsilon=-1$ and of type well-folded foci when $\epsilon=+1$.

To show that the configurations in each stratum are topologically equivalent and equivalent to those of the model, we proceed as in the proofs of Theorems 3.1 and 3.2. Consider, for example, the case where $\epsilon=-1$ and the discriminant $f_{(u,v)}=0$ has a Morse singularity at a given point and the IDE has well-folded saddle at another point on the discriminant (see Figure 7 left). In Figure 7 all the curves except the thick ones are solution curves. We assume that the continuous (resp. dashed) curves are the solution curves of the direction field α (resp. β) associated to $p=\sqrt{-f_{u,v}(x,y)}$ (resp. $p=-\sqrt{-f_{u,v}(x,y)}$).

We construct a neighbourhood U of the discriminant as follows. We choose a continuous curve $BP_0\dots P_n$ so that B and P_0 are points on the separatrices of the well-folded saddle at S_1, P_{n-2} and P_{n-1} are on the separatrices at the singularity of the discriminant (Figure 7 left). The vertices P_i and P_{i+1} are linked by sliding along the solution curve of α from P_i to the discriminant and back along the solution curve of β to P_{i+1}. We choose another continuous curve $CQ_0\dots Q_n$ in the same way (Figure 7 left). We construct a neighbourhood U' of the discriminant of the model in an analogous way, making sure that we have the same number of vertices on the curves (this can always be done).

Then any pair of increasing homeomorphisms $h: AB \to A'B'$ and $k: AC \to A'C'$ induce a homeomorphism $H: U \to U'$ that preserves the integral curves. This homeomorphism is constructed as follows.

A point p in the sector delimited by ABS_1C is the intersection of a solution curve of α from a point $p_1 \in AB$ and an solution curve of β from a point $p_2 \in AC$. We define $H(p)$ in the sector delimited by $A'B'S_1'C'$ as the point of intersection of the solution curve of α' from $h(p_1) \in A'B'$ and the solution curve of β' from $k(p_2) \in A'C'$.

A point p in a triangular sector $P_iL_iP_{i+1}$ (the vertex L_i being the endpoint on the discriminant of the solution curves through P_i and P_{i+1}) is the intersection of a solution curve of α and of β from distinct points on P_iP_{i+1}. To each of these points corresponds a unique point on AC, say p_1 and p_2 respectively, obtained by zigzagging along solution curves of α and β (with turning points on the thick curves). We define $H(p)$ as the point of intersection of the solution curves of α' and β', in the triangular sector $P_i'L_i'P_{i+1}'$, obtained by zigzagging back from $k(p_1)$ and $k(p_2)$ on $A'C'$. The map H is defined similarly in the remaining sectors. It is clear that the resulting map H is a homeomorphism taking the solution curves of one IDE to the other.

The cases where (u, v) belongs to the remaining strata are delt with in the same way. For example when (u, v) belongs to the open region where the discriminant has two components, the homeomorphism H is also completely determined by two increasing homeomorphisms $h : AB \to A'B'$ and $k : AC \to A'C'$; see Figure 7 right. $\square$

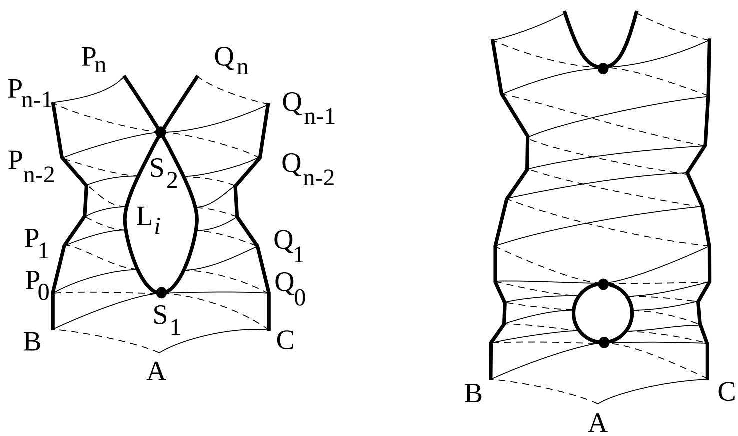

FIG. 7. Neighbourhood of the discriminant.

4. APPLICATION TO THE BIFURCATIONS OF THE ASYMPTOTIC AND CHARACTERISTIC CURVES

Given an oriented surface M in $\mathbb{R}^3$ with a family of normals N we have the Gauss map $N : M \to S^2$. At a point q the map $-dN(q) : T_qM \to T_{N(q)}S^2$ can be thought of as an automorphism of T_qM, this is the classical shape operator S_q, or simply S. If M is parametrised by $\mathbf{r}(x, y)$ with shape operator S, the coefficients of the first (resp. second) fundamental form E, F, G (resp. l, m, n) are given by

$$E = \mathbf{r}_x.\mathbf{r}_x,\ F = \mathbf{r}_x.\mathbf{r}_y,\ G = \mathbf{r}_y.\mathbf{r}_y$$
$$l = S(\mathbf{r}_x).\mathbf{r}_x = N.\mathbf{r}_{xx},\ m = S(\mathbf{r}_x).\mathbf{r}_y = N.\mathbf{r}_{xy},\ n = S(\mathbf{r}_y).\mathbf{r}_y = N.\mathbf{r}_{yy}.$$

Given a direction v in T_qM consider the conjugate direction $\bar{v}$ (this is a direction that satisfies $S_q(v).\bar{v} = 0$). At hyperbolic points there are two directions that coincide with their conjugate. These are the asymptotic directions and are given by the IDE

$$ndy^2 + 2mdxdy + ldx^2 = 0.$$

The discriminant of this IDE is precisely the parabolic set.

At elliptic points there is a unique pair of conjugate directions for which the included angle (i.e the angle between these directions) is minimal. These directions are called characteristic directions and are determined, in terms of the coefficients of the first and second fundamental forms, by the following IDE (see [21], [12]):

$$(2m(Gm-Fn)-n(Gl-En))dy^2+2(m(Gl+En)-2Fln)dydx+ \\ (l(Gl-En)-2m(Fl-Em))dx^2=0.$$

The discriminant of the characteristic directions IDE consists of the parabolic set and the umbilic points ([12]).

Away from flat umbilic points, i.e points where the principal curvatures $\kappa_1=\kappa_2=0$ (resp. umbilic, i.e points where $\kappa_1=\kappa_2$), the equations of the asymptotic (resp. characteristic) directions are IDE's is of order 2.

The singularities of both IDE's are intimately related to the singularities of the hight function of the surface. Recall that the family of height functions is given by

$$\begin{array}{rcl} H: M\times S^2 & \to & \mathbb{R} \\ (q,u) & \mapsto & q.u \end{array}$$

where $q.u$ denotes the dot product. The singularities of the function H_u at q measure the local contact of the surface with the plane normal to u. The parabolic curve is the set of points where the height function, in the normal direction, has an $A_{\geq 2}$-singularities. (Here we are using Arnold's notation [1]: an A_k-singularity is one equivalent to $x^2\pm y^{k+1}$. This means that we can change coordinates in the source and target and write the function in the form $x^2\pm y^{k+1}$.) Cusps of Gauss correspond to A_3-singularities of the height function. There are two types of cusp of Gauss, the hyperbolic type corresponding to (x^2-y^4) and the elliptic type (x^2+y^4). At a hyperbolic cusp of Gauss the asymptotic (resp. characteristic) directions IDE has a well-folded saddle (resp. node or focus) singularity and at an elliptic cusp of Gauss it has a well-folded node or focus (resp. saddle) singularity (see [12]).

Away from flat umbilics, the transitions in gereric 1-parameter families of height functions occur in two ways (see [5]):

(i) A non-versal A_3, here the family of height functions fails to versally unfold the A_3-singularity. This occurs at points where the parabolic set has a Morse singularity and both the asymptotic and characteristic directions IDE's undergo generically the Morse Type 1 transitions.

(ii) An A_4-singularity: both the asymptotic and characteristic directions IDE's undergo generically the well-folded saddle-node bifurcations.

In 2-parameter families of surfaces, we expect the following singularities of the height function to occur: (i) an A_5-singularity; (ii) an A_4-singularity at a Morse singularity the parabolic set; (iii) an A_3-singularity at a cusp singularity of the parabolic set.

We apply here the results of the previous sections to deduce the configurations of the asymptotic and characteristic directions IDE's at the above singularities, and the way they bifurcate in generic 2-parameter families of surfaces. To carry out the calculations we write the surface locally at a parabolic point and away from flat umbilics, in Monge form

$$f(x,y) \ = a_0x^2 + \Sigma_{i=0}^3 b_i x^{3-i}y^i + \Sigma_{i=0}^4 c_i x^{4-i}y^i + \Sigma_{i=0}^5 d_i x^{5-i}y^i \\ +\Sigma_{i=0}^6 e_i x^{6-i}y^i + \cdots$$

(i) *At an A_5-singularity of the height function*: this occurs on a smooth parabolic set if and only if

$$b_3 = 0, \ \ b_2^2 - 4c_4 = 0, \ d_5 - \frac{1}{2}c_3b_2 + \frac{1}{4}b_1b_2^2 = 0, b_2 \neq 0$$

$$A = -\frac{1}{4}c_3^2 + \frac{1}{2}c_3b_2b_1 - \frac{1}{4}b_2^2b_1^2 - \frac{1}{8}b_0b_2^3 + \frac{1}{4}c_2b_2^2 - \frac{1}{2}d_4b_2 + e_6 \neq 0.$$

A calculation (using Maple) shows that the 5-jet of the asymptotic (resp. characteristic) IDE can be transformed to $p^2 + (y + g(x,y)) = 0$, with $g(x,0) = \frac{A}{b_2^2}x^4 + \cdots$ (resp. $g(x,0) = -\frac{A}{b_2^2}x^4 + \cdots$).

(ii) *At an A_4-singularity of the height function where the parabolic set has a Morse singularity*: this singularity occurs if and only if

$$b_2 = b_3 = c_4 = 0, \ c_3d_5 \neq 0.$$

In this case the 3-jet of the asymptotic (resp. characteristic) directions IDE is equivalent to $p^2+8(3c_3xy+40d_5x^3) = 0$ (resp. $p^2-8(3c_3xy+40d_5x^3) = 0$). So we have a non-transverse Morse singularity.

(ii) *At an A_3-singularity of the height function where the parabolic set has a cusp singularity*: This occurs if and only if

$$b_2 = b_3 = 3c_3^2 + 8c_4(b_1^2 - c_2) = 0, \ c_4 \neq 0, C \neq 0.$$

where

$C = -\frac{4}{27c_3^3}(80d_5b_1^6 - 18c_3^2b_1^5 - 240d_5b_1^4c_2 + 72c_3d_4b_1^4 - 18c_3^2b_1^3c_2 - 144c_3d_4b_1^2c_2 + 240d_5b_1^2c_2^2 + 81b_1^2c_3^3b_0 + 54c_3^2b_1^2d_3 - 81b_1c_3^3c_1 + 36c_3^2b_1c_2^2 - 80d_5c_2^3 - 54c_3^2c_2d_3 + 72c_3d_4c_2^2 + 27c_3^3d_2)$.

In this case the 3-jet of the asymptotic (resp. characteristic) directions IDE is equivalent to $p^2 + (\frac{36c_3^2}{c_2-b_1^2}x^2 + Cy^3) = 0$ (resp. $p^2 + (-\frac{36c_3^2}{c_2-b_1^2}x^2 + Cy^3) = 0$).

We observe that a generic 2-parameter family of surfaces (that yields a versal deformation of the height function with one of the above singularities) induces a generic 2-parameter family of the singularity of the IDE's of the asymptotic and characteristic directions. We have thus the following.

PROPOSITION 4.1. *In generic 2-parameter families of surfaces, the asymptotic and characteristic curves undergo* *(i) the bifurcations in Theorem 3.1 and Figure 3 (left for one and right for the other, or vice-versa) at an A_5-singularity of the height function;*

(ii) the bifurcations in Theorem 3.2 and Figure 4 at an A_4-singularity of the height function where the parabolic set has a Morse singularity (A_1^-);

(iii) the bifurcations in Theorem 3.3 and Figure 6 (left for one and right for the other, or vice-versa) at an A_3-singularity of the height function where the parabolic set has a cusp singularity.

5. APPENDIX: PROOF OF THEOREM 2.1

Proof We take, without loss of generality, the IDE of the form $p^2 + f(x, y) = 0$, where f is a germ, at the origin, of a smooth function. When f is analytic, the hypothesis of the theorem are equivalent to the IDE having finite multiplicity ([9]). Indeed, the multiplicity of the equation is given by $m = \mu(f) + \mu(f(x, 0))$ (μ denotes the Milnor number), and this is finite if and only if $\mu(f)$ and $\mu(f(x, 0))$ are finite.

To prove the theorem, we first show that the IDE defines a finite number of regions (a region here is a connected component of the set $(f \leq 0) - \{0\}$), each region is divided into a finite number of sectors and these are topologically determined by the s-jet of f, where $s = \max(\mu(f), \mu(f(x, 0)))$. The sectors form a partition of the regions where $f \leq 0$ and $j^s f \leq 0$ (these are the regions where the IDE's have solutions). Since f and $j^s f$ have the same $\mathcal{K}$-singularity, they have the same number of branches and there exists a germ of a diffeomorphism taking the regions where $f \leq 0$ to the regions where $j^s f \leq 0$.

We proceed by blowing-up the singularity at the origin. We shall denote by $a_{l,0}x^l$ and $a_{k-1,1}x^{k-1}y$ the terms in the Taylor expansion of $-f$, where l (resp. k) indicates the lowest degree such that $a_{l,0} \neq 0$ (resp. $a_{k-1,1} \neq 0$). We need to analyse the cases $l < 2k$, $l > 2k$ and $l = 2k$ separately.

(1) $l < 2k$.

Case $l = 2m$. We consider the following blowing up

$$\begin{cases} x = X \\ y = X^{m+1}v. \end{cases}$$

and still write x for X. Then the IDE $p^2 + f(x, y) = 0$ yields two ODE's

$$(m+1)v + x\frac{dv}{dx} = \pm\sqrt{a_{l,0} + a_{k-1,1}x^{k-m}v + xg(x, v)}$$

for some smooth function g. If $a_{l,0} > 0$, the vector fields $\eta_\pm$ associated to the above ODE's are C^∞ in a neighbourhood of the exceptional fibre $x = 0$ and are singular on the exceptional fibre if and only if $(m+1)v = \pm\sqrt{a_{l,0}}$. So we have one singularity for each vector field, and these occur at distinct points. These singularities are saddles and one of their separatrices is the exceptional fibre. Therefore each field $\eta_\pm$ has one separatrix transverse to the exceptional fibre (see Figure 8(i) where the integral curves of η_+ (resp. η_-) are in continuous (resp. dashed) lines; the thick lines represent the blown-up discriminant). Blowing down, yields an IDE with two regions having two (semi) separatrices each or one region having four (semi) separatrices; the remaining region are without separatrices.

If $a_{l,0} < 0$ then $\eta_\pm$ are defined in the region where $a_{l,0} + xg(x,v) \geq 0$ which does not contain the exceptional fibre (see Figure 8(ii)). Therefore blowing down, yields an IDE with regions where there are no separatrices.

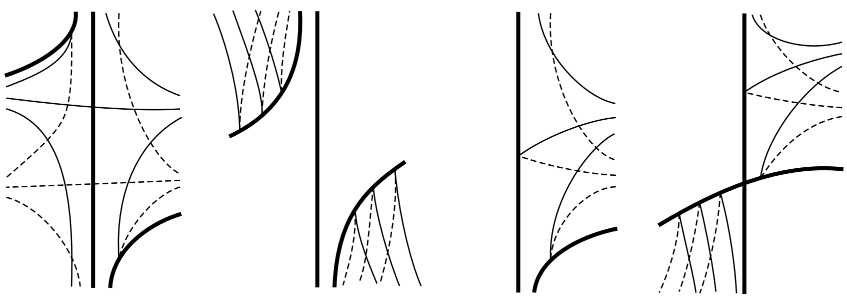

FIG. 8. The case $l < 2k$, l even (i) and (ii), l odd (iii) and (iv).

Case $l = 2m+1$. We consider the same blowing up as above and obtain the following ODE's

$$(m+1)v + x\frac{dv}{dx} = \pm\sqrt{a_{l,0}x + a_{k-1,1}x^{k-m}v + x^2 g(x,v)}.$$

for some smooth function g. We consider the cases $k > m+1$ and $k = m+1$ separately. When $k > m+1$ we can write $-f(x, x^{m+1}v)x^{-2m} = a_{l,0}x(1 + \frac{a_{k-1,1}}{a_{l,0}}x^{k-m-1}v + \frac{1}{a_{l,0}}xg(x,v))$, so we need $a_{l,0}x \geq 0$. We set $x = a_{l,0}u^2$, $u \geq 0$, and obtain the following ODE's

$$(m+1)v + \frac{u}{2}\frac{dv}{du} = \pm|a_{l,0}|u\sqrt{1 + a_{k-1,1}a_{l,0}^{k-m-2}u^{2(k-m-1)}v + u^2\tilde{g}(a_{l,0}u^2, v)}.$$

The associated fields to these new ODE's have a saddle singularity at the origin. The set $u = 0$ (corresponding to the exceptional fibre) is a separatrix and the second separatrices are of the form $v_\pm = u\phi_\pm(u)$, $u \geq 0$, where $\phi_\pm$ are smooth functions with $\phi_\pm(0) \neq 0$ (Figure 8(iii)). Blowing down yields a region with two separatrices given by $x = a_{l,0}u^2$, $y = a_{l,0}^{m+1}u^{2m+3}\phi_\pm(u)$, and the remaining regions are without separatrices.

When $k = m+1$ we can write $-f(x, x^{m+1}v)x^{-2m} = x(a_{l,0} + a_{k-1,1}v + xg(x,v))$. The smooth curve $\gamma : a_{l,0} + a_{k-1,1}v + xg(x,v) = 0$ intersects the exceptional fibre transversally at $v_0 = -\frac{a_{l,0}}{a_{k-1,1}} \neq 0$. At this point we can change coordinates (set $t^2 = a_{l,0} + a_{k-1,1}v + ug(u,v)$) and show that the integral curves of both fields form a family of smooth curves ending transversally on γ (Figure 8(iv)). Away from v_0 we proceed as follows.

Suppose $a_{k-1,1} > 0$ (the case $a_{k-1,1} < 0$ is analogous and is omitted). We do the following change of variable

$$\begin{cases} x = u^2 & \text{if} \quad v \geq v_0 \\ x = -u^2 & \text{if} \quad v \leq v_0 \end{cases}$$

with $u \geq 0$. When $v \geq v_0$, we obtain the following ODE's

$$(m+1)v + \frac{u}{2}\frac{dv}{du} = \pm u\sqrt{a_{l,0} + a_{k-1,1}v + ug(u,v)}$$

for some new smooth function g. If $v_0 < 0$ then the associated vector fields have saddle type singularities at the origin (away from v_0). If $v_0 > 0$ then we have no separatrices. When $v \leq v_0$, one can show that there are two separatrices if $v_0 > 0$ and none if $v_0 < 0$ (Figure 8(iv)). So blowing down yields one region with two separatrices, and the remaining regions without.

(2) $2k < l$

We consider the following blowing up

$$\begin{cases} x = X \\ y = a_{k-1,1}X^{k+1}v. \end{cases}$$

and obtain two ODE's

$$(k+1)v + x\frac{dv}{dx} = \pm\sqrt{v + xg(x,v)}$$

for some smooth function g. The smooth curve $v + xg(x,v) = 0$ intersects the exceptional fibre at the origin and the ODE's are defined in the region where $v + xg(x,v) \geq 0$.

To analyse the behaviour of the integral curves at the origin we make the following change of variable $u = v + xg(x,v)$ so that $v = u + \psi(x,u)$ for some germ of a smooth function ψ with a zero 1-jet and with $\psi(x,0) = -a_l x^{l-2k}$. This yields two new ODE's

$$(k+1)(u + \psi(x,u)) + x\psi_x(x,u) + x(1 + \psi_u(x,u))\frac{du}{dx} = \pm\sqrt{u}, \quad u \geq 0.$$

We now set $u = t^2$, with $t \geq 0$ and obtain

$$(k+1)(t^2 + \psi(x,t^2)) + x\psi_x(x,t^2) + 2tx(1 + \psi_u(x,t^2))\frac{dt}{dx} = \pm t,\ t \geq 0.$$

It is not difficult to show that the $(l-2k)$-jets of theses ODE's are smoothly equivalent to $tdx \pm x^{l-2k}dt = 0$ and hence the ODE's are topologically equivalent to $tdx \pm x^{l-2k}dt = 0$. There are three distinct topological behaviours depending on the parity of $l-2k$ (see Figure 9). So we have two separatrices for each field one of which is the exceptional fibre. The second is transverse to the exceptional fibre and is given by $t = \alpha x^p + h.o.t.$ Substituiting in the ODE's we get $\alpha = \mp\frac{a_{l,0}}{a_{k-1,1}^2}$ and $p = l - 2k$. We observe that for the original ODE's

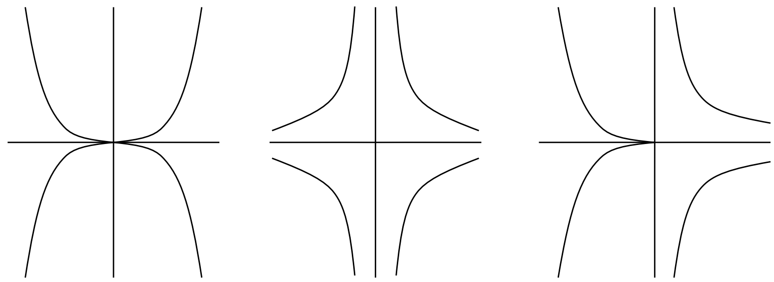

FIG. 9. Integral curves of $tdx \pm x^{l-2k}dt = 0$.

if one of the separatrices is in the region where $v + xg(x, y) \geq 0$ the other is in the region where $v+xg(x, y) \leq 0$. We have then three possibilities for the original ODE's as illustrated in Figure 10.

Away from the origin, the vector field η_+ associated to

$$(k+1)v + x\frac{dv}{dx} = \sqrt{v + xg(x, v)}$$

has a saddle singularity on the exceptional fibre at $v = \frac{1}{(k+1)^2}$, with the exceptional fibre being one of the separatrices. The other separatrix is given by $v = \frac{1}{(k+1)^2} + x\tilde{v}(x)$ for some smooth function $\tilde{v}$ with $\tilde{v}(0) \neq 0$. The vector field η_- is not singular along the exceptional fibre.

Blowing down, yields an IDE with two regions having two separatrices each or one region having four separatrices; the remaining regions are without separatrices.

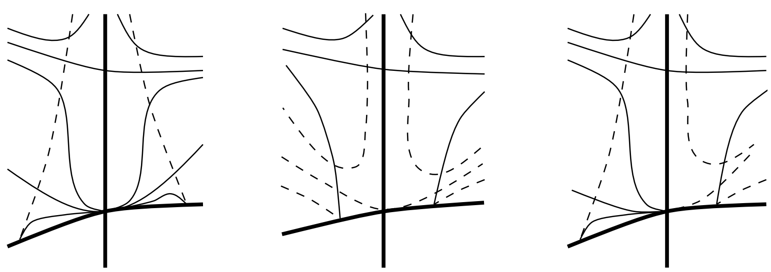

FIG. 10. The case $l > 2k$.

(3) $l = 2k$

We proceed as above by considering the blowing up

$$\begin{cases} x = X \\ y = X^{k+1}v. \end{cases}$$

We obtain two ODE's

$$(k+1)v + x\frac{dv}{dx} = \pm\sqrt{a_{2k,0} + a_{k-1,1}v + xg(x,v)}$$

for some smooth function g. (We need $a_{2l,0} + a_{k-1,1}v + xg(x,v) \geq 0$.) The associated fields $\eta_{\pm}$ are singular exceptional on the exceptional fibre if and only if $(k+1)v = \pm\sqrt{a_{2k,0} + a_{k-1,1}v}$. So we obtain a quadratic equation $(k+1)^2v^2 - a_{k-1,1}v - a_{2k,0} = 0$ (with $v > 0$ for the $+$ case and $v < 0$ for the $-$ case). If the discriminant $a_{k-1,1}^2 + 4(k+1)^2a_{2k,0} > 0$, $\eta_{\pm}$ has 2,1 or 0 singularities (in the right regions) with total number of singularities of both fields equal to 2. When η_+ or η_- has two singularities, one is a saddle and another is a node. When they have one singularity each, they are both of type saddle or node (Figure 11, first three figures). The integral curves of both fields form a family of smooth curves ending transversally on the curve $a_{2k,0} + a_{k-1,1}v + ug(u,v) = 0$ near $(u,v) = (0, v_0)$. Blowing down, yields an IDE with two sectors having two separatrices each or one region having four separatrices; the remaining regions are without separatrices.

When $a_{k-1,1}^2 + 4(k+1)^2a_{2k,0} = 0$, η_+ (resp. η_-) has a saddle-node singularity if $a_{k-1,1} > 0$ (resp. $a_{k-1,1} < 0$) (Figure 11, last figure), so blowing down yields an IDE with sectors with a unique separatrix.

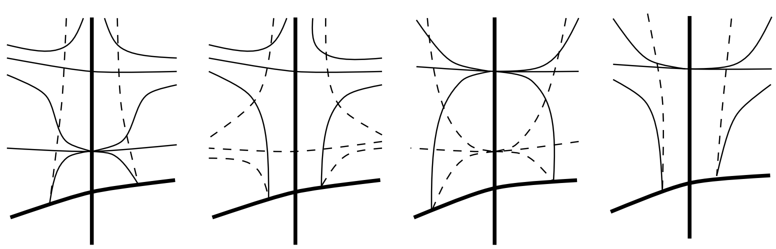

FIG. 11. The case $l = 2k$.

We conclude from the above that there are 0,1,2 or 4 separatrices in any region and the number of regions and separatrices in each one is determined by j^sf with $s = \max(\mu(f), \mu(f(x,0)))$. We shall now show that two regions with the same number of separatrices (0,1,2 or 4) are topologically equivalent. We observe that each region is divided into sectors and in each sector the solution curves of the two directional fields determined by the IDE have one of the configuration in Figure 12, where the thick curves which are branches of the discriminant. (The integral curves in the last case in Figure 12 are the blow-down of integral curves that start from a singularity and finish at a singularity. This does not occur.)

If the discriminant is an isolated point, then the solutions of the IDE can be an isolated point (and we assume that all such IDE's are equivalent) or are defined in a neighbourhood of the origin. In this case, following the above calculations, we have four (semi)-separatrices and the topological model is given by the horizontal and vertical lines. In particular all IDE's with finite multiplicity and an isolated point discriminant are topologically equivalent.

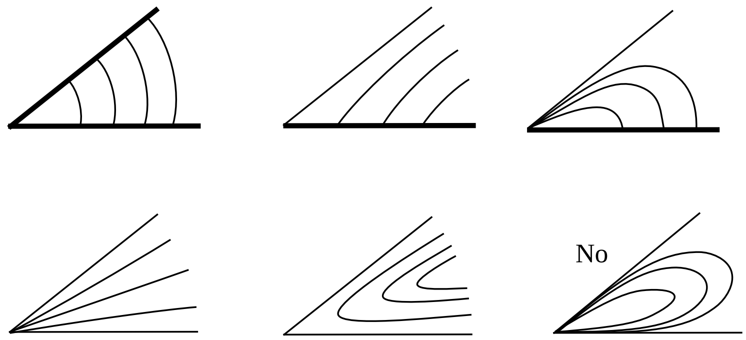

FIG. 12. The possible sector configurations for one directional field.

Suppose now that the discriminant is not an isolated point.

(i) There are no separatrices in the region. Then the configuration of the solution curves is as in Figure 13. It is not hard to show that any two such configurations are homeomorphic. If we choose the neighbourhood as in Figure 13, then any point in this neighbourhood is the intersection of an integral curve of one directional field defined by the IDE starting from the segment OA and an integral curve of the other directional field starting from the segment OB. A homeomorphism taking the integral curves in this region to the corresponding one of the model is determined by a pair of homeomorphism $h : OA \to O'A'$ and $k : OB \to O'B'$.

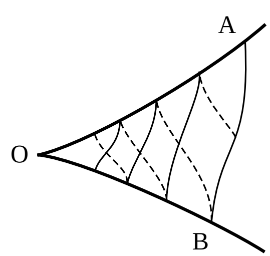

FIG. 13. Region without a separatrix.

(ii) There is one separatrix in the region. We have three possibilities as shown in Figure 14. However, the solution curves of the IDE are projections of integral curves of a smooth vector field on the surface of the equation $M = \{(x, y, p) : p^2 + f(x, y) = 0\}$, so they inherit the orientation of this vector field. It follows then that only one of the configurations in Figure 14 is possible. It is not hard to show that any two such configurations are homeomorphic.

(iii) There are two separatrices in the region. These separatrices are both of η^+ or η^- or one of each (this is completely determined by the s-jet of f as can be seen from the above

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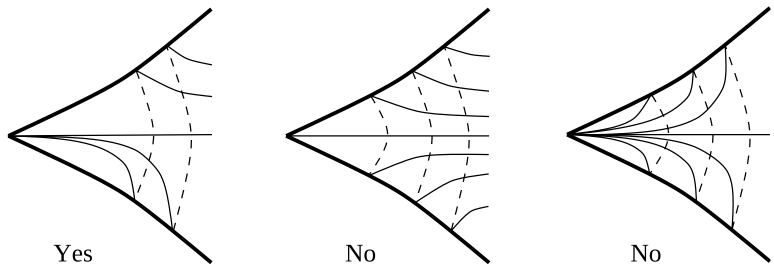

FIG. 14. Regions with one separatrix.

calculations). Following the same arguments as above, and using the orientation we show that the only possible configurations are those in Figure 15. Any two given configurations of the same type are homeomorphic.

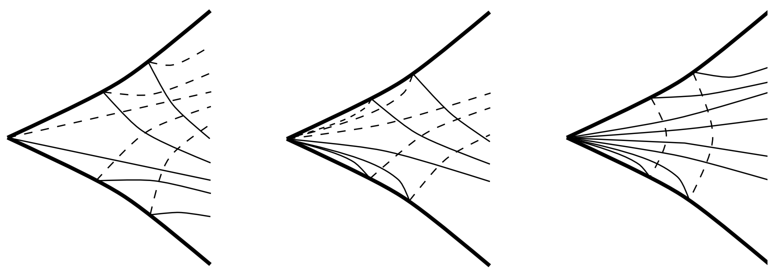

FIG. 15. Regions with two separatrises.

(iii) There are four separatrices in the region. Following the same arguments as above, and using the orientation we show that the only possible configurations are those in Figure 16. Any two given configurations of the same type are homeomorphic. □

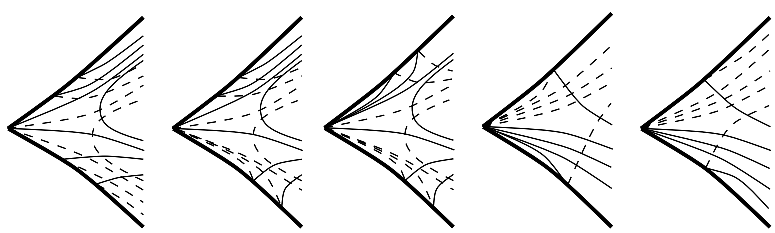

FIG. 16. Regions with four separatrises.

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