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A remark on second order differential equations with nonlocal conditions

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In this work we discuss the existence of mild solution for the abstract Cauchy problem (1)-(3). Also in this paper we make some remarks with respect to several recent papers reporting existence results for second order partial differential systems. October, 2003 ICMC-USP

1. INTRODUCTION

In this paper we study the the existence of mild solutions for a class of partial second order differential equations with nonlocal conditions modelled by mean of the abstract Cauchy problem

$$\frac{d}{dt}(x'(t)+g(t,x(t),x'(t))) = Ax(t)+f(t,x(t),x'(t)), \qquad t\in I=[0,a], \tag{1}$$

$$x(0) = x_0+p(x,x'), \tag{2}$$

$$x'(0) = y_0+q(x,x'), \tag{3}$$

where A is the infinitesimal generator of an strongly continuous cosine function of bounded linear operators on a Banach space X and $f,g: I\times X^2\to X$; $p,q: C(I:X)^2\to X$ are appropriate functions.

The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations and it's equally appealing devoted their generality and simplicity. For basic concepts and applications of this theory, we refer to the reader to Fattorini [5], Travis & Weeb [19].

Our purpose in this work is consider a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Weeb in [18, 19] and of some recent developments by Staněk in [14, 15, 16, 17]. Additionally, we observe that the ideas and techniques in this paper permit the reformulation of the problems studied in [1], [2], [3], [7], [9], [11] to the context of " partial " second order differential equations, see [18] pp. 557 and the referred papers for details.

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Next we mention a few results and notations respect of the cosine function theory which are needed to establish our results. Along of this paper, A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators, $(C(t))_{t\in\mathbb{R}}$, on a Banach space X and $S(t)$ represents the sine function associated with $(C(t))_{t\in\mathbb{R}}$, which is defined by $S(t)x = \int_0^t C(s)xds$, $x \in X$, $t \in \mathbb{R}$. We designate by $N, \tilde{N}$ certain constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \tilde{N}$ for every $t \in I$. In this paper, $[D(A)]$ is the space

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable}\},$$

endowed with the norm $\| x \|_A = \| x \| + \| Ax \|$, $x \in D(A)$. Moreover, the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisińsky [6], that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0\leq t\leq 1} \|AS(t)x\|, \qquad x \in E,$$

is a Banach space.

The operator valued function $G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of linear operators on $E \times X$ generated by $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this it follows that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$, as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is locally integrable then $y(t) = \int_0^t S(t-s)x(s)ds$ is a E-valued continuous function. This is an consequence of the fact that

$$\int_0^t g(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s)\,ds \\ \int_0^t C(t-s)x(s)\,ds \end{bmatrix}$$

defines an $E \times X$-valued continuous function.

The existence of solutions of the second order abstract Cauchy problem

$$\begin{cases} x''(t) &= Ax(t) + h(t), \quad 0 \leq t \leq a, \\ x(0) &= x_0, \quad x'(0) = x_1, \end{cases} \tag{4}$$

where $h : [0, a] \to X$ is an integrable function, has been discussed in [18]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [19]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)\,ds, \quad 0 \leq t \leq a, \tag{5}$$

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is called, mild solution of (4) and that when $x_0 \in E$, $x'(\cdot)$ is continuous and

$$x'(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s)\,ds.$$

For a complementary literature respect of the cosine function theory, we refer to the reader to [5].

The terminologies and notations are those generally used in functional analysis. In particular, $C(I;X)$ is the space of the continuous function from I into X endowed with the norm of the uniform convergence and $B_r(x:Z)$ denotes the closed ball with center at x and radius $r>0$ in an metric space Z. Additionally, for a bounded function $\xi:[0,a]\to[0,\infty)$ and $0\le t\le a$ we will employ the notation ξ_t for

$$\xi_t = \sup\{\xi(s) : s\in[0,t]\}. \tag{6}$$

The paper has two sections. In section 2 we prove some existence results of mild solutions of the abstract Cauchy problem (1)-(3) using the following fixed point results for condensing operators.

THEOREM 1.1. *[4, pp. 61] Let D be a convex subset of a Banach space X and assume that $0\in D$. Let $F:D\to D$ be a completely continuous map. Then the map F has a fix point in D or the set $\{x\in D : x=\lambda F(x),\ 0<\lambda<1\}$ is unbounded.*

THEOREM 1.2. *(Sadovskii) [13]. Let D be a convex, closed and bounded subset of a Banach space X. If $F:D\to D$ is a condensing operator, then F has a fix point in D.*

2. EXISTENCE RESULTS

Initially we study the second order nonlocal abstract Cauchy problem

$$\frac{d}{dt}[x'(t)+g(t,x(t))] = Ax(t)+f(t,x(t)), \qquad t\in I=[0,a], \tag{7}$$

$$x(0) = x_0 + p(x), \tag{8}$$

$$x'(0) = y_0 + q(x), \tag{9}$$

where $f,g: I\times X\to X$ and $p,q: C(I:X)\to X$ are appropriate functions.

If $u(\cdot)$ is a solution of (7)-(9), g is $D(A)$-valued and $Ag\in C(I:X)$ then

$$\begin{aligned} u(t) &= C(t)(x_0+p(u)) + S(t)[y_0 - g(0,x_0+p(u)) + q(u)] - \int_0^t g(s,u(s))ds \\ &\quad - \int_0^t AS(t-s)\int_0^s g(\tau,u_\tau)d\tau ds + \int_0^t S(t-s)f(s,u(s))ds. \end{aligned} \tag{10}$$

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The expression (10) is the motivation of the following definition.

Definition 2.1. A function $u \in C(I : X)$ is a mild solution of the abstract nonlocal Cauchy problem (7)-(9) if $u(0) = (x_0 + p(u))$ and

$$u(t) = C(t)(x_0 + p(u)) + S(t)[y_0 - g(0, x_0 + p(u)) + q(u)] - \int_0^t C(t-s)g(s, u(s))ds + \int_0^t S(t-s)f(s, u(s))ds, \quad t \in I.$$

To study the existence of mild solutions of (7)-(9), we introduce the following conditions.

$\mathbf{H_1}$ The function $f : I \times X \to X$ satisfies the following conditions.

(i) The function $f(t, \cdot) : X \to X$ is continuous $a.e.$ $t \in I$;

(ii) For each $x \in X$, the function $f(\cdot, x) : I \to X$ is strongly measurable;

(iii) There exist a continuous function $m_f : I \to [0, \infty)$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, x) \| \leq m_f(t) W_f(\| x \|), \qquad (t, x) \in I \times X.$$

$\mathbf{H_2}$ The function $g : I \times X \to X$ satisfy the following conditions:

(i) $g(t, \cdot) : X \to X$ is continuous $a.e.$ $t \in I$;

(ii) For each $x \in X$, the function $g(\cdot, x) : I \to X$ is strongly measurable;

(iii) There exists a continuous function $m_g(\cdot) : [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $W_g(\cdot) : [0, \infty) \to (0, \infty)$ such that

$$\| g(t, x) \| \leq m_g(t) W_g(\| x \|), \qquad (t, x) \in I \times X.$$

Theorem 2.1. *Assume that the conditions* $\mathbf{H_1}$-$\mathbf{H_2}$ *are verified and that* $g(\cdot)$ *is completely continuous. Suppose, furthermore, that the following conditions hold.*

(**a**)*For every* $0 < t' \leq t$ *and* $r > 0$, *the set* $U(t, t', r) = \{S(t')f(s, x) : s \in [0, t], \| x \| \leq r\}$ *is relatively compact in X.*

(**b**)$p(\cdot)$ *is completely continuous and there is* $N_p > 0$ *such that* $\| p(u) \| \leq N_p$ *for every* $u \in C(I : X)$.

(**c**)*For every* $s \in I$ *and every* $r > 0$, *the set* $V(s, r) = \{S(s)q(x) : \| x \| \leq r\}$ *is relatively compact in X and there is* $N_q > 0$ *such that* $\| q(u) \| \leq N_q$ *for every* $u \in C(I : X)$.

If

$$\int_0^a (N m_g(s) + \tilde{N} m_f(s)) ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)},$$

where $c = N(\| x_0 \| + N_p) + \tilde{N}(\| y_0 \| + N_q + \sup_{\|x\| \leq N_p} \| g(0, x_0 + x) \|)$*, then there exists a mild solution of (7)-(9).*

Proof. On the space $C(I : X)$ we define the map $\Gamma : C(I : X) \to C(I : X)$ by

$$\begin{aligned}\Gamma u(t) &= C(t)(x_0 + p(u)) + S(t)[y_0 + q(u) - g(0, x_0 + p(u))] - \int_0^t C(t-s)g(s, u(s))ds \\ &+ \int_0^t S(t-s)f(s, u(s))ds, \qquad t \in I. \end{aligned} \tag{11}$$

In order to use Leray Schauder Alternative, we obtain an *a priori* bound for the solution of the integral equation $u = \lambda\Gamma(u)$, $\lambda \in (0, 1)$. If u^λ is a solution of $u = \lambda\Gamma(u)$, $\lambda \in (0, 1)$, we get

$$\begin{aligned}\| u^\lambda(t) \| &\leq N(\| x_0 \| + N_p) + \tilde{N}(\| y_0 \| + N_q + \sup_{\|x\| \leq N_p} \| g(0, x_0 + x) \|) \\ &+ N \int_0^t m_g(s) W_g(\| u^\lambda(s) \|)ds + \tilde{N} \int_0^t m_f(s) W_f(\| u^\lambda(s) \|)ds.\end{aligned}$$

Denoting by $\beta_\lambda(t)$ to the right hand side of the last inequality we find that

$$\beta'_\lambda(t) \leq (N m_g(t) + \tilde{N} m_f(t))(W_g(\beta_\lambda(t)) + W_f(\beta_\lambda(t))),$$

and hence

$$\int_c^{\beta_\lambda(t)} \frac{ds}{W_g(s) + W_f(s)} \leq \int_0^t (N m_g(s) + \tilde{N} m_f(s))ds < \int_c^\infty \frac{ds}{W_g(s) + W_f(s)},$$

which permit to conclude that the set $\{\beta_\lambda : \lambda \in (0, 1)\}$ is bounded in $C(I : X)$ and hence that $\{u^\lambda : \lambda \in (0, 1)\}$ is bounded in $C(I : X)$.

Next we prove that Γ is completely continuous. To this end, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned}\Gamma_1 u(t) &= C(t)(x_0 + p(u)) + S(t)[y_0 + q(u) - g(0, x_0 + p(u))], \qquad t \in I, \\ \Gamma_2 u(t) &= -\int_0^t C(t-s)g(s, u(s))ds + \int_0^t S(t-s)f(s, u(s))ds, \qquad t \in I.\end{aligned}$$

It's easy to show that Γ_1 is completely continuous and that Γ_2 is continuous. Next, by using Azcoli Arzela we prove that $\Gamma_2(B_r(0, C(I : X)))$ is relatively compact in $C(I : X)$. In the sequel $B_r = B_r(0, C(I : X))$.

Step 1 The set $\Gamma_2(B_r) = \{\Gamma_2 u : u \in B_r\}$ is equicontinuous on I.

Let $t \in I$ and $\epsilon > 0$. Since $C(\cdot)$ is strongly continuous and $g(\cdot)$ is completely continuous, there exists $\delta > 0$ such that

$$\| (C(s+h) - C(s))g(s', x) \| < \epsilon, \qquad \| x \| \leq r,\ (s, s') \in I^2,$$

when $| h | \leq \delta$. For $u \in B_r$ and $|h| \leq \delta$ with $t+h \in I$, we get

$$\begin{aligned}
&\| \Gamma_2 u(t+h) - \Gamma_2 u(t) \| \\
&\quad \leq \int_0^t \| (C(t+h-s) - C(t-s))g(s,u(s)) \| ds + N \int_t^{t+h} \| g(s,u(s)) \| ds \\
&\qquad + \int_0^t \| S(t+h-s) - S(t-s) \| \| f(s,u(s)) \| ds + \tilde{N} \int_t^{t+h} \| f(s,u(s)) \| ds \\
&\quad \leq \epsilon t + N W_g(r) \int_t^{t+h} m_g(s) ds + \tilde{N} h W_f(r) \int_0^t m_f(s) ds + \tilde{N} W_f(r) \int_t^{t+h} m_f(s) ds,
\end{aligned}$$

which prove the assertion.

Step 2 The set $\Gamma_2(B_r)(t) = \{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for every $t \in I$.

Let $t \in I$ and $\epsilon > 0$. If $u \in B_r$, from the estimate

$$\| f(\theta, u(\theta)) \| \leq m_f(\theta) W_f(\| u(\theta) \|) \leq m_f(\theta) W_f(r),$$

follows that the set $U = \{f(t-s, u(t-s)) : s \in [0,t], u \in B_r\}$ is bounded in X. Using that $S : I \to \mathcal{L}(X)$ is uniformly Lipschitz on I, we can chose $0 = s_1 < s_2 < ... < s_k = t$ such that

$$\| S(s')y - S(s)y \| < \epsilon, \qquad y \in U,$$

when $s, s' \in [s_i, s_{i+1}]$ for some $i = 1,..,k-1$. Let $u \in B_r$. From the mean value theorem for the Bochner integral, see [12, Lemma 2.1.3], and the fact that $V = \{C(s)g(s', x) : (s, s', x) \in I^2 \times B_r(0, X)\}$ is relatively compact in X, follows that

$$\begin{aligned}
\Gamma_2 u(t) &= -\int_0^t C(t-s) g(s,u(s)) ds + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} (S(s) - S(s_i)) f(t-s, u(t-s)) ds \\
&\quad + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} S(s_i) f(t-s, u(t-s)) ds \\
&\in t\overline{co(V)} + \epsilon B_a(0,X) + \sum_{i=1}^{k-1} (s_{i+1} - s_i) \overline{co(U(t, s_i, r))},
\end{aligned}$$

where $co(Q)$ denote the convex hull of a set Q. Thus $\Gamma_2(B_r)(t)$ is relatively compact X.

From the steps 1 and 2, follows that $\Gamma_2(B_r)$ is relatively compact in $C(I; X)$ and so that Γ_2 is completely continuous.

Finally, the Theorem 1.1 assert that Γ has a fixed in $C(I : X)$. The proof is complete. ■

If the maps g, p, q fulfill some Lipschitz conditions instead of the compactness proprieties considered in the preceding theorem, we also can establish a result of existence.

THEOREM 2.2. *Assume that* $\mathbf{H_1}$ *is verified and that the followings conditions hold.*

(**a**)*For every* $0 < t' \leq t$ *and* $r > 0$*, the set* $U(t,t',r) = \{S(t')f(s,x) : s \in [0,t], \| x \| \leq r\}$ *is relatively compact in X.*

(**b**)*There exist positive constants* l_g, l_p, l_q *such that*

$$\begin{aligned} \| g(t,x_1) - g(t,x_2) \| &\leq l_g \| x_1 - x_2 \|, \qquad (t,x_i) \in I \times X, \\ \| p(u) - p(v) \| &\leq l_p \| u - v \|, \qquad u,v \in C(I : X), \\ \| q(u) - q(v) \| &\leq l_q \| u - v \|, \qquad u,v \in C(I : X). \end{aligned}$$

and

$$N(l_p + l_g a) + \tilde{N}(l_q + l_g) + \tilde{N} \liminf_{r \to +\infty} \frac{W_f(r)}{r} \int_0^a m_f(s) ds < 1.$$

Then there exists a mild solution of (7)-(9).

Proof. Let $Y = C(I : X)$ and $\Gamma = \Gamma_1 + \Gamma_2 : Y \to Y$ be the map defined by

$$\begin{aligned} \Gamma_1 u(t) &= C(t)(x_0 + p(u)) + S(t)[y_0 - g(0,u(0)) + q(u)] - \int_0^t C(t-s) g(s,u(s)) ds, \\ \Gamma_2 u(t) &= \int_0^t S(t-s) f(s,u(s)) ds. \end{aligned}$$

We affirm that there exist $r > 0$ such that $\Gamma(B_r(0,Y)) \subset B_r(0,Y)$. In fact, if we assume that the affirmation is false, then for each $r > 0$ there exist $u^r \in B_r(0,Y)$ such that $\| \Gamma u^r \| > r$ which imply that

$$\begin{aligned} r \leq \| \Gamma u^r \| &\leq N[\| x_0 \| + l_p r + \| p(0) \|] + \tilde{N}(\| y_0 \| + l_g r + \| g(0,0) \| + l_q r + \| q(0) \|) \\ &\quad + N \int_0^a [l_g \| u^r(s) \| + \| g(s,0) \|] ds + \tilde{N} \int_0^a m_f(s) W_f(\| u^r(s) \|) ds \end{aligned}$$

and so that

$$1 \leq N(l_p + l_g a) + \tilde{N}(l_g + l_q) + \tilde{N} \liminf_{r \to +\infty} \frac{W_f(r)}{r} \int_0^a m_f(s) ds,$$

which is an absurd.

Let $r_0 > 0$ such that $\Gamma(B_{r_0}(0,Y)) \subset B_{r_0}(0,Y)$. Using the steps in the proof of Theorem 2.1, follows that Γ_2 is completely continuous and from the estimate

$$\| \Gamma_1 u - \Gamma_1 v \| \leq \Big(N(l_p + l_g a) + \tilde{N}(l_q + l_g) \Big) \| u - v \|,$$

that Γ_1 is a contraction. Thus, Γ is a condensing map on $B_{r_0}(0,Y)$. The assertion is now consequence of the Sadovskii point fixed Theorem, see [13]. The proof is finished. ■

Next we discuss existence of solution for the abstract Cauchy problem (1)-(3). In general, the results are similar to those in the first part of this section and for this reason, the proof of the next results will be summarized or simply omitted. To study these problem we introduce the next technical conditions.

$\mathbf{H_1}$ The function $f : I \times X \times X \to X$ satisfy the following conditions:

(i) $f(t, \cdot) : X \times X \to X$ is continuous a.e. $t \in I$;

(ii) For each $(x, y) \in X \times X$, the function $f(\cdot, x, y) : I \to X$ is strongly measurable;

(iii) There exists a continuous function $m_f : [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, x, y) \| \leq m_f(t) W_f(\| x \| + \| y \|), \qquad (t, x, y) \in I \times X \times X.$$

$\mathbf{H_4}$ $g : I \times X \to E$ is continuous and verifies the following conditions

(i) There exists a continuous function $m_g : [0, \infty) \to [0, \infty)$ and a continuous nondecreasing function $W_g : [0, \infty) \to (0, \infty)$ such that

$$\| g(t, x, y) \|_1 \ \leq \ m_g(t) W_g(\| x \| + \| y \|), \qquad (t, x, y) \in I \times X^2.$$

(ii) There exist positive constants c_1, c_2 such that

$$\| g(t, x, y) \| \leq c_1(\| x \| + \| y \|) + c_2, \qquad (t, x, y) \in I \times X^2.$$

$(\mathbf{H_g})$ For every $r > 0$, the set of functions $\{t \to g(t, u(t), v(t)) : u, v \in B_r(0, C(I : X))\}$ is equicontinuous on I.

THEOREM 2.3. *Let assumption* $\mathbf{H_3}$, $\mathbf{H_4}$, $\mathbf{H_g}$ *be satisfied and assume that the following conditions hold.*

(a)For each $r > 0$, *the set* $g(I \times B_r(0, X^2)) \times f(I \times B_r(0, X^2))$ *is relatively compact in* $E \times X$.

(**b**)$p(\cdot)$ *is* E *valued,* $p(\cdot) : C(I : X) \to E$ *and* $q(\cdot) : C(I : X) \to X$ *are completely continuous and there are constants* $N_p, N_q > 0$ *such that* $\| p(u) \|_1 \leq N_p$ *and* $\| q(u) \| \leq N_q$ *for every* $u \in C(I : X)$.

If $\mu = 1 - c_1 > 0$ *and*

$$\frac{1}{\mu} \int_0^t \Big[(N+1) m_g(s) + (\tilde{N} + N) m_f(s) \Big] ds < \int_{c\mu^{-1}}^{\infty} \frac{ds}{W_g(s) + W_f(s)}$$

where

$$c = N(\| x_0 \| + N_p) + \| x_0 \|_1 + N_p + (\tilde{N} + N)[\| y_0 \| + c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2] + c_2,$$

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then there exists a mild solution of (1)-(3).

Proof. Let $Y = C(I : X)$ and $\Gamma(\cdot) : Y^2 \to Y^2$ be the map $\Gamma(u,v) = (\Gamma_1(u,v), \Gamma_2(u,v))$ where

$$\begin{aligned}
\Gamma_1(u,v)(t) &= C(t)(x_0 + p(u,v)) + S(t)[y_0 + q(u,v) - g(0, x_0 + p(u,v), y_0 + q(u,v))] \\
&\quad - \int_0^t C(t-s)g(s,u(s),v(s))ds + \int_0^t S(t-s)f(s,u(s),v(s))ds, \\
\Gamma_2(u,v)(t) &= AS(t)(x_0 + p(u,v)) + C(t)[y_0 - g(0, x_0 + p(u,v), y_0 + q(u,v)) + q(u,v)] \\
&\quad - g(t,u(t),v(t)) - \int_0^t AS(t-s)g(s,u(s),v(s))ds \\
&\quad + \int_0^t C(t-s)f(s,u(s),v(s))ds.
\end{aligned}$$

From the assumptions follows that Γ is well defined and continuous. If $(u^\lambda, v^\lambda) = \lambda\Gamma(u^\lambda, v^\lambda)$, $\lambda \in (0,1)$, we find that

$$\begin{aligned}
\| u^\lambda(t) \| &\leq N(\| x_0 \| + N_p) + \tilde{N}[\| y_0 \| + c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2] \\
&\quad + N\int_0^t m_g(s)W_g(\| u^\lambda(s) \| + \| v^\lambda(s) \|)ds \\
&\quad + \tilde{N}\int_0^t m_f(s)W_f(\| u^\lambda(s) \| + \| v^\lambda(s) \|)ds
\end{aligned}$$

and that

$$\begin{aligned}
\| v^\lambda(t) \| &\leq \| x_0 \|_1 + N_p + N[\| y_0 \| + c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2] \\
&\quad + c_1(\| u^\lambda(t) \| + \| v^\lambda(t) \|) + c_2 + \int_0^t m_g(s)W_g(\| u^\lambda(s) \| + \| v^\lambda(s) \|)ds \\
&\quad + N\int_0^t m_f(s)W_f(\| u^\lambda(s) \| + \| v^\lambda(s) \|)ds.
\end{aligned}$$

Using the notation $\alpha_\lambda(t) = \| u^\lambda(t) \| + \| v^\lambda(t) \|$ and the previous inequalities, we see that

$$\begin{aligned}
\alpha_\lambda(t) &\leq N(\| x_0 \| + N_p) + \| x_0 \|_1 + N_p \\
&\quad + (\tilde{N} + N)[\| y_0 \| + c_1(\| x_0 \| + \| y_0 \| + N_p + N_q) + c_2] + c_1\alpha_\lambda(t) + c_2 \\
&\quad + (N+1)\int_0^t m_g(s)W_g(\alpha_\lambda(s))ds + (\tilde{N} + N)\int_0^t m_f(s)W_f(\alpha_\lambda(s))ds
\end{aligned}$$

which imply that

$$\alpha_\lambda(t) \leq \frac{c}{\mu} + \frac{1}{\mu}\left[(N+1)\int_0^t m_g(s)W_g(\alpha_\lambda(s))ds + (\tilde{N} + N)\int_0^t m_f(s)W_f(\alpha_\lambda(s))ds\right].$$

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Defining as $\beta_\lambda(t)$ the right hand side of the last inequality, we see that

$$\beta'_\lambda(t) \leq \mu^{-1}((N+1)m_g(t) + (\tilde{N}+N)m_f(t))(W_g(\beta_\lambda(t)) + W_f(\beta_\lambda(t)))$$

and hence

$$\int_{c\mu^{-1}}^{\beta_\lambda(t)} \frac{ds}{W_g(s)+W_f(s)} \leq \frac{1}{\mu}\int_0^t \Big[(N+1)m_g(s) + (\tilde{N}+N)m_f(s)\Big]\, ds,$$

which prove that the set $\{z^\lambda = (x^\lambda, y^\lambda) : z^\lambda = \lambda\Gamma z^\lambda,\ \lambda \in (0,1)\}$ is bounded in Y^2.

From the proof of Theorem 2.1, we infer that Γ_1 is completely continuous. Moreover, using the compactness assumption on the functions g, f, the condition $\mathbf{H_g}$, the fact that the X-valued function $t \to AS(t)x$ is continuous on I for every $x \in E$ and the ideais in the steep $1, 2$ in the proof of Theorem 2.1, we can prove that Γ_2 is completely continuous.

Theses remarks prove that Γ verifies the assumptions of Theorem 1.1. Thus, there exists a mild solution of (1)-(3). The proof is complete. ■

The proof of the next result, follows using the ideas in the proof of Theorem 2.2.

THEOREM 2.4. *Let* $(\mathbf{H_3})$ *be satisfied. Assume that that functions* g, p *are* E*-valued and that there are constants* l_g, l_p, l_q *such that*

$$\begin{aligned}
\| g(t,x_1,y_1) - g(t,x_2,y_2) \|_1 &\leq l_g \| (x_1,y_1) - (x_2,y_2) \|, && x_i, y_i \in X.\\
\| p(u_1,v_1) - p(u_2,v_2) \|_1 &\leq l_p \| (u_1,v_1) - (u_2,v_2) \|, && u_i, v_i \in C(I:X),\\
\| q(u_1,v_1) - q(u_2,v_2) \| &\leq l_q \| (u_1,v_1) - (u_2,v_2) \|, && u_i, v_i \in C(I:X),
\end{aligned}$$

If $(N+1)l_p + (N+\tilde{N})(l_g+l_q) + l_g((N+1)a+1) + (N+\tilde{N})\liminf_{r\to+\infty} \frac{W_f(r)}{r}\int_0^a m_f(s)ds < 1,$ *then there exists a mild solution of (1)-(3).*

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