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A remark on neutral partial differential equations

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In this work we study the existence of mild solution for a class of evolution equation described in the form $\frac{d}{dt}(x(t) + g(t, x(t))) = Ax(t) + g(t, x(t))$ where A is the infinitesimal generator of an analytic semigroup of linear operators. Our results are applicable to partial neutral functional differential systems.
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1. INTRODUCTION

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades. A good guide concerning the literature for ordinary neutral functional differential equations is the Hale and Verduyn Lunel book [14] and the references therein. On the other hand, for partial neutral functional differential equations we refer to the reader to Addimy [1], Datko [11], Hale [16], Hernández & Henríquez [17, 18] and Wu [24].

Using the semigroup theory of bounded linear operators on infinite dimensional Banach spaces, Hernández & Henríquez studied in [18] the existence of mild, strong and classical solutions for a class of partial neutral differential equation with unbounded delay modelled in the form

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \qquad t \in I = [0, a], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{2}$$

where A is the infinitesimal generator of an analytic semigroup defined on a Banach space X and $\mathcal{B}$ is a phase defined axiomatically as Hale and Kato in [15]. In general, the results was proved using the theory of fractional power of closed operators; the ideas and techniques in [21] and the Sadovkii fixed point theorem, see [22].

Motivated by the Hernández & Henríquez papers, recently several works reported existence results of mild solutions for different partial neutral differential problems related to the neutral equation (1) (see for example [2], [3], [4], [5], [6], [7], [8], [9], [23]). However, in these works their authors impose a severe compactness assumptions on the operator

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family generated by A, which imply that the underlying space X has finite dimension. As consequence, the equations treated in these works are really ordinary and not partial equations.

Our purpose in this paper is by mean of the study of the first order abstract Cauchy problem

$$\frac{d}{dt}(x(t)+g(t,x(t))) = Ax(t)+f(t,x(t)), \qquad t\in I=[0,a], \tag{3}$$

$$x(0) = x_0 \in X, \tag{4}$$

where A is the infinitesimal generator of analytic semigroup, give some ideas and techniques which can be used in the study of the existence of mild solution of partial neutral differential systems.

We remark, that the system system (3) is a generalization of the classical problems studied in [21] and [13] and that our results are applicable to partial neutral differential equations.

Along of this work, $A : D(A) \subset X \to X$ is the infinitesimal generator of an uniformly bounded analytic semigroup of linear operators, $(T(t))_{t\geq 0}$, on a Banach space X. Now we mention a few results and notations needed to establish our results. We will assume that $0 \in \rho(A)$ and that $\| T(t) \| \leq \tilde{M}$ for all $t \in I$. Under these conditions, it's possible to define the fractional power $(-A)^{\alpha}$, $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^{\alpha}$. Furthermore, $D(-A)^{\alpha}$ is dense in X and the function $\| x \|_{\alpha} = \| (-A)^{\alpha} x \|$ defines a norm in $D(-A)^{\alpha}$. If X_{α} is the space $D(-A)^{\alpha}$ endowed with the norm $\| \cdot \|_{\alpha}$, then the following properties hold, see [21].

Lemma 1.1. *Let $0 < \beta \leq \alpha \leq 1$. Under the previous conditions, the following properties hold:*

(i) *X_{α} is a Banach space and there exists $C_{\alpha} > 0$ such that $\| (-A)^{\alpha} T(t) \| \leq \dfrac{C_{\alpha}}{t^{\alpha}}$, for $t > 0$.*

(ii) *The inclusion $X_{\alpha} \to X_{\beta}$ is continuous.*

For additional details respect of semigroup theory, we refer to the reader to Pazy [21].

The terminologies and notations are those generally used in functional analysis. In particular, $C(I;X)$ is the space of the continuous function from I into X endowed with the topology of the uniform convergence and $B_r(x : Z)$ denotes the closed ball with center at x and radius $r > 0$ in a metric space Z. Additionally, for a bounded function $\xi : [0,a] \to [0,\infty)$ and $0 \leq t \leq a$ we will employ the notation ξ_t for

$$\xi_t = \sup\{\xi(s) : s \in [0,t]\}. \tag{5}$$

The paper has two sections. In section 2 we establish some existence results of mild solution of the system (3)-(4) using the following well known results.

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THEOREM 1.1. *[12, pp. 61] Let D be a convex subset of a Banach space X and assume that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the map F has a fix point in D or the set $\{x \in D : x = \lambda F(x),\ 0 < \lambda < 1\}$ is unbounded.*

THEOREM 1.2. *(Sadovskii [22]) Let D be a convex, closed and bounded subset of a Banach space X. If $F : D \to D$ is a condensing operator, then F has a fix point in D.*

2. EXISTENCE RESULTS

In this section we discuss the existence of mild solutions of the system (3)-(4). At first we introduce the following definition.

DEFINITION 2.1. A function $x : [0, a] \to X$ is a mild solution of (3)-(4), if $x(0) = x_0$; the function $s \to AT(t-s)g(s, x(s))$ is integrable on $[0, t)$ for each $0 \leq t < a$ and

$$\begin{aligned} x(t) \ &= \ T(t)(x_0 + g(0, x_0)) - g(t, x(t)) - \int_0^t AT(t-s)g(s, x(s))ds \\ &\quad + \int_0^t T(t-s)f(s, x(s))ds, \qquad t \in I. \end{aligned}$$

To establish our results we consider the following conditions.

$\mathbf{H_1}$ The function $f : I \times X \to X$ satisfies the following conditions:

(i) The function $f(t, \cdot) : X \to X$ is continuous $a.e.\ t \in I$.

(ii) For each $x \in X$, the function $f(\cdot, x) : I \to X$ is strongly measurable.

(iii) There exist a continuous function $m : I \to [0, \infty)$ and a continuous nondecreasing function $W : [0, \infty) \to (0, \infty)$ such that

$$\| f(t, x) \| \ \leq \ m(t)W(\| x \|), \qquad (t, x) \in I \times X.$$

$\mathbf{H_2}$ There exist constants $0 < \beta < 1$, c_1, c_2 such that g is X_β valued, $(-A)^\beta g$ is continuous and

$$\| (-A)^\beta g(t, x) \| \ \leq \ c_1 \| x \| + c_2, \qquad (t, x) \in I \times X.$$

$\mathbf{H_3}$ The set of functions $\{t \to g(t, x(t)) : x \in B_r(0, C(I : X))\}$ is equicontinuous on I for every $r > 0$.

THEOREM 2.1. *Let assumptions $\mathbf{H_1}$-$\mathbf{H_3}$ be satisfied and assume that*

(a) *For every $\epsilon > 0$ and all $r > 0$ there are compact sets $W^i_{\epsilon,r} \subset X$, $i = 1, 2$, such that $T(\epsilon)(-A)^\beta g(s, x) \in W^1_{\epsilon,r}$ and $T(\epsilon)f(s, x) \in W^2_{\epsilon,r}$ for every $(s, x) \in I \times B_r(0, X)$.*

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If $\mu = c_1(\| (-A)^{-\beta} \| + \frac{C_{1-\beta}a^{\beta}}{\beta}) < 1$ *and* $\frac{\tilde{M}}{1-\mu}\int_0^a m(s)ds < \int_c^{\infty} \frac{1}{W(s)}ds$ *where*

$$c(1-\mu) = \left[\tilde{M}(\| x_0 \| (1 + c_1 \| (-A)^{-\beta} \|) + c_2 \| (-A)^{-\beta} \|) + c_2(\| (-A)^{-\beta} \| + C_{1-\beta}\frac{a^{\beta}}{\beta})\right],$$

then there exists a mild solution of the abstract Cauchy problem (3)-(4).

Proof: Let $\Gamma : C(I : X) \to C(I : X)$ be the map defined by

$$\begin{aligned}\Gamma x(t) &= T(t)(x_0 + g(0, x_0)) - g(t, x(t)) - \int_0^t AT(t-s)g(s, x(s))ds \\ &\quad + \int_0^t T(t-s)f(s, x(s))ds, \qquad t \in I.\end{aligned}$$

From the assumptions, we know that the function $s \to (-A)^{\beta}g(s, x(s))$ is continuous. In addition, in view of the fact that $(T(t))_{t\geq 0}$ is an analytic semigroups (see [21]), the operator function $s \to AT(t-s)$ is continuous in the uniform operator topology on $[0, t)$ which from the estimate

$$\| (-A)T(t-s)g(s, x(s)) \| = \| (-A)^{1-\beta}T(t-s)(-A)^{\beta}g(s, x(s)) \| \leq \frac{C_{1-\beta}Cte}{(t-s)^{1-\beta}},$$

and the Bochner's Theorem implies that $\| AT(t-s)g(s, x(s)) \|$ is integrable on $[0, t)$. Now, it's clear that Γ is well defined and with values in $C(I : X)$. Moreover, from the Lebesgue dominated convergence theorem we infer that Γ is continuous.

In order to use the Theorem 1.1, we obtain *a priori* estimates for the solutions of the integral equation $x = \lambda\Gamma x$, $\lambda \in (0, 1)$. Let $\lambda \in (0, 1)$ and $x^{\lambda}(\cdot)$ be a solution of $x = \lambda\Gamma x$. From the definition of Γ we get

$$\begin{aligned}\| x^{\lambda}(t) \| &\leq \tilde{M}(\| x_0 \| (1 + \| (-A)^{-\beta} \| c_1) + c_2 \| (-A)^{-\beta} \|) + c_1 \| (-A)^{-\beta} \| \| x^{\lambda}(t) \| \\ &\quad + c_2 \| (-A)^{-\beta} \| + C_{1-\beta}c_1 \int_0^t \frac{\| x^{\lambda} \|_s}{(t-s)^{1-\beta}}ds + \frac{C_{1-\beta}c_2a^{\beta}}{\beta} \\ &\quad + \tilde{M}\int_0^t m(s)W(\| x^{\lambda} \|_s)ds,\end{aligned}$$

where the notation introduced in (5) has been used. Consequently,

$$\begin{aligned}\| x^{\lambda} \|_t &\leq \tilde{M}(\| x_0 \| (1 + \| (-A)^{-\beta} \| c_1) + c_2 \| (-A)^{-\beta} \|) + c_2(\| (-A)^{-\beta} \| + \frac{C_{1-\beta}a^{\beta}}{\beta}) \\ &\quad + \mu \| x^{\lambda} \|_t + \tilde{M}\int_0^t m(s)W(\| x^{\lambda} \|_s)ds,\end{aligned}$$

and then

$$\| x^{\lambda} \|_t \leq c + \frac{\tilde{M}}{1-\mu}\int_0^t m(s)W(\| x^{\lambda} \|_s)ds, \qquad t \in I.$$

Designating by $\beta^{\lambda}(t)$ the right hand side of the above expression follows that

$$\beta'_{\lambda}(t) \leq \frac{\tilde{M}}{1-\mu} m(t) W(\beta_{\lambda}(t))$$

and so that

$$\int_{c}^{\beta_{\lambda}(t)} \frac{ds}{W(s)} \leq \frac{\tilde{M}}{1-\mu} \int_{0}^{t} m(s)ds < \int_{c}^{\infty} \frac{ds}{W(s)},$$

which imply that the functions $\beta^{\lambda}(\cdot)$, $\lambda \in (0,1)$, are uniformly bounded on I. Thus, the functions $x^{\lambda}(\cdot)$, $\lambda \in (0,1)$, are uniformly bounded on I.

Next we prove that Γ is completely continuous. To this end, by using the relation $A\int_{0}^{t} T(s)xds = T(t)x - x$, we rewrite Γ in the form $\Gamma = \sum_{i=1}^{3} \Gamma_i$ where

$$\begin{aligned}
\Gamma_1 x(t) &= T(t)(x_0 + g(0,x_0) - g(t,x(t))), \\
\Gamma_2 x(t) &= -\int_{0}^{t} AT(t-s)g(s,x(s))ds, \\
\Gamma_3 x(t) &= \int_{0}^{t} T(t-s)f(s,x(s))ds + \int_{0}^{t} AT(t-s)g(t,x(t))ds.
\end{aligned}$$

From the Lebesgue dominated convergence theorem and the conditions $\mathbf{H_1}, \mathbf{H_2}$, follows that each Γ_i is continuous. In the sequel we prove that $\Gamma_i(B_r(0, C(I:X)))$ is relatively compact in $C(I:X)$ for every $i = 1,..,3$. In what follow $B_r = B_r(0, C(I:X))$.

Step 1. The set $\Gamma_1(B_r)$ is relatively compact in $C(I:X)$.

Firstly we study the equicontinuity of $\Gamma_1(B_r)$. Let $0 < \epsilon < t_0 < t \leq a$ and $W^1_{\epsilon,r}$ be the compact set in (**a**). Using the condition $\mathbf{H_3}$ and the fact that set of functions $\{s \to T(s)x : x \in W^1_{\epsilon,r}\}$ is equicontinuous on $[0,a]$, we can choose $0 < \delta < \epsilon$ such that

$$\begin{aligned}
\| T(t)x - T(t_0)x \| &< \epsilon, \qquad x \in W^1_{\epsilon,r}, \\
\| g(t,x(t)) - g(t_0,x(t_0)) \| &< \epsilon, \qquad x(\cdot) \in B_r,
\end{aligned}$$

when $| t - t_0 |< \delta$. Under these conditions, for $x(\cdot) \in B_r$ and $| t - t_0 |< \delta$ we get

$$\begin{aligned}
\| \Gamma_1 x(t_0) - \Gamma_1 x(t) \| &\leq \| (T(t_0) - T(t))(x_0 + g(0,x_0)) \| \\
&\quad + \| T(t_0) \| \| g(t_0,x(t_0)) - g(t,x(t)) \| \\
&\quad + \| (-A)^{-\beta}(T(t_0-\epsilon) - T(t-\epsilon))T(\epsilon)(-A)^{\beta} g(t,x(t)) \| \\
&\leq \| (T(t_0) - T(t))(x_0 + g(0,x_0)) \| + (\| T(t_0) \| + \| (-A)^{-\beta} \|)\epsilon,
\end{aligned}$$

which imply that $\Gamma_1(B_r)$ is equicontinuous from the right hand side at t_0. Similarly we can prove the equicontinuity from the left hand side at t_0 and the right equicontinuity at zero. Thus, the set $\Gamma_1(B_r)$ is equicontinuous on I.

It's clear that $\Gamma_1(B_r)(0)$ is compact in X. Moreover, if $0 < \epsilon < t$, we get

$$\Gamma_1(B_r)(t) = \{\Gamma_1 u(t) : u \in B_r\} \ \subset \ T(t)(x_0 + g(0, x_0)) + (-A)^{-\beta} T(t-\epsilon) W^1_{\epsilon,r},$$

which proves that $\Gamma_1(B_r)(t)$ is relatively compact in X since $(-A)^{-\beta}T(t-\epsilon)$ is continuous. Thus, the set $\Gamma_1(B_r)$ is relatively compact in $C(I : X)$.

Step 2. The set $\Gamma_2(B_r)$ is relatively compact in $C(I : X)$

At first we prove that $\Gamma_2(B_r)(t) = \{\Gamma_2 u(t) : u \in B_r\}$ is relatively compact in X for every $t \in I$. The case $t = 0$ is trivial. Let $0 < \epsilon < t \leq a$ and $W^1_{\frac{\epsilon}{2},r}$ be the compact set in (**a**). Since the function $t \to (-A)^{\beta}T(t)$ is continuous in the uniform convergence topology on $[\frac{\epsilon}{2}, a]$, follows that the set $U_\epsilon = \{(-A)^{1-\beta}T(s)x : s \in [\frac{\epsilon}{2}, a],\ x \in W^1_{\frac{\epsilon}{2},r}\}$ is precompact in X. If $x(\cdot) \in B_r$, from the mean value theorem for the Bochner integral, see [20] pp. 25, we get

$$\begin{aligned}
\Gamma_2 x(t) \ &= \ \int_0^{t-\epsilon} (-A)^{1-\beta} T(t-s-\frac{\epsilon}{2}) T(\frac{\epsilon}{2}) (-A)^{\beta} g(s, x(s)) ds \\
&\quad + \int_{t-\epsilon}^{t} (-A)^{1-\beta} T(t-s) (-A)^{\beta} g(s, x(s)) ds \\
&\in \ (t-\epsilon)\overline{co(U_\epsilon)} + C_\epsilon,
\end{aligned}$$

where $co(U_\epsilon)$ denote the convex hull of U_ϵ and $diam(C_\epsilon) \leq 2C_{1-\beta}(c_1 r + c_2)\dfrac{\epsilon^\beta}{\beta}$. These remarks permit infer that $\Gamma_2(B_r)(t)$ is relatively compact in X for every $t \in I$.

Now we prove that $\Gamma_2(B_r)$ is equicontinuous on I. Let $0 \leq t_0 \leq t \leq a$ and $x \in B_r$. Then

$$\begin{aligned}
\| \Gamma_2 x(t) - \Gamma_2 x(t_0) \| \ &= \ \| \int_0^t AT(t-s) g(s, x(s)) ds - \int_0^{t_0} AT(t_0-s) g(s, x(s)) ds \| \\
&= \ \| (T(t-t_0) - I)\Gamma_2 x(t_0) \| + \int_{t_0}^{t} \| AT(t-s) g(s, x(s)) \| \, ds \\
&\leq \ \| (T(t-t_0) - I)\Gamma_2 x(t_0) \| + C_{1-\beta}(c_1 r + c_2)\frac{(t-t_0)^\beta}{\beta}.
\end{aligned}$$

Since $\Gamma_2(B)(t_0)$ is relatively compact, it is clear that the first term on the right hand side converges to zero when $t \to t_0$ uniformly on B_r. This shows that $\Gamma_2(B_r)$ is equicontinuous from the right hand side at t_0. Similarly we can prove that $\Gamma_2(B_r)$ is equicontinuous from the left hand side at t_0 and equicontinuous at $t = 0$. Thus, $\Gamma_2(B_r)$ is equicontinuous on $[0, a]$ and relatively compact in $C(I : X)$.

THEOREM 2.2. *Assume that* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *are verified and that the following conditions hold.*

(**a**)*For each* $\epsilon > 0$ *and all* $r > 0$ *there is a compact* $W_{\epsilon,r} \subset X$ *such that* $T(\epsilon)f(s, x) \in W_{\epsilon,r}$ *for every* $(s, x) \in I \times B_r(0, X)$.

(b) *There exist constants $0 < \beta < 1$ and $L_g > 0$ such that such that g is X_β-valued and*

$$\| (-A)^\beta g(t, x_1) - (-A)^\beta g(t, x_2) \| \leq L_g \| x_1 - x_2 \|, \qquad (t, x_i) \in I \times X.$$

If $L_g(\| (-A)^{-\beta} \| + \frac{C_{1-\beta} a^\beta}{\beta}) + \liminf_{r \to +\infty} \frac{W(r)}{r} \int_0^a m(s)ds < 1$, then there exists a mild solution of (3)-(4).

¶ Let $\Gamma : C(I : X) \to C(I : X)$ be the map defined in the proof of Theorem 2.1 and consider the decomposition $\Gamma = \tilde{\Gamma}_1 + \tilde{\Gamma}_2$ where

$$\begin{aligned} \tilde{\Gamma}_1 x(t) &= T(t)(x_0 + g(0, x_0)) - g(t, x(t)) - \int_0^t AT(t-s) g(s, x(s)) ds, \qquad t \in I, \\ \tilde{\Gamma}_2 x(t) &= \int_0^t T(t-s) f(s, x(s)) ds, \qquad t \in I. \end{aligned}$$

We affirm that there exists $r > 0$ such that $\Gamma(B_r(0, C(I : X))) \subset B_r(0, C(I : X))$. In fact, if we assume that the assertion is false, then for every $r > 0$ there exists $x^r \in B_r(0, C(I : X))$ and $t^r \in I$ such that $r < \| \Gamma x^r(t^r) \|$. This yields that

$$\begin{aligned} r < \| \Gamma x^r(t^r) \| &\leq \tilde{M} \| x_0 + g(0, x_0) \| + \| (-A)^{-\beta} \| L_g r + \| g(t^r, 0) \| + \frac{L_g C_{1-\beta} a^\beta r}{\beta} \\ &+ \int_0^{t^r} \frac{C_{1-\beta} \| g(s, 0) \|}{(t^r - s)^{1-\beta}} ds + \int_0^{t^r} m(s) W(\| x^r(s) \|) ds, \end{aligned}$$

and so that

$$1 \leq L_g(\| (-A)^{-\beta} \| + \frac{C_{1-\beta} a^\beta}{\beta}) + \liminf_{r \to +\infty} \frac{W(r)}{r} \int_0^a m(s) ds,$$

which is contradictory with our assumptions.

Let $r_0 > 0$ such that $\Gamma(B_{r_0}(0, C(I : X)) \subset B_{r_0}(0, C(I : X))$. It follows from the proof of Theorem 2.1 that $\tilde{\Gamma}_2$ is completely continuous on $B_{r_0}(0, C(I : X))$ and from the estimate

$$\| \tilde{\Gamma}_1 x - \tilde{\Gamma}_1 y \|_a \leq L_g(\| (-A)^{-\beta} \| + \frac{C_{1-\beta} a^\beta}{\beta}) \| x - y \|_a,$$

that $\tilde{\Gamma}_1$ is a contraction on $B_{r_0}(0, C(I : X))$. Now, the assertion is consequence of the Sadovskii point fixed Theorem. ■

Arguing as in step 2, it follows that $\Gamma_3(B_r)$ is relatively compact $C(I : X)$.

These remarks and the Theorem 1.1 assert that there exists a mild solution of the problem (3). The proof is complete. ■

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