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Topological dynamics of retarded functional differential equations

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We prove that a local flow can be constructed for a general class of nonautonomous retarded functional differential equations (RFDE). This is an extension to a result of Artstein [1] and fits in the classical theory of R. Miller and G. Sell. The main tool in this paper are generalized ordinary differential equations according to Kurzweil [7]. In obtaining our results we must prove the space of RFDE's can be embedded in a space of generalized ordinary differential equations. In opposition to the technical hypotheses of Oliva and Vorel [15], this auxiliary result, as we present, is advantageous in the sense that our assumptions have an explanatory character. Applications based on topological dynamics techniques follow naturally from our results. As an illustration of this fact we show how to achieve in this setting a theorem on continuous dependence on initial data of solutions of RFDE's. May, 2003 ICMC-USP

Key Words: Retarded differential equations, Topological dynamics, Local flow

1. INTRODUCTION

In order to generalize certain results on continuous dependence of solutions of ordinary differential equations (ODE) with respect to parameters, J. Kurzweil introduced, in 1957, what he called generalized ordinary differential equations (GODE) for euclidean and Banach space-valued functions (see [7] - [11]).

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Among other applications, the theory of GODE's has been shown to be useful in the investigation of topological dynamics of ODE's of type

$$\dot{x} = f(x,t), \quad \left(\dot{x} = \frac{dx}{dt}\right), \tag{1.1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}^n$, with $\Omega \subset \mathbb{R}^n$ open, is measurable in t and continuous in x. As usual, given $s \in \mathbb{R}$, the translate f_s of f is defined by

$$f_s(x,t) = f(x,t+s); \quad x \in \Omega,\ t \in \mathbb{R}.$$

It is known the limit points of f_s, when $|s| \to \infty$, define limiting equations and one can relate the behavior of the solutions of (1.1) with the behavior of the solutions of its limiting equations. However it may happen that a limiting equation of an ODE is no longer an ODE. In fact, as observed by Artstein in [2], when we consider the topology characterized by the convergence

$$f_k \to f_0, \qquad \text{if} \quad \int_0^t f_k(x,s)ds \to \int_0^t f_0(x,s)ds, \ \ (x,t) \in \Omega \times \mathbb{R}$$

in a space of functions $f(x,t)$ satisfying local Lipchitz- and Charathéodory-type conditions, the following situation can be considered: let F_0 be a continuous nowhere differentiable function. Since C^1, endowed with the C^0 topology, is dense in C^0, there exists a sequence F_j, $j = 1, 2, \ldots$, of C^1 functions uniformly convergent to F_0. If f_j is the derivative of F_j, then $\int_0^t f_j(x,s)ds$ converges for all (x,t), but the limit

$$F_0(x,t) = \lim \int_0^t f_j(x,s)ds$$

does not have an integral representation as $F_0(x,t) = \int_0^t f(x,s)ds$. Since we can associate an equation of type (1.1) with each f_j, it turns out that the solution of the equation is, up to a constant, the primitive F_j of f_j. Therefore the limiting solution F_0 is not a solution of an ODE.

A GODE is defined in terms of a primitive of the ODE (1.1), that is, $F(x,t) = \int_0^t f(x,s)ds$, but it also applies even if F is not a primitive. In [2], Artstein proved that, when a class of ODE's of type (1.1) is embedded in a compact space of GODE's, a local flow can be constructed and hence the techniques of topological dynamics can be applied. We are concerned with this subject in the setting of retarded functional differential equations (RFDE).

In [15], F. Oliva and Z. Vorel proved that RFDE's can be regarded as Banach space-valued GODE's. In the present paper, we prove the same fact assuming different conditions. In opposition to [15], our assumptions are self-explanatory. Moreover, neglecting the delay, they are weaker than those assumed by Artstein in [2] and still they are enough for the construction of a local flow.

Since the starting point of many results of [17] and [18] is the existence of a local flow, it is natural that we accomplish some applications. To show this, we present some immediate results on continuous dependence of solutions.

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This paper is organized as follows. We begin by giving the basic definitions and properties of the theory of GODE's for Banach space-valued functions in Section 2. In the same section we state the relation between RFDE's and GODE's in Banach spaces. Section 3 is dedicated to defining a space of GODE's in which the space of RFDE's will be embedded. Then the proof that the former space is compact is a straightforward adaptation of [2]. The construction of a local flow is discussed in Section 4. Applications to continuous dependence of solutions are given in Section 5.

2. GENERALIZED EQUATIONS IN BANACH SPACES

2.1. Basic definitions and properties

A *tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection (τ_i, s_i), where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a,b]$ and $\tau_i \in [s_{i-1}, s_i]$, $i = 1, 2, \ldots, k$. A *gauge* of a set $E \subset [a,b]$ is any function $\delta : E \to \,]0, +\infty[$. Given a gauge δ of $[a,b]$, a tagged division $d = (\tau_i, s_i)$ is *δ-fine* if, for every i,

$$[s_{i-1}, s_i] \subset \{t \in [a,b]\,;\, |t - \tau_i| < \delta(\tau_i)\}\,.$$

Alternatively we can write $d = (\tau_i, [s_{i-1}, s_i])$ instead of $d = (\tau_i, s_i)$.

Let X be a Banach space.

Definition 2.2.1. *A function $U(\tau, t) : [a,b] \times [a,b] \to X$ is* Kurzweil integrable *if there is a unique element $I \in X$ ($I = (K)\int_{[a,b]} DU(\tau,t)$) such that given $\varepsilon > 0$, there is a gauge δ of $[a,b]$ such that for every δ-fine tagged division $d = (\tau_i, s_i)$ of $[a,b]$,*

$$\|S(U,d) - I\| < \varepsilon,$$

where $S(U,d) = \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})]$.

Remark 2.2.1. *If the integral $(K)\int_{[a,b]} DU(\tau,t)$ exists, we define*

$$(K)\int_{[b,a]} DU(\tau,t) = -(K)\int_{[a,b]} DU(\tau,t)\,.$$

Note that although $(K)\int_{[a,b]} DU(\tau,t)$ exists, $DU(\tau,t)$ does not need to be defined.

A more restricted integral based on McShane's Riemann-type integral ([12]) is defined below. Instead of tagged divisions of $[a,b]$, McShane considered what we call semi-tagged divisions of $[a,b]$. A *semi-tagged division* of a compact interval $[a,b] \subset \mathbb{R}$ is a finite collection (τ_i, s_i), where $a = s_0 \leq s_1 \leq \ldots \leq s_k = b$ is a division of $[a,b]$ and $\tau_i \in [a,b]$, $i = 1, 2, \ldots, k$. Note that it is not required that $\tau_i \in [s_{i-1}, s_i]$ for any i. Then, given a gauge δ of $[a,b]$, a semi-tagged division $d = (\tau_i, s_i)$ is *δ-fine* if, for every i,

$$[s_{i-1}, s_i] \subset \{t \in [a,b]\,;\, |t - \tau_i| < \delta(\tau_i)\}\,.$$

When this elegant modification is applied to Kurzweil's definition, we obtain an integral known as Kurzweil-McShane or McShane integral. We have

Definition 2.2.2. *A function* $U(\tau,t):[a,b]\times[a,b]\to X$ *is* McShane integrable *if there is a unique element* $I\in X$ $\left(I=\int_{[a,b]}DU(\tau,t)\right)$ *such that given* $\varepsilon>0$*, there is a gauge* δ *of* $[a,b]$ *such that for every* δ*-fine semi-tagged division* $d=(\tau_i,s_i)$ *of* $[a,b]$*,*

$$\|S(U,d)-I\|<\varepsilon,$$

where $S(U,d)=\sum_i[U(\tau_i,s_i)-U(\tau_i,s_{i-1})]$.

In the real-valued case, the McShane integral gives a constructive definition for the Lebesgue integral (see [12] and [3]) which means that the McShane and Lebesgue integrals are equivalent and the integrals coincide when defined. The reader may want to consult [5] for special aspects of such integrals for Banach space-valued functions. See also [4].

The lemma presented below is known as Saks-Henstock Lemma and it can be proved by following the standard steps as in [16], Proposition 16, for instance. In the following section, the Saks-Henstock lemma will be crucial in simplifying the proof of the fact that RFDE's can be regarded as GODE's in Banach spaces (see [15] for comparison).

Lemma 2.2.1 (Saks-Henstock Lemma). *Let* $U(\tau,t):[a,b]\times[a,b]\to X$*. If for every* $\varepsilon>0$*,* δ *is a gauge of* $[a,b]$ *such that for every* δ*-fine semi-tagged division* $d=(\tau_i,s_i)$ *of* $[a,b]$*,*

$$\left\|\sum_i[U(\tau_i,s_i)-U(\tau_i,s_{i-1})]-\int_{[a,b]}DU(\tau,t)\right\|<\varepsilon$$

then, for $a\le c_1\le\eta_1\le d_1\le c_2\le\eta_2\le d_2\le\ldots\le c_l\le\eta_l\le d_l\le b$*, with* $\eta_j\in[c_j,d_j]\subset[\eta_j-\delta(\eta_j),\eta_j+\delta(\eta_j)]$*,* $j=1,2,\ldots,l$*,*

$$\left\|\sum_j\left[U(\eta_j,d_j)-U(\eta_j,c_j)-\int_{[c_j,d_j]}DU(\tau,t)\right]\right\|<\varepsilon.$$

Corollary 2.2.1. *Given* $\varepsilon>0$*, let* δ *be the gauge of* $[a,b]$ *from Definition 2.2.2. Let* $[t_1,t_2]\subset[a,b]$*. Then*
(i) $(t_2-t_1)<\delta(t_1)$ *implies*

$$\left\|U(t_1,t_2)-U(t_1,t_1)-\int_{[t_1,t_2]}DU(\tau,t)\right\|<\varepsilon;$$

(ii) $(t_2-t_1)<\delta(t_2)$ *implies*

$$\left\|U(t_2,t_2)-U(t_2,t_1)-\int_{[t_1,t_2]}DU(\tau,t)\right\|<\varepsilon.$$

Proof. (i) follows from the fact that $(t_1, [t_1, t_2])$ is a δ-fine tagged division of $[t_1, t_2]$ and the Saks-Henstock Lemma. A similar argument applies to (ii). ∎

Remark 2.2.2. *Lemma 2.2.1 and Corollary 2.2.1 also hold if the McShane integral is replaced by the Kurzweil integral.*

The proof of the next result follows the steps of [2], Lemma A.1, with appropriate adaptations for Banach space-valued functions.

Lemma 2.2.2. *Suppose $U(\tau, t) : [a, b] \times [a, b] \to X$ is continuous in t for each τ. If $\int_{[a,b]} DU(\tau, t)$ exists, then for every $t_1 < t_2$ in $[a, b]$, $\int_{[t_1,t_2]} DU(\tau, t)$ exists. Moreover, $\int_{[a,s]} DU(\tau, t)$ is continuous in s.*

Let $X_1 \subset X$ be open and $F(x, t) : X_1 \times]T_1, T_2[\to X$ be a function.

Definition 2.2.3. *A function $x(t)$ defined in $]t_1, t_2[\subset]T_1, T_2[$ is a solution of the generalized ordinary differential equation (GODE)*

$$\frac{dx}{d\tau} = DF(x, t) \tag{2.2}$$

in the interval $]t_1, t_2[$, with initial condition $x(t_0) = x_0 \in X_1$, $t_0 \in]t_1, t_2[$, if for every $t_3, t_4 \in]t_1, t_2[$,

$$x(t_4) - x(t_3) = \int_{[t_3,t_4]} DF(x(\tau), t).$$

Remark 2.2.3. *Let $U(\tau, t) = F(x(\tau), t)$. In the definition of $\int_{[a,b]} DF(x(\tau), t)$ there are only differences as*

$$U(\tau_i, s_i) - U(\tau_i, s_{i-1}) = F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}).$$

Thus, adding to $F(x, t)$ a function varying only in x, the solutions of (2.2) do not change. In particular, subtracting $F(x, 0)$ from $F(x, t)$, we obtain a normalized representation F_1 *of F fulfilling $F_1(x, 0) = 0$ for every x.*

2.2. The relation with retarded functional differential equations

Let r, a, σ be non-negative numbers. By $C(a, r, \sigma)$ we mean the space of all continuous functions from $[a - r, a + \sigma]$ to $\mathbb{R}^n$ and by $C(r)$ we mean the space of continuous functions from $[-r, 0]$ to $\mathbb{R}^n$; both spaces are equipped with the usual supremum topology. For every $x \in C(a, r, \sigma)$ and every $a \le t \le a + \sigma$, let $x_t \in C(r)$ be given by

$$x_t(\theta) = x(t + \theta), \ \theta \in [-r, 0].$$

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Let $C_1 \subset C(a,r,\sigma)$ be an open set with the following property: if $x = x(t)$, $t \in [a-r, a+\sigma]$, is an element of C_1 and $\bar{t} \in [a-r, a+\sigma]$, then $\bar{x}$ given by

$$\bar{x}(t) = \begin{cases} x(t), & a - r \le t \le \bar{t} \\ x(\bar{t}), & \bar{t} \le t \le a + \sigma \end{cases}$$

is also an element of C_1. In particular, any open ball in $C(a,r,\sigma)$ has this property.

Let $H_1 \subset C(r)$ be such that $\{x_t \mid t \in [a, a+\sigma], x \in C_1\} \subset H_1$ and consider the following RFDE:

$$\dot{x}(t) = f(x_t, t), \quad \left(\dot{x} = \frac{dx}{dt}\right), \tag{2.3}$$

where $f(\varphi, t) : H_1 \times [a, a+\sigma] \to \mathbb{R}^n$, $t \mapsto f(x_t, t)$ is Lebesgue integrable and the following conditions are fulfilled:

(A) for each compact $A \subset C_1$, there is a locally Lebesgue integrable function $M_A(t)$ such that $x \in A$ implies

$$\left| \int_{[t_1,t_2]} f(x_s, s)\, ds \right| \le \int_{[t_1,t_2]} M_A(s)\, ds,$$

where $t_1, t_2 \in [a, a+\sigma]$ and $\int_{[t,t+h]} M_A(s)\, ds$ is uniformly continuous in t, $t \in [a, a+\sigma]$;

(B) for each compact $A \subset C_1$, there is a locally Lebesgue integrable function $L_A(t)$ such that $x_1, x_2 \in A$ implies

$$\left| \int_{[t_1,t_2]} \left[f\left((x_1)_s, s\right) - f\left((x_2)_s, s\right) \right] ds \right| \le \int_{[t_1,t_2]} L_A(s) \left\| (x_1)_s - (x_2)_s \right\| ds,$$

where $t_1, t_2 \in [a, a+\sigma]$ and $\int_{[t,t+1]} L_A(s)\, ds \le N_A$ for every t and for $N_A \le 1$ fixed.

Notice that conditions (A) and (B) above are, respectively, Carathéodory and Lipschitz-types conditions with respect to the McShane integral of a certain function f. Usually these types of conditions are imposed to the function itself rather to its integral (see [2], p. 226). Thus, even when we neglect the delay and deal with ODE's only, it is reasonable to expect more general results.

Definition 2.2.4. *Let $(\varphi, a) \in H_1 \times [a, a+\sigma]$. If there is a function $x \in C(a,r,\sigma)$ such that $(x_t, t) \in H_1 \times [a, a+\sigma]$ for $a \le t < a+\sigma$ and moreover*
(i) $x_a = \varphi$ and
(ii) $\dot{x}(t) = f(x_t, t)$, $a \le t < a+\sigma$,
then x is a solution of (2.3) in $]a, a+\sigma[$ with initial condition (a, φ).

Let $f(x_t, t)$ satisfy the conditions required for (2.3). For each $x \in C_1$, define

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$$F(x,t)(\tau)=\begin{cases}\int_{[a,\tau]} f(x_s,s)\,ds, & a\le\tau\le t\le a+\sigma;\\ \int_{[a,t]} f(x_s,s)\,ds, & a\le t\le\tau\le a+\sigma;\\ 0, & a-r\le\tau\le a \text{ ou } a-r\le t\le a.\end{cases} \tag{2.4}$$

Then given (x,t), (2.4) defines an element $F(x,t)$ of $C(a,r,\sigma)$ and $F(x,t)(\tau)\in\mathbb{R}^n$ is the value of $F(x,t)$ at a point $\tau\in[a,a+\sigma]$.

PROPOSITION 2.2.1. *Under the conditions above, $F(x,t)$ given by (2.4) is continuous on $C_1\times[a-r,a+\sigma]$.*

Proof. Let $A\subset C_1$ be compact and suppose $(x_1,t_1),(x_2,t_2)\in A\times[a-r,a+\sigma]$ are such that $\|x_1-x_2\|$ and $|t_1-t_2|$ are small enough, with $t_1\le t_2$. Hence,

$$\left|\int_{[a,t_1]} f((x_1)_s,s)\,ds-\int_{[a,t_2]} f((x_2)_s,s)\,ds\right|\le$$

$$\le\left|\int_{[a,t_1]}[f((x_1)_s,s)-f((x_2)_s,s)]\,ds\right|+\left|\int_{[t_1,t_2]} f((x_2)_s,s)\,ds\right|\le$$

$$\le\int_{[t_1,t_2]} L_A(s)\,\|(x_1)_s-(x_2)_s\|\,ds+\int_{[t_1,t_2]} M_A(s)\,ds$$

which can be made sufficiently small by properties (A) and (B) of f. Then the continuity of $F(x,t)$ follows. ∎

The next two theorems (Theorems 2.2.1 and 2.2.2) imply equation (2.3) is a special case of certain GODE's in Banach spaces. This fact was first proved by Oliva and Vorel in [15] under different conditions. We follow the main ideas of [15], but the use of the Saks-Henstock Lemma allowed us to contribute with more simple proofs.

THEOREM 2.2.1. *Let $y(t)$ be a solution of (2.3) in the interval $[t_1,t_2]\subset[a,a+\sigma]$. Given $t\in[a-r,a+\sigma]$, let*

$$x(t)(\tau)=\begin{cases} y(\tau), & \tau\in[a-r,t]\\ y(t), & \tau\in[t,a+\sigma].\end{cases} \tag{2.5}$$

Then $x(t)\in C(a,r,\sigma)$ is a solution of the GODE (2.2) in $[t_1,t_2]$.

Proof. Let $[t_3,t_4]\subset[t_1,t_2]$. We will show that $\int_{[t_3,t_4]} DF(x(\tau),t)$ exists and

$$x(t_4)-x(t_3)=\int_{[t_3,t_4]} DF(x(\tau),t).$$

In view of Definition 2.2.2 and Remark 2.2.3, it is enough to prove that for every $\varepsilon > 0$, there is a gauge δ of $[t_3, t_4]$ such that if $d = (\tau_i, s_i)$ is a δ-fine semi-tagged division of $[t_3, t_4]$ such that $(s_i - s_{i-1}) \leq 1$, for every i, then

$$\left\| x(t_4) - x(t_3) - \sum_i \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \right] \right\| < \varepsilon. \tag{2.6}$$

For each i, it follows from (2.4) that

$$[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})](\tau) = \begin{cases} 0, \ \tau \in [a-r, s_{i-1}] \\ \int_{[s_{i-1},\tau]} f\left(x(\tau_i)_s, s\right) ds, \ \tau \in [s_{i-1}, s_i] \\ \int_{[s_{i-1},s_i]} f\left(x(\tau_i)_s, s\right) ds, \ \tau \in [s_i, a+\sigma]. \end{cases} \tag{2.7}$$

Since $y(t)$ is a solution of (2.3), we have

$$y(u_2) - y(u_1) = \int_{[u_1,u_2]} f(y_s, s)\, ds,$$

for arbitrary $u_1, u_2 \in [t_1, t_2]$. Then by (2.5),

$$[x(s_i) - x(s_{i-1})](\tau) = \begin{cases} 0, \ \tau \in [a-r, s_{i-1}] \\ \int_{[s_{i-1},\tau]} f(y_s, s)\, ds, \ \tau \in [s_{i-1}, s_i] \\ \int_{[s_{i-1},s_i]} f(y_s, s)\, ds, \ \tau \in [s_i, a+\sigma]. \end{cases} \tag{2.8}$$

Thus (2.7) and (2.8) imply

$$[x(s_i) - x(s_{i-1}) - F(x(\tau_i), s_i) + F(x(\tau_i), s_{i-1})](\tau) =$$

$$= \begin{cases} 0, \ \tau \in [a-r, s_{i-1}] \\ \int_{[s_{i-1},\tau]} \left[f(y_s, s) - f\left(x(\tau_i)_s, s\right)\right] ds, \ \tau \in [s_{i-1}, s_i] \\ \int_{[s_{i-1},s_i]} \left[f(y_s, s) - f\left(x(\tau_i)_s, s\right)\right] ds, \ \tau \in [s_i, a+\sigma]. \end{cases} \tag{2.9}$$

Let $A \subset C_1$ be a compact neighborhood of y such that the distance ρ of $y \in C_1$ to the complement of A is larger than $2\|y\|$. Then for all i,

$$|y(\tau) - x(\tau_i)(\tau)| = \begin{cases} 0, \ \tau \in [a-r, \tau_i] \\ |y(\tau) - y(\tau_i)|, \ \tau \in [\tau_i, a+\sigma] \end{cases}$$

and therefore $x(\tau_i) \in A$ for every i, since

$$\|y - x(\tau_i)\| \leq 2\|y\| < \rho.$$

For an arbitrary i, let $\tau \in [s_{i-1}, s_i]$. Thus condition (B) implies there is a locally Lebesgue integrable function $L_A(s)$ such that

$$\left\| \int_{[s_{i-1},\tau]} \left[f(y_s, s) - f\left(x(\tau_i)_s, s\right)\right] ds \right\| \leq \int_{[s_{i-1},\tau]} L_A(s) \left\|y_s - x(\tau_i)_s\right\| ds.$$

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We assert that $\|y_s - x(\tau_i)_s\| \le \varepsilon/i$ which then implies

$$\left\| \int_{[s_{i-1},\tau]} \left[f(y_s, s) - f(x(\tau_i)_s, s) \right] ds \right\| \le \frac{\varepsilon}{i} N_A, \tag{2.10}$$

by (B). Since

$$x(\tau_i)(\nu) = \begin{cases} y(\nu), \ \nu \in [a-r, \tau_i] \\ y(\tau_i), \ \nu \in [\tau_i, a+\sigma] \end{cases}$$

and s varies in $[s_{i-1}, \tau]$, we have

$$\|y_s - x(\tau_i)_s\| = \sup_{\nu \in [s-r,s]} |y(\nu) - x(\tau_i)(\nu)| \le \sup_{\nu \in [s_{i-1}-r,\tau]} |y(\nu) - x(\tau_i)(\nu)| =$$

$$= \sup_{\nu \in [s_{i-1}-r,\tau]} \begin{cases} 0, \ \nu \in [a-r, \tau_i] \\ |y(\nu) - y(\tau_i)|, \ \nu \in [\tau_i, a+\sigma] \end{cases} = I.$$

But for $\nu \in [\tau_i, a+\sigma]$,

$$|y(\nu) - x(\tau_i)(\nu)| = \left| \int_{[\tau_i,\nu]} f(y_t, t)\, dt \right| \le \int_{[\tau_i,\nu]} M_A(t)\, dt,$$

where $M_A(t)$ is a locally Lebesgue integrable function obtained from condition (A). Hence

$$I \le \int_{[\tau_i,\tau]} M_A(t)\, dt.$$

Since $\int_{[t,t+h]} M_A(s)\, ds$ is uniformly continuous in t, then there is a positive function $\mu_A(\varepsilon)$ such that $\int_{[t,t+h]} M_A(s)\, ds \le \varepsilon$, whenever $|h| \le \mu_A(\varepsilon)$. But we can refine the gauge δ, without loss of generality, in the following manner: take δ such that it refines the gauge of $[t_3, t_4]$ corresponding to $\varepsilon > 0$ in the riemannian definition of the Lebesgue integral $\int_{[t_3,t_4]} M_A(s)\, ds$ (see [12]) and such that $\delta(\tau_i) \le \mu_A(\varepsilon)$ for every i. Notice that the compact A does not depend on the choice of i. In this manner we can take $\int_{[\tau_i,\tau]} M_A(t)\, dt \le \varepsilon/i$ and the assertion is proved. Hence (2.10) follows.

It can be proved similarly, with adequate adaptations, that given i, if $\tau \in [s_i, a+\sigma]$, then

$$\left\| \int_{[s_{i-1},s_i]} \left[f(y_s, s) - f(x(\tau_i)_s, s) \right] ds \right\| \le \frac{\varepsilon}{i} N_A.$$

Then, adding all terms in i in (2.9), we obtain

$$\left| \left[x(t_4) - x(t_3) - \sum_i \left[F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1}) \right] \right] (\tau) \right| \le$$

$$\leq N_A \sum_i \frac{\varepsilon}{i} = \varepsilon \cdot N_A \leq \varepsilon$$

and hence (2.6) follows. ∎

Theorem 2.2.2. *Let $x(t)$ be a solution of (2.2), with F given by (2.4), in the interval $[t_1, t_2] \subset [a, a+\sigma]$ satisfying the initial condition*

$$x(t_1)(\tau) = x(t_1)(t_1), \quad \tau \in [t_1, a+\sigma]$$

For every $\tau \in [a-r, a+\sigma]$, let

$$y(\tau) = \begin{cases} x(a)(\tau), \ a - r \leq \tau \leq a \\ x(\tau)(\tau), \ a \leq \tau \leq a + \sigma. \end{cases} \tag{2.11}$$

Then $y(\tau)$ is a solution of (2.3) in $[t_1, t_2]$ and $y(\tau) = x(t_2)(\tau)$, $\tau \in [a-r, a+\sigma]$.

In order to prove Theorem 2.2.2, we need the following lemma whose proof can be found in [15], Lemma 2.1.

Lemma 2.2.3. *Under the hypothesis of Theorem 2.2.2,*

$$x(t)(\tau) = x(t)(t), \quad \tau \geq t, \ \tau \in [a-r, a+\sigma], \ t \in [t_1, t_2] \tag{2.12}$$

and

$$x(t)(\tau) = x(\tau)(\tau), \quad t \geq \tau, \ \tau \in [a-r, a+\sigma], \ t \in [t_1, t_2]. \tag{2.13}$$

Now we are able to prove Theorem 2.2.2

Proof. (*of Theorem 2.2.2*) It is enough to show that, given $\varepsilon > 0$ and a sufficiently small interval $[t_3, t_4] \subset [t_1, t_2]$, we have

$$\left| y(t_4) - y(t_3) - \int_{[t_3, t_4]} f(y_s, s)\, ds \right| < \varepsilon. \tag{2.14}$$

Since $y(t_4) = x(t_4)(t_4)$ and $y(t_3) = x(t_3)(t_3)$ (by (2.11)) and $x(t_3)(t_4) = x(t_3)(t_3)$ (by (2.12)), we have

$$y(t_4) - y(t_3) = [x(t_4) - x(t_3)](t_4).$$

By hypothesis,

$$x(t_4) - x(t_3) = \int_{[t_3, t_4]} DF(x(\tau), t).$$

Therefore

$$y(t_4) - y(t_3) = \left[\int_{[t_3, t_4]} DF(x(\tau), t) \right](t_4).$$

On the other hand, (2.11) and (2.13) imply $y(\tau) = x(t)(\tau)$, $t \leq \tau$, and therefore

$$\int_{[t_3,t_4]} f(y_s, s)\, ds = \int_{[t_3,t_4]} f\left(x(t_4)_s, s\right) ds.$$

By (2.4),

$$[F(x(t_4), t_4) - F(x(t_4), t_3)](t_4) = \int_{[a,t_4]} f\left(x(t_4)_s, s\right) ds - \int_{[a,t_4]} f\left(x(t_4)_s, s\right) ds =$$

$$= \int_{[t_3,t_4]} f\left(x(t_4)_s, s\right) ds.$$

Hence,

$$\int_{[t_3,t_4]} f(y_s, s)\, ds = [F(x(t_4), t_4) - F(x(t_4), t_3)](t_4).$$

If δ is a gauge of $[t_3, t_4]$ from the definition of $\int_{[t_3,t_4]} DF(x(\tau), t)$ corresponding to $\varepsilon > 0$ and if $(t_4 - t_3) < \delta(t_4)$, then

$$\left| y(t_4) - y(t_3) - \int_{[t_3,t_4]} f(y_s, s)\, ds \right| =$$

$$= \left| \left[\int_{[t_3,t_4]} DF(x(\tau), t) - F(x(t_4), t_4) + F(x(t_4), t_3) \right](t_4) \right| < \varepsilon,$$

by Corollary 2.2.1. ∎

REMARK 2.2.4. *It worths mentioning that Theorems 2.2.2 and 2.2.2 also hold when the McShane integral is replaced by the Kurzweil integral.*

3. A COMPACT SPACE OF GENERALIZED EQUATIONS

Motivated by conditions (A) and (B) (before Definition 2.2.4), we will now define a family $\mathcal{F}$ of continuous functions

$$F(x,t) : C_1 \times [a - r, a + \sigma] \to C(a, r, \sigma).$$

The definition below was borrowed from [2]. Its form is adapted to suit our purposes.

DEFINITION 3.3.1. *For every compact $A \subset C_1$, let N_A be a positive number and $\mu_A(\varepsilon)$ be a positive function $\mu_A :]0, \infty[\to]0, \infty[$. Let $\mathcal{F}$ be a collection of all continuous functions $F(x,t) : C_1 \times [a - r, a + \sigma] \to C(a, r, \sigma)$ such that $F(x, 0) = 0$ for every x and fulfilling the conditions:*

(A')if $t_1 = s_0 < s_1 < \ldots < s_k = t_2$, with $t_2 - t_1 < \mu_A(\varepsilon)$, and if $\tau_1, \ldots, \tau_k$ are elements of A, then

$$\left\| \sum_{i=1}^{k} \left[F(\tau_i, s_i) - F(\tau_i, s_{i-1}) \right] \right\| \leq \varepsilon;$$

(B')for every compact $A \subset C_1$, there is a continuous non-decreasing function $K = K_{A,F}(t)$, with $K(0) = 0$, such that if $x, y \in A$, then

$$\| F(x,t) - F(x,s) - [F(y,t) - F(y,s)] \| \leq \|x - y\| \, |K(t) - K(s)|$$

and

$$K(t+1) - K(t) \leq N_A \leq 1, \quad \text{for every } t.$$

Remark 3.3.1. *The space $\mathcal{F}$ defined above is a topological space where the convergence is the uniform convergence in compact subsets of $C_1 \times [a-r, a+\sigma]$. Such a topology makes $\mathcal{F}$ a metric space under the metric*

$$d(F,G) = \sum_{i,j=1}^{\infty} 2^{-(i+j)} \min \{1, \|F(x_i, t_j) - G(x_i, t_j)\|\}.$$

The following result is borrowed from [2] (see Lemma 5.5) with obvious adaptations.

Lemma 3.3.1. *$\mathcal{F}$ is equicontinuous on compact subsets of $C_1 \times [a-r, a+\sigma]$. In particular, the uniform convergence on compact subsets is equivalent to pointwise convergence on a dense set that is, the sequence F^k converges to F^0 in $\mathcal{F}$ if and only if the sequence $F^k(x_j, t_j)$ converges to $F^0(x_j, t_j)$ in $C(a, r, \sigma)$, for each (x_j, t_j) in a prescribed dense sequence.*

Proof. Let $A \subset C_1$ and $C \subset C_1$ be compact sets and suppose $(x,t), (y,s) \in A \times C$ such that $|t - s| < \mu_A(\varepsilon)$. Then

$$\|F(x,t) - F(y,s)\| \leq \|F(x,t) - F(y,t)\| + \|F(y,t) - F(y,s)\|.$$

Since $F(z, 0) = 0$ for every z, then

$$\|F(x,t) - F(y,t)\| = \|F(x,t) - F(x,0) - [F(y,t) - F(y,0)]\| \leq$$

$$\leq \|x - y\| \, |K(t) - K(0)| = \|x - y\| \, |K_{A,F}(t)|,$$

where we applied condition (B') and $K(0) = 0$. By condition (A') and since $|t-s| < \mu_A(\varepsilon)$, we have

$$\|F(y,t) - F(y,s)\| \leq \varepsilon.$$

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Thus

$$\|F(x,t) - F(y,s)\| \leq \|x-y\| \, |K_{A,F}(t)| + \varepsilon.$$

As usual, if $[t] = \max\{n \in \mathbb{Z};\ n \leq t\}$, then for every $t \geq 0$,

$$K(t) \leq K([t]+1) \leq ([t]+1)N_A \leq (t+1)N_A,$$

where we used condition (B'). Thus $K_{A,F}(t) = K(t) \leq (|t|+1)N_A$ for all t, since K is non-decreasing and $K(0) = 0$. Then, if B_A is a bound for all $|t|+1$ for $(z,t) \in A \times C$ and if $\|x-y\| \leq \varepsilon B_A^{-1} N_A^{-1}$ and $|t-s| \leq \mu_A(\varepsilon)$, it follows that

$$\|F(x,t) - F(y,s)\| \leq 2\varepsilon, \quad \text{for every } F \in \mathcal{F}$$

and the proof is complete. ∎

THEOREM 3.3.1. *The space $\mathcal{F}$ is compact.*

A proof of Theorem 3.3.1 is nothing but a straightforward adaptation of [2], Theorem 5.6. It is too extensive and somewhat involved to be reproduced here.

4. THE EXISTENCE OF A LOCAL FLOW

This section consists of the results of Section 6 of [2]. The proofs are obvious adaptations.

4.1. Local flows

Let E be a metric space. For each $p \in E$, let $I_p =]a_p, b_p[\subset \mathbb{R}$ be an open interval such that $0 \in I_p$ and

$$S = \{(t,p) \in I_p \times E\}.$$

DEFINITION 4.4.1. *Consider the mapping $\pi : S \to E$. For every fixed p, $\pi(t,p) : I_p \to E$ is the **motion** through p and*

$$\gamma(p) = \{\pi(t,p);\ t \in I_p\},$$
$$\gamma^+(p) = \{\pi(t,p);\ 0 \leq t < b_p\},$$
$$\gamma^-(p) = \{\pi(t,p);\ a_p < t \leq 0\}$$

*are respectively the **orbit**, the **positive orbit** and the **negative orbit** through p*

DEFINITION 4.4.2. *A mapping $\pi : S \to E$ is called a **local flow** on E if the following properties are fulfilled:*

(i)$\pi(0,p) = p$, for every $p \in E$;
(ii)if $t \in I_p$ and $s \in I_{\pi(t,p)}$, then $t+s \in I_p$ and $\pi(s, \pi(p,t)) = \pi(t+s, p)$;

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(iii)π is continuous;

(iv)$I_p =]a_p, b_p[$ is maximal in the following sense: either $I_p = \mathbb{R}$ or, if $b_p \neq +\infty$ (resp. if $a_p \neq -\infty$), then the positive orbit (resp. the negative orbit) is not precompact;

(v)if $p_k \to p$, then $I_p \subset \liminf I_{p_k}$.

Let $F \in \mathcal{F}$. For each $t \in \mathbb{R}$ we define the translation F_s of F by

$$F_s(x,t) = F(x, s+t) - F(x,s),$$

where we subtract $F(x,s)$ in view of the normalization in $\mathcal{F}$. It is not difficult to check that the translations F_s of F belong to $\mathcal{F}$ and therefore $\mathcal{F}$ is translation invariant. In addition,

(a) $F_{s+\tau} = (F_s)_\tau$

and

(b) the mapping $(s,F) \to F_s$ is continuous. Indeed, let $s_k \to s_0$, $F^k \to F^0$ and take a fixed (x,t). By Lemma 3.3.1, it is enough to show that $F^k(x, t+s_k) - F^k(x, s_k)$ converges to $F^0(x, t+s_0) - F^0(x, s_0)$. But $\mathcal{F}$ is equicontinuous on compact subsets of $C_1 \times [a-r, a+\sigma]$ (Lemma 3.3.1). Hence $F^k(z,s) \to F^0(z,s)$ uniformly on compact subsets of $C_1 \times [a-r, a+\sigma]$. The conclusion follows easily.

The next result defines a local flow on $C_1 \times \mathcal{F}$. Its proof follows the steps of [2], Theorem 6.3.

Theorem 4.4.1. *Let $x(t,u,F)$ be the only maximal solution of*

$$\begin{cases} \frac{dx}{d\tau} = DF(x(\tau), t) \\ x(0) = u, \end{cases} \tag{4.15}$$

where $F \in \mathcal{F}$ and $u \in C_1$. Let $J_{u,F} =]\alpha(u,F), \omega(u,F)[$ be the maximal interval of definition of $x(\cdot, u, F)$ and π be the mapping given by

$$\pi(t,u,F) = (x(t,u,F), F_t).$$

Then π is a local flow on $C_1 \times \mathcal{F}$.

The theorem below is a consequence of the definition of π in Theorem 4.4.1.

Theorem 4.4.2. *Let $(u,F) \in C_1 \times \mathcal{F}$. The motion $\pi(t,u,F)$ is precompact in $C_1 \times \mathcal{F}$ if and only if the orbit $x(\cdot,u,F)$ is precompact. The same holds for the positive and negative orbits.*

4.2. An application

As an immediate consequence of Theorems 2.2.1, 4.4.2 and 2.2.2 we obtain the following result.

THEOREM 4.4.3. *Let $\mathcal{D}$ be the set of functions $f(\varphi, t) : H_1 \times [a, a+\sigma] \to \mathbb{R}^n$ such that $t \mapsto f(x_t, t)$ is Lebesgue integrable and conditions (A) and (B) are satisfied. Let $y(t, \upsilon, f)$ be a solution of the RFDE*

$$\frac{dy}{dt} = f(y_t, t), \tag{4.16}$$

$(t, \upsilon, f) \in [a, a+\sigma] \times \mathbb{R}^n \times \mathcal{D}$. Let $(u, F) \in C_1 \times \mathcal{F}$. If the motion $\pi(t, u, F)$ corresponding to the GODE

$$\frac{dx}{d\tau} = DF(x(\tau), t)$$

associated to (4.16) is precompact in $C_1 \times \mathcal{F}$, then the orbit $y(\cdot, \upsilon, f)$ as well as its positive and negative orbits are precompact .

5. CONTINUOUS DEPENDENCE OF SOLUTIONS

The results of this section are in the spirit of [1], [2] and [6]. Theorems 5.5.3 and 5.5.4 below state continuous dependence results for RFDE's. In particular, Theorem 5.5.3 clarifies the continuous dependence result of [6].

Let $F \in \mathcal{F}$ and consider the following GODE:

$$\frac{dx}{d\tau} = DF(x(\tau), t). \tag{5.17}$$

We shall establish a result on continuous dependence of solutions of (5.17) with respect to initial conditions and parameters. Together with equation (5.17), we consider equations

$$\frac{dx_n}{d\tau} = DF_n(x_n(\tau), t), \quad n = 1, 2, 3, \ldots, \tag{5.18}$$

where $F_n \in \mathcal{F}$ and $F_n(y, t) \to F(y, t)$ in $C(a, r, \sigma)$ as $n \to \infty$.

LEMMA 5.5.1. *Under the conditions above, let $\varepsilon > 0$ and $A \subset C_1$ be a compact such that $x(t_1), x_n(t_1) \in A$, where $x(t)$ and $x_n(t)$ are solutions of (5.17) and (5.18) respectively in a sufficiently small interval $[t_1, t_2] \subset [a - r, a + \sigma]$. Then, for every $t_1 \leq t \leq t_2$,*

$$\|x(t) - x_n(t)\| < \|x(t_1) - x_n(t_1)\| (1 + N_A) + 2\varepsilon.$$

Proof. Given $\varepsilon > 0$ and $n \in \{1, 2, 3, \ldots\}$, let δ be a gauge of $[a-r, a+\sigma]$ from the definition of $\int_{[a-r,a+\sigma]} D\left[F\left(x\left(\tau\right), t\right) - F_n\left(x\left(\tau\right), t\right)\right]$. If $(t_2 - t_1) < \delta\left(t_1\right)$, then Corollary 2.2.1 implies

$$\left\| \int_{[t_1,t_2]} D\left[F\left(x\left(\tau\right), t\right) - F_n\left(x_n\left(\tau\right), t\right)\right] - F\left(x\left(t_1\right), t_2\right) + F\left(x\left(t_1\right), t_1\right) + \right.$$

$$\left. + F_n\left(x_n\left(t_1\right), t_2\right) - F_n\left(x_n\left(t_1\right), t_1\right) \right\| < \varepsilon \tag{5.19}$$

Because $F_n\left(y, t\right) \to F\left(y, t\right)$ in $C\left(a, r, \sigma\right)$, then for sufficiently large n,

$$\left\| F_n\left(x\left(t_1\right), t_2\right) - F_n\left(x\left(t_1\right), t_1\right) + F\left(x\left(t_1\right), t_2\right) - F\left(x\left(t_1\right), t_1\right) \right\| < \varepsilon. \tag{5.20}$$

For every appropriate compact $A \subset C_1$ such that $x\left(t_1\right), x_n\left(t_1\right) \in A$, it comes by (B') that

$$\left\| F_n\left(x_n\left(t_1\right), t_2\right) - F_n\left(x_n\left(t_1\right), t_1\right) - F_n\left(x\left(t_1\right), t_2\right) + F_n\left(x\left(t_1\right), t_1\right) \right\| \leq$$

$$\leq \left\| x_n\left(t_1\right) - x\left(t_1\right) \right\| \left| K_{A,F_n}\left(t_2\right) - K_{A,F_n}\left(t_1\right) \right| \leq \left\| x_n\left(t_1\right) - x\left(t_1\right) \right\| \cdot N_A. \tag{5.21}$$

Thus, combining (5.19), (5.20) and (5.21) we get

$$\left\| \int_{[t_1,t_2]} D\left[F\left(x\left(\tau\right), t\right) - F_n\left(x\left(\tau\right), t\right)\right] \right\| < 2\varepsilon + \left\| x_n\left(t_1\right) - x\left(t_1\right) \right\| \cdot N_A$$

for sufficiently large n. Now, since

$$x\left(t_2\right) = x\left(t_1\right) + \int_{[t_1,t_2]} DF\left(x\left(\tau\right), s\right)$$

and

$$x_n\left(t_2\right) = x_n\left(t_1\right) + \int_{[t_1,t_2]} DF_n\left(x_n\left(\tau\right), s\right),$$

we obtain

$$\left\| x\left(t_2\right) - x_n\left(t_2\right) \right\| \leq \left\| x\left(t_1\right) - x_n\left(t_1\right) \right\| +$$

$$+ \left\| \int_{[t_1,t_2]} D\left[F\left(x\left(\tau\right), s\right) - F_n\left(x\left(\tau\right), s\right)\right] \right\| < \left\| x\left(t_1\right) - x_n\left(t_1\right) \right\| \left(1 + N_A\right) + 2\varepsilon. \tag{5.22}$$

As a matter of fact, t_2 in estimate (5.22) can be replaced by any $t \in [t_1, t_2]$. ∎

The next theorem is a consequence of Lemma 5.5.1.

Theorem 5.5.1. *Let $x(t)$ and $x_n(t)$ be solutions of (5.17) and (5.18) respectively in a sufficiently small interval $[t_1, t_2] \subset [a-r, a+\sigma]$, with $x(t_1) = x^1$ and $x_n(t_1) = x_n^1$, $n = 1, 2, 3, \ldots$. Then $x_n(t) \to x(t)$ uniformly in $[t_1, t_2]$ as $n \to \infty$.*

The proof of the next theorem is analogous to that of [2], Theorem A.8, with the appropriate adaptations.

Theorem 5.5.2. *Let $x(t, u, \mathcal{F})$ be the unique maximal solution of*

$$\begin{cases} \frac{dx}{d\tau} = DF(x,t) \\ x(0) = u \end{cases}$$

defined in the maximal interval $]\alpha(u, \mathcal{F}), \omega(u, \mathcal{F})[$. Then $x(t, u, \mathcal{F})$ is continuous in $(t, u, \mathcal{F}) \in [a, a+\sigma] \times C_1 \times \mathcal{F}$ when defined. Besides, $\omega(u, \mathcal{F})$ is lower semicontinuous on $C_1 \times \mathcal{F}$ and $\alpha(u, \mathcal{F})$ is upper semicontinuous on $C_1 \times \mathcal{F}$.

Now we consider the following RFDE's

$$\frac{dy}{dt} = f(y_t, t) \tag{5.23}$$

and

$$\frac{d(y^n)}{dt} = f_n\left((y^n)_t, t\right), \quad n = 1, 2, 3, \ldots, \tag{5.24}$$

where $f, f_n(\varphi, t) : H_1 \times [a, a+\sigma] \to \mathbb{R}^n$ are such that $t \mapsto f_n(x_t, t)$ and $t \mapsto f(x_t, t)$ are Lebesgue integrable and conditions such as (A) and (B) are fulfilled for f_n, $n \in \mathbb{N}$, and f. Suppose

$$\int_{[t_1,t_2]} f_n\left((y^n)_s, s\right) ds \to \int_{[t_1,t_2]} f(y_s, s)\, ds$$

uniformly in all variables as $n \to \infty$, where $[t_1, t_2] \subset [a-r, a+\sigma]$. Under these conditions and by means of Theorems 2.2.1, 5.5.1, 2.2.2 and the continuation of solutions, we have

Theorem 5.5.3. *If $y(t)$ and $y_n(t)$ are solutions of (5.23) and (5.24) respectively in an interval $[t_1, t_2] \subset [a-r, a+\sigma]$, with $y(t_1) = \varphi$ and $y_n(t_1) = \varphi_n$, $n = 1, 2, 3, \ldots$, and if $\varphi_n \to \varphi$, then $y_n(t) \to y(t)$ uniformly in $[t_1, t_2]$ as $n \to \infty$.*

The following result is a particular case of Theorem 5.5.2 for RFDE's. It can be proved by applying Theorems 2.2.1, 5.5.2 and 2.2.2.

Theorem 5.5.4. *Let $\mathcal{D}$ be the set of functions $f(\varphi, t) : H_1 \times [a, a+\sigma] \to \mathbb{R}^n$ that are continuous in φ, measurable in t, and such that conditions like (A) and (B) are satisfied. Let $y(t, \upsilon, f)$ be the unique maximal solution of*

$$\begin{cases} \frac{dy}{dt} = f(y_t, t) \\ y(0) = \upsilon \end{cases}$$

defined in the maximal interval $]\alpha(\upsilon,f),\omega(\upsilon,f)[$, *where* f *satisfies conditions (A) and (B). Then* $y(t,\upsilon,f)$ *is continuous in* $(t,\upsilon,f)\in[a,a+\sigma]\times\mathbb{R}^n\times\mathcal{D}$ *when defined. Besides,* $\omega(\upsilon,f)$ *is lower semicontinuous on* $\mathbb{R}^n\times\mathcal{D}$ *and* $\alpha(\upsilon,f)$ *is upper semicontinuous on* $\mathbb{R}^n\times\mathcal{D}$.

REFERENCES

1. Z. Artstein, Topological dynamics of an ordinary differential equation, *J. Diff. Eq.* 23 (1977), 216-223.
2. Z. Artstein, Topological dynamics of an ordinary differential equation and Kurzweil equations, *J. Diff. Eq.*23 (1977), 224-243.
3. B. Bongiorno, Relatively weakly compact sets in the Denjoy Space, *J. Math. Study*, 27 (1994), 37-43.
4. L. Di Piazza and K. Musial, A characterization of variationally McShane integrable Banach-space valued functions, *Illinois J. Math.*, 45(1), (2001), 279-289.
5. M. Federson, Some peculiarities of the Henstock and Kurzweil integrals of Banach space-valued functions, *Real Analysis Exchange*, "to appear".
6. C. Imaz and Z. Vorel, Generalized ordinary differential equations in Banach spaces and applications to functional equations, *Bol. Soc. Mat. Mexicana* 11 (1966), 47-59.
7. J. Kurzweil, Generalized ordinary differential equations and continuous dependence on a parameter, *Czech. Math. J.* 7 (82), (1957), 418-448.
8. J. Kurzweil, Generalized ordinary differential equations, *Czechoslovak Math. J.* 8 (83), (1958), 360-388.
9. J. Kurzweil, Unicity of solutions of generalized differential equations, *Czechoslovak Math. J.* 8 (83), (1958), 502-509.
10. J. Kurzweil, Addition to my paper "Generalized ordinary differential equations and continuous dependence on a parameter", *Czech. Math. J.* 9 (84), (1959), 564-573.
11. J. Kurzweil, Problems which lead to a generalization of the concept of an ordinary nonlinear differential equation. 1963 *Differential Equations and Their Applications*, (Proc. Conf., Prague, 1962), pp. 65–76. Publ. House Czechoslovak Acad. Sci., Prague; Academic Press, New York.
12. E. J. McShane, A unified theory of integration, *Am. Math. Monthly*, 80 (1973), 349-359.
13. R. K. Miller, Almost periodic differential equations as dynamical systems with applications to the existence of a.p. solutions, *J. Diff. Eq.*, 1, (1965), 337-345.
14. R. K. Miller and G. R. Sell, Topological dynamics and its relation to integral equations and nonautonomous systems, *in* "Dynamical Systems, An International Symposium" vol I, 223-249, Academic Press, New York, 1976.
15. F. Oliva and Z. Vorel, Functional equations and generalized ordinary differential equations, *Bol. Soc. Mat. Mexicana* 11 (1966), 40-46.
16. S. Schwabik, Abstract Perron-Stieltjes integral, *Math. Bohem.*, 121 (4), (1996), 425-447.
17. G. R. Sell, Nonautonomous differential equations and topological dynamics II. Limiting equations, *Trans. Am. Math. Soc.* 127 (1967), 263-283.
18. G. R. Sell, *Lectures on Topological Dynamics and Differential Equations*, Von Nostrand-Reinhold, London, 1971.

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