

Spatial Homogeneity in Parabolic Problems With Nonlinear Boundary Conditions

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Abstract. In this work we prove that global attractors of systems of weakly coupled parabolic equations with nonlinear boundary conditions and large diffusivity are close to attractors of an ordinary differential equation. The limiting ordinary differential equation is given explicitly in terms of the reaction, boundary flux, the n -dimensional Lebesgue measure of the domain and the $n-1$ -Hausdorff measure of its boundary. The tools are invariant manifold theory and comparison results. May, 2003 ICMC-USP

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$, be a bounded domain with smooth boundary $\Gamma := \partial\Omega$. Consider the following problem

$$\begin{cases} u_t(t, x) = D\Delta u(t, x) + F(u(t, x)) & t > 0, \quad x \in \Omega, \\ D\frac{\partial u}{\partial \vec{n}}(t, x) = G(u(t, x)) & t > 0, \quad x \in \Gamma, \end{cases} \quad (1.1)$$

where $u = (u_1, u_2, \dots, u_n)^\top$, $n \geq 1$, $\frac{\partial u}{\partial \vec{n}} = (\langle \nabla u_1, \vec{n} \rangle, \dots, \langle \nabla u_n, \vec{n} \rangle)^\top$, \vec{n} is the outward normal vector and D is the matrix

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$$D := \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}_{n \times n} \quad (1.2)$$

with $d_i > 0$, $i = 1, \dots, n$. The nonlinearities

$$F = (F_1, \dots, F_n), G = (G_1, \dots, G_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

are locally Lipschitz functions.

Our aim is to show that, for suitably chosen matrix D , the asymptotic behavior of (1.1) is essentially the same as the asymptotic behavior of the following system of ordinary differential equations:

$$\dot{v}(t) = F(v(t)) + |F|G(v(t)). \quad (1.3)$$

More precisely, we show, under some hypothesis on the nonlinearities F and G , that the problem (1.1) has a global attractor and that this attractor is contained in a small neighborhood of the global attractor of the ordinary differential equation (1.3), for matrices D with suitably large diagonal entries. This is saying that for large times the solutions of (1.1) are almost independent of the space variable. This is what we are calling *spatial homogeneity*.

These results are seen in [9] for the case $G \equiv 0$ and later in [10] for the case G linear. The results that we obtain here generalize, partially, these results and offer a unified approach to these kind of problems. See also [5] for related results.

We will obtain the existence of an invariant manifold for the solutions of (1.1). We note that in the problem treated in [9], the space of constant functions is already an invariant manifold for (1.1) with $G \equiv 0$, fact that does not happen when $G \not\equiv 0$ (in this case we show that there is an invariant manifold which is close to the space of constant functions).

This work is organized as follows: In Section 2 we state the needed hypotheses, introduce some notation and state the main results of this work; in Section 3 we extend some results on positivity and comparison of solutions obtained in [4] and [6], that can be used for systems even more general than (1.1); in Section 4 we obtain the global existence of solutions and the existence of global attractors for (1.1), as well as some uniform bounds relatively to the matrix D for this attractor; finally in Section 5 we prove the main results.

2. STATEMENT OF THE RESULTS

In this section we introduce some notation and state the main results of this paper. Throughout this work, $X = L^q(\Omega, \mathbb{R}^n)$ or $X = W^{1,q}(\Omega, \mathbb{R}^n)$. To properly state our results we first need to introduce some notation and basic results.

Following [3], consider the operator $A_D = \text{diag}(A_1, \dots, A_n)$, where A_i , for every $i = 1, \dots, n$, is defined as follows: let $A_i : D(A_i) \subset L^q(\Omega) \rightarrow L^q(\Omega)$, with

$$\begin{aligned} D(A_i) &= \{ \phi \in W^{2,q}(\Omega); \frac{\partial \phi}{\partial \bar{n}} = 0, \text{ em } \Gamma \} := W_{\mathcal{N}}^{2,q}(\Omega); \\ A_i \phi &= -d_i \Delta \phi. \end{aligned} \tag{2.1}$$

Let $(E_i)_q^1 := D(A_i)$, $i = 1, 2, \dots, n$ and $E := (E)_q^0 := L^q(\Omega)$, $1 < q < \infty$. Taking $[\cdot, \cdot]_\theta$ as the complex interpolation functor of exponent θ , A_i , $i = 1, 2, \dots, n$ has a scale of Banach spaces E_i^α , $0 \leq \alpha \leq 1$ satisfying

$$(E_i)_q^\alpha = [(E_i)_q^1, (E_i)_q^0]_\alpha, \quad 1 < q < \infty.$$

Define, $E_q^\alpha := (E_1)_q^\alpha \times (E_2)_q^\alpha \times \dots \times (E_n)_q^\alpha$, for $\alpha \in [0, 1]$, $1 < q < \infty$. We have that, for $\alpha > \beta$ and $1 < q < \infty$, the embedding $E_q^\alpha \subset E_q^\beta$ is compact, $E_q^{\frac{1}{2}} = W^{1,q}(\Omega, \mathbb{R}^n)$ and we obtain from [2] that

$$E_q^\alpha := [W_{\mathcal{N}}^{2,q}(\Omega, \mathbb{R}^n), L^q(\Omega, \mathbb{R}^n)]_\alpha \hookrightarrow H_q^{2\alpha}(\Omega, \mathbb{R}^n), \tag{2.2}$$

where $\alpha \in (0, 1)$, $W_{\mathcal{N}}^{2,q}(\Omega, \mathbb{R}^n) = \prod_{i=1}^n W_{\mathcal{N}}^{2,q}(\Omega)$ and $H_q^{2\alpha}(\Omega, \mathbb{R}^n)$ are the Bessel potential spaces (see [2, 7] for details). We can extend this scale of Banach Spaces to negative exponents by $E_q^{-\alpha} := (E_q^\alpha)'$, for $\alpha > 0$. The operator A_D can be extended to these spaces in such a way that its realization, $A_{\alpha-1}$, in $E_q^{\alpha-1}$ satisfy $A_{\alpha-1} : E_q^\alpha \rightarrow E_q^{\alpha-1}$ and is a sectorial operator, for $0 \leq \alpha \leq 1$.

We then consider problems of the form

$$\begin{aligned} \dot{u}(t) &= -A_D u(t) + H(u(t)), \quad t > 0; \\ u(0) &= u_0, \end{aligned} \tag{2.3}$$

where A_D is the operator defined above and the function H is defined by $H := F_\Omega + G_\Gamma$, acting in suitable test functions, ϕ , in the following way:

$$\langle H(u), \phi \rangle = \langle F_\Omega(u), \phi \rangle + \langle G_\Gamma(u), \phi \rangle = \int_\Omega (F(u(x)) \cdot \phi(x)) dx + \int_\Gamma (G(u(x)) \cdot \phi(x)) dx.$$

To obtain the local existence and uniqueness for the problem (1.1), we need to impose some growth conditions on F and G in (1.1), these conditions are the same obtained in [3]. Next we describe the restrictions.

(C1) Suppose that F_i, G_i , $i = 1, 2, \dots, n$ satisfy

$$|h(u) - h(v)| \leq c|u - v|(|u|^{\rho-1} + |v|^{\rho-1} + 1), \quad u, v \in \mathbb{R}^n, \tag{2.4}$$

with exponents ρ_i and $\bar{\rho}_i$, $i = 1, 2, \dots, n$ respectively, such that, with $N \geq 2$,

$$\rho_i \leq \rho_f := 1 + \frac{2q}{N} \quad \text{and} \quad \bar{\rho}_i \leq \rho_g := 1 + \frac{q}{N}, \quad i = 1, 2,$$

with the second inequality being strict when $N = 1$.

(C2) Suppose that the following conditions are satisfied

1. $q > N$,
2. $q = N$ and for all $\eta > 0$, there is a constant $c_\eta > 0$ such that F_i and G_i , $i = 1, 2, \dots, n$ satisfy

$$|h(u) - h(v)| \leq c_\eta |u - v| (e^{\eta|u|^{\frac{N}{N-1}}} + e^{\eta|v|^{\frac{N}{N-1}}}), \quad u, v \in \mathbb{R}^n, \quad (2.5)$$

3. $1 < q < N$, F_i and G_i , $i = 1, 2, \dots, n$ satisfy (2.4) with exponents ρ_i and $\bar{\rho}_i$, $i = 1, 2, \dots, n$ respectively, such that,

$$\rho_i \leq \rho_f := 1 + \frac{2q}{N-q} \quad \text{and} \quad \bar{\rho}_i \leq \rho_g := 1 + \frac{q}{N-q}, \quad i = 1, 2, \dots, n.$$

The proof of the following theorem can be found in [3].

THEOREM 2.1. *Suppose that F_i and G_i satisfy the growth conditions (C1), for every $i = 1, 2, \dots, n$. Then, for all $u_0 \in L^q(\Omega, \mathbb{R}^n)$, there is a unique local solution, $u(\cdot; u_0)$, to (1.1), satisfying $u(0; u_0) = u_0$, and depending continuously on $u_0 \in L^q(\Omega, \mathbb{R}^n)$. Furthermore, this solution is classic for every $t > 0$.*

If for $i=1, 2, \dots, n$, F_i and G_i satisfy (C2), then for $u_0 \in W^{1,q}(\Omega, \mathbb{R}^n)$, there is a unique local solution, $u(\cdot; u_0)$, to (1.1), satisfying $u(0; u_0) = u_0$, and depending continuously on $u_0 \in W^{1,q}(\Omega, \mathbb{R}^n)$. Furthermore, this solution is classic for $t > 0$.

An important remark is that in any of the cases of the Theorem 2.1, the solution satisfies the variation of constants formula given by

$$u(t; u_0) = e^{-ADt}u_0 + \int_0^t e^{-AD(t-s)}H(u(s; u_0))ds. \quad (2.6)$$

To show global existence, following the ideas in [4], we need to assume some restrictions on the sign of the nonlinearities in (1.1). Suppose that there are constants $B_0, C_0 \in \mathbb{R}$ and $B_1, C_1 \geq 0$ such that for $u \in \mathbb{R}^n$,

$$\begin{aligned} u_i F_i(u) &\leq -C_0 u_i^2 + C_1 |u_i|, \\ u_i G_i(u) &\leq -B_0 u_i^2 + B_1 |u_i|, \end{aligned} \quad (2.7)$$

$i = 1, 2, \dots, n$. Furthermore, with the same reasoning as in [4] and [13], to obtain the existence of global attractors for (1.1), suppose that the first eigenvalue, $\lambda_1(D)$, of the problem

$$\begin{cases} -D\Delta u + C_0 u = \lambda u & \text{em } \Omega, \\ D\frac{\partial u}{\partial \bar{n}} + B_0 u = 0 & \text{em } \Gamma, \end{cases} \quad (2.8)$$

be positive, where B_0, C_0 are given in (2.7).

Under these conditions we prove the following result:

THEOREM 2.2. *Under the above conditions, the solutions found in Theorem 2.1 are globally defined. Furthermore, the problem (1.1) has a global attractor $\mathcal{A}_X(D)$, in $X = L^q(\Omega, \mathbb{R}^n)$ or $X = W^{1,q}(\Omega, \mathbb{R}^n)$, such that*

$$\sup_{v \in \mathcal{A}_X(D)} \|v\|_{L^\infty(\Omega, \mathbb{R}^N)} \leq K_0,$$

where $K_0 = K_0(\Omega, \Gamma, \lambda_1(D), C_0, C_1, B_0, B_1)$. If $d > 1$, where

$$d := \min\{d_1, d_2, \dots, d_n\}, \tag{2.9}$$

then K_0 can be taken uniform with respect to D given in (1.2).

In order to prove the properties about the global attractors for (1.1), we need to introduce the limiting ordinary differential equation (1.3). To do this, for every $i = 1, 2, \dots, n$, let $\lambda_i(d_i)$ be the first eigenvalue of the operator A_i , defined in (2.1). We know that $\lambda_i(d_i) = 0$, $i = 1, 2, \dots, n$. Consider $\varphi_i(d_i)$, $i = 1, 2, \dots, n$, the corresponding normalized eigenfunction. In this case, $\varphi_i(d_i) = |\Omega|^{-\frac{1}{2}}$, $i = 1, \dots, n$. We can, without loss of generality, suppose that $|\Omega| = 1$, then $\varphi_i(d_i) = 1$, for $i = 1, 2, \dots, n$.

Now, with the above notation, fix $q \in (1, \infty)$ and $E^\alpha := E_q^\alpha$, $\alpha \in [-2, 2]$. Consider the following decomposition of E^α :

$$E^\alpha = U \oplus U_\alpha^\perp, \tag{2.10}$$

where $U = \mathbb{R}^n$ and $U_\alpha^\perp = \{\varphi \in E^\alpha; \langle \psi, \varphi \rangle = 0, \psi \in U\}$ with

$$\langle \psi, \varphi \rangle = \int_\Omega \varphi(x)^\top \psi(x) dx,$$

for $\psi \in U$ and $\varphi \in E^\alpha$. Observe that if $\psi \in U$, then $\psi \in L^\infty(\Omega, \mathbb{R}^n)$ and, therefore, the above integral is well defined for $\varphi \in E^\alpha$. Also note that U can be viewed as a n -dimensional subspace of E^α , containing only constant functions.

Let $u(t, \cdot)$, be a solution of (1.1) in X . We can write

$$u(t, x) = v(t) + w(t, x), \quad x \in \Omega, \tag{2.11}$$

where $v(t) \in U \equiv \mathbb{R}^n$ and $w(t, \cdot) \in U_\alpha^\perp$, $t > 0$. Integrating (2.11) we obtain that

$$\int_\Omega u(t, x) dx = v(t) + \int_\Omega w(t, x) dx = v(t),$$

since $1 \in U$. Hence, $v(t) = \int_\Omega u(t, x) dx$ and $w(t, x) = u(t, x) - v(t)$, for $t > 0$ and $x \in \Omega$. Thus,

$$\begin{aligned} \frac{d}{dt} v(t) &= \frac{d}{dt} \left(\int_\Omega u(t, x) dx \right) = \int_\Omega D \Delta u(t, x) dx + \int_\Omega F(u(t, x)) dx \\ &= \int_\Omega F(v(t) + w(t, x)) dx + \int_\Gamma G(v(t) + w(t, x)) dx = P(v(t), w(t, \cdot)), \end{aligned}$$

where, for $v \in \mathbb{R}^n$ and $w \in U_\alpha^\perp$

$$P(v, w) = \int_{\Omega} F(v + w(x))dx + \int_{\Gamma} G(v + w(x))dx.$$

In addition, for $t > 0$ and $x \in \Omega$, we have

$$\begin{aligned} \frac{\partial}{\partial t} w(t, x) &= \frac{\partial}{\partial t} (u(t, x) - v(t)) = D\Delta u(t, x) + F(u(t, x)) - (P(v(t), w(t, \cdot))) \\ &= D\Delta w(t, x) + F(u(t, x)) - (P(v(t), w(t, \cdot))). \end{aligned}$$

Also, for $x \in \Gamma$ and $t > 0$,

$$D \frac{\partial}{\partial \vec{n}} w(t, x) = D \frac{\partial}{\partial \vec{n}} u(t, x) = G(v(t) + w(t, x)).$$

Therefore, using the decomposition (2.10), we can write every solution of (1.1) as a solution of the problem

$$\begin{cases} \frac{d}{dt} v(t) = P(v(t), w(t, \cdot)), & t > 0, \\ \frac{\partial}{\partial t} w(t, x) = D\Delta w(t, x) + Q(v(t), w(t, x)), & t > 0, x \in \Omega, \\ D \frac{\partial w}{\partial \vec{n}} = G(v(t) + w(t, x)) & t > 0, x \in \Gamma, \end{cases} \quad (2.12)$$

with $P : U + U_\alpha^\perp \rightarrow U$ and $Q : U + U_\alpha^\perp \rightarrow L^q(\Omega, \mathbb{R}^n)$, defined, for $v \in \mathbb{R}^n$ and $w \in U_\alpha^\perp$, by

$$\begin{aligned} P(v, w) &= \int_{\Omega} F(v + w(x))dx + \int_{\Gamma} G(v + w(x))dx, \\ (Q(v, w))(x) &= (F_\Omega(v + w))(x) - P(v, w), \quad x \in \Omega. \end{aligned} \quad (2.13)$$

With all of this we can prove the main result of this work:

THEOREM 2.3. *Let \mathcal{A}_0 be the global attractor for the ODE (1.3) and $V \subset \mathbb{R}^n$ be a neighborhood of \mathcal{A}_0 such that $w(V) \subset \mathcal{A}_0$. Then, for all neighborhood, W , of \mathcal{A}_0 such that $\overline{W} \subset V$, there exists a $d_0 > 0$ and a function $\sigma_d : \overline{W} \rightarrow U_\alpha^\perp$, $d > d_0$, such that $\sigma_d(v) \rightarrow 0$, as $d \rightarrow \infty$, uniformly for $v \in \overline{W}$, d as in (2.9). The set*

$$M_d = \{u = v + \sigma_d(v), v \in W\}$$

is an exponentially attracting invariant manifold for (1.1). Furthermore, the flow on this manifold is given by

$$u(t, x) = v(t) + \sigma_d(v(t))(x),$$

where $\frac{d}{dt} v(t) = P(v(t), \sigma_d(v(t)))$.

3. COMPARISON AND POSITIVITY RESULTS

In this section we extend the results of [4] on comparison and positivity of solutions to problems including (1.1). These results will be useful to show global well posedness and asymptotic properties of solutions to (1.1). The *abstract comparison* results in this section can be found in [4].

DEFINITION 3.1. An ordered Banach space is a pair (X, \leq) , where X is a Banach space and \leq is an order relation in X such that

- $x \leq y$ implies $x + z \leq y + z$, for all $x, y, z \in X$;
- $x \leq y$ implies $\lambda x \leq \lambda y$ for $\lambda \in \mathbb{R}^+$ and $x, y \in X$;
- The "positive cone" $C = \{x \in X; 0 \leq x\}$ is closed in X .

DEFINITION 3.2. Let (X, \leq) be an ordered Banach space. We say that the map $T : X \rightarrow X$ is increasing $T(x) \leq T(y)$, whenever $x \leq y$, $x, y \in X$ and we say that it is positive if $0 \leq T(x)$, whenever $0 \leq x \in X$.

DEFINITION 3.3. A sectorial operator A in an ordered Banach space (X, \leq) has positive resolvent in X , if there is $\lambda_0 \in \mathbb{R}$ such that $(A + \lambda)^{-1}$ is an increasing map in X , for all $\lambda > \lambda_0$.

Let (X, \leq) be an ordered Banach space and $A : D(A) \subset X \rightarrow X$ be a sectorial operator. Consider $\lambda_0 \in \mathbb{R}$ such that $Re(\sigma(A + \lambda_0)) > 0$. Additionally, suppose that A has positive resolvent.

We consider now problems of the form

$$\begin{cases} u_t + Au = f(u), & t > t_0, \\ u(t_0) = u_0 \in X, \end{cases} \tag{3.1}$$

where A and X satisfy the above conditions. Suppose that we have constructed, as in Section 2, a scale of interpolation spaces, which we denote by X^α , $\alpha \geq 0$, with $X^0 = X$ and $X^1 = D(A)$. In each X^α , consider the order induced by X , and suppose that the scale X^α , $\alpha \geq 0$ is an ordered scale of spaces in the sense of the definition below.

DEFINITION 3.4. A scale of Banach spaces X^α , $\alpha \geq 0$ is an Ordered Scale of Spaces if the inclusions $X^\alpha \hookrightarrow X^\beta$, $\alpha \geq \beta$ are positive, the α -realization of A in X^α , $A_\alpha : X^{\alpha+1} \rightarrow X^\alpha$, $\alpha \geq 0$, have positive resolvent and the positive cone of X^α is dense in the positive cone of X^β for all $\alpha \geq \beta \geq 0$.

Assume that the nonlinearity f in (3.1), is subcritical in a certain space $X^{1+\varepsilon}$, $\varepsilon \geq 0$, that is, there exists $\gamma > 0$ with $0 \leq 1 + \varepsilon - \gamma < 1$ such that $f : X^{1+\varepsilon} \rightarrow X^\gamma$ is locally Lipschitz.

Furthermore, suppose that the problem (3.1) is locally well posed in X^1 . Therefore, we have the following result whose proof can be found in [4].

THEOREM 3.1. *Let A and X be as above and suppose that the nonlinearities f, g and h satisfy the conditions described above for f .*

(i) *Suppose that for all $r > 0$ there exists a constant $\beta = \beta(r) > 0$ such that $f(\cdot) + \beta I$ is positive when restricted to the ball of radius r in $X^{1+\varepsilon}$. If $0 \leq u_0 \in X^1$ then, the solution, $u(\cdot; u_0, f)$, of (3.1) is positive for as long as it exists.*

(ii) *Suppose that for all $r > 0$ there is a constant $\beta = \beta(r) > 0$ such that $f(\cdot) + \beta I$ is increasing in the ball of radius r of $X^{1+\varepsilon}$. If $u_0, u_1 \in X^1$, with $u_0 \leq u_1$ then $u(\cdot; u_0, f) \leq u(\cdot; u_1, f)$ for as long as both solutions exist.*

(iii) *Suppose that f and g satisfy $f(\cdot) \leq g(\cdot)$. Then, for all $u_0 \in X^1$,*

$$u(\cdot; u_0, f) \leq u(\cdot; u_0, g)$$

for as long as both solutions exist.

(iv) *Suppose that f and g are such that for all $r > 0$, there is a constant $\beta = \beta(r) > 0$ and an increasing function, h , such that*

$$f(\cdot) + \beta I \leq h(\cdot) \leq g(\cdot) + \beta I$$

in the ball of radius r in $X^{1+\varepsilon}$. If $u_0, u_1 \in X^1$ with $u_0 \leq u_1$, then

$$u(\cdot; u_0, f) \leq u(\cdot; u_0, g)$$

for as long as both solutions exist.

Now the idea is to apply this result to the problem (1.1). In fact, we will obtain results on comparison and positivity for problems a little more general than (1.1). First, we define an order, \preceq , in $L^q(\Omega, \mathbb{R}^n)$, $n \in \mathbb{N}$, $1 < q < \infty$. Let $f = (f_1, f_2, \dots, f_n)$, $g = (g_1, g_2, \dots, g_n) \in L^q(\Omega, \mathbb{R}^n)$, we will say that $f \preceq g$ if $f_i \leq g_i$, $i = 1, 2, \dots, n$, where \leq is the usual order in $L^q(\Omega)$. Clearly $(L^q(\Omega, \mathbb{R}^n), \preceq)$ is an ordered Banach space.

The next two results are adapted from [8], and will be used to show that the linear operators associated to problems like (1.1) have positive resolvent in $L^q(\Omega, \mathbb{R}^n)$, in the sense of Definition 3.3.

LEMMA 3.1. *Let H be a Hilbert space and $f \in H$. If there exists $\tilde{f} \in H$ such that*

$$\|\tilde{f}\| \leq \|f\| \text{ and } \langle \tilde{f}, f \rangle \geq |\langle f, f \rangle|,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in H , then $\tilde{f} = f$.

THEOREM 3.2. *Let (H, \leq) be an ordered Hilbert space, C its positive cone and $A : D(A) \subset H \rightarrow H$ be a positive self adjoint operator; that is, $\langle Au, u \rangle \geq 0$, for all $u \in D(A)$. Suppose that there exists a dense subset $\mathcal{D} \subset H$ such that:*

1. $(A + \alpha)^{-1}\mathcal{D} \subset \mathcal{D}$, for all $\alpha \geq 0$;
2. for each $d \in \mathcal{D}$, we can define $|d| \in \mathcal{D} \cap C$ such that $\|d\| = \||d|\|$. Furthermore, if $d \in \mathcal{D}$, then $d \in C$ if and only if $d = |d|$;
3. for all $d \in \mathcal{D}$ and $g \in C$, $\langle |d|, g \rangle \geq |\langle d, g \rangle|$;
4. if $u \in D(A^{1/2})$, then $|u| \in D(A^{1/2})$ and $\langle A^{1/2}|u|, A^{1/2}|u| \rangle \leq \langle A^{1/2}u, A^{1/2}u \rangle$.

Then, A has positive resolvent in H .

Proof. Consider in $D(A^{\frac{1}{2}})$ the inner product given by

$$\langle f, g \rangle_1 = \langle A^{\frac{1}{2}}f, A^{\frac{1}{2}}g \rangle + \alpha \langle f, g \rangle,$$

for $f, g \in D(A^{\frac{1}{2}})$ and for all $\alpha > 0$. Let $X^{\frac{1}{2}}$ be the Hilbert space defined by $X^{\frac{1}{2}} := (D(A^{\frac{1}{2}}), \langle \cdot, \cdot \rangle_1)$.

Let $g \in \mathcal{D}$ be such that $g \in C$ and $c = (A + \alpha)^{-1}g$. Then, since $A^{\frac{1}{2}}$ is self adjoint, we have that

$$\langle |c|, c \rangle_1 = \langle (A + \alpha)|c|, (A + \alpha)^{-1}g \rangle \geq |\langle c, g \rangle|.$$

Additionally,

$$\||c|\|_1^2 = \langle A^{\frac{1}{2}}|c|, A^{\frac{1}{2}}|c| \rangle + \||c|\|^2 \leq \|c\|_1^2.$$

Using Lemma 3.1 with $f = c$ and $\tilde{f} = |c|$, we conclude that if $g \in \mathcal{D} \cap C$, then

$$|(A + \alpha)^{-1}g| = |c| = c = (A + \alpha)^{-1}g$$

and, therefore $(A + \alpha)^{-1}g \in C$.

Now, the density of \mathcal{D} in H and the continuity of $(A + \alpha)^{-1}$, implies that for all $g \in C$, $(A + \alpha)^{-1}g \in C$. This shows that A has positive resolvent in H . \blacksquare

For $n \geq 1$ and $1 < q < \infty$, consider the operator \tilde{B} defined by

$$\tilde{B} : D(\tilde{B}) \subset L^q(\Omega, \mathbb{R}^n) \rightarrow L^q(\Omega, \mathbb{R}^n),$$

with

$$\begin{aligned} D(\tilde{B}) &= \{ \phi \in W^{2,q}(\Omega, \mathbb{R}^n) ; D \frac{\partial \phi}{\partial \bar{n}} = -K_n \phi \text{ em } \Gamma \}; \\ \tilde{B}\phi &= -D\Delta\phi + c\phi, \end{aligned} \quad (3.2)$$

for $\phi \in D(\tilde{B})$, where $K_n := \{k_{ij}\}_{n \times n}$, hereafter called coupling matrix, is a symmetric matrix of order n , such that for all $u \in \mathbb{R}^n$,

$$K_n u \cdot u \geq 0, \quad (3.3)$$

$$K_n |u| \cdot |u| \leq K_n u \cdot u, \quad (3.4)$$

where $|(u_1, \dots, u_n)| = (|u_1|, \dots, |u_n|)$, “ \cdot ” is the usual inner product in \mathbb{R}^n , D is an $n \times n$ matrix given by (1.2), with $d_i > 0$ for $i = 1, 2, \dots, n$ and $c \geq 0$ is a constant taken such that the first eigenvalue, $\mu_1(K)$, associated to the operator \tilde{B} be positive. We have the following result:

PROPOSITION 3.1. *The operator \tilde{B} defined in (3.2) is a sectorial operator and has positive resolvent in $L^q(\Omega, \mathbb{R}^n)$, for all $n \geq 1$.*

Proof. Firstly, we will use Theorem 3.2 to show that \tilde{B} has positive resolvent in $L^2(\Omega, \mathbb{R}^n)$.

After integrating by parts we have that, for all $\phi \in D(\tilde{B})$,

$$\langle \tilde{B}\phi, \phi \rangle_{L^2(\Omega, \mathbb{R}^n)} = \int_{\Omega} |\nabla \phi(x)|^2 + \int_{\partial\Omega} K_n \phi \cdot \phi + c \|\phi\|_{L^2(\Omega, \mathbb{R}^n)}^2,$$

hence (3.3) shows that \tilde{B} is an positive operator. The symmetry of the matrix K_n and a computation similar to the above show that

$$\langle \tilde{B}\phi, \psi \rangle_{L^2(\Omega, \mathbb{R}^n)} = \langle \phi, \tilde{B}\psi \rangle_{L^2(\Omega, \mathbb{R}^n)},$$

for every $\psi, \phi \in \Delta(\tilde{B})$. But 0 is in the resolvent set of \tilde{B} , then we conclude that \tilde{B} is self adjoint.

Now, we consider the set \mathcal{D} , in the Theorem 3.2, as $L^2(\Omega, \mathbb{R}^n)$ itself. Therefore the condition 1 of Theorem 3.2 is clearly satisfied. Let $f = (f_1, \dots, f_n)$ be a function in $L^2(\Omega, \mathbb{R}^n)$ and consider $|f| = (|f_1|, \dots, |f_n|)$, where for each $i = 1, 2, \dots, n$, $|f_i|(x) = |f_i(x)|$, $x \in \Omega$. With the definition of order in $L^2(\Omega, \mathbb{R}^n)$, given above, the condition 2 follows immediately. The condition 3 is easily verified. It remains to verify the condition 4. But, for every $\phi \in D(\tilde{B})$, integrating by parts and using (3.4), we have

$$\langle \tilde{B}|\phi|, |\phi| \rangle_{L^2(\Omega, \mathbb{R}^n)} \leq \langle \tilde{B}\phi, \phi \rangle_{L^2(\Omega, \mathbb{R}^n)},$$

since $D(\tilde{B})$ is dense in $D(\tilde{B}^{1/2})$ and using that \tilde{B} is closed operator, a density argument shows the condition 4.

Hence, Theorem 3.2 implies that $(\tilde{B} + \alpha)^{-1}$ is increasing for all $\alpha > 0$, in $L^2(\Omega, \mathbb{R}^n)$.

Now, for $q \geq 2$, we have that $L^q(\Omega, \mathbb{R}^n) \stackrel{d}{\hookrightarrow} L^2(\Omega, \mathbb{R}^n)$, \tilde{B} has positive resolvent in $L^2(\Omega, \mathbb{R}^n)$ and the positive cone of $L^q(\Omega, \mathbb{R}^n)$ is contained in the positive cone of $L^2(\Omega, \mathbb{R}^n)$. Thus, simple computations show that \tilde{B} has positive resolvent in $L^q(\Omega, \mathbb{R}^n)$, $q \geq 2$.

Finally, if $1 < q \leq 2$, since the positive cone of $L^2(\Omega, \mathbb{R}^n)$ is densely embedded in the positive cone of $L^q(\Omega, \mathbb{R}^n)$, \tilde{B} has positive resolvent in $L^2(\Omega, \mathbb{R}^n)$ and $(\tilde{B} + \alpha)^{-1}$ is continuous in $L^q(\Omega, \mathbb{R}^n)$, a density argument shows that \tilde{B} has positive resolvent in $L^q(\Omega, \mathbb{R}^n)$. ■

As in Section 1, we can construct a scale of Banach spaces, X_q^α , $\alpha \in [-1, 1]$, $1 < q < \infty$, for the operator \tilde{B} defined in (3.2), the Proposition 3.1, the Theorem 2.7.2 in [2] and the fact that $\mu_1(k)$ is positive show the following result:

THEOREM 3.3. *For all $1 < q < \infty$ and $\alpha \in [0, 1]$, X_q^α is an ordered Banach space with the order induced by $L^q(\Omega, \mathbb{R}^n)$. The space $X_q^{-\alpha}$ is an ordered Banach space with the dual order; that is, $0 \leq u$ in $X_q^{-\alpha}$ if and only if $0 \leq \langle u, \phi \rangle$ for all ϕ in the positive cone of X_q^α .*

For $\gamma \geq \beta$, the embedding $X_q^\gamma \hookrightarrow X_q^\beta$ is positive and the positive cone of X_q^γ is dense in the positive cone of X_q^β .

Furthermore, $B_{\alpha-1} : X_q^\alpha \rightarrow X_q^{\alpha-1}$ is a sectorial operator with positive resolvent.

With all of this, we are prepared to obtain the results on comparison and positivity for solutions of the following semilinear parabolic problem, with boundary coupling:

$$\begin{cases} u_t(t, x) = D\Delta u(t, x) + F(u(t, x)) & t > 0, \quad x \in \Omega, \\ D \frac{\partial u}{\partial \vec{n}}(t, x) = -K_n u(t, x) + G(u(t, x)) & t > 0, \quad x \in \Gamma, \end{cases} \quad (3.5)$$

where u, k, Ω, Γ and D and $F = (F_1, F_2, \dots, F_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $G = (G_1, G_2, \dots, G_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are as before.

For smooth initial data, an immediate consequence of the abstract results in this section we have the next result on comparison and positivity of solutions to (1.1). Since the solutions of (1.1) depend continuously on the initial data in $W^{1,q}(\Omega, \mathbb{R}^n)$ or in $L^q(\Omega, \mathbb{R}^n)$, we obtain the next proposition.

PROPOSITION 3.2. *Let \tilde{B} be the operator defined in (3.2). Suppose that for $i = 1, \dots, n$, $F_i, G_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are locally Lipschitz functions satisfying the conditions (C1) and (C2) defined in Section 2.*

(i) Suppose that $F_i(0, \dots, 0), G_i(0, \dots, 0) \geq 0$, $i = 1, 2, \dots, n$. If $0 \preceq u_0$ then, $0 \preceq u(t; u_0)$, for as long as they exist.

(ii) If $u_0 \preceq u_1$ then $u(t; u_0) \preceq u(t; u_1)$, for as long as they exist.

(iii) Suppose that for all $u \in \mathbb{R}^n$, $F_0^i(u) \leq F_i(u)$ and $G_0^i(u) \leq G_i(u)$, for $i = 1, 2, \dots, n$. If $u_0 \preceq u_1$ then $u(t; u_0, F_0, G_0) \preceq u(t; u_1, F, G)$, for as long as they exist, where $F_0 = (F_0^1, \dots, F_0^n)$ and $G_0 = (G_0^1, \dots, G_0^n)$, F and G are as before.

Where “ \preceq ” denotes the order defined above for the space where the initial data u_i , $i = 0, 1$ belongs; that is, $L^q(\Omega, \mathbb{R}^n)$ or $W^{1,q}(\Omega, \mathbb{R}^n)$.

4. GLOBAL EXISTENCE AND EXISTENCE OF GLOBAL ATTRACTORS

In this section we will show that the local solutions to the problem (1.1) found in Theorem 2.1 are globally defined, to that end, we use the comparison and positivity results given

in Proposition 3.2. We also obtain the existence of global attractors for (1.1), as well as a $L^\infty(\Omega, \mathbb{R}^N)$ bounds for the attractors, uniformly with respect to the matrix D .

Now consider the problem

$$\begin{cases} u_t(t, x) = D\Delta u(t, x) + F(u(t, x)) & t > 0, \quad x \in \Omega, \\ D\frac{\partial u}{\partial n}(t, x) = G(u(t, x)) & t > 0, \quad x \in \Gamma, \end{cases} \quad (4.1)$$

where u, k, Ω, Γ, D are as before and $F = (F_1, \dots, F_n), G = (G_1, \dots, G_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies the previously imposed conditions.

Taking $K_n \equiv 0$, in the definition of \tilde{B} , the Proposition 3.2 ensures comparison and positivity results for the solutions to (4.1), and with the same reasoning used as in [4], we obtain the following result:

THEOREM 4.1. *Suppose that for $i = 1, 2, \dots, n$ F_i, G_i satisfy the growth conditions **(C1)**, or **(C2)** and (2.7). Then*

(i) *For each $u_0 \in X$, ($X = L^q(\Omega, \mathbb{R}^n)$ or $X = W^{1,q}(\Omega, \mathbb{R}^n)$), the solutions found for (4.1), are globally defined.*

(ii) *Suppose, in addition, that the first eigenvalue, $\lambda_1(D)$, of (2.8) be positive. Then, the problem (4.1) has a global attractor $\mathcal{A}_X(D)$ in X such that*

$$\mathcal{A}_X(D) \subset \Sigma(\phi) := \{\phi \in L^\infty(\Omega, \mathbb{R}^n) : |u_i(x)| \leq \phi_i(x), \quad i = 1, 2, \dots, n \quad x \in \Omega\},$$

where ϕ is the solution to the elliptic problem:

$$\begin{cases} -D\Delta\phi + C_0\phi = C_1 & \text{in } \Omega, \\ D\frac{\partial\phi}{\partial n} + B_0\phi = B_1 & \text{in } \Gamma, \end{cases} \quad (4.2)$$

where C_0, C_1, B_0 and B_1 are given in (2.7). Furthermore, for $2\alpha < 1 + \frac{1}{r}$, and $r \geq q$, $\mathcal{A}_X(D) \subset E_r^\alpha$, is compact in this space and attracts bounded subsets of X , in the norm of E_r^α .

The next result ensures an estimate to the attractor $\mathcal{A}_X(D)$ of (4.1), uniform with respect to the matrix D , if $d_i > 1$, $i = 1, 2, \dots, n$. The idea is to use **(ii)** of Theorem 4.1. We start with the following simple lemma.

LEMMA 4.1. *Let $a \geq 1$ and $\lambda_1(a)$ be the first eigenvalue of the problem*

$$\begin{cases} -a\Delta u + C_0u = \lambda u & \text{in } \Omega, \\ a\frac{\partial u}{\partial n} + B_0u = 0 & \text{in } \Gamma. \end{cases}$$

Then, $\lambda_1(a) \geq \lambda_1 > 0$, where λ_1 is the first eigenvalue of the above problem with $a = 1$.

An immediate consequence of Lemma 4.1 is that if, $d_i \geq 1, i = 1, 2, \dots, n$, then the first eigenvalue, $\lambda_1(D)$, of (2.8) satisfies $\lambda_1(D) \geq \lambda_1 > 0$, where λ_1 is the first eigenvalue of the problem

$$\begin{aligned} -\Delta u + C_0 u &= \lambda u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \bar{n}} + B_0 u &= 0 \quad \text{in } \Gamma. \end{aligned} \tag{4.3}$$

Additionally, the constants C_0 and B_0 in (2.7), can be taken independently of $d_i \geq 1, i = 1, 2, \dots, n$.

LEMMA 4.2. *Let $\phi = (\phi_1, \phi_2, \dots, \phi_n) \in H^1(\Omega, \mathbb{R}^n)$ be a weak solution to the elliptic problem (4.2), d given in (2.9) and $\lambda_1 > 0$ be the first eigenvalue of (4.3). If $d \geq 1$, then*

$$\|\phi\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq K,$$

where $K = K(\Omega, \Gamma, C_0, B_0, C_1, B_1, N, \lambda_1)$ is a positive constant independently of $d \geq 1$.

Proof. Throughout this proof, let us fix $i \in \{1, 2, \dots, n\}$ and show first that $\|\phi_i\|_{L^2(\Omega)} \leq K_1$, where $K_1 = K_1(\Omega, \Gamma, B_1, B_0, C_1, C_0, \lambda_1)$ and ϕ_i is the solution of

$$\begin{aligned} -d_i \Delta \phi_i + C_0 \phi_i &= C_1 \quad \text{em } \Omega, \\ d_i \frac{\partial \phi_i}{\partial \bar{n}} + B_0 \phi_i &= B_1 \quad \text{em } \Gamma. \end{aligned} \tag{4.4}$$

Multiplying the first equation in (4.4) by ϕ_i and integrating by parts, we obtain that

$$d_i \int_{\Omega} |\nabla \phi_i|^2 + C_0 \int_{\Omega} \phi_i^2 - C_1 \int_{\Omega} \phi_i + B_0 \int_{\Gamma} \phi_i^2 - B_1 \int_{\Gamma} \phi_i = 0. \tag{4.5}$$

Taking $\psi = \phi_i - \frac{B_1}{2B_0}$, and using that $\lambda_1(d_i) > \lambda_1 > 0$, where $\lambda_1(d_i)$ is the first eigenvalue of the linear problem associated to (4.4), we have that

$$\begin{aligned} \lambda_1(d_i) \int_{\Omega} \psi^2 + \left(\frac{C_0 B_1}{B_0} - C_1\right) \int_{\Omega} \psi &\leq \int_{\Omega} (d_i |\nabla \psi|^2 \\ &+ C_0 \psi^2) + B_0 \int_{\Gamma} \psi^2 + \left(\frac{C_0 B_1}{B_0} - C_1\right) \int_{\Omega} \psi. \end{aligned}$$

Substituting ψ and using (4.5), it follows that

$$\lambda_1(d_i) \int_{\Omega} \psi^2 + K_2 \int_{\Omega} \psi + K_3 \leq 0,$$

where

$$K_2 := -C_1 + \frac{C_0 B_1}{B_0} \quad \text{e} \quad K_3 := |\Gamma| \frac{B_0 B_1^2}{4B_0^2} + \left(\frac{C_1 B_1}{2B_0} - \frac{3}{4} \frac{C_0 B_1^2}{B_0^2}\right) |\Omega|.$$

But, the Young inequality implies that for all $\varepsilon > 0$,

$$-K_2 \int_{\Omega} \psi - K_3 \leq |K_3| + |K_2| |\Omega| \varepsilon / 2 + \frac{|K_2|}{2\varepsilon} \int_{\Omega} |\psi|^2.$$

Hence, since $d_i \geq 1$, it follows from Lemma 4.1 that

$$\left(\lambda_1 - \frac{|K_2|}{2\varepsilon}\right) \int_{\Omega} |\psi|^2 \leq \left(\lambda_1(d_i) - \frac{|K_2|}{2\varepsilon}\right) \int_{\Omega} |\psi|^2 \leq |K_3| + \frac{|K_2||\Omega|}{2}\varepsilon,$$

where λ_1 is the first eigenvalue of (4.3). Taking $\varepsilon = |K_2|/\lambda_1$, we obtain that

$$\|\psi\|_{L^2(\Omega)} \leq \frac{1}{\lambda_1} \sqrt{2|K_3| \sqrt{\lambda_1} + |K_2|^2 |\Omega|}$$

and, therefore,

$$\|\phi_i\|_{L^2(\Omega)} \leq K_1,$$

where $K_1 = K_1(\Omega, \Gamma, C_0, C_1, B_0, B_1, \lambda_1) > 0$.

Next we show the limitation in $L^\infty(\Omega)$. Suppose that $\frac{2N}{N+2} \leq p < \frac{N}{2}$. We consider

$$r := \frac{(N-2)p}{N-2p} \geq 2 \quad \text{and} \quad k \geq 1.$$

We define $\phi = (\phi_i - k)^+$. Multiplying the first equation in (4.4) by $|\phi|^{r-2}\phi$ and integrating by parts, we obtain

$$\begin{aligned} & \frac{4(r-1)}{r^2} \int_{\Omega} d_i |\nabla |\phi|^{r/2}|^2 + C_0 \int_{\Omega} (|\phi|^{r/2})^2 + B_0 \int_{\Gamma} (|\phi|^{r/2})^2 \\ &= \int_{\Omega} (C_1 - kC_0) |\phi|^{r-2} \phi + \int_{\Gamma} (B_1 - kB_0) |\phi|^{r-2} \phi. \end{aligned}$$

Calling $\tilde{C} := C_1 - kC_0$ and $\tilde{B} := B_1 - kB_0$, the Hölder inequality implies that

$$\begin{aligned} & \frac{4(r-1)}{r^2} \int_{\Omega} d_i |\nabla |\phi|^{r/2}|^2 + C_0 \int_{\Omega} (|\phi|^{r/2})^2 + B_0 \int_{\Gamma} (|\phi|^{r/2})^2 \\ & \leq |\tilde{C}| |\Omega|^{1/p} \|\phi|^{r/2}\|_{L^{2p'/r'}(\Omega)}^{2/r'} + |\tilde{B}| |\Gamma|^{\frac{N-p}{(N-1)p}} \|\phi|^{r/2}\|_{L^{\frac{2p'(N-1)}{Nr'}}(\Gamma)}^{2/r'}, \end{aligned}$$

making $v := |\phi|^{r/2}$ and since that $p'/r' = N/(N-2)$, we obtain that

$$\begin{aligned} & \frac{4(r-1)}{r^2} \int_{\Omega} d_i |\nabla v|^2 + (C_0 + \lambda) \int_{\Omega} |v|^2 + B_0 \int_{\Gamma} |v|^2 \\ & \leq |\bar{C}| \|v\|_{L^{\frac{2N}{N-2}}(\Omega)}^{2/r'} + |\bar{B}| \|v\|_{L^{\frac{2(N-1)}{N-2}}(\Gamma)}^{2/r'} + \lambda \int_{\Omega} |v|^2, \end{aligned}$$

where $\bar{C} = |\tilde{C}| |\Omega|^{1/p}$ and $\bar{B} = |\tilde{B}| |\Gamma|^{\frac{N-p}{(N-1)p}}$.

Now, using Sobolev embeddings

$$H^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega) \quad \text{and} \quad H^1(\Omega) \hookrightarrow L^{2(N-1)/(N-2)}(\Gamma),$$

with embedding constant $C = C(\Omega, \Gamma, N)$, for details see [1]. Then

$$\begin{aligned} & \frac{4(r-1)}{r^2} \int_{\Omega} d_i |\nabla v|^2 + (C_0 + \lambda) \int_{\Omega} |v|^2 + B_0 \int_{\Gamma} |v|^2 \\ & \leq C|\overline{C}| \|v\|_{H^1(\Omega)}^{2/r'} + C|\overline{B}| \|v\|_{H^1(\Omega)}^{2/r'} + \lambda \int_{\Omega} |v|^2. \end{aligned}$$

We choose $\lambda = \lambda(\Omega, B_0, C_0) > 0$ such that

$$\frac{4(r-1)}{r^2} \int_{\Omega} d_i |\nabla v|^2 + (C_0 + \lambda) \int_{\Omega} |v|^2 + B_0 \int_{\Gamma} |v|^2 \geq \frac{4(r-1)}{r^2} d_i (\|v\|_{H^1(\Omega)})^2.$$

Hence,

$$\frac{(r-1)}{r^2} d_i (\|v\|_{H^1(\Omega)})^2 \leq (C|\overline{C}| + C|\overline{B}|) \|v\|_{H^1(\Omega)}^{2/r'} + \lambda \|v\|_{L^2(\Omega)}^{2/r} \|v\|_{H^1(\Omega)}^{2/r'},$$

and so,

$$\frac{r-1}{r^2} d_i \|v\|_{H^1(\Omega)}^{2/r} \leq C|\overline{C}| + C|\overline{B}| + \lambda \|v\|_{L^2(\Omega)}^{2/r}.$$

Since $d_i \geq 1$, then

$$\begin{aligned} & \frac{r-1}{r^2} \|\phi\|_{L^{Np/(N-2p)}(\Omega)} = \frac{r-1}{r^2} \|v\|_{L^{2N/(N-2)}(\Omega)}^{2/r} \leq \frac{r-1}{r^2} d_i \|v\|_{H^1(\Omega)}^{2/r} \\ & \leq C(C|\overline{C}| + C|\overline{B}| + \lambda \|v\|_{L^2(\Omega)}^{2/r}) = C(C|\overline{C}| + C|\overline{B}| + \lambda \|\phi\|_{L^r(\Omega)}) \end{aligned}$$

But $2 \leq r \leq \frac{Np}{N-2p}$, then, for each $\varepsilon > 0$, there exists $C_\varepsilon(\Omega, N, p)$ such that

$$\|\phi\|_{L^r(\Omega)} \leq \varepsilon \|\phi\|_{L^{\frac{Np}{N-2p}}(\Omega)} + C_\varepsilon \|\phi\|_{L^2(\Omega)}.$$

Thus,

$$\left(\frac{r-1}{r^2} - \lambda C_\varepsilon\right) \|\phi\|_{L^{Np/(N-2p)}(\Omega)} \leq C(C|\overline{C}| + C|\overline{B}| + \lambda C_\varepsilon \|\phi\|_{L^2(\Omega)}).$$

Therefore,

$$\|\phi\|_{L^{Np/(N-2p)}(\Omega)} \leq K_4 [C_1 + |B_1| + k + \|\phi\|_{L^2(\Omega)}],$$

where $K_4 := K_4(\Omega, \Gamma, N, p, B_0, C_0) > 0$.

Now, for all $q \geq \frac{2N}{N-2}$, there is $p_0 \in [\frac{2N}{N+2}, \frac{N}{2})$ such that $q = \frac{Np_0}{N-2p_0}$. Hence,

$$\|\phi\|_{L^q(\Omega)} \leq K_4[|C_1| + |B_1| + k + \|\phi_i\|_{L^2(\Omega)}], \quad (4.6)$$

with $K_4 := K_4(\Omega, \Gamma, N, q, B_0, C_0) > 0$.

For $k \geq 1$, consider the set $A_k := \{x \in \Omega : \phi_i(x) > k\}$. As before, for $\phi = (\phi_i - k)^+$, there exists $\lambda = \lambda(\Omega, B_0, C_0, N) > 0$ such that

$$\int_{\Omega} d_i |\nabla \phi|^2 + \int_{\Gamma} B_0 \phi^2 + \int_{\Omega} (C_0 + \lambda) \phi^2 \geq d_i \|\phi\|_{H^1(\Omega)}^2 \geq \|\phi\|_{H^1(\Omega)}^2. \quad (4.7)$$

Also, multiplying the first equation in (4.4) by ϕ and integrating by parts, we obtain that

$$\begin{aligned} & \int_{\Omega} d_i |\nabla \phi|^2 + \int_{\Gamma} B_0 \phi^2 + \int_{\Omega} (C_0 + \lambda) \phi^2 \\ &= \int_{\Omega} (C_1 - kC_0) \phi + \int_{\Gamma} (B_1 - kB_0) \phi + \lambda \int_{\Omega} \phi^2. \end{aligned} \quad (4.8)$$

Therefore, (4.7), (4.8), the Sobolev embeddings above and Holder inequality imply that

$$\begin{aligned} (\|\phi\|_{L^1(\Omega)} |A_k|^{-\frac{N+2}{2N}}) \|\phi\|_{H^1(\Omega)} &\leq \|\phi\|_{L^{\frac{2N}{N-2}}(\Omega)} \|\phi\|_{H^1(\Omega)} \\ &\leq C \left(\int_{\Omega} |C_1 - kC_0| |\phi| + \int_{\Gamma} |B_1 - kB_0| |\phi| + \lambda \int_{\Omega} |\phi|^2 \right), \end{aligned}$$

where $C = C(\Omega, \Gamma, N) > 0$. Estimating each one of the terms in the above inequality it follows that

$$\begin{aligned} \int_{\Omega} |C_1 - kC_0| |\phi| &\leq |C_1 - kC_0| C \|\phi\|_{H^1(\Omega)} |A_k|^{\frac{1}{2}}, \\ \int_{\Gamma} |B_1 - kB_0| |\phi| &\leq |B_1 - kB_0| C \|\phi\|_{H^1(\Omega)} |A_k|^{\frac{1}{2}}. \end{aligned}$$

Finally

$$\lambda \int_{\Omega} |\phi|^2 \leq \lambda \|\phi\|_{L^q(\Omega)} \|\phi\|_{L^{\frac{2N}{N-2}}(\Omega)} |A_k|^{\frac{1}{2} + \frac{1}{N} - \frac{1}{q}} \leq \lambda C \|\phi\|_{L^q(\Omega)} \|\phi\|_{H^1(\Omega)} |A_k|^{\frac{1}{2} + \frac{1}{N} - \frac{1}{q}}.$$

Taking $q = 2N$ and using (4.6), it follows that

$$\lambda \int_{\Omega} |\phi|^2 \leq \lambda C K_4 (|C_1| + |B_1| + k + \|\phi_i\|_{L^2(\Omega)}) \|\phi\|_{H^1(\Omega)} |A_k|^{1/2 + 1/2N},$$

with $K_4 = K_4(\Omega, \Gamma, B_0, C_0, N)$. Therefore,

$$\begin{aligned} (\|\phi\|_{L^1(\Omega)} |A_k|^{-\frac{N+2}{2N}}) \|\phi\|_{H^1(\Omega)} &\leq (C |A_k|^{1/2} ((|C_1| + k|C_0|) + (|B_1| + k|B_0|)) \\ &+ \lambda C |A_k|^{1/2N} (|C_1| + |B_1| + k + \|\phi_i\|_{L^2(\Omega)})) \|\phi\|_{H^1(\Omega)}, \end{aligned}$$

that is,

$$\|\phi\|_{L^1(\Omega)} \leq |A_k|^{1+\varepsilon} K_5 k (|C_1| + |B_1| + k + \|\phi_i\|_{L^2(\Omega)}),$$

where $K_5 = K_5(\Omega, \Gamma, B_0, C_0, N) > 0$, $\lambda = \lambda(\Omega, B_0, C_0, N) > 0$ and $\varepsilon = \varepsilon(N)$.

Thus, Lemma 5.1 in [12] implies that

$$\max\{\phi_i(x); x \in \Omega\} \leq K_6 (|C_1| + |B_1| + \|\phi_i\|_{L^2(\Omega)}),$$

where $K_6 = K_6(\Omega, \Gamma, C_0, B_0, N)$. Exchanging ϕ_i by $-\phi_i$, with a similar reasoning we obtain that

$$\|\phi\|_{L^\infty(\Omega)} \leq K_6 (|C_1| + |B_1| + \|\phi_i\|_{L^2(\Omega)}),$$

The first part of this lemma implies that

$$\|\phi\|_{L^\infty(\Omega)} \leq K,$$

where $K = K(\Omega, \Gamma, B_0, B_1, C_0, C_1, N, \lambda_1) > 0$ and λ_1 is the first eigenvalue of (4.3), completing the proof of the lemma. ■

Finally, the next result ensures an estimate of the attractor $\mathcal{A}_X(D)$, uniform with respect to D if $d_i \geq 1$, $i = 1, 2, \dots, n$.

COROLLARY 4.1. *Let d be as before. If $d \geq 1$, then there exists $\mathcal{M} > 0$, independently of D , such that*

$$\mathcal{A}_X(D) \subset \mathcal{B} := \{\phi \in L^\infty(\Omega, \mathbb{R}^n) ; \|\phi\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq \mathcal{M}\},$$

and attracts bounded subsets of X , through the flow defined by (4.1), in the norm of E_r^α , $2\alpha < 1 + \frac{1}{r}$ and $r \geq q$.

Proof. The Lemmas 4.1, 4.2 and the item (ii) of Theorem 4.1 prove the corollary. ■

5. PROOF OF THEOREM 2.3

In this section we prove Theorem 2.3. Firstly recall that with the notations of Section 2, the operator A_D defined in (2.1), generates an analytic semigroup, $\{e^{-A_D t}; t \geq 0\}$, in E_q^α , satisfying the following estimates

$$\begin{aligned} \|e^{-A_D t} w\|_{E_q^\alpha} &\leq K_1 e^{-(d\delta)t} \|w\|_{E_q^\alpha}, \quad w \in U_\alpha^\perp, t \geq 0; \\ \|e^{-A_D t} w\|_{E_q^\alpha} &\leq K_1 e^{-(d\delta)t} t^{-(\alpha+r)} \|w\|_{E_q^{-r}}, \quad w \in U_{-r}^\perp, t > 0, \end{aligned} \tag{5.1}$$

where $K_1 > 0$, $\delta > 0$, d are as in (2.9), $q \in (1, \infty)$, α and r are taken in such a way that $2r > 1/q'$ and $\alpha + r < 1$. We observe that these restrictions imply that $\alpha < 1/2 + 1/2q$.

LEMMA 5.1. *Let $F, G : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be functions satisfying the conditions of previous sections. Consider the decomposition of E^α given in (2.10), $P : U + U_\alpha^\perp \rightarrow U$ and $Q : U + U_\alpha^\perp \rightarrow L^q(\Omega, \mathbb{R}^n)$ defined in (2.13). Suppose that α and r satisfy*

$$\alpha > 1/2q, \quad \alpha + r < 1 \quad \text{and} \quad \frac{1}{2q'} < r < \frac{N}{2q'}, \quad (5.2)$$

Define $H : U + U_\alpha^\perp \rightarrow E_q^{-r}$, acting in functions $\phi \in E_q^r$ in the following way:

$$\langle H(v, w), \phi \rangle = \int_\Omega Q(v + w(x))\phi(x)dx + \int_\Gamma G(v + w(y))\phi(y)dy. \quad (5.3)$$

Then, fixed $R > 0$, there are positive constants N_H, L_H, N_P, L_P such that

$$\begin{aligned} \|H(v, w)\|_{E_q^{-r}} &\leq N_H \\ \|H(v_1, w_1) - H(v_2, w_2)\|_{E_q^{-r}} &\leq L_H(|v_1 - v_2| + \|w_1 - w_2\|_{U_\alpha^\perp}) \\ |P(v, w)|_{\mathbb{R}^n} &\leq N_P \\ |P(v_1, w_1) - P(v_2, w_2)|_{\mathbb{R}^n} &\leq L_P(|v_1 - v_2| + \|w_1 - w_2\|_{U_\alpha^\perp}), \end{aligned}$$

for all $\|v_1 + w_1\|_{E^\alpha}, \|v_2 + w_2\|_{E^\alpha}, \|v + w\|_{E^\alpha} \leq R$.

Proof. Let us consider $u_1 = v_1 + w_1, u_2 = v_2 + w_2, u = v + w \in E^\alpha$, satisfying the conditions of the lemma. Then we have

$$\begin{aligned} |P(v, w)|_{\mathbb{R}^n} &\leq \int_\Omega |F(v + w(x))|_{\mathbb{R}^n} dx + \int_\Gamma |G(v + w(y))|_{\mathbb{R}^n} dy \\ &\leq \|F_\Omega(v + w)\|_{L^q(\Omega, \mathbb{R}^n)} + |\Gamma|^{1/q'} \|G_\Gamma\|_{L^q(\Gamma, \mathbb{R}^n)} \leq N_P, \end{aligned}$$

where $N_P := N_F(R, q, N) + K(\Gamma, q, N)N_G(R, q, N)$, and for every $\phi \in E_q^r$,

$$\begin{aligned} |\langle H(v, w), \phi \rangle| &\leq (\|F_\Omega(v + w)\|_{L^q(\Omega, \mathbb{R}^n)} + |P(v, w)|_{\mathbb{R}^n}) \|\phi\|_{L^{q'}(\Omega, \mathbb{R}^n)} \\ &\quad + \|G_\Gamma(v + w)\|_{L^q(\Gamma, \mathbb{R}^n)} \|T(\phi)\|_{L^{q'}(\Gamma, \mathbb{R}^n)}, \end{aligned}$$

where T denotes the trace operator. Now, with the values of α and r given in (5.2), we obtain that

$$\begin{aligned} |\langle H(v, w), \phi \rangle| &\leq (N_F(R, q, N) + N_P + N_G(R, q, N)K(r, q, N, \Omega, \Gamma)) \|\phi\|_{E_q^r} \\ &\leq N_H \|\phi\|_{E_q^r}. \end{aligned}$$

Hence,

$$\|H(v, w)\|_{E_q^{-r}} \leq N_H,$$

where $N_H := N_F(R, q, N) + N_P + N_G(R, q, N)K(r, q, N, \Omega, \Gamma)$.

With the same reasoning, we show the other two inequalities of the lemma. ■

Let \mathcal{A}_0 be the compact global attractor of the ordinary differential equation (1.3), then as in [9], there is $\Sigma : \mathbb{R}^n \rightarrow \mathbb{R}$ a Lipschitz continuous function with Lipschitz constant L_Σ , such that

1. $\Sigma(v) = 0$, for all $v \in \mathcal{A}_0$;
2. $a(d(v, \mathcal{A}_0)) \leq \Sigma(v) \leq b(d(v, \mathcal{A}_0))$, where $a(r)$ is a continuous non-decreasing function, $a(r) > 0$ if $r > 0$, $a(r) \rightarrow +\infty$ when $r \rightarrow +\infty$ and $b(r)$ is a continuous function with $b(0) = 0$;
3. For all $v \in \mathbb{R}^n$,

$$\dot{\Sigma}_{(1.3)}(v) \leq -\Sigma(v),$$

where $\dot{\Sigma}_{(1.3)}(v) = \limsup_{h \rightarrow 0^+} (\Sigma(x(h; v)) - \Sigma(v))/h$, and $x(\cdot; v)$ is a solution of the ODE (1.3), with initial data $x(0; v) = v$.

Fix α and r satisfying (5.2), and define $V_c := \{v \in \mathbb{R}^n; \Sigma(v) < c\}$, for $c > 0$. The property 2 above implies that for all $c > 0$, \bar{V}_c is compact. Furthermore, for every $c > 0$ and $\eta > 0$, consider the set

$$W_{c,\eta} := \{u = v + w \in E^\alpha ; v \in V_c \text{ and } \|w\|_{E^\alpha} < \eta\}. \tag{5.4}$$

The next result shows that for d large enough, the flow of (5.1) stays in $W_{c,\eta}$ if it attained $W_{c,\eta}$, for some $t_0 > 0$. Precisely we have:

LEMMA 5.2. *Let us consider $u_0 = v_0 + w_0 \in X$, α and r satisfying (5.2) and $u(t, \cdot; u_0)$ the solution (1.1) with initial data u_0 . If there exists $t_0 > 0$ such that $u(t_0, \cdot; u_0) \in W_{c,\eta}$, then there is $d_0 > 0$, sufficiently large, such that $u(t, \cdot) := v(t) + w(t, \cdot) \in W_{c,\eta}$, for $t \geq t_0$ and $d > d_0$, where $W_{c,\eta}$ is defined in (5.4).*

Proof. Let $u(t, \cdot; u_0)$ be the solution of (1.1) with initial condition $u_0 \in X$ ($X = L^q(\Omega, \mathbb{R}^n)$ or $X = W^{1,q}(\Omega, \mathbb{R}^n)$). For α satisfying (5.2), consider $u(t, \cdot) = v(t) + w(t, \cdot) \in U + U_\alpha^\perp$, where $v(t) \in \mathbb{R}^n$ and $w(t, \cdot) \in U_\alpha^\perp$, for all $t > 0$. Suppose that $v(t_0) + w(t_0, \cdot) \in W_{c,\eta}$. Then, as before

$$\begin{cases} \frac{d}{dt}v(t) = P(v(t), w(t, \cdot)) & t > 0, \\ \frac{\partial}{\partial t}w(t, x) = D\Delta w(t, x) + Q(v(t), w(t, x)), & t > 0, x \in \Omega, \\ D\frac{\partial w}{\partial \vec{n}} = G(v(t) + w(t, x)) & t > 0, x \in \Gamma, \end{cases} \tag{5.5}$$

with the functions $P : U + U_\alpha^\perp \rightarrow U$ and $Q : U + U_\alpha^\perp \rightarrow L^q(\Omega, \mathbb{R}^n)$ defined as in (2.13). If $H : U + U_\alpha^\perp \rightarrow E_q^{-r}$ acts in functions $\phi \in E_q^r$, as in (5.3), with α and r satisfying (5.2). We

can write the problem (5.5), using the variation of constants formula

$$\begin{cases} \frac{d}{dt}v(t) = F(v(t)) + |\Gamma|G(v(t)) + [P(v(t), w(t)) - F(v(t)) - |\Gamma|G(v(t))], & t > 0, \\ v(0) = v_0, \\ w(t, \cdot) = e^{-A_D t}w_0 + \int_0^t e^{-A_D(t-s)}[H(v(s), w(s, \cdot))]ds, & t > 0, \end{cases} \quad (5.6)$$

with A_D as defined in Section 2.

Now, for all $v \in \mathbb{R}^n$ and $w \in U_\alpha^\perp$, we have

$$\begin{aligned} |P(v, w) - F(v) - |\Gamma|G(v)| &= \left| \int_\Omega F(v + w(x))dx \right. \\ &\quad \left. + \int_\Gamma G(v + w(y))dy - F(v) - |\Gamma|G(v) \right| \\ &\leq \|F_\Omega(v + w) - F_\Omega(v)\|_{L^q(\Omega, \mathbb{R}^n)} + |\Gamma|^{1/q'} \|G_\Gamma(v + w) - G_\Gamma(v)\|_{L^q(\Gamma, \mathbb{R}^n)} \\ &\leq (L_f + K_\Omega L_g |\Gamma|^{1+\frac{1}{q'}}) \|w\|_{E^\alpha}. \end{aligned}$$

Hence, for all $(v(t), w(t, \cdot))$ satisfying (5.5) and such that

$$v(s) \in V_c \quad \text{and} \quad \|w(s, \cdot)\|_{E^\alpha} < \eta,$$

for all $t_0 \leq s \leq t$, (5.6) implies that

$$\begin{aligned} \dot{\Sigma}(v(t)) &\leq \limsup_{h \rightarrow 0} \frac{\Sigma(v(t+h)) - \Sigma(v(t))}{h} \\ &\leq \limsup_{h \rightarrow 0} \frac{\Sigma(v(t+h)) - \Sigma(x(t+h; v(t)))}{h} + \dot{\Sigma}_{(1.3)}(v(t)) \\ &\leq L_\Sigma \limsup_{h \rightarrow 0} \frac{|v(t+h) - x(t+h; v(t))|}{h} - \Sigma(v(t)) \\ &\leq -\Sigma(v(t)) + L_\Sigma |P(v(t), w(t)) - F(v(t)) - |\Gamma|G(v(t))| \\ &\leq -\Sigma(v(t)) + L_\Sigma (L_f + K_\Omega L_g |\Gamma|^{1+\frac{1}{q'}}) \|w(t, \cdot)\|_{E^\alpha}. \end{aligned}$$

Therefore, using the estimates (5.1), Lemma 5.1 and making $w(t) := w(t, \cdot)$ in (5.5), we obtain that

$$\begin{aligned} \|w(t)\|_{E^\alpha} &\leq K_1[e^{-d\delta t}\|w(t_0)\|_{E^\alpha} + \int_{t_0}^t (t-s)^{-\alpha-r} e^{-d\delta(t-s)} \|H(v(s), w(s))\|_{E_q^-} ds] \\ &\leq K_1 e^{-(d\delta)t} \|w(t_0)\|_{E^\alpha} + K_1 \int_{t_0}^t (t-s)^{-\alpha-r} e^{-(d\delta)(t-s)} L_H \|w(s)\|_{E^\alpha} ds \\ &\quad + K_1 \int_{t_0}^t (t-s)^{-\alpha-r} e^{-(d\delta)(t-s)} N_H ds. \end{aligned}$$

Taking $0 < \sigma < \delta$, and multiplying the above expression by $e^{d\sigma t}$ we obtain that

$$\begin{aligned} \|w(t)e^{d\sigma t}\|_{E^\alpha} &\leq K_1 e^{-d(\delta-\sigma)t} \|w(t_0)\|_{E^\alpha} \\ &\quad + K_1 N_H \int_{t_0}^t (t-s)^{-\alpha-r} e^{d\sigma t} e^{-(d\delta)(t-s)} ds \\ &\quad + K_1 L_H \int_{t_0}^t (t-s)^{-\alpha-r} e^{-(d\delta)(t-s)} e^{d\sigma t} \|w(s)\|_{E^\alpha} ds \\ &\leq K_1 e^{-(\delta-\sigma)dt} \|w(t_0)\|_{E^\alpha} + \frac{K_1 N_H L e^{d\sigma t}}{(d\delta)^{1-\alpha-r}} + \frac{K_1 L_H L y(t)}{(d\delta)^{1-\alpha-r}}, \end{aligned}$$

where $L := \int_0^\infty s^{-\alpha-r} e^{-(1-\frac{\sigma}{\delta})s} ds$ and

$$y(t) = \sup\{e^{d\sigma s} \|w(s)\|_{E^\alpha}; t_0 \leq s \leq t\}.$$

Making $z(t) := e^{d\sigma t} \|w(t)\|_{E^\alpha}$, $t \geq t_0$, we obtain that

$$z(t) \leq K_1 e^{-d(\delta-\sigma)t} z(t_0) + \frac{K_1 N_H L e^{d\sigma t}}{(d\delta)^{1-\alpha-r}} + \frac{K_1 L_H L}{(d\delta)^{1-\alpha-r}} y(t).$$

If

$$\xi := (K_1 L_H L) / ((d\delta)^{1-\alpha-r}),$$

then applying the same arguments as above for $t_0 \leq s \leq t$, recalling that $\delta - \sigma > 0$, we obtain that

$$z(s) \leq K_1 e^{-d(\delta-\sigma)s} z(t_0) + \xi y(s) + \frac{K_1 N_H L e^{d\sigma s}}{(d\delta)^{1-\alpha-r}} \leq K_1 z(t_0) + \xi y(t) + \frac{K_1 N_H L e^{d\sigma t}}{(d\delta)^{1-\alpha-r}}.$$

Thus,

$$y(t) \leq K_1 z(t_0) + \xi y(t) + \frac{K_1 N_H L e^{d\sigma t}}{(d\delta)^{1-\alpha-r}}.$$

Since $\alpha + r < 1$, there is $d_0 > 0$ sufficiently large such that $1 - \xi > 0$ for $d \geq d_0$. Then,

$$y(t) \leq K_1(1 - \xi)^{-1} [z(t_0) + \frac{N_H L e^{d\sigma t}}{(d\delta)^{1-\alpha-r}}].$$

Therefore, if $v(s) \in V_c$ and $\|w(s)\|_{E^\alpha} < \eta$, $t_0 \leq s \leq t$, then

$$\dot{\Sigma}(v(t)) \leq -\Sigma(v(t)) + L_\Sigma(L_f + K_\Omega L_g |\Gamma|) \|w(t)\|_{E^\alpha},$$

and

$$\|w(t)\|_{E^\alpha} \leq K_1(1 - \xi)^{-1} [e^{-d\sigma t} \|w(t_0)\|_{E^\alpha} + \frac{N_H L}{(d\delta)^{1-\alpha-r}}].$$

Defining $K_3 := (L_f K_1 + |\Gamma| K_\Omega L_g) L_\Sigma$ and

$$K_4(d) := K_1(1 - \xi)^{-1} [e^{-d\sigma t} \|w(t_0)\|_{E^\alpha} + \frac{N_H L}{(d\delta)^{1-\alpha-r}}],$$

it follows that, for $t_0 \leq s \leq t$,

$$\|w(t)\|_{E^\alpha} \leq K_4(d) \tag{5.7}$$

and

$$\dot{\Sigma}(v(t)) \leq -\Sigma(v(t)) + K_3 K_4(d). \tag{5.8}$$

Using the fact that $1 - \alpha - r > 0$, we can take $d_0 > 0$, sufficiently large such that $c - K_3 K_4(d) > 0$ and $K_4(d) < \eta$, for all $d \geq d_0$. Therefore, (5.7) and (5.8) imply that if $v(t_0) \in V_c$ and $\|w(t_0)\|_{E^\alpha} < \eta$, then

$$v(t) \in V_c \quad \text{e} \quad \|w(t)\|_{E^\alpha} < \eta,$$

for all $t \geq t_0$ e $d \geq d_0$, and this concludes the proof of the lemma. \blacksquare

Now define the set $\mathcal{K} := \{\phi \in X; \|\phi\|_X \leq K\}$. Theorem 4.1 implies that there are positive constants c, η and t_0 such that for all $u_0 \in \mathcal{K}$, $u(t, \cdot; u_0) \in W_{c,\eta}$, for $t \geq t_0$, where $W_{c,\eta}$ is given by (5.4). Lemma 5.2 implies that

$$\gamma^+(W_{c,\eta}) \subset W_{c,\eta}, \tag{5.9}$$

and is relatively compact for all $c > 0$. Also, Lemma 5.2 shows that for $t \geq t_0$,

$$\|w(t)\|_{E^\alpha} \rightarrow 0, \quad \text{as } d \rightarrow +\infty. \tag{5.10}$$

But if $v \in \mathcal{A}_0$, then $\Sigma(v) = 0$. Thus, $W_{c,\eta}$ is a bounded neighborhood of \mathcal{A}_0 and there is a neighborhood, $\mathcal{N}(\mathcal{A}_0)$, of \mathcal{A}_0 such that $W_{c,\eta} \subset \mathcal{N}(\mathcal{A}_0)$ and $\omega(W_{c,\eta}) \subset \mathcal{N}(\mathcal{A}_0)$.

Therefore, making $\eta > 0$ and $c > 0$ large, if necessary, in such a way that $\mathcal{B} \subset W_{c,\eta}$, where \mathcal{B} is given in Corollary 4.1, we obtain that for all $d > d_0$, the global attractor, $\mathcal{A}_X(D)$, of (1.1) satisfies

$$\mathcal{A}_X(D) \subset \mathcal{N}(\mathcal{A}_0).$$

The next lemma shows the existence of the invariant manifold claimed in Theorem 2.3.

LEMMA 5.3. For α and r fixed as in (5.2) and d as in (2.9), consider the decomposition of E^α as in (2.10) and the functions $P : U + U_\alpha^\perp \rightarrow U$ and $H : U + U_\alpha^\perp \rightarrow E_q^{-r}$, as in Lemma 5.1. Suppose also that the estimates (5.1) are satisfied. Consider the problem

$$\begin{cases} \frac{d}{dt}v(t) = P(v(t), w(t, \cdot)), & t > 0, \\ \dot{w}(t) = -A_D w(t) + H(v(t), w(t)), & t > 0. \end{cases} \quad (5.11)$$

Then for d large enough, there is an exponentially attracting invariant manifold, $S_d := \{(v, w); w = \sigma_d(v), v \in \mathbb{R}^n\}$, for (5.11), where $\sigma_d : U \rightarrow U_\alpha^\perp$ satisfies

$$s(d) = \sup_{v \in \mathbb{R}^n} \|\sigma_d(v)\|_{U_\alpha^\perp} \rightarrow 0, \quad \|\sigma_d(v_1) - \sigma_d(v_2)\|_{U_\alpha^\perp} \leq l(d) |v_1 - v_2|_{\mathbb{R}^n}$$

with $l(d) \rightarrow 0$, as $d \rightarrow \infty$.

Proof. The proof of this lemma can be adapted from the proofs in [11], Chapter 6. ■

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