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Morse decompositions in the absence of uniqueness, II

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This paper is a sequel to our previous work [3]. We first extend the concept of $\mathcal{T}$-Morse decompositions to the partially ordered case and prove a generalization of a result by Franzosa and Mischaikow characterizing partially ordered $\mathcal{T}$-Morse decompositions by the so-called $\mathcal{T}$-attractor semifiltrations. Then we extend the (regular) continuation result for Morse decompositions from [3] to the partially ordered case. We also define singular convergence of sequences of 'solution' sets in the spirit of our previous paper [4] and prove various singular continuation results for attractor-repeller pairs and Morse decompositions. We give a few applications of our results, e.g. to thin domain problems. The results of this paper are a main ingredient in the proof of regular and singular continuation results for the homology braid and the connection matrix in infinite dimensional Conley index theory. These topics are considered in the forthcoming publications [6] and [7]. May, 2003 ICMC-USP

1. INTRODUCTION

Let (X, d) be a metric space and $\mathcal{C} = C(\mathbb{R} \to X)$ be the set of all continuous maps from $\mathbb{R}$ to X. Let $\mathcal{T}$ be an arbitrary subset of $\mathcal{C}$. We view $\mathcal{T}$ as a set of full solutions of a semiflow on X or of a differential equation on X which might not define a semiflow. In our previous paper [3] we defined the concepts of $\mathcal{T}$-attractors, $\mathcal{T}$-repellers and $\mathcal{T}$-Morse decompositions $(M_i)_{i\in[\![1,m]\!]}$ (of the first and second kind), extending the corresponding concepts introduced in [8] for the flow case and in [15] and [18] for the semiflow case. We showed that all the main results about Morse decompositions from, e.g., [18] continue to hold in the more general setting. We also defined a concept of convergence of sequences in $\mathcal{C}$ and established various *continuation* results, i.e. perturbation results, for $\mathcal{T}$-attractor-repeller pairs, $\mathcal{T}$-attractor filtrations and $\mathcal{T}$-Morse decompositions (Theorems 2.19, 3.14 and 3.15 in [3]), which were new even in the semiflow case. We applied the latter result to give an alternative proof of a multiplicity result for a variational problem previously

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obtained in [1] by the use of Floer homology. Our continuation result was later applied in the paper [11] to prove a multiplicity conjecture made in [1].

Replacing, in the definition of Morse decompositions (of the second kind), the set $[\![1, m]\!]$ by an arbitrary finite set P and the standard ordering of the integers by an arbitrary (strict) partial order $\prec$ on P we arrive at the more general concept of a partially ordered Morse decomposition $(M_i)_{i \in P}$. This was done for the first time in [9] for the flow case and in [10] for the semiflow case. Partially ordered Morse decompositions are more appropriate for proving the non-existence of certain connections: if i and $j \in P$, but neither $i \prec j$ nor $j \prec i$, then there is no connection between M_i and M_j.

In the present paper we similarly extend the concept of $\mathcal{T}$-Morse decompositions to the partially ordered case. Analogously as in [10], we show that partially ordered $\mathcal{T}$-Morse decompositions can be characterized by the so-called $\mathcal{T}$-attractor semifiltrations (Theorems 2.16 and 2.17).

We then extend the continuation result from [3] to the partially ordered case (Theorem 3.3). Again this is new even in the semiflow case (cf. Corollaries 3.5 and 3.6). We illustrate this result by extending Theorems 4.5 and 4.15 from [3] to the partially ordered case.

In the last section of this paper we define singular convergence of families $(\mathcal{T}_\varepsilon)_{\varepsilon \in]0, \varepsilon_0]}$ of 'solution sets' in the setting of our previous article [4]. We establish various properties of this convergence concept leading to the main singular continuation result for partially ordered $\mathcal{T}$-Morse decompositions (Theorem 4.12). After specializing to the semiflow case we apply this result to some thin domain problems. Some other applications of the abstract results of this paper to parabolic and singularly perturbed hyperbolic equations are given in [17].

The results of this paper are fundamental for proving the regular and singular continuation of the (co)homology index braid and the resulting connection matrix in infinite dimensional Conley index theory. These topics are considered in the forthcoming publications [6] and [7].

In this paper we use the notation and results of [3] without any further comment.

2. PARTIALLY ORDERED MORSE DECOMPOSITIONS

In this section we will define partially ordered $\mathcal{T}$-Morse decompositions and establish some of their properties. In particular, we will extend a result from [10] which characterizes such $\mathcal{T}$-Morse decompositions in terms of certain $\mathcal{T}$-attractor semifiltrations.

Recall that a *strict partial order* on a set P is a relation $\prec \subset P \times P$ which is irreflexive and transitive. As usual, we write $x \prec y$ instead of $(x, y) \in \prec$. The symbol $<$ will be reserved for the less-than-relation on $\mathbb{R}$.

For the rest of this paper, unless specified otherwise, let P be a fixed set and $\prec$ be a strict partial order on P.

A set $I \subset P$ is called *a $\prec$-interval* if whenever i, j, $k \in P$, i, $k \in I$ and $i \prec j \prec k$, then $j \in I$. By $I(\prec)$ we denote the set of all $\prec$-intervals in P. A set I is called *a $\prec$-attracting interval* if whenever i, $j \in P$, $j \in I$ and $i \prec j$, then $i \in I$. By $A(\prec)$ we denote the set of all $\prec$-attracting intervals in P. Of course, $A(\prec) \subset I(\prec)$. The following result is obvious.

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Proposition 2.1. *Let $J \in I(\prec)$ be arbitrary. Define K to be the set of all $k \in P$ for which there is a $j \in J$ such that $k = j$ or $k \prec j$. Then $K \in A(\prec)$ and $I := K \setminus J \in A(\prec)$.*

Let us also note the following known results.

Proposition 2.2. *Let I be an arbitrary $\prec$-attracting interval. Then $\prec$ can be extended to a total order $\prec^* = \prec_I^*$ on P such that I is an $\prec^*$-attracting interval.*

Proof. Let $\prec' \subset P \times P$ be defined by $\prec' = \prec \cup \{ (i,j) \mid i \in I,\, j \in P \setminus I \text{ and } (i,j) \notin \prec \}$. It is easily seen that $\prec'$ is a (strict) partial order on P. By Zorn's Lemma there is a total order $\prec^*$ extending $\prec'$. It is clear that $\prec^*$ has the desired property. ∎

Proposition 2.3. *Let $k \in \mathbb{N}_0$ be arbitrary and suppose P has $k+1$ elements. Then there is a bijective map $\varphi \colon [\![0,k]\!] \to P$ such that whenever i, $j \in [\![0,k]\!]$ and $\varphi(i) \prec \varphi(j)$ then $i < j$.*

Proof. This is proved by induction on $k \in \mathbb{N}_0$. The result is obvious for $k = 0$. Assume the proposition for $k-1$ and let P have $k+1$ elements. There is a $\prec$-maximal element $a \in P$. By the induction hypothesis there is a bijective map $\varphi' \colon [\![0,k-1]\!] \to P \setminus \{a\}$ such that whenever i and $j \in [\![0,k-1]\!]$ and $\varphi'(i) \prec \varphi'(j)$ then $i < j$. Let φ be the extension of φ' to $[\![0,k]\!]$ obtained by setting $\varphi(k) = a$. The map $\varphi \colon [\![0,k]\!] \to P$ has the desired properties. ∎

For the rest of this paper we assume that P is a finite set.

Definition 2.4. *Let $\mathcal{T}$ be a subset of $\mathcal{C}$. A family $(M_i)_{i \in P}$ is called a $\prec$-ordered $\mathcal{T}$-Morse decomposition if the following properties hold:*

(1) The sets M_i, $i \in P$, are closed, $\mathcal{T}$-invariant and pairwise disjoint.
(2) For every $\sigma \in \mathcal{T}$ either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are k, $l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

Remark 2.5. *If $\prec$ is a strict total order on P and P has m elements, then there is a unique order isomorphism $\varphi \colon ([\![1,m]\!], <) \to (P, \prec)$. In this case, a family $(M_i)_{i \in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition if and only if the sequence $(M'_r)_{r=1}^m$ is a $\mathcal{T}$-Morse-decomposition of the second kind (cf. Definition 3.3 in [3]). Here, $M'_r = M_{\varphi(r)}$, $r \in [\![1,m]\!]$.*

Remark 2.6. *Let π be a local semiflow on X and S be a compact invariant set relative to π. Let $\mathcal{T} := \mathcal{T}_{\pi,S}$ be the set of all full bounded solutions of π lying in S. In this case the concept of a $\mathcal{T}$-attractor is equivalent to the concept of an π-attractor in S and the concept of a $\mathcal{T}$-attractor-repeller pair is equivalent the concept of an attractor-repeller pair in S (relative to π) as introduced in [15]. Moreover the concept of a $\prec$-ordered $\mathcal{T}$-Morse decomposition is equivalent to the concept of a Morse decomposition for S (relative to π) in the sense of [10]. More explicitly, a family $(M_i)_{i \in P}$ is called a $\prec$-ordered Morse decomposition of S (relative to π) if the following properties hold:*

(1) The sets M_i, $i \in P$, are closed, π-invariant and pairwise disjoint.
(2) For every full solution σ of π lying in S either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are k, $l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

We have the following proposition.

Proposition 2.7. *Let π, S and $\mathcal{T}$ be as in Remark 2.6. Then $\mathcal{T}$ is compact in $C(\mathbb{R} \to X)$, translation and cut-and-glue invariant.*

Proof. Let $\sigma\colon \mathbb{R} \to S$ be a full solution of π and $t \in \mathbb{R}$. We claim that $\mathrm{tsl}_t\sigma$ is a full solution of π. In fact, notice that $\mathrm{tsl}_t\sigma(\mathbb{R}) \subset S$. Now let $h \geq 0$. Since $\sigma \in \mathcal{T}$, it follows that $\sigma(s+h) = \sigma(s)\pi h$ for all $s \in \mathbb{R}$. As $\mathrm{tsl}_t\sigma(s+h) = \sigma(t+s+h)$ and $\mathrm{tsl}_t\sigma(s)\pi h = \sigma(s+t)\pi h$, we have that $\mathrm{tsl}_t\sigma(s+h) = \mathrm{tsl}_t(s)\pi h$.

Let $\sigma_1\colon \mathbb{R} \to S$ and $\sigma_2\colon \mathbb{R} \to S$ be full solutions of π with $\sigma_1(0) = \sigma_2(0)$. Define $\sigma := \sigma_1 \triangleright \sigma_2$. Since σ_1 and σ_2 are solutions of π, it follows that $\sigma_1(s) = \sigma_2(s) = \sigma_1(0)\pi s$, for all $s \geq 0$. Therefore, $\sigma_1 \triangleright \sigma_2 = \sigma_1$ is a full solution of π and so we have proved that $\mathcal{T}$ is translation and cut-and-glue invariant.

In order to prove the compactness of $\mathcal{T}$, let $(\sigma_n)_n$ be an arbitrary sequence in $\mathcal{T}$. Using the standard Cantor diagonalization procedure and the fact that S is compact, we obtain a subsequence of $(\sigma_n)_n$, which it is denoted again by $(\sigma_n)_n$, such that for all $k \in \mathbb{N}_0$

$$\sigma_n(-k) \to x_{-k} \in S \text{ as } n \to \infty. \tag{2.1}$$

For each $k \in \mathbb{N}_0$ and for $t \in [-k, \infty[$ define

$$\tilde{\sigma}_{-k}(t) := x_{-k}\pi(t+k).$$

The compactness of S implies that $\omega_y = \infty$ for all $y \in S$ so $\tilde{\sigma}_{-k}\colon [-k, \infty[\to S$ is well-defined.

Note that if $k < k'$ and $t \in [-k, \infty[$ then $\sigma_n(-k)\pi(t+k) = \sigma_n(t) = \sigma_n(-k')\pi(t+k')$ for all $n \in \mathbb{N}$, hence

$$\tilde{\sigma}_{-k}(t) = x_{-k}\pi(t+k) = \lim_{n\to\infty} \sigma_n(-k)\pi(t+k)$$

and

$$\tilde{\sigma}_{-k'}(t) = x_{-k'}\pi(t+k') = \lim_{n\to\infty} \sigma_n(-k')\pi(t+k')$$

so $\tilde{\sigma}_{-k}$ and $\tilde{\sigma}_{-k'}$ coincide on $[-k, \infty[$. Thus there is a unique map $\sigma\colon \mathbb{R} \to S$ such that $\sigma(t) = \tilde{\sigma}_{-k}(t)$ for all $k \in \mathbb{N}_0$ and $t \in [-k, \infty[$. It follows that σ is a full solution of π lying in S.

To complete the proof we need to show that $\sigma_n \to \sigma$ in $\mathcal{C}$. This is equivalent to showing that whenever $(t_n)_n$ is a sequence in $\mathbb{R}$ such that $t_n \to t \in \mathbb{R}$ as $n \to \infty$, then $\sigma_n(t_n) \to \sigma(t)$ as $n \to \infty$. Thus let $(t_n)_n$ be a sequence in $\mathbb{R}$ such that $t_n \to t \in \mathbb{R}$ as $n \to \infty$.

There is a $k \in \mathbb{N}$ such that t, $t_n \in [-k, \infty[$ for all $n \in \mathbb{N}$. Therefore, $\sigma(t) = \tilde{\sigma}_{-k}(t) = x_{-k}\pi(t+k)$ and $\sigma_n(t_n) = \sigma_n(-k)\pi(t_n+k)$. Now, the continuity of π and formula (2.1) imply that $\sigma_n(t_n) \to \sigma(t)$ as $n \to \infty$. This completes the proof. ∎

If A, $B \subset X$ then the *$\mathcal{T}$-connection set $\mathrm{CS}_{\mathcal{T}}(A, B)$ from A to B* is the set of all points $x \in X$ for which there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$, $\alpha(\sigma) \subset A$ and $\omega(\sigma) \subset B$.

Definition 2.8. *Let $(M_i)_{i\in P}$ be a $\prec$-ordered $\mathcal{T}$-Morse decomposition. For an arbitrary $\prec$-interval I set*

$$M(I) = \bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\mathcal{T}}(M_i, M_j).$$

Note that $M(I)$ also depends on $\mathcal{T}$ and the family $(M_i)_{i\in P}$. Sometimes we need to stress this dependence and then we write $M(I, \mathcal{T}, (M_i)_{i\in P})$ instead of just $M(I)$. If π, S and $\mathcal{T}$ are as in Remark 2.6 then we write $M_{\pi,S}(I) := M(I, \mathcal{T}, (M_i)_{i\in P})$.

We have the following simple result:

Proposition 2.9. *Let $(M_i)_{i\in P}$ be a $\prec$-ordered $\mathcal{T}$-Morse decomposition and I be a $\prec$-interval. Then*

$$M(I) = \bigcup_{i\in I} M_i \cup \bigcup_{(i,j)\in I\times I,\,(i,j)\in\prec} \mathrm{CS}_{\mathcal{T}}(M_j, M_i).$$

Moreover, if I is a $\prec$-attracting interval, then

$$M(I) = \{\, x\in X \mid\ \exists\sigma\in\mathcal{T},\ \exists i\in I,\ \text{with } \sigma(0)=x \text{ and } \alpha(\sigma)\subset M_i \,\}.$$

Remark 2.10. *In the situation of Remark 2.5 the set $I' = \varphi^{-1}(I)$ is a $<$-attracting interval in $[\![1,m]\!]$ and so there is a $j\in[\![1,m]\!]$ such that $I' = [\![1,j]\!]$. Consequently, by Proposition 2.9,*

$$M(I) = \{\, x\in X \mid\ \exists\sigma\in\mathcal{T},\ \exists r\in[\![1,j]\!],\ \text{with } \sigma(0)=x \text{ and } \alpha(\sigma)\subset M_{\varphi(r)} \,\}.$$

If $\mathcal{T}$ is compact, translation and cut-and-glue invariant, Theorem 3.10 in [3] implies that $M(I)$ is a $\mathcal{T}$-attractor.

We now obtain the following results.

Proposition 2.11. *Let $\mathcal{T}$ be compact, translation and cut-and-glue invariant and let $(M_i)_{i\in P}$ be a $\prec$-ordered $\mathcal{T}$-Morse decomposition. Whenever $I\in A(\prec)$, then the pair $(M(I), M(P\setminus I))$ is a $\mathcal{T}$-attractor-repeller pair.*

Proof. Let $A = M(I)$ and $A^* = M(P\setminus I)$. By Proposition 2.2 there is a strict total order $\prec^*$ extending $\prec$ such that $I\in A(\prec^*)$. Remark 2.10 implies that $A = M(I)$ is a $\mathcal{T}$-attractor.

We claim that

$$A\cap A^* = \emptyset. \tag{2.2}$$

In fact, if there is an $x\in A\cap A^*$ then there are σ_1 and $\sigma_2\in\mathcal{T}$ with $\sigma_1(0)=\sigma_2(0)=x$, $\alpha(\sigma_1)\subset M_j$ and $\omega(\sigma_2)\subset M_i$ for some $j\in I$ and $i\in P\setminus I$. Letting $\sigma=\sigma_1\triangleright\sigma_2$ we see that $\sigma\in\mathcal{T}$, $\sigma(0)=x$, $\alpha(\sigma)\subset M_j$ and $\omega(\sigma)\subset M_i$. It follows that either $i=j$ or else $i\prec j$. Since $I\cap(P\setminus I)=\emptyset$ and $I\in A(\prec)$, both possibilities lead to a contradiction, proving (2.2). We now prove that $A^* = A^*_{\mathcal{T}}$. If $x \in A^*$ is arbitrary then there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\omega(\sigma) \subset M_i$ for some $i \in P \setminus I$. Since $M_i \subset M(P \setminus I)$ it follows that $\omega(\sigma) \subset X \setminus A$ by (2.2), so $x \in A^*_{\mathcal{T}}$. Conversely, let $x \in A^*_{\mathcal{T}}$ be arbitrary. Then there is a $\sigma \in \mathcal{T}$ such that $\sigma(0) = x$ and $\omega(\sigma) \subset X \setminus A$. Moreover, $\alpha(\sigma) \subset M_j$ and $\omega(\sigma) \subset M_i$ for some i and $j \in P$ with $j = i$ or $i \prec j$. If $i \in I$ then $\omega(\sigma) \subset M_i \subset A$, a contradiction. Thus $i \in P \setminus I$, so $j \in P \setminus I$ and so $x \in \mathrm{CS}_{\mathcal{T}}(M_j, M_i) \subset M(P \setminus I)$. ▌

Proposition 2.12. *Let $\mathcal{T}$ be compact, translation and cut-and-glue invariant and let $(M_i)_{i\in P}$ be a $\prec$-ordered $\mathcal{T}$-Morse decomposition. Let $K \in A(\prec)$ be arbitrary. Define $\mathcal{T}^K$ to be the set of all $\sigma \in \mathcal{T}$ with $\sigma(\mathbb{R}) \subset M(K)$. Then $\mathcal{T}^K$ is compact in $\mathcal{C}$, translation and cut-and-glue invariant. Moreover, the family $(M_i)_{i\in K}$ is a $\prec$-ordered $\mathcal{T}^K$-Morse decomposition. Furthermore, whenever $I \in I(\prec)$ with $I \subset K$, then $M(I, \mathcal{T}, (M_i)_{i\in P}) = M(I, \mathcal{T}^K, (M_i)_{i\in K})$. Finally, whenever $I \in A(\prec)$ with $I \subset K$, then $(M(I), M(K \setminus I))$ is a $\mathcal{T}^K$-attractor-repeller pair.*

Proof. It is easy to show that $\mathcal{T}^K$ is compact in $\mathcal{C}$, translation and cut-and-glue invariant. Since the family $(M_i)_{i\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition, it follows from Proposition 2.11 that

$$M(K) \cap M(P \setminus K) = \emptyset. \tag{2.3}$$

Moreover, the sets M_i, $i \in K$, are closed, $\mathcal{T}$-invariant and pairwise disjoint. Now let $\sigma \in \mathcal{T}^K \subset \mathcal{T}$. Thus either $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$ or else there are $k, l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$.

Suppose $\sigma(\mathbb{R}) \subset M_k$ for some $k \in P$. If $k \notin K$, then $M_k \subset M(P \setminus K)$ and so $\sigma(\mathbb{R}) \subset M(K) \cap M(P \setminus K)$ which contradicts (2.3).

Now assume that there are $k, l \in P$ with $k \prec l$, $\alpha(\sigma) \subset M_l$ and $\omega(\sigma) \subset M_k$. Let $t \in \mathbb{R}$ and $x := \sigma(t)$. Define $\tau := \mathrm{tsl}_t\sigma$. Hence, $\tau(0) = x$ and $\alpha(\tau) \subset M_l$. Since $\tau(\mathbb{R}) \subset M(K)$, it follows that $\alpha(\tau) \subset M(K)$ and so formula (2.3) implies that $l \in K$. Recall that K is a $\prec$-attracting interval. Thus, $k \in K$. This completes the proof that $(M_i)_{i\in K}$ is a $\prec$-ordered $\mathcal{T}^K$-Morse decomposition. Let $I \in I(\prec)$ with $I \subset K$. Since $\mathcal{T}^K \subset \mathcal{T}$, it follows that

$$\bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\mathcal{T}^K}(M_i, M_j) \subset \bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\mathcal{T}}(M_i, M_j).$$

Now let $x \in \mathrm{CS}_{\mathcal{T}}(M_i, M_j)$ for some i and j in I. Therefore, there exists a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$, $\alpha(\sigma) \subset M_i$ and $\omega(\sigma) \subset M_j$. We claim that $\sigma(\mathbb{R}) \subset M(K)$. Let $t \in \mathbb{R}$ and $y := \sigma(t)$. Define $\tau := \mathrm{tsl}_t\sigma$. Hence, $\tau(0) = y$ and $\alpha(\tau) \subset M_i$. Proposition 2.9 implies that $y \in M(K)$. This concludes the proof of our claim which in turn implies that $M(I, \mathcal{T}, (M_i)_{i\in P}) = M(I, \mathcal{T}^K, (M_i)_{i\in K})$. The last part of the proposition follows from Proposition 2.11. ▌

Proposition 2.13. *Let $\mathcal{T}$ be compact, translation and cut-and-glue invariant and let $(M_i)_{i\in P}$ be an $\prec$-ordered $\mathcal{T}$-Morse decomposition. Then there are families $(V_i)_{i\in P}$ and $(V_I)_{I\in I(\prec)}$ of closed subsets of X such that $M_i = \mathrm{Inv}_{\mathcal{T}}(V_i) \subset \mathrm{Int}_X(V_i)$ and $M(I) = \mathrm{Inv}_{\mathcal{T}}(V_I) \subset \mathrm{Int}_X(V_I)$ for all $i \in P$ and $I \in I(\prec)$.*

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Proof. For every $i \in P$ the set $I := \{i\}$ is an $\prec$-interval and $M_i = M(I)$. We thus only have to show that for every $I \in I(\prec)$ there is a closed set V_I with $M(I) = \mathrm{Inv}_{\mathcal{T}}(V_I) \subset \mathrm{Int}_X(V_I)$. If $I \in A(\prec)$ then $M(I)$ is a $\mathcal{T}$-attractor so the existence of V_I follows from Theorem 2.8 in [3]. Now let $J \in I(\prec)$ be arbitrary. By Proposition 2.1 there are I and $K \in A(\prec)$ such that $I \subset K$ and $J = K \setminus I$. By what we have proved so far there is a closed set V_K with $M(K) = \mathrm{Inv}_{\mathcal{T}}(V_K) \subset \mathrm{Int}_X(V_K)$. By Proposition 2.12 the set $M(J) = M(K \setminus I)$ is a $\mathcal{T}^K$-repeller i.e. a $(\mathcal{T}^K)^-$-attractor so there is a closed set V such that $M(J) = \mathrm{Inv}_{(\mathcal{T}^K)^-}(V) \subset \mathrm{Int}_X(V)$. Since $\mathrm{Inv}_{(\mathcal{T}^K)^-}(V) = \mathrm{Inv}_{\mathcal{T}^K}(V)$ and $M(J) \subset M(K)$ we see that $V_J := V \cap V_K$ is closed and $M(J) = \mathrm{Inv}_{\mathcal{T}}(V_J) \subset \mathrm{Int}_X(V_J)$. ∎

Let us introduce the following concept.

Definition 2.14. *A finite collection $\mathcal{A}$ of $\mathcal{T}$-attractors is called a $\mathcal{T}$-attractor semifiltration if*

(1) $\emptyset$, $S_{\mathcal{T}} \in \mathcal{A}$,
(2) whenever A, $B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ and $\omega_{\mathcal{T}}(A \cap B) \in \mathcal{A}$.

Remark. In the special case of a local semiflow π on X and S and $\mathcal{T} = \mathcal{T}_{\pi,S}$ as above, the concept of $\mathcal{T}$-attractor semifiltration boils down to the one introduced in [10].

We will now show that, under certain hypotheses on $\mathcal{T}$, $\prec$-ordered $\mathcal{T}$-Morse decompositions can be characterized by certain $\mathcal{T}$-attractor semifiltrations. We require the following technical proposition.

Proposition 2.15. *Let $\mathcal{T}$ be compact and translation-invariant and (A, A^*) be a $\mathcal{T}$-attractor-repeller pair. Let $U \subset X$ be such that $A = \omega_{\mathcal{T}}(U) \subset \mathrm{Int}_X(U)$. Then the following properties hold:*

(1) If V is open in X with $A \subset V$, then there is a $t_0 = t_0(V)$ such that $\mathcal{T}(U, [t_0, \infty[) \subset V$.
(2) If $B \subset X$ is closed in X, $A \subset X \setminus B$, $V^* \subset X$ is open in X and $A^* \subset V^*$ then there is a $t_0 = t_0(B, V^*)$ such that whenever $\sigma \in \mathcal{T}$ and $t \in [t_0, \infty[$ are such that $\sigma(t) \in B$, then $\sigma(0) \in V^*$.
(3) If $C \subset X$ is closed in X with $A \subset C \subset X \setminus A^*$, then $A = \omega_{\mathcal{T}}(C)$.

Proof. If (1) is not true then there is an open set V in X with $A \subset V$ and there are sequences $(\sigma_n)_n$ in $\mathcal{T}$ and $(t_n)_n$ in $\mathbb{R}$ such that $t_n \to \infty$, $\sigma_n(0) \in U$ and $\sigma_n(t_n) \in X \setminus V$ for all n. Since $\mathcal{T}$ is compact and translation-invariant we may assume that $\sigma_n(t_n) \to x$ for some $x \in X$. It follows that $x \in X \setminus V$. Moreover, Proposition 2.1 in [3] implies that $x \in \omega_{\mathcal{T}}(U) = A$. Thus $x \in A \setminus V = \emptyset$, a contradiction, proving (1).

If (2) is not true, then there is a closed set B, $A \subset X \setminus B$, and an open set V^* such that $A^* \subset V^*$ and there are sequences $(\sigma_n)_n$ in $\mathcal{T}$ and $(t_n)_n$ in $\mathbb{R}$ such that $t_n \to \infty$, $\sigma_n(t_n) \in B$ and $\sigma_n(0) \in X \setminus V^*$ for all n. By compactness and translation-invariance of $\mathcal{T}$ we may assume that $\sigma_n \to \sigma$ for some $\sigma \in \mathcal{T}$. It follows that $\sigma(0) \in X \setminus V^*$ so, by Theorem 2.11 in [3], we obtain that $\omega(\sigma) \subset A$. Hence there is an $s_1 \in \mathbb{R}$ with $\sigma(s_1) \in \mathrm{Int}_X(U)$. Thus there is an n_0 such that $\sigma_n(s_1) \in \mathrm{Int}_X(U)$ for all $n \geq n_0$. By part (1) there is an $s_2 \in \mathbb{R}$ such that $\mathcal{T}(U, [s_2, \infty[) \subset X \setminus B$. Since $\mathcal{T}$ is translation-invariant, we thus see that $\sigma_n(t) \in X \setminus B$ for all $t \geq s_1 + s_2$ and all $n \geq n_0$. Thus $\sigma_n(t_n) \in X \setminus B$ for all n large enough, a contradiction, proving (2).

Let us now prove (3). It is clear that $\omega_{\mathcal{T}}(A) = A$ so $A \subset C$ implies $A = \omega_{\mathcal{T}}(A) \subset \omega_{\mathcal{T}}(C)$. Therefore, if (3) is not true, then there is a closed set C in X with $A \subset C \subset X \setminus A^*$, and there is a $y \in \omega_{\mathcal{T}}(C) \setminus A$. There are sequences $(\sigma_n)_n$ in $\mathcal{T}$ and $(s_n)_n$ in $\mathbb{R}$ with $s_n \to \infty$, $\sigma_n(0) \in C$ and $\sigma_n(s_n) \to y$. Set $V = \mathrm{Int}_X(U)$. We claim that for every $t \in \mathbb{R}$ there is an $m \in \mathbb{N}$ such that $\sigma_n(t) \in X \setminus V$ for all $n \geq m$. In fact, otherwise there is a $t \in \mathbb{R}$ and a sequence $(n_m)_m$ in $\mathbb{N}$ such that $n_m \to \infty$ as $m \to \infty$ and such that $\sigma_{n_m}(t) \in V$. Thus $s_{n_m} - t \to \infty$ as $m \to \infty$. Since $\mathrm{tsl}_t\sigma_{n_m}(0) \in V$ and $\mathrm{tsl}_t\sigma_{n_m}(s_{n_m} - t) = \sigma_{n_m}(s_{n_m}) \to y$ as $m \to \infty$, we have $y \in \omega_{\mathcal{T}}(U) = A$ which is a contradiction. This proves our claim. Let $V^* = X \setminus C$. Then V^* is an open set in X such that $A^* \subset V^*$ and $V^* \cap C = \emptyset$. Let $t_0 = t_0(B, V^*)$ be as in part (2), where $B = X \setminus V$. By the above claim $\sigma_n(t_0) \in B$ for all n large enough so, by part (2), we have that $\sigma_n(0) \in V^*$ for all such n. However, this contradicts the fact that C is disjoint from V^*. Part (3) is proved. ∎

We can now prove the first characterization result, which extends Theorem 2.4 in [10].

Theorem 2.16. *Let $\mathcal{T}$ be compact, translation and cut-and-glue invariant. Suppose $(M_i)_{i\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition and define*

$$\mathcal{A} = \{\, M(I) \mid I \in A(\prec) \,\}.$$

Then $\mathcal{A}$ is a $\mathcal{T}$-attractor semifiltration.

Proof. We proceed as in the proof of Theorem 2.4 in [10]. Let $I \in A(\prec)$ be arbitrary. By Proposition 2.2 there is a strict total order $\prec^*$ extending $\prec$ such that $I \in A(\prec^*)$. Remark 2.10 implies that $M(I)$ is a $\mathcal{T}$-attractor. Now note that $\emptyset = M(\emptyset) \in \mathcal{A}$ and $S_{\mathcal{T}} = M(P) \in \mathcal{A}$. Next, let I and $J \in A(\prec)$ be arbitrary. Then $I \cup J \in A(\prec)$. An application of Proposition 2.9 clearly shows that $M(I) \cup M(J) = M(I \cup J) \in \mathcal{A}$. Finally, note that $I \cap J \in A(\prec)$. Therefore, since $M(I \cap J) \in \mathcal{A}$, the theorem will be proved if we show that

$$\omega_{\mathcal{T}}(M(I) \cap M(J)) = M(I \cap J). \tag{2.4}$$

Since $M(I)$ and $M(J)$ are $\mathcal{T}$-attractors hence closed it follows that $C := M(I) \cap M(J)$ is closed. Clearly, $C \supset A := M(I \cap J)$. We show that

$$C \cap A^*_{\mathcal{T}} = \emptyset. \tag{2.5}$$

Formula (2.5) together with Proposition 2.15 implies (2.4) and completes the proof. Suppose (2.5) is not true and let $x \in C \cap A^*_{\mathcal{T}}$. Then there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\omega(\sigma) \subset X \setminus A$. Moreover, there are σ_1 and $\sigma_2 \in \mathcal{T}$ such that $\sigma_1(0) = \sigma_2(0) = x$, $\alpha(\sigma_1) \subset M_i$ and $\alpha(\sigma_2) \subset M_j$ for some $i \in I$ and $j \in J$. Since $\mathcal{T}$ is cut-and-glue invariant it follows that $\tau_k := \sigma_k \triangleright \sigma \in \mathcal{T}$, $\tau_k(0) = x$ and $\omega(\tau_k) = \omega(\sigma) \subset M_l$ for some $l \in P$ and $k = 1, 2$. Since $\alpha(\tau_1) = \alpha(\sigma_1) \subset M_i$ it follows that $l \prec i$ or $l = i$ so $l \in I$. Similarly, $l \in J$. Thus $\omega(\sigma) \subset M_l \subset M(I \cap J) = A$, a contradiction proving (2.5). ∎

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Theorem 2.16 has the following converse, which extends Theorem 2.6 in [10].

THEOREM 2.17. *Let $\mathcal{T}$ be compact, translation and cut-and-glue invariant. Let $\mathcal{A}$ be a $\mathcal{T}$-attractor semifiltration with $\#\mathcal{A} = m+1$ for some $m \in \mathbb{N}_0$. Then there is a partial order $\prec$ on $P := [\![1,m]\!]$ and a $\prec$-ordered Morse decomposition $(M_i)_{i\in P}$ such that*

$$\mathcal{A} = \mathcal{S} := \{\, M(I) \mid I \in A(\prec)\,\}. \tag{2.6}$$

Proof. We proceed as in the proof of Theorem 2.6 in [10]. By Proposition 2.3 there is a bijection $\phi\colon [\![0,m]\!] \to \mathcal{A}$, $i \mapsto A_i$, such that whenever i, $j \in [\![0,m]\!]$, $i \neq j$ and $A_i \subset A_j$ then $i < j$. In particular, $A_0 = \emptyset$. Set $\overline{A}_i = \bigcup_{\nu=1}^{i} A_\nu$, $i \in P$. Then $\overline{A}_i$ is a $\mathcal{T}$-attractor for every $i \in P$ and $i < j$ implies $\overline{A}_i \subset \overline{A}_j$. Moreover, $\overline{A}_0 = \emptyset$ and $\overline{A}_m = S_{\mathcal{T}}$. Define $M_i = \overline{A}_i \cap (\overline{A}_{i-1})^*_{\mathcal{T}}$. Now Definition 3.2 and Theorem 3.8 in [3] imply that $(M_i)_{i\in P}$ is a $<$-ordered $\mathcal{T}$-Morse decomposition.

LEMMA 2.18. *Let $i \in P$ and $A \in \mathcal{A}$ be such that $M_i \cap A \neq \emptyset$. Then $M_i \subset A$.*

Proof. Since $M_i \subset \overline{A}_i \setminus \overline{A}_{i-1}$ it follows that $M_i \subset A_i$. Choose an $x \in M_i \cap A$. Then $x \in (A_i \cap A) \cap M_i$. Since M_i is $\mathcal{T}$-invariant, it follows that there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\sigma(\mathbb{R}) \subset M_i$. Since $\mathcal{T}$ is compact and translation invariant there is a sequence $(t_n)_n$ with $t_n \to \infty$ and a $y \in X$ with $\sigma(t_n) \to y$. Since M_i is closed it follows that $y \in M_i \cap \omega_{\mathcal{T}}(A_i \cap A)$. Hence $M_i \cap \omega_{\mathcal{T}}(A_i \cap A) \neq \emptyset$. Now $\omega_{\mathcal{T}}(A_i \cap A) \in \mathcal{A}$ so $\omega_{\mathcal{T}}(A_i \cap A) = A_l$ for some $l \in P$. Since $M_i \cap A_j = \emptyset$ for all $j \leq i-1$ we have that $i \leq l$. But $A_l = \omega_{\mathcal{T}}(A_i \cap A) \subset \omega_{\mathcal{T}}(A_i) = A_i$ and so $l \leq i$ by our ordering property. Thus $l = i$ and so $A_i = \omega_{\mathcal{T}}(A_i \cap A) \subset A$. Thus $M_i \subset A_i \subset A$. The lemma is proved. ▮

Now define a relation R on P by $(i,j) \in R$ if and only if whenever $A \in \mathcal{A}$ and $M_j \subset A$ then $M_i \subset A$. It is clear that R is transitive. This implies that the relation $\prec$ defined by

$$i \prec j \text{ if and only if } (i,j) \in R \text{ and } (j,i) \notin R$$

is a strict partial order on P. We show that $(M_i)_{i\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition. Let $\sigma \in \mathcal{T}$ be arbitrary. We have to show that one of the following cases holds:

(1) $\sigma(\mathbb{R}) \subset M_i$ for some $i \in P$.
(2) $\alpha(\sigma) \subset M_j$ and $\omega(\sigma) \subset M_i$ for some i, $j \in P$ with $i \prec j$.

Suppose that (1) does not hold. Since $(M_i)_{i\in P}$ is a $<$-ordered $\mathcal{T}$-Morse decomposition it follows that $\alpha(\sigma) \subset M_j$ and $\omega(\sigma) \subset M_i$ for some i, $j \in P$ with $i < j$. In particular, this implies that $M_j \neq \emptyset$. To show that $i \prec j$ let $A \in \mathcal{A}$ be arbitrary with $M_j \subset A$. Then $\alpha(\sigma) \subset A$ so Theorem 2.11 in [3] implies that $\sigma(\mathbb{R}) \subset A$ and so $\omega(\sigma) \subset A$. Thus $M_i \cap A \neq \emptyset$ which, by Lemma 2.18 implies that $M_i \subset A$. It follows that $(i,j) \in R$. Now $M_j \subset X \setminus A_i$ and $M_i \subset A_i$. If $(j,i) \in R$ then $M_j \subset A_i$ so $M_j = \emptyset$, a contradiction proving that $(j,i) \notin R$. It follows that $i \prec j$, as claimed. To complete the proof we only have to check formula (2.6). We need a lemma.

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Lemma 2.19. *If* $(i,j) \in R$, $(j,i) \in R$ *and* $j \neq i$ *then* $M_i = \emptyset$ *or* $M_j = \emptyset$.

Proof. We may assume that $i < j$. Then we obtain $M_i \subset A_i$ and $M_j \subset X \setminus A_i$. Since $(j,i) \in R$ we conclude $M_j \subset A_i$ so $M_j = \emptyset$. ▌

Now let $A \in \mathcal{A}$ be arbitrary. Define I_A to be the set of all $i \in P$ with $M_i \subset A$. It follows from the definition of $\prec$ that $I_A \in A(\prec)$. We claim that $A = M(I_A)$. This claim implies that $A \in \mathcal{S}$ and so, since $A \in \mathcal{A}$ is arbitrary, it follows that $\mathcal{A} \subset \mathcal{S}$. To prove the claim let $x \in A$ be arbitrary. Since A is $\mathcal{T}$-invariant, we obtain a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\sigma(\mathbb{R}) \subset A$. Moreover, there is a $j \in P$ with $\alpha(\sigma) \subset M_j$, so $M_j \cap A \neq \emptyset$. Lemma 2.18 now implies that $M_j \subset A$ so $j \in I_A$. It follows that $x \in M(I_A)$ and so $A \subset M(I_A)$. Conversely, let $x \in M(I_A)$ be arbitrary. Then there is a $\sigma \in \mathcal{T}$ with $\sigma(0) = x$ and $\alpha(\sigma) \subset M_j$ for some $j \in I_A$. Thus $M_j \subset A$ and so $\alpha(\sigma) \subset A$. Hence, by Theorem 2.11 in [3], we obtain that $\sigma(\mathbb{R}) \subset A$ so $x \in A$. It follows that $M(I_A) \subset A$. The claim follows. Now let us prove that $\mathcal{S} \subset \mathcal{A}$. Let $I \in A(\prec)$ be arbitrary. We only need to prove that $M(I) \in \mathcal{A}$. Let A be a minimal element of $\mathcal{A}$ containing $M(I)$. To complete the proof we only need to show that $A \subset M(I)$. Using Lemma 2.18 together with the closedness and $\mathcal{T}$-invariance of A we only need to show that $i \in P$, $M_i \subset A$ and $M_i \neq \emptyset$ imply $i \in I$. Suppose this is not true. Then there is an $i \in P \setminus I$ with $M_i \subset A$ and $M_i \neq \emptyset$. Now, whenever $j \in P$ and $(i,j) \notin R$ then there is a $B_j \in \mathcal{A}$ such that $M_j \subset B_j$ and $M_i \not\subset B_j$. By Lemma 2.18 we thus obtain $M_i \cap B_j = \emptyset$. Let B be the union of all the sets B_j such that $j \in I$ and $(i,j) \notin R$. It follows that $B \in \mathcal{A}$ and $B \cap M_i = \emptyset$. We claim that

$$M(I) \subset B. \tag{2.7}$$

This implies that $M(I) \subset C := \omega_{\mathcal{T}}(A \cap B) \subset A$. Since $C \subset B$, M_i is nonempty, B is disjoint from M_i and A contains M_i it follows that C is strictly included in A. Moreover, $C \in \mathcal{A}$, so this contradicts the minimality of A and completes the proof of the theorem. To prove (2.7) we only have to show that $j \in I$ and $M_j \neq \emptyset$ imply $M_j \subset B$. Now if $(i,j) \notin R$ then $M_j \subset B$ by the definition of B. Thus assume $(i,j) \in R$. We cannot have $(j,i) \notin R$ since otherwise $i \prec j$ so $j \in I$ implies that $i \in I$, a contradiction to our choice of i. Thus $(j,i) \in R$. However, this implies, by Lemma 2.19, that at least one of the sets M_i or M_j is empty, which again is a contradiction. The theorem is proved. ▌

3. CONTINUATION OF PARTIALLY ORDERED MORSE DECOMPOSITIONS

In this section we will extend the continuation result for $\mathcal{T}$-Morse decompositions from our previous paper [3] to the partially ordered case. We apply this to the semiflow case and to the nonuniqueness case considered in [3].

Recall the following definition from [3].

Definition 3.1. *Let* $(\mathcal{T}_\kappa)_{\kappa\in\mathbb{N}}$ *be a sequence of subsets of* $\mathcal{C}$ *and* $\mathcal{T} \subset \mathcal{C}$ *be arbitrary. We say that* $(\mathcal{T}_\kappa)_{\kappa\in\mathbb{N}}$ *converges to* $\mathcal{T}$, *and we write* $\mathcal{T}_\kappa \to \mathcal{T}$ *(as* $\kappa \to \infty$*), if for every sequence* $(\kappa_n)_{n\in\mathbb{N}}$ *in* $\mathbb{N}$ *with* $\kappa_n \to \infty$ *as* $n \to \infty$ *and every sequence* $(\sigma_n)_{n\in\mathbb{N}}$ *such that* $\sigma_n \in \mathcal{T}_{\kappa_n}$

for all $n \in \mathbb{N}$ there is a subsequence $(\sigma_{n_m})_{m\in\mathbb{N}}$ and a $\sigma \in \mathcal{T}$ such that $\sigma_{n_m} \to \sigma$ in $\mathcal{C}$ as $m \to \infty$.

We saw in [3] that this convergence concept is applicable, e.g., to ordinary differential equations in Banach spaces and their Galerkin approximations. Another sufficient condition for $\mathcal{T}_\kappa \to \mathcal{T}$ is furnished by the following proposition.

Proposition 3.2. *Let N be a closed set in X and let π_κ, $\kappa \in \mathbb{N}_0$, be local semiflows on X. Assume that $\pi_\kappa \to \pi_0$ as $\kappa \to \infty$. Furthermore, suppose that for each $\kappa \in \mathbb{N}_0$, N is strongly π_κ-admissible and N is $(\pi_{\kappa_n})_n$-admissible for every subsequence $(\pi_{\kappa_n})_n$ of $(\pi_\kappa)_\kappa$. For each $\kappa \in \mathbb{N}_0$, define $\mathcal{T}_\kappa := \mathcal{T}_{\pi_\kappa}$ to be the set of all full solutions of π_κ lying in $S_\kappa = \mathrm{Inv}_{\pi_\kappa}(N)$. Then $\mathcal{T}_\kappa \to \mathcal{T}_0$ as $\kappa \to \infty$.*

Proof. Let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with $\kappa_n \to \infty$ as $n \to \infty$ and $(\sigma_n)_{n\in\mathbb{N}}$ be sequence such that $\sigma_n \in \mathcal{T}_{\kappa_n}$ for all $n \in \mathbb{N}$. We need to show that there is a subsequence $(\sigma_{n_m})_{m\in\mathbb{N}}$ and a $\sigma \in \mathcal{T}$ such that $\sigma_{n_m} \to \sigma$ in $\mathcal{C}$ as $m \to \infty$. Using the standard Cantor diagonalization procedure and the $(\pi_{\kappa_n})_n$-admissibility of N, we obtain a subsequence of $(\sigma_n)_n$, which it is denoted again by $(\sigma_n)_n$, such that for all $k \in \mathbb{N}_0$

$$\sigma_n(-k) \to x_{-k} \in N \text{ as } n \to \infty. \tag{3.1}$$

We claim that for each $k \in \mathbb{N}_0$ the solution of π_0 through x_{-k} is defined for all $t \in [0, \infty[$. In fact, if this is not true for some $k \in \mathbb{N}_0$, then the assumption that π_0 does not explode in N implies the existence of a $t \in [0, \infty[$ such that $x_{-k}\pi_0 t$ is defined and $x_{-k}\pi_0 t \notin N$. Since $\pi_\kappa \to \pi_0$, we see that, for all $n \in \mathbb{N}$ large enough, $\sigma_n(-k)\pi_{\kappa_n} t$ is defined and $\sigma_n(-k)\pi_{\kappa_n} t \notin N$, a contradiction which proves our claim.

For each $k \in \mathbb{N}_0$ and for $t \in [-k, \infty[$ define

$$\tilde{\sigma}_{-k}(t) := x_{-k}\pi_0(t + k).$$

The above claim implies that $\tilde{\sigma}_{-k} \colon [-k, \infty[\to N$ is well-defined.

Note that if $k < k'$, and $t \in [-k, \infty[$ then $\sigma_n(-k)\pi_{\kappa_n}(t+k) = \sigma_n(t) = \sigma_n(-k')\pi_{\kappa_n}(t+k')$ for all $n \in \mathbb{N}$, hence

$$\tilde{\sigma}_{-k}(t) = x_{-k}\pi_0(t + k) = \lim_{n\to\infty} \sigma_n(-k)\pi_{\kappa_n}(t + k)$$

and

$$\tilde{\sigma}_{-k'}(t) = x_{-k'}\pi_0(t + k') = \lim_{n\to\infty} \sigma_n(-k')\pi_{\kappa_n}(t + k')$$

so $\tilde{\sigma}_{-k}$ and $\tilde{\sigma}_{-k'}$ coincide on $[-k, \infty[$. Thus there is a unique map $\sigma \colon \mathbb{R} \to N$ such that $\sigma(t) = \tilde{\sigma}_{-k}(t)$ for all $k \in \mathbb{N}_0$ and $t \in [-k, \infty[$. It follows that σ is a full solution of π_0 lying in N.

To complete the proof we need to show that $\sigma_n \to \sigma$ in $\mathcal{C}$. This is equivalent to showing that whenever $(t_n)_n$ is a sequence in $\mathbb{R}$ such that $t_n \to t \in \mathbb{R}$ as $n \to \infty$, then $\sigma_n(t_n) \to \sigma(t)$ as $n \to \infty$. Thus let $(t_n)_n$ be a sequence in $\mathbb{R}$ such that $t_n \to t \in \mathbb{R}$ as $n \to \infty$.

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There is a $k \in \mathbb{N}$ such that t, $t_n \in [-k, \infty[$ for all $n \in \mathbb{N}$. Therefore, $\sigma(t) = \tilde{\sigma}_{-k}(t) = x_{-k}\pi_0(t+k)$ and $\sigma_n(t_n) = \sigma_n(-k)\pi_{\kappa_n}(t_n+k)$. Now, the fact that $\pi_\kappa \to \pi_0$ and formula (3.1) imply that $\sigma_n(t_n) \to \sigma(t)$ as $n \to \infty$. This completes the proof. ▌

We state our continuation result for partially ordered Morse decompositions.

THEOREM 3.3. *Suppose $\mathcal{T}_\kappa \to \mathcal{T}$, where $\mathcal{T}$ and $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$, are compact (in $\mathcal{C}$), translation and cut-and-glue invariant. Suppose $(M_i)_{i\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition. For each $i \in P$, let V_i be closed in X such that*

$$M_i = \mathrm{Inv}_{\mathcal{T}}(V_i) \subset \mathrm{Int}_X(V_i). \tag{3.2}$$

Moreover, for every $I \in I(\prec)$, let V_I be closed in X such that

$$M(I) = \mathrm{Inv}_{\mathcal{T}}(V_I) \subset \mathrm{Int}_X(V_I).$$

(In view of Proposition 2.13 such sets V_i, $i \in P$, and V_I, $I \in I(\prec)$, always exist.) For $\kappa \in \mathbb{N}$ and $i \in P$ set $M_i(\kappa) := \mathrm{Inv}_{\mathcal{T}_\kappa}(V_i)$. Then there is a κ_0 such that for every $\kappa \geq \kappa_0$ the family $(M_i(\kappa))_{i\in P}$ is a $\prec$-ordered $\mathcal{T}_\kappa$-Morse decomposition. Moreover, for every $I \in I(\prec)$,

$$M_I(\kappa) := \bigcup_{(i,j)\in I\times I} \mathrm{CS}_{\mathcal{T}_\kappa}(M_i(\kappa), M_j(\kappa)) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_I) \subset \mathrm{Int}_X(V_I).$$

To prove this theorem we need the following lemma.

LEMMA 3.4. *Under the hypotheses of the theorem there is a κ' such that for every $\kappa \geq \kappa'$ the family $(M_i(\kappa))_{i\in P}$ is a $\prec$-ordered $\mathcal{T}_\kappa$-Morse decomposition. Moreover, for every $I \in A(\prec)$*

$$M_I(\kappa) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_I) \subset \mathrm{Int}_X(V_I),\ \kappa \geq \kappa' \tag{3.3}$$

and

$$M_{P\setminus I}(\kappa) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_{P\setminus I}) \subset \mathrm{Int}_X(V_{P\setminus I}),\ \kappa \geq \kappa'. \tag{3.4}$$

Proof. We will consider two cases.

Case 1. We first suppose that $\prec$ is a total order. In view of Remark 2.5 we may assume that $P = [\![1, m]\!]$ for some $m \in \mathbb{N}$ and $\prec$ is $<$. It follows that $(M_i)_{i\in P}$ is a $\mathcal{T}$-Morse decomposition of the second kind. Therefore, Theorem 3.15 in [3] implies that there exists a $\tilde{\kappa}_0$ such that for every $\kappa \geq \tilde{\kappa}_0$, $(M_i(\kappa))_{i=1}^m$ is a $\mathcal{T}_\kappa$-Morse decomposition.

Let $r \in [\![0, m]\!]$ and consider the sets

$$A_r = \{\, x \mid \exists\sigma \in \mathcal{T} \text{ with } \sigma(0) = x \text{ and } \alpha(\sigma) \subset \textstyle\bigcup_{i=1}^r M_i \,\}.$$

Theorem 3.10 in [3] implies that $(A_r)_{r=0}^m$ is a $\mathcal{T}$-attractor filtration and $M_r = A_r \cap (A_{r-1})^*_{\mathcal{T}}$ for $r \in [\![1, m]\!]$. Recall that $A_0 = \emptyset$. Now, Remark 2.10 implies that, for $r \in [\![1, m]\!]$, $A_r = M([\![1, r]\!])$. Moreover, Proposition 2.11 implies that $M([\![r + 1, m]\!]) = (A_r)^*_{\mathcal{T}}$.

For each $r \in [\![0, m]\!]$ define $W_r := V_{[\![1,r]\!]}$ and $W_r^* := V_{[\![r+1,m]\!]}$. It follows that

$$\begin{aligned} A_r &= \mathrm{Inv}_{\mathcal{T}}(W_r) \subset \mathrm{Int}_X(W_r) \\ (A_r)^*_{\mathcal{T}} &= \mathrm{Inv}_{\mathcal{T}^-}(W_r^*) = \mathrm{Inv}_{\mathcal{T}}(W_r^*) \subset \mathrm{Int}_X(W_r^*). \end{aligned}$$

Since the set M_r is $\mathcal{T}$-invariant and $M_r \subset W_r \cap W_{r-1}^*$ we obtain

$$\begin{aligned} M_r &\subset \mathrm{Inv}_{\mathcal{T}}(W_r \cap W_{r-1}^*) \subset \mathrm{Inv}_{\mathcal{T}}(W_r) \cap \mathrm{Inv}_{\mathcal{T}}(W_{r-1}^*) \\ &= \mathrm{Inv}_{\mathcal{T}}(W_r) \cap \mathrm{Inv}_{\mathcal{T}^-}(W_{r-1}^*) = A_r \cap (A_{r-1})^*_{\mathcal{T}} = M_r \end{aligned}$$

so

$$\begin{aligned} M_r &= \mathrm{Inv}_{\mathcal{T}}(W_r \cap W_{r-1}^*) = \mathrm{Inv}_{\mathcal{T}}(W_r) \cap \mathrm{Inv}_{\mathcal{T}^-}(W_{r-1}^*) \\ &\subset \mathrm{Int}_X(W_r) \cap \mathrm{Int}_X(W_{r-1}^*) \subset \mathrm{Int}_X(W_r \cap W_{r-1}^*). \end{aligned} \tag{3.5}$$

For each $r \in [\![0, m]\!]$ and for each $\kappa \in \mathbb{N}$, define $A_r^\kappa = \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r)$ and $\widetilde{A}_r^\kappa = \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r^*)$. Theorem 3.14 in [3] implies that there exists a $\kappa_1 \geq \tilde{\kappa}_0$ such that, for every $\kappa \geq \kappa_1$ and $r \in [\![0, m]\!]$, $A_r^\kappa \subset \mathrm{Int}_X(W_r)$, $\widetilde{A}_r^\kappa \subset \mathrm{Int}_X(W_r^*)$, $(A_r^\kappa)_{r=0}^m$ is a $\mathcal{T}_\kappa$-attractor filtration and $(\widetilde{A}_r^\kappa)_{r=0}^m$ is its dual $\mathcal{T}_\kappa$-repeller filtration.

For all $r \in [\![1, m]\!]$ and for all $\kappa \in \mathbb{N}$ with $\kappa \geq \kappa_1$, define $\widetilde{M}_r^\kappa = A_r^\kappa \cap \widetilde{A}_{r-1}^\kappa$. It follows that the set $\widetilde{M}_r^\kappa$ is $\mathcal{T}_\kappa$-invariant and $\widetilde{M}_r^\kappa \subset W_r \cap W_{r-1}^*$. Hence

$$\begin{aligned} \widetilde{M}_r^\kappa &\subset \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r \cap W_{r-1}^*) \subset \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r) \cap \mathrm{Inv}_{\mathcal{T}_\kappa}(W_{r-1}^*) \\ &= \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r) \cap \mathrm{Inv}_{\mathcal{T}_\kappa^-}(W_{r-1}^*) = A_r^\kappa \cap (A_{r-1}^\kappa)^*_{\mathcal{T}_\kappa} = \widetilde{M}_r^\kappa \end{aligned}$$

and so

$$\begin{aligned} \widetilde{M}_r^\kappa &= \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r \cap W_{r-1}^*) = \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r) \cap \mathrm{Inv}_{\mathcal{T}_\kappa^-}(W_{r-1}^*) \\ &\subset \mathrm{Int}_X(W_r) \cap \mathrm{Int}_X(W_{r-1}^*) \subset \mathrm{Int}_X(W_r \cap W_{r-1}^*). \end{aligned} \tag{3.6}$$

Now, formulas (3.2) and (3.5) and Proposition 2.17 in [3] implies that there exists a $\kappa' \geq \kappa_1$ such that for all $r \in [\![1, m]\!]$

$$\mathrm{Inv}_{\mathcal{T}_\kappa}(W_r \cap W_{r-1}^*) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_r), \text{ for all } \kappa \geq \kappa'.$$

Thus, for all $r \in [\![1, m]\!]$ and for all $\kappa \geq \kappa'$, $\widetilde{M}_r^\kappa = M_r(\kappa)$. Let $\kappa \geq \kappa'$ and $I \in A(<)$ be arbitrary. Then there is an $r \in [\![0, m]\!]$ such that $I = [\![1, r]\!]$. Now Proposition 3.9 in [3] implies that

$$\begin{aligned} \mathrm{Inv}_{\mathcal{T}_\kappa}(W_r) = A_r^\kappa &= \{\, x \mid \exists \sigma \in \mathcal{T}_\kappa \text{ with } \sigma(0) = x \text{ and } \alpha(\sigma) \subset \textstyle\bigcup_{i=1}^r M_i(\kappa) \,\} \\ &= M_{[\![1,r]\!]}(\kappa). \end{aligned}$$

Since $(A_r^\kappa, \widetilde{A}_r^\kappa)$ and $(M_{[\![1,r]\!]}(\kappa), M_{[\![r+1,m]\!]}(\kappa))$ are $\mathcal{T}_\kappa$-attractor-repeller pairs, it follows that $\widetilde{A}_r^\kappa = M_{[\![r+1,m]\!]}(\kappa)$ and this completes the proof of the first case.

Case 2. Now suppose that $\prec$ is an arbitrary strict partial order on P. It follows from case 1 that for each total order $\prec^*$ extending $\prec$, $(M_i(\kappa))_{i\in P}$ is a $\prec^*$-ordered $\mathcal{T}_\kappa$-Morse decomposition for all κ large enough. Since there is a finite number of such extensions, it also follows that there exists a $\tilde{\kappa}_0$ such that for every extension $\prec^*$ of $\prec$ and for all $\kappa \geq \tilde{\kappa}_0$, $(M_i(\kappa))_{i\in P}$ is a $\prec^*$-ordered $\mathcal{T}_\kappa$-Morse decomposition.

Let $\kappa \geq \tilde{\kappa}_0$ be arbitrary. It follows that the sets $M_i(\kappa)$, $i \in P$, are closed, $\mathcal{T}_\kappa$-invariant and pairwise disjoint. Moreover, for each $\sigma \in \mathcal{T}_\kappa$ either

(1) $\sigma(\mathbb{R}) \subset M_k(\kappa)$ for some $k \in P$

or else

(2) there are i, $j \in P$ such that $i \neq j$, $\alpha(\sigma) \subset M_j(\kappa)$ and $\omega(\sigma) \subset M_i(\kappa)$.

Assume the second alternative. We shall prove that $i \prec j$. Indeed, define

$$J := \{\, k \in P \mid k = j \text{ or } k \prec j \,\}.$$

It is clear that $j \in J$. Moreover, $J \in A(\prec)$. By Proposition 2.2, there is a total order $\prec^*$ extending $\prec$ such that $J \in A(\prec) \cap A(\prec^*)$. We also have that $(M_i(\kappa))_{i\in P}$ is a $\prec^*$-ordered $\mathcal{T}_\kappa$-Morse decomposition. Thus $i \prec^* j$ and so $i \in J$. This implies, as $i \neq j$, that $i \prec j$. This concludes the proof of our claim. Hence, we have proved that $(M_i(\kappa))_{i\in P}$ is a $\prec$-ordered $\mathcal{T}_\kappa$-Morse decomposition for all $\kappa \geq \tilde{\kappa}_0$.

Let $I \in A(\prec)$. By Proposition 2.2, there is a total order $\prec^*$ extending $\prec$ such that $I \in A(\prec) \cap A(\prec^*)$.

It follows from case 1 that

$$M_I(\kappa) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_I) \subset \mathrm{Int}_X(V_I), \quad \kappa \geq \kappa'$$

and

$$M_{P\setminus I}(\kappa) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_{P\setminus I}) \subset \mathrm{Int}_X(V_{P\setminus I}), \quad \kappa \geq \kappa'.$$

This completes the proof the lemma. ∎

Proof of Theorem 3.3. Let κ' be as in Lemma 3.4 and $K \in A(\prec)$ be arbitrary. Using the notation of Proposition 2.12 we will now show that

(3.7) the assumptions of Theorem 3.3 hold with P replaced by K and $\mathcal{T}$, $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$, replaced by $\mathcal{T}^K$, $\mathcal{T}_\kappa^K$, $\kappa \in \mathbb{N}$.

First we claim that $\mathcal{T}_\kappa^K \to \mathcal{T}^K$ as $\kappa \to \infty$. Indeed, let $(\kappa_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with $\kappa_n \to \infty$ as $n \to \infty$ and $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence such that $\sigma_n \in \mathcal{T}_{\kappa_n}^K$ for all $n \in \mathbb{N}$. Since $\mathcal{T}_\kappa \to \mathcal{T}$, there is a subsequence $(\sigma_{n_m})_{m\in\mathbb{N}}$ and a $\sigma \in \mathcal{T}$ such that $\sigma_{n_m} \to \sigma$ in $\mathcal{C}$ as $m \to \infty$. We only need to show that $\sigma \in \mathcal{T}^K$. It follows from Lemma 3.4 that for m large enough

$$M_K(\kappa_{n_m}) = \mathrm{Inv}_{\mathcal{T}_{\kappa_{n_m}}}(V_K) \subset \mathrm{Int}_X(V_K).$$

Since $\sigma_{n_m}(t) \in M_K(\kappa_{n_m})$ for all $t \in \mathbb{R}$ and for all $m \in \mathbb{N}$, it follows that for m large enough, $\sigma_{n_m}(t) \in V_K$ for all $t \in \mathbb{R}$. Hence, $\sigma(t) \in V_K$ for all $t \in \mathbb{R}$, that is, $\sigma(t) \in M(K)$ for all $t \in \mathbb{R}$. The proof of our claim is complete.

For each $i \in K$ we have $M_i \subset M(K)$ and $\mathcal{T}^K \subset \mathcal{T}$, hence

$$M_i \subset \mathrm{Inv}_{\mathcal{T}^K}(V_i) \subset \mathrm{Inv}_{\mathcal{T}}(V_i) = M_i$$

so

$$M_i = \mathrm{Inv}_{\mathcal{T}^K}(V_i) \subset \mathrm{Int}_X(V_i).$$

Let $I \in I(\prec)$, $I \subset K$, be arbitrary. It follows that $M(I) \subset M(K)$ and so

$$M(I) \subset \mathrm{Inv}_{\mathcal{T}^K}(V_I) \subset \mathrm{Inv}_{\mathcal{T}}(V_I) = M(I).$$

Thus

$$M(I) = \mathrm{Inv}_{\mathcal{T}^K}(V_I) \subset \mathrm{Int}_X(V_I).$$

Now Proposition 2.12 implies that, indeed, (3.7) holds.

Now let $J \in I(\prec)$ be arbitrary. By Proposition 2.1 there are I and $K \in A(\prec)$ with $I \subset K$ and $J = K \setminus I$. Therefore, (3.7) and Lemma 3.4 imply that

$$M_J(\kappa) = M_{K\setminus I}(\kappa) = \mathrm{Inv}_{\mathcal{T}_\kappa^K}(V_J) \subset \mathrm{Int}_X(V_J), \text{ for all } \kappa \geq \tilde{\kappa}_0(J),$$

where $\tilde{\kappa}_0(J)$ is chosen large enough. We claim that

$$\mathrm{Inv}_{\mathcal{T}_\kappa^K}(V_J) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_J) \text{ for all } \kappa \text{ large enough.} \tag{3.8}$$

Suppose that (3.8) is not true. Since $\mathrm{Inv}_{\mathcal{T}_\kappa^K}(V_J) \subset \mathrm{Inv}_{\mathcal{T}_\kappa}(V_J)$ for all κ, it follows that there exist a sequence $(\kappa_n)_n$ in $\mathbb{N}$ with $\kappa_n \geq \kappa'$ for all $n \in \mathbb{N}$, $\kappa_n \to \infty$ as $n \to \infty$ and a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $\sigma_n \in \mathcal{T}_{\kappa_n}$, $\sigma_n(\mathbb{R}) \subset V_J$ and $\sigma_n(\mathbb{R}) \not\subset M_{\kappa_n}(K) = \mathrm{Inv}_{\mathcal{T}_{\kappa_n}}(V_K)$. By the translation invariance of $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}$, we may thus assume that $\sigma_n(0) \notin V_K$ for all $n \in \mathbb{N}$. Taking subsequences if necessary we may assume that there is a $\sigma \in \mathcal{T}$ such that $\sigma_n \to \sigma$ in $\mathcal{C}$. It follows that $\sigma_n(0) \to \sigma(0) =: x$. Moreover, $\sigma(\mathbb{R}) \subset V_J$ and so $x \in \mathrm{Inv}_{\mathcal{T}}(V_J) = M(J) \subset M(K) \subset \mathrm{Int}_X(V_K)$. Therefore, for all n large enough we have $\sigma_n(0) \in \mathrm{Int}_X(V_K)$ which is a contradiction. Thus formula (3.8) holds. Therefore, for each $J \in I(\prec)$, there exists a $\kappa_0(J) \geq \tilde{\kappa}_0(J)$ such that

$$M_J(\kappa) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_J) \subset \mathrm{Int}_X(V_J), \text{ for all } \kappa \geq \kappa_0(J). \tag{3.9}$$

Since the set $I(\prec)$ is finite, formula (3.9) implies that there is a κ_0 such that, for all $\kappa \geq \kappa_0$ and for all $J \in I(\prec)$,

$$M_J(\kappa) = \mathrm{Inv}_{\mathcal{T}_\kappa}(V_J) \subset \mathrm{Int}_X(V_J).$$

The theorem is proved. ∎

Specializing to the semiflow case we thus arrive at the following corollaries.

Corollary 3.5. *Assume the following hypotheses:*

(1) $\pi_\kappa \to \pi_0$, where π_κ, $\kappa \in \mathbb{N}_0$, are local semiflows on X. N is a closed subset of X which is strongly π_κ-admissible for every $\kappa \in \mathbb{N}_0$ and $(\pi_{\kappa_n})_{n\in\mathbb{N}}$-admissible for every subsequence $(\pi_{\kappa_n})_n$ of $(\pi_\kappa)_\kappa$. $(M_i)_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S_0 := \mathrm{Inv}_{\pi_0}(N)$ relative to π_0.

(2) For each $i \in P$, $V_i \subset N$ is closed in X such that

$$M_i = \mathrm{Inv}_{\pi_0}(V_i) \subset \mathrm{Int}_X(V_i).$$

Moreover, for every $I \in I(\prec)$, $V_I \subset N$ is closed in X such that

$$M_{\pi_0,S_0}(I) = \mathrm{Inv}_{\pi_0}(V_I) \subset \mathrm{Int}_X(V_I).$$

For $\kappa \in \mathbb{N}$ and $i \in P$ set $M_i(\kappa) := \mathrm{Inv}_{\pi_\kappa}(V_i)$. Then there is a κ_0 such that for every $\kappa \geq \kappa_0$ the family $(M_i(\kappa))_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S_\kappa := \mathrm{Inv}_{\pi_\kappa}(N)$ relative to π_κ. Moreover, for every $I \in I(\prec)$,

$$M_{\pi_\kappa,S_\kappa}(I) = \mathrm{Inv}_{\pi_\kappa}(V_I) \subset \mathrm{Int}_X(V_I).$$

Proof. For $\kappa \in \mathbb{N}_0$ let $\mathcal{T}_\kappa := \mathcal{T}_{\pi_\kappa,S_\kappa}$. By Proposition 2.7 the set $\mathcal{T}_\kappa$ is compact in $\mathcal{C}$, translation- and cut-and-glue-invariant for every $\kappa \in \mathbb{N}_0$. Moreover, by Proposition 3.2 we have that $\mathcal{T}_\kappa \to \mathcal{T}_0$. Finally, if $W \subset N$ and $\kappa \in \mathbb{N}_0$, then $\mathrm{Inv}_{\pi_\kappa}(W) = \mathrm{Inv}_{\mathcal{T}_\kappa}(W)$. Now the corollary follows immediately from Theorem 3.3. ∎

Corollary 3.6. *Assume hypothesis (1) of Corollary 3.5. Moreover, suppose that $S_0 \subset \mathrm{Int}_X(N)$. Then there are sets V_i, $i \in P$ and sets V_I, $I \in I(\prec)$, such that hypothesis (2) of Corollary 3.5 (and, consequently, its assertion) holds.*

Proof. Let $\mathcal{T}_0$ be as in the proof of Corollary 3.5. For every $V \subset X$ it is clear that

$$\mathrm{Inv}_{\mathcal{T}_0}(V \cap N) = \mathrm{Inv}_{\pi_0}(V \cap N).$$

If, addition, $\mathrm{Inv}_{\mathcal{T}_0}(V) \subset \mathrm{Int}_X(V)$, then

$$\mathrm{Inv}_{\pi_0}(V \cap N) \subset \mathrm{Int}_X(V \cap N).$$

By Proposition 2.13 there are families $(V_i')_{i\in P}$ and $(V_I')_{I\in I(\prec)}$ of closed subsets of X such that

$$M_i = \mathrm{Inv}_{\mathcal{T}_0}(V_i') \subset \mathrm{Int}_X(V_i'), \quad i \in P$$

$$M_{\pi_0,S_0}(I) = \mathrm{Inv}_{\mathcal{T}_0}(V_I') \subset \mathrm{Int}_X(V_I'), \quad I \in I(\prec).$$

Setting $V_i := V_i' \cap N$, $i \in P$ and $V_I := V_I' \cap N$, $I \in I(\prec)$, we thus conclude the proof. ∎

We will now extend some results from [3] to the partially ordered case. We use the notation of section 4 in [3]. In particular, for the rest of this section, let $(E, \|\cdot\|)$ is a Banach space, set $X = E$ and define the metric d on X by $d(x, y) = \|x - y\|$ for x and $y \in X$. Let $U \subset X$ be and $f \in C(U \to X)$ be arbitrary. Suppose S is invariant relative to f (i.e. invariant with respect to the ordinary differential equation $\dot{x} = f(x)$ on U) and let $\mathcal{T} = \mathcal{T}_{(f,S)}$ be the set of all (full) solutions of f lying in S. We say that $(M_i)_{i\in P}$ is a *$\prec$-ordered Morse decomposition of S (relative to f)* if $(M_i)_{i\in P}$ is a $\prec$-ordered $\mathcal{T}$-Morse decomposition. If $I \in I(\prec)$ we write $M_{f,S}(I) := M(I, \mathcal{T}_{(f,S)}, (M_i)_{i\in P})$.

We now have the following extension of Theorem 4.5 in [3] (with a slightly different notation).

Corollary 3.7. *Assume the following hypotheses:*

(1) $X = E$ is a finite dimensional Banach space, U is open in X, N is bounded and closed in X with $N \subset U$ and $\sup_{x\in N} \|f_\kappa(x) - f_0(x)\|_E \to 0$ as $\kappa \to \infty$ where $(f_\kappa)_{\kappa\in\mathbb{N}_0}$ is a sequence in $C(U \to X)$. $(M_i)_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S_0 := \mathrm{Inv}(f_0, N)$ relative to f_0.

(2) For each $i \in P$, $V_i \subset N$ is closed in X such that

$$M_i = \mathrm{Inv}(f_0, V_i) \subset \mathrm{Int}_X(V_i).$$

Moreover, for every $I \in I(\prec)$, $V_I \subset N$ is closed in X such that

$$M_{f_0,S_0}(I) = \mathrm{Inv}(f_0, V_I) \subset \mathrm{Int}_X(V_I).$$

For $\kappa \in \mathbb{N}$ and $i \in P$ set $M_i(\kappa) := \mathrm{Inv}(f_\kappa, V_i)$. Then there is a κ_0 such that for every $\kappa \geq \kappa_0$ the family $(M_i(\kappa))_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S_\kappa := \mathrm{Inv}(f_\kappa, N)$ relative to f_κ. Moreover, for every $I \in I(\prec)$,

$$M_{f_\kappa,S_\kappa}(I) = \mathrm{Inv}(f_\kappa, V_I) \subset \mathrm{Int}_X(V_I).$$

Proof. For $\kappa \in \mathbb{N}_0$, let $\mathcal{T}_\kappa := \mathcal{T}_{f_\kappa,S_\kappa}$. By Propositions 4.1 and 4.2 in [3] the sets $\mathcal{T}_\kappa$, $\kappa \in \mathbb{N}_0$ are compact in $\mathcal{C}$, translation- and cut-and-glue-invariant and $\mathcal{T}_\kappa \to \mathcal{T}_0$. Finally, if $W \subset N$ and $\kappa \in \mathbb{N}_0$, then $\mathrm{Inv}(f_\kappa, W) = \mathrm{Inv}_{\mathcal{T}_\kappa}(W)$. Now the corollary follows immediately from Theorem 3.3. ∎

Corollary 3.8. *Assume hypothesis (1) of Corollary 3.7. Moreover, suppose that $S_0 \subset \mathrm{Int}_X(N)$. Then there are sets V_i, $i \in P$ and sets V_I, $I \in I(\prec)$, such that hypothesis (2) of Corollary 3.7 (and, consequently, its assertion) holds.*

Proof. Let $\mathcal{T}_0$ be as in the proof of Corollary 3.7. For every $V \subset X$ it is clear that

$$\mathrm{Inv}_{\mathcal{T}_0}(V \cap N) = \mathrm{Inv}(f_0, V \cap N).$$

If, addition, $\mathrm{Inv}_{\mathcal{T}_0}(V) \subset \mathrm{Int}_X(V)$, then

$$\mathrm{Inv}(f_0, V \cap N) \subset \mathrm{Int}_X(V \cap N).$$

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By Proposition 2.13 there are families $(V_i')_{i\in P}$ and $(V_I')_{I\in I(\prec)}$ of closed subsets of X such that

$$M_i = \mathrm{Inv}_{\mathcal{T}_0}(V_i') \subset \mathrm{Int}_X(V_i'), \quad i \in P$$

$$M_{f_0,S_0}(I) = \mathrm{Inv}_{\mathcal{T}_0}(V_I') \subset \mathrm{Int}_X(V_I'), \quad I \in I(\prec).$$

Setting $V_i := V_i' \cap N$, $i \in P$ and $V_I := V_I' \cap N$, $I \in I(\prec)$, we thus conclude the proof. ▮

We now extend Theorem 4.15 in [3] to the partially ordered case (again with a slightly different notation).

Corollary 3.9. *Assume the following hypotheses:*

(1) $X = E$ is an infinite dimensional Banach space, Hypothesis 4.9 in [3] is satisfied and L, L^ℓ, P^ℓ, E^ℓ, $\ell \in \mathbb{N}$, be as in that hypothesis. U is open in X, N is bounded and closed in X with $N \subset U$ and $K \in C(U \to X)$ is such that $K(N)$ is relatively compact in X. The maps $f_0 \colon U \to X$ and $f_\ell \colon U \cap E^\ell \to E^\ell$, $\ell \in \mathbb{N}$, are defined by

$$f_0(x) = Lx + K(x), \quad x \in U$$

and

$$f_\ell(x) = L^\ell x + P^\ell K(x), \quad \ell \in \mathbb{N}, x \in U \cap E^\ell.$$

$(M_i)_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S_0 := \mathrm{Inv}(f_0, N)$ relative to f_0.

(2) For each $i \in P$, $V_i \subset N$ is closed in X such that

$$M_i = \mathrm{Inv}(f_0, V_i) \subset \mathrm{Int}_X(V_i).$$

Moreover, for every $I \in I(\prec)$, $V_I \subset N$ is closed in X such that

$$M_{f_0,S_0}(I) = \mathrm{Inv}(f_0, V_I) \subset \mathrm{Int}_X(V_I).$$

For $\ell \in \mathbb{N}$ and $i \in P$ set $M_i(\ell) := \mathrm{Inv}(f_\ell, V_i \cap E^\ell)$. Then there is a ℓ_0 such that for every $\ell \geq \ell_0$ the family $(M_i(\ell))_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S_\ell := \mathrm{Inv}(f_\ell, N \cap E^\ell)$ relative to f_ℓ. Moreover, for every $I \in I(\prec)$,

$$M_{f_\ell,S_\ell}(I) = \mathrm{Inv}(f_\ell, V_I) \subset \mathrm{Int}_X(V_I).$$

Proof. For $\ell \in \mathbb{N}_0$, let $\mathcal{T}_\ell := \mathcal{T}_{f_\ell,S_\ell}$. By Proposition 4.11 in [3] the sets $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$ are compact in $\mathcal{C}$, translation- and cut-and-glue-invariant and $\mathcal{T}_\ell \to \mathcal{T}_0$ as $\ell \to \infty$. Finally, if $W \subset N$ and $\ell \in \mathbb{N}_0$, then $\mathrm{Inv}(f_\ell, W) = \mathrm{Inv}_{\mathcal{T}_\ell}(W)$. Now the corollary follows immediately from Theorem 3.3. ▮

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Corollary 3.10. *Assume hypothesis (1) of Corollary 3.9. Moreover, suppose that* $S_0 \subset \mathrm{Int}_X(N)$. *Then there are sets* V_i, $i \in P$ *and sets* V_I, $I \in I(\prec)$, *such that hypothesis (2) of Corollary 3.9 (and, consequently, its assertion) holds.*

Proof. The proof is identical to the proof of Corollary 3.8. ▮

4. SINGULAR CONVERGENCE AND TOTALLY ORDERED MORSE DECOMPOSITIONS

In this section we will discuss perturbations of Morse decompositions within the framework introduced in [4] for the study of singular perturbation problems. After recalling some concepts from [4] we define singular convergence for sequence of 'solution sets'. Then we prove a few properties of this convergence concept, in particular we obtain a few basic continuation results for singularly perturbed $\mathcal{T}$-attractor-repeller pairs, (totally ordered) $\mathcal{T}$-attractor filtrations and (totally ordered) $\mathcal{T}$-Morse decompositions. We then extend the latter result to the partially ordered case. We end this section by specializing to the semiflow case and considering some thin domain problems.

Let us recall the basic concepts related to singular perturbation problems, introduced in [4]. For the rest of this paper, unless specified otherwise, let (X_0, d_0) be a metric space, ε_0 be a positive number and, for each $\varepsilon \in \,]0, \varepsilon_0]$, $(Y_\varepsilon, d_\varepsilon)$ be a metric space and $\theta_\varepsilon \in Y_\varepsilon$ be a distinguished point of Y_ε.

The open ball in Y_ε of center in v and radius $\beta > 0$ is denoted by $B_\varepsilon(v, \beta)$.

For each $\varepsilon \in \,]0, \varepsilon_0]$ define the set $Z_\varepsilon := X_0 \times Y_\varepsilon$. Endow Z_ε with the metric

$$\Gamma_\varepsilon((u,v),(u',v')) := \max\{\, d_0(u,u'), d_\varepsilon(v,v')\,\},\ (u,v),\ (u',v') \in Z_\varepsilon.$$

Let $\mathcal{C}_0$ denote the set of all continuous functions from $\mathbb{R}$ to X_0 endowed with the metric introduced in [3], replacing d by d_0. For every $\varepsilon \in \,]0, \varepsilon_0]$, denote by $\mathcal{C}_\varepsilon$ the set of all continuous functions from $\mathbb{R}$ to Z_ε endowed with the metric introduced in [3], replacing d by Γ_ε.

Given an $\varepsilon \in \,]0, \varepsilon_0]$ and $\sigma \in \mathcal{C}_\varepsilon$ we denote, for each $t \in \mathbb{R}$, the components of $\sigma(t)$ by $(\phi(t), \psi(t))$, where $\phi(t) \in X_0$ and $\psi(t) \in Y_\varepsilon$. This notation will also be used if the symbol σ carries an index, e.g. σ_n. Then the components of σ_n are written with the same index, e.g. ϕ_n and ψ_n.

Definition 4.1. *Given a subset* V *of* X_0, $\beta > 0$ *and* $\varepsilon \in \,]0, \varepsilon_0]$ *define the 'inflated' subsets* $]V[_{\varepsilon,\beta}$ *and* $[V]_{\varepsilon,\beta}$ *of* Z_ε *as follows:*

$$]V[_{\varepsilon,\beta} := \{\, (u,v) \in Z_\varepsilon \mid u \in V \text{ and } v \in B_\varepsilon(\theta_\varepsilon, \beta)\,\};$$
$$[V]_{\varepsilon,\beta} := \{\, (u,v) \in Z_\varepsilon \mid u \in V \text{ and } v \in \mathrm{Cl}_\varepsilon\, B_\varepsilon(\theta_\varepsilon, \beta)\,\}.$$

Now we introduce a solution convergence concept in the context of singular perturbations.

Definition 4.2. *For each* $\varepsilon \in \,]0, \varepsilon_0]$, *let* $\mathcal{T}_\varepsilon$ *be a subset of* $\mathcal{C}_\varepsilon$ *and let* $\mathcal{T}_0$ *be a subset of* $\mathcal{C}_0$. *We say that* $(\mathcal{T}_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ *converges singularly to* $\mathcal{T}_0$, *and we write* $\mathcal{T}_\varepsilon \stackrel{\mathrm{sg}}{\to} \mathcal{T}_0$, *if for every* *sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *in* $]0,\varepsilon_0]$ *with* $\varepsilon_n \to 0$ *as* $n\to\infty$ *and every sequence* $(\sigma_n)_{n\in\mathbb{N}}$ *such that* $\sigma_n \in \mathcal{T}_{\varepsilon_n}$ *for all* $n\in\mathbb{N}$ *there is a subsequence* $(\sigma_{n_m})_{m\in\mathbb{N}}$ *and a* $\sigma\in\mathcal{T}_0$ *such that*

$$\Gamma_{\varepsilon_{n_m}}\left(\sigma_{n_m}(t),(\sigma(t),\theta_{\varepsilon_{n_m}})\right)\to 0 \text{ as } m\to\infty$$

uniformly on compact subsets of $\mathbb{R}$.

The next proposition gives sufficient condition for $\mathcal{T}_\varepsilon \overset{\text{sg}}{\to} \mathcal{T}_0$. We first recall two basic definitions introduced in [4].

Let π_0 be a local semiflow on X_0 and for every $\varepsilon\in\,]0,\varepsilon_0]$ let π_ε denote a local semiflow on Z_ε.

We say that the family $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ of local semiflows *converges singularly to* the local semiflow π_0 if whenever $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences of positive numbers such that $\varepsilon_n\to 0$, $t_n\to t_0$ as $n\to\infty$, for some $t_0\in[0,\infty[$ and whenever $u_0\in X_0$ and $w_n\in Z_{\varepsilon_n}$, $n\in\mathbb{N}$, are such that $\Gamma_{\varepsilon_n}(w_n,(u_0,\theta_{\varepsilon_n}))\to 0$ as $n\to\infty$ and $u_0\pi_0 t_0$ is defined, then there exists an $n_0\in\mathbb{N}$ such that for all $n\ge n_0$, $w_n\pi_{\varepsilon_n}t_n$ is defined and

$$\Gamma_{\varepsilon_n}(w_n\pi_{\varepsilon_n}t_n,(u_0\pi_0t_0,\theta_{\varepsilon_n}))\to 0 \text{ as } n\to\infty.$$

Let η be a positive number and N be a closed subset of X_0. We say that N is a *singularly strongly admissible set with respect to* η *and the family* $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ of local semiflows if the following conditions are satisfied:

(1) N is a strongly π_0-admissible set;
(2) for each $\varepsilon\in\,]0,\varepsilon_0]$ the set $[N]_{\varepsilon,\eta}$ is strongly π_ε-admissible;
(3) whenever $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ are sequences of positive numbers such that $\varepsilon_n\to 0$, $t_n\to\infty$ as $n\to\infty$ and whenever $w_n\in Z_{\varepsilon_n}$, $n\in\mathbb{N}$, are such that $w_n\pi_{\varepsilon_n}[0,t_n]\subset[N]_{\varepsilon_n,\eta}$, $n\in\mathbb{N}$, then there exist a $u_0\in N$ and a subsequence of the sequence $(w_n\pi_{\varepsilon_n}t_n)_{n\in\mathbb{N}}$ of endpoints, denoted again by $(w_n\pi_{\varepsilon_n}t_n)_{n\in\mathbb{N}}$, such that

$$\Gamma_{\varepsilon_n}(w_n\pi_{\varepsilon_n}t_n,(u_0,\theta_{\varepsilon_n}))\to 0 \text{ as } n\to\infty.$$

PROPOSITION 4.3. *Let* η *be a positive number. Suppose* $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ *is a family of local semiflows that converges singularly to the local semiflow* π_0 *and* N *is a singularly strongly admissible set with respect to* η *and* $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$. *For each* $\varepsilon\in\,]0,\varepsilon_0]$, *define* $\mathcal{T}_\varepsilon:=\mathcal{T}_{\pi_\varepsilon}$ *be the set of all full solutions of* π_ε *lying in* $S_\varepsilon=\mathrm{Inv}_{\pi_\varepsilon}([N]_{\varepsilon,\eta})$ *and* $\mathcal{T}_0:=\mathcal{T}_{\pi_0}$ *be the set of all full solutions of* π_0 *lying in* $S_0=\mathrm{Inv}_{\pi_0}(N)$. *Then* $\mathcal{T}_\varepsilon\overset{\text{sg}}{\to}\mathcal{T}_0$.

Proof. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ with $\varepsilon_n\to 0$ as $n\to\infty$ and $(\sigma_n)_{n\in\mathbb{N}}$ be sequence such that $\sigma_n\in\mathcal{T}_{\varepsilon_n}$ for all $n\in\mathbb{N}$. We need to show that there is a subsequence $(\sigma_{n_m})_{m\in\mathbb{N}}$ and a $\sigma\in\mathcal{T}_0$ such that

$$\Gamma_{\varepsilon_{n_m}}\left(\sigma_{n_m}(t),(\sigma(t),\theta_{\varepsilon_{n_m}})\right)\to 0 \text{ as } m\to\infty$$

uniformly on compact subsets of $\mathbb{R}$. Using the standard Cantor diagonalization procedure and the singular admissibility of N, we obtain a subsequence of $(\sigma_n)_n$, which it is denoted again by $(\sigma_n)_n$, and a sequence $(u_{-k})_{k\in\mathbb{N}_0}$ in N such that for all $k \in \mathbb{N}_0$

$$\Gamma_{\varepsilon_n}(\sigma_n(-k), (u_{-k}, \theta_{\varepsilon_n})) \text{ as } n \to \infty. \tag{4.1}$$

We claim that for each $k \in \mathbb{N}_0$ the solution of π_0 through u_{-k} is defined for all $t \in [0, \infty[$. In fact, if this is not true for some $k \in \mathbb{N}_0$, then the assumption that π_0 does not explode in N implies the existence of a $t \in [0, \infty[$ such that $u_{-k}\pi_0 t$ is defined and $u_{-k}\pi_0 t \notin N$. Since $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ converges singularly to π_0, we see that, for all $n \in \mathbb{N}$ large enough, $\sigma_n(-k)\pi_{\varepsilon_n} t$ is defined and $\sigma_n(-k)\pi_{\varepsilon_n} t \notin [N]_{\varepsilon_n,\eta}$, a contradiction which proves our claim.

For each $k \in \mathbb{N}_0$ and for $t \in [-k, \infty[$ define

$$\tilde{\sigma}_{-k}(t) := u_{-k}\pi_0(t + k).$$

The above claim implies that $\tilde{\sigma}_{-k} \colon [-k, \infty[\to N$ is well-defined.

Note that if $k < k'$ and $t \in [-k, \infty[$ then $\sigma_n(-k)\pi_{\varepsilon_n}(t+k) = \sigma_n(t) = \sigma_n(-k')\pi_{\varepsilon_n}(t+k')$ for all $n \in \mathbb{N}$, hence, by the singular convergence of $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ to π_0,

$$\Gamma_{\varepsilon_n}(\sigma_n(-k)\pi_{\varepsilon_n}(t + k), (\tilde{\sigma}_{-k}(t), \theta_{\varepsilon_n})) \to 0, \text{ as } n \to \infty$$

and

$$\Gamma_{\varepsilon_n}(\sigma_n(-k')\pi_{\varepsilon_n}(t + k'), (\tilde{\sigma}_{-k'}(t), \theta_{\varepsilon_n})) \to 0, \text{ as } n \to \infty$$

so $\tilde{\sigma}_{-k}$ and $\tilde{\sigma}_{-k'}$ coincide on $[-k, \infty[$. Thus there is a unique map $\sigma \colon \mathbb{R} \to N$ such that $\sigma(t) = \tilde{\sigma}_{-k}(t)$ for all $k \in \mathbb{N}_0$ and $t \in [-k, \infty[$. It follows that σ is a full solution of π_0 lying in N.

To complete the proof we need to show that whenever $(t_n)_n$ is a sequence in $\mathbb{R}$ such that $t_n \to t \in \mathbb{R}$ as $n \to \infty$, then

$$\Gamma_{\varepsilon_n}(\sigma_n(t_n), (\sigma(t), \theta_{\varepsilon_n})) \to 0. \tag{4.2}$$

Thus let $(t_n)_n$ be a sequence in $\mathbb{R}$ such that $t_n \to t \in \mathbb{R}$ as $n \to \infty$.

There is a $k \in \mathbb{N}$ such that t, $t_n \in [-k, \infty[$ for all $n \in \mathbb{N}$. Therefore, $\sigma(t) = \tilde{\sigma}_{-k}(t) = u_{-k}\pi_0(t+k)$ and $\sigma_n(t_n) = \sigma_n(-k)\pi_{\kappa_n}(t_n+k)$. Now, the singular convergence of $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ to π_0 and formula (4.1) imply (4.2). This completes the proof. ▮

Some elementary properties of the above concepts are proved in the next propositions.

Proposition 4.4. *Suppose N is a closed set in X_0, $\mathcal{T}_\varepsilon \xrightarrow{\text{sg}} \mathcal{T}_0$ and $\text{Inv}_{\mathcal{T}_0}(N) \subset \text{Int}_{X_0}(N)$. Assume also that each $\mathcal{T}_\varepsilon$, $\varepsilon \in]0, \varepsilon_0]$, is translation-invariant. Let $\eta > 0$. Then there is an $\tilde{\varepsilon} \in]0, \varepsilon_0]$ such that $\text{Inv}_{\mathcal{T}_\varepsilon}([N]_{\varepsilon,\eta}) \subset \text{Int}_{Z_\varepsilon}([N]_{\varepsilon,\eta})$ for all $\varepsilon \in]0, \tilde{\varepsilon}]$.*

Proof. If the proposition is not true, it follows from Definition 4.2 and the translation-invariance of $\mathcal{T}_\varepsilon$ that there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n \to \infty$, a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\varepsilon_n}$ for all $n \in \mathbb{N}$ and a $\sigma \in \mathcal{T}_0$ such that for each $t \in \mathbb{R}$

$$\Gamma_{\varepsilon_n}(\sigma_n(t), (\sigma(t), \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty, \tag{4.3}$$

$\sigma_n(\mathbb{R}) \subset [N]_{\varepsilon_n,\eta}$ and $\sigma_n(0) \in \partial_{Z_{\varepsilon_n}}([N]_{\varepsilon_n,\eta})$ for every $n \in \mathbb{N}$. It follows from (4.3) that for each $t \in \mathbb{R}$

$$d_0(\phi_n(t), \sigma(t)) \to 0 \text{ as } n \to \infty \tag{4.4}$$

and

$$d_{\varepsilon_n}(\psi_n(t), \theta_{\varepsilon_n}) \to 0 \text{ as } n \to \infty. \tag{4.5}$$

Letting $t = 0$ in (4.5), it follows that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d_{\varepsilon_n}(\psi_n(0), \theta_{\varepsilon_n}) < \eta. \tag{4.6}$$

Since $\sigma_n(0) \in \partial_{Z_{\varepsilon_n}}([N]_{\varepsilon_n,\eta})$ for every $n \in \mathbb{N}$, formula (4.6) implies that $\sigma(0) \in \partial_{X_0}(N)$. On the other hand, since N is closed and $\phi_n(t) \in N$ for every $t \in \mathbb{R}$, formula (4.4) implies that $\sigma(\mathbb{R}) \subset N$. Our assumption implies that $\sigma(\mathbb{R}) \subset \mathrm{Int}_{X_0}(N)$. Thus, $\sigma(0) \in \partial_{X_0}(N) \cap \mathrm{Int}_{X_0}(N)$ which is a contradiction. ▮

Proposition 4.5. *Suppose N is closed in X_0 and U is open in X_0, $\mathcal{T}_\varepsilon \xrightarrow{\text{sg}} \mathcal{T}_0$ and $\mathrm{Inv}_{\mathcal{T}_0}(N) \subset \mathrm{Inv}_{\mathcal{T}_0}(U)$. Assume also that each $\mathcal{T}_\varepsilon$, $\varepsilon \in]0, \varepsilon_0]$, is translation-invariant. Let $\eta > 0$. Then there is an $\tilde{\varepsilon} \in]0, \varepsilon_0]$ such that $\mathrm{Inv}_{\mathcal{T}_\varepsilon}([N]_{\varepsilon,\eta}) \subset \mathrm{Inv}_{\mathcal{T}_\varepsilon}(]U[_{\varepsilon,\eta})$ for all $\varepsilon \in]0, \tilde{\varepsilon}]$.*

Proof. If the proposition is not true, it follows from Definition 4.2 and the translation-invariance of $\mathcal{T}_\varepsilon$ that there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n \to \infty$, a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\varepsilon_n}$ for all $n \in \mathbb{N}$ and a $\sigma \in \mathcal{T}_0$ such that for each $t \in \mathbb{R}$

$$\Gamma_{\varepsilon_n}(\sigma_n(t), (\sigma(t), \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty, \tag{4.7}$$

$\sigma_n(\mathbb{R}) \subset [N]_{\varepsilon_n,\eta}$ and $\sigma_n(0) \in Z_{\varepsilon_n} \setminus]U[_{\varepsilon_n,\eta}$ for every $n \in \mathbb{N}$. It follows from (4.7) that for each $t \in \mathbb{R}$

$$d_0(\phi_n(t), \sigma(t)) \to 0 \text{ as } n \to \infty \tag{4.8}$$

and

$$d_{\varepsilon_n}(\psi_n(t), \theta_{\varepsilon_n}) \to 0 \text{ as } n \to \infty. \tag{4.9}$$

Letting $t = 0$ in (4.9), it follows that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d_{\varepsilon_n}(\psi_n(0), \theta_{\varepsilon_n}) < \eta. \tag{4.10}$$

Since $\sigma_n(0) \notin]U[_{\varepsilon_n,\eta}$ for all $n \in \mathbb{N}$, formula (4.10) implies that $\phi_n(0) \notin U$ for all $n \geq n_0$. Thus, we obtain that $\sigma(0) \in X \setminus U$. However, since N is closed it follows from (4.8) that $\sigma(\mathbb{R}) \subset N$ and so $\sigma(0) \in \mathrm{Inv}_{\mathcal{T}_0}(N) \subset \mathrm{Inv}_{\mathcal{T}_0}(U) \subset U$, a contradiction. The proof is complete. ▮

COROLLARY 4.6. *Suppose that N is a closed set in X_0, $N' \subset X$ is a arbitrary set, $\mathcal{T}_\varepsilon \overset{\text{sg}}{\to} \mathcal{T}_0$ and $\text{Inv}_{\mathcal{T}_0}(N) \subset \text{Inv}_{\mathcal{T}_0}(N') \subset \text{Int}_{X_0}(N')$. Assume also that each $\mathcal{T}_\varepsilon$ is translation-invariant. Let $\eta > 0$. Then there is an $\tilde{\varepsilon} \in \,]0, \varepsilon_0]$ such that $\text{Inv}_{\mathcal{T}_\varepsilon}([N]_{\varepsilon,\eta}) \subset \text{Inv}_{\mathcal{T}_\varepsilon}([N']_{\varepsilon,\eta})$ for all $\varepsilon \in \,]0, \tilde{\varepsilon}]$.*

Proof. Define $U := \text{Int}_{X_0}(N')$. Since $\text{Inv}_{\mathcal{T}_0}(N') \subset U$, we obtain that $\text{Inv}_{\mathcal{T}_0}(U) = \text{Inv}_{\mathcal{T}_0}(N')$ so the corollary follows from Proposition 4.5. ▮

COROLLARY 4.7. *Suppose N and N' are closed in X_0, $\mathcal{T}_\varepsilon \overset{\text{sg}}{\to} \mathcal{T}_0$, $\text{Inv}_{\mathcal{T}_0}(N) \subset \text{Int}_{X_0}(N)$, $\text{Inv}_{\mathcal{T}_0}(N') \subset \text{Int}_{X_0}(N')$ and $\text{Inv}_{\mathcal{T}_0}(N) = \text{Inv}_{\mathcal{T}_0}(N')$. Assume also that each $\mathcal{T}_\varepsilon$ is translation-invariant. Let $\eta > 0$. Then there is an $\tilde{\varepsilon} \in \,]0, \varepsilon_0]$ such that $\text{Inv}_{\mathcal{T}_\varepsilon}([N]_{\varepsilon,\eta}) = \text{Inv}_{\mathcal{T}_\varepsilon}([N']_{\varepsilon,\eta})$ for all $\varepsilon \in \,]0, \tilde{\varepsilon}]$.*

Proof. This is an immediate consequence of Corollary 4.6. ▮

PROPOSITION 4.8. *Suppose that N is a closed set in X_0, $\mathcal{T}_\varepsilon \overset{\text{sg}}{\to} \mathcal{T}_0$, $t' \in \,]0, \infty[$ and $\mathcal{T}_0(N, t') \subset \text{Int}_{X_0}(N)$. Let $\eta > 0$. Then there is an $\tilde{\varepsilon} \in \,]0, \varepsilon_0]$ such that $\mathcal{T}_\varepsilon([N]_{\varepsilon,\eta}, t') \subset \text{Int}_{Z_\varepsilon}([N]_{\varepsilon,\eta})$ for all $\varepsilon \in \,]0, \tilde{\varepsilon}]$.*

Proof. If the proposition is not true then, by Definition 4.2, there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n \to \infty$, a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\varepsilon_n}$ for all $n \in \mathbb{N}$ and a $\sigma \in \mathcal{T}_0$ such that for each $t \in \mathbb{R}$

$$\Gamma_{\varepsilon_n}(\sigma_n(t), (\sigma(t), \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty, \tag{4.11}$$

$\sigma_n(t') \notin \text{Int}_{Z_{\varepsilon_n}}([N]_{\varepsilon_n,\eta})$ and $\sigma_n(0) \in [N]_{\varepsilon_n,\eta}$ for every $n \in \mathbb{N}$. Formula (4.11) implies that for each $t \in \mathbb{R}$

$$d_0(\phi_n(t), \sigma(t)) \to 0 \text{ as } n \to \infty \tag{4.12}$$

and

$$d_{\varepsilon_n}(\psi_n(t), \theta_{\varepsilon_n}) \to 0 \text{ as } n \to \infty. \tag{4.13}$$

Letting $t = t'$ in (4.13), it follows that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d_{\varepsilon_n}(\psi_n(t'), \theta_{\varepsilon_n}) < \eta. \tag{4.14}$$

Since $\sigma_n(t') \notin \text{Int}_{Z_{\varepsilon_n}}([N]_{\varepsilon_n,\eta})$, formula (4.14) implies that $\phi_n(t') \notin \text{Int}_{X_0}(N)$ for all $n \geq n_0$ and so $\sigma(t') \notin \text{Int}_{X_0}(N)$. Now, notice that $\phi_n(0) \in N$ for every $n \in \mathbb{N}$ and N is closed. Therefore, it follows from (4.12) that $\sigma(0) \in N$. Hence, $\sigma(t') \in \mathcal{T}_0(N, t') \setminus \text{Int}_{X_0}(N)$ which is a contradiction. ▮

In the next theorem we prove the stability of attractor-repeller pairs under singular perturbations.

THEOREM 4.9. *For each $\varepsilon \in \,]0, \varepsilon_0]$, let $\mathcal{T}_\varepsilon$ be a compact and translation-invariant subset of $\mathcal{C}_\varepsilon$ and $\mathcal{T}_0 \subset \mathcal{C}_0$ be compact and translation-invariant. Suppose $\mathcal{T}_\varepsilon \overset{\text{sg}}{\to} \mathcal{T}_0$ and let (A, A^*)*

be a $\mathcal{T}_0$-attractor-repeller pair. Let V (resp. V^*) be closed in X_0 and such that $A = \mathrm{Inv}_{\mathcal{T}_0}(V) \subset \mathrm{Int}_{X_0}(V)$ (resp. $A^* = \mathrm{Inv}_{\mathcal{T}_0^-}(V^*) \subset \mathrm{Int}_{X_0}(V^*)$). Let $\eta > 0$. Then there is an $\tilde{\varepsilon} \in]0, \varepsilon_0]$ such that, for all $\varepsilon \in]0, \tilde{\varepsilon}]$, the pair $(\mathrm{Inv}_{\mathcal{T}_\varepsilon}([V]_{\varepsilon,\eta}), \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V^*]_{\varepsilon,\eta}))$ is a $\mathcal{T}_\varepsilon$-attractor-repeller pair.

Proof. Let N and N^* be closed and such that $A = \omega_{\mathcal{T}_0}(N) \subset \mathrm{Int}_{X_0}(N)$ and $A^* = \omega_{\mathcal{T}_0^-}(N^*) \subset \mathrm{Int}_{X_0}(N^*)$. Since A and A^* are disjoint and closed by Theorem 2.11 in [3] we may use Proposition 2.7 in [3] and choose N and N^* smaller, if necessary, to ensure that N and N^* are disjoint. For $\varepsilon \in]0, \varepsilon_0]$ set

$$A_\varepsilon = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([N]_{\varepsilon,\eta}) \text{ and } \tilde{A}_\varepsilon = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([N^*]_{\varepsilon,\eta}).$$

By Theorem 2.8 in [3] there is a $t_0 \in]0, \infty[$ such that $\mathcal{T}_0(N, t_0) \subset \mathrm{Int}_{X_0}(N)$. Furthermore, by Proposition 4.8 there is an $\varepsilon_1 \in]0, \varepsilon_0]$ such that $\mathcal{T}_\varepsilon([N]_{\varepsilon,\eta}, t_0) \subset \mathrm{Int}_{Z_\varepsilon}([N]_{\varepsilon,\eta})$ for all $\varepsilon \in]0, \varepsilon_1]$. Now Theorem 2.8 in [3] implies that

$$A_\varepsilon = \omega_{\mathcal{T}_\varepsilon}([N]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([N]_{\varepsilon,\eta}) \text{ for all } \varepsilon \in]0, \varepsilon_1] \tag{4.15}$$

so A_ε is a $\mathcal{T}_\varepsilon$-attractor for all $\varepsilon \in]0, \varepsilon_1]$. Set

$$A^*_\varepsilon = (A_\varepsilon)^*_{\mathcal{T}_\varepsilon}.$$

If $\varepsilon \in]0, \varepsilon_1]$ and $(u, v) \in \tilde{A}_\varepsilon$, then there is a $\sigma \in \mathcal{T}_\varepsilon$ with $\sigma(0) = (u, v)$ and $\sigma(\mathbb{R}) \subset [N^*]_{\varepsilon,\eta}$. Since $[N^*]_{\varepsilon,\eta}$ is closed, we conclude that $\omega(\sigma) \subset [N^*]_{\varepsilon,\eta} \subset Z_\varepsilon \setminus [N]_{\varepsilon,\eta}$, so $\omega(\sigma) \cap A_\varepsilon = \emptyset$. Hence $(u, v) \in A^*_\varepsilon$ which proves that

$$\tilde{A}_\varepsilon \subset A^*_\varepsilon.$$

Now suppose that it is not true that $A^*_\varepsilon \subset \tilde{A}_\varepsilon$ for all ε small enough. Then there are sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ as $n \to \infty$ and $((u_n, v_n))_{n\in\mathbb{N}}$ such that $(u_n, v_n) \in A^*_{\varepsilon_n} \setminus \tilde{A}_{\varepsilon_n}$ for all $n \in \mathbb{N}$. Thus there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ with $\sigma_n \in \mathcal{T}_{\varepsilon_n}$, $\sigma_n(0) = (u_n, v_n)$ and $\omega(\sigma_n) \cap A_{\varepsilon_n} = \emptyset$ for all $n \in \mathbb{N}$. Proposition 2.10 in [3] and (4.15) imply that $\sigma_n(\mathbb{R}) \cap [N]_{\varepsilon_n,\eta} = \emptyset$ for all $n \in \mathbb{N}$ large enough. On the other hand, for every $n \in \mathbb{N}$ we have $\sigma_n(\mathbb{R}) \not\subset [N^*]_{\varepsilon_n,\eta}$ since otherwise $(u_n, v_n) \in \mathrm{Inv}_{\mathcal{T}_{\varepsilon_n}}([N^*]_{\varepsilon_n,\eta}) = \tilde{A}_{\varepsilon_n}$, a contradiction. It follows that for every $n \in \mathbb{N}$ there is a $t_n \in \mathbb{R}$ with $\sigma_n(t_n) \notin [N^*]_{\varepsilon_n,\eta}$. Let $\tau_n = \mathrm{tsl}_{t_n}\sigma_n$, $n \in \mathbb{N}$. Taking subsequences if necessary we may assume that there is a $\tau \in \mathcal{T}_0$ such that for each $t \in \mathbb{R}$

$$\Gamma_{\varepsilon_n}(\tau_n(t), (\tau(t), \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty,$$

which implies that for each $t \in \mathbb{R}$

$$d_0(\phi_n(t), \tau(t)) \to 0 \text{ as } n \to \infty \tag{4.16}$$

and

$$d_{\varepsilon_n}(\psi_n(t), \theta_{\varepsilon_n}) \to 0 \text{ as } n \to \infty, \tag{4.17}$$

where $\tau_n(t) = (\phi_n(t), \psi_n(t))$ with $\phi_n(t) \in X_0$ and $\psi_n(t) \in Y_{\varepsilon_n}$. Letting $t = 0$ in formula (4.17), it follows that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d_{\varepsilon_n}(\psi_n(0), \theta_{\varepsilon_n}) < \eta. \tag{4.18}$$

Since $\tau_n(0) = \sigma_n(t_n) \notin [N^*]_{\varepsilon_n,\eta}$ for all $n \in \mathbb{N}$, formula (4.18) implies that

$$\phi_n(0) \notin N^* \text{ for all } n \geq n_0. \tag{4.19}$$

Formulas (4.19) and (4.16) imply that $\tau(0) \notin \mathrm{Int}_{X_0}(N^*)$. This fact together with Theorem 2.11 in [3] imply that $\omega(\tau) \subset A$ and so $\tau(t') \in \mathrm{Int}_{X_0}(N)$ for some $t' \in \mathbb{R}$. However, $\tau_n(\mathbb{R}) = \sigma_n(\mathbb{R}) \subset Z_{\varepsilon_n} \setminus [N]_{\varepsilon_n,\eta}$.

Now, letting $t = t'$ in formula (4.17), it follows that there exists an $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$

$$d_{\varepsilon_n}(\psi_n(t'), \theta_{\varepsilon_n}) < \eta.$$

Thus, $\phi_n(t') \notin N$ for all $n \geq n_1$ and so $\tau(t') \notin \mathrm{Int}_{X_0}(N)$, a contradiction. This proves that

$$A^*_\varepsilon \subset \tilde{A}_\varepsilon$$

so $A^*_\varepsilon = \tilde{A}_\varepsilon$ for all ε sufficiently small. Thus, for all such ε, the pair $(A_\varepsilon, \tilde{A}_\varepsilon)$ is a $\mathcal{T}_\varepsilon$-attractor-repeller pair. Now, since $A = \mathrm{Inv}_{\mathcal{T}_0}(V) \subset \mathrm{Int}_{X_0}(V)$ and $A = \mathrm{Inv}_{\mathcal{T}_0}(N) \subset \mathrm{Int}_{X_0}(N)$, Corollary 4.7 implies that

$$\mathrm{Inv}_{\mathcal{T}_\varepsilon}([V]_{\varepsilon,\eta}) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([N]_{\varepsilon,\eta}) = A_\varepsilon \text{ for all } \varepsilon \text{ sufficiently small.}$$

Similarly, $\mathrm{Inv}_{\mathcal{T}_\varepsilon^-}([V^*]_{\varepsilon,\eta}) = \mathrm{Inv}_{\mathcal{T}_\varepsilon^-}([N^*]_{\varepsilon,\eta}) = \tilde{A}_\varepsilon$ for all ε sufficiently small. This completes the proof. ▮

We will now state and prove two continuation (i.e. 'stability') results for attractor filtrations and Morse decompositions under singular perturbations.

Theorem 4.10. *For each $\varepsilon \in \,]0, \varepsilon_0]$, let $\mathcal{T}_\varepsilon$ be a compact and translation- and cut-and-glue-invariant subset of $\mathcal{C}_\varepsilon$ and $\mathcal{T}_0 \subset \mathcal{C}_0$ be compact and translation- and cut-and-glue-invariant. Suppose that $\mathcal{T}_\varepsilon \xrightarrow{\text{sg}} \mathcal{T}_0$. Let $(A_r)_{r=0}^m$ be a $\mathcal{T}_0$-attractor filtration. For every $r \in [\![0, m]\!]$ let V_r and V_r^* be closed sets with $A_r = \mathrm{Inv}_{\mathcal{T}_0}(V_r) \subset \mathrm{Int}_{X_0}(V_r)$ and $(A_r)^*_{\mathcal{T}_0} = \mathrm{Inv}_{\mathcal{T}_0^-}(V_r^*) \subset \mathrm{Int}_{X_0}(V_r^*)$. Let $\eta > 0$. For $\varepsilon \in \,]0, \varepsilon_0]$ and $r \in [\![0, m]\!]$ set*

$$A_r^\varepsilon = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_r]_{\varepsilon,\eta}), \quad \tilde{A}_r^\varepsilon = \mathrm{Inv}_{\mathcal{T}_\varepsilon^-}([V_r^*]_{\varepsilon,\eta}).$$

Then there is an $\tilde{\varepsilon} \in \,]0, \varepsilon_0]$ such that, for all $\varepsilon \in \,]0, \tilde{\varepsilon}]$, the sequence $(A_r^\varepsilon)_{r=0}^m$ is a $\mathcal{T}_\varepsilon$-attractor filtration and $(\tilde{A}_r^\varepsilon)_{r=0}^m$ is its dual $\mathcal{T}_\varepsilon$-repeller filtration.

Proof. An application of Theorem 4.9 shows that $(A_r^\varepsilon, \tilde{A}_r^\varepsilon)$ is a $\mathcal{T}_\varepsilon$-attractor-repeller pair for all $r \in [\![0, m]\!]$ and all ε sufficiently small. Furthermore, we conclude from Corollary 4.6

that $A^\varepsilon_r \subset A^\varepsilon_{r+1}$ for all $r \in [\![0, m-1]\!]$ and all ε sufficiently small. Thus we only have to show that

$$A^\varepsilon_0 = \emptyset \text{ and } A^\varepsilon_m = S_{\mathcal{T}_\varepsilon} \text{ for all } \varepsilon \text{ sufficiently small.}$$

If there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$ and $A^{\varepsilon_n}_0 \neq \emptyset$, then there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\varepsilon_n}$ and $\sigma_n(\mathbb{R}) \subset [V_0]_{\varepsilon_n,\eta}$ for all $n \in \mathbb{N}$. Then, taking a subsequence if necessary, we may assume that there exists a $\sigma \in \mathcal{T}_0$ such that for each $t \in \mathbb{R}$

$$\Gamma_{\varepsilon_n}(\sigma_n(t), (\sigma(t), \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty.$$

In particular, for each $t \in \mathbb{R}$

$$d_0(\phi_n(t), \sigma(t)) \to 0 \text{ as } n \to \infty. \tag{4.20}$$

Since $\sigma_n(\mathbb{R}) \subset [V_0]_{\varepsilon_n,\eta}$, formula (4.20) implies that $\sigma(\mathbb{R}) \subset V_0$ and so $A_0 = \mathrm{Inv}_{\mathcal{T}_0}(V_0) \neq \emptyset$, a contradiction.

Now clearly $A_\varepsilon \subset S_{\mathcal{T}_\varepsilon}$ for every $\varepsilon \in]0, \varepsilon_0]$. Consequently, if there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $]0, \varepsilon_0]$ with $\varepsilon_n \to 0$ and $A^{\varepsilon_n}_m \neq S_{\mathcal{T}_{\varepsilon_n}}$, then there is a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\sigma_n \in \mathcal{T}_{\varepsilon_n}$ and $\sigma_n(0) \notin [V_m]_{\varepsilon_n,\eta}$ for all $n \in \mathbb{N}$. Taking a subsequence if necessary, we may assume that there exists a $\sigma \in \mathcal{T}_0$ such that for each $t \in \mathbb{R}$

$$\Gamma_{\varepsilon_n}(\sigma_n(t), (\sigma(t), \theta_{\varepsilon_n})) \to 0 \text{ as } n \to \infty.$$

Therefore, $d_0(\phi_n(t), \sigma(t)) \to 0$ as $n \to \infty$ and

$$d_{\varepsilon_n}(\psi_n(t), \theta_{\varepsilon_n}) \to 0 \text{ as } n \to \infty. \tag{4.21}$$

Letting $t = 0$ in formula (4.21) we have that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$d_{\varepsilon_n}(\psi_n(0), \theta_{\varepsilon_n}) < \eta.$$

Since $\sigma_n(0) \notin [V_m]_{\varepsilon_n,\eta}$ for all $n \in \mathbb{N}$, we see that $\phi_n(0) \notin V_m$ for n large enough. Thus, $\sigma(0) \in X_0 \setminus \mathrm{Int}_{X_0}(V_m)$ so $A_m \neq S_{\mathcal{T}_0}$, a contradiction. ∎

Theorem 4.11. *For each $\varepsilon \in]0, \varepsilon_0]$, let $\mathcal{T}_\varepsilon$ be a compact and translation- and cut-and-glue-invariant subset of $\mathcal{C}_\varepsilon$ and $\mathcal{T}_0 \subset \mathcal{C}_0$ be compact and translation- and cut-and-glue-invariant. Suppose that $\mathcal{T}_\varepsilon \xrightarrow{\text{sg}} \mathcal{T}_0$. Let $(M_r)_{r=1}^m$ be a $\mathcal{T}_0$-Morse decomposition. Let $(W_r)_{r=1}^m$ be a finite sequence of closed sets such that*

$$M_r = \mathrm{Inv}_{\mathcal{T}_0}(W_r) \subset \mathrm{Int}_{X_0}(W_r), \quad r \in [\![1, m]\!].$$

Let $\eta > 0$. For $\varepsilon \in]0, \varepsilon_0]$ and $r \in [\![1, m]\!]$ set

$$M^\varepsilon_r = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r]_{\varepsilon,\eta}). \tag{4.22}$$

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Then there is an $\tilde{\varepsilon} \in]0, \varepsilon_0]$ such that for all $\varepsilon \in]0, \tilde{\varepsilon}]$ the sequence $(M_r^\varepsilon)_{r=1}^m$ is a $\mathcal{T}_\varepsilon$-Morse decomposition.

Proof. Choose a $\mathcal{T}_0$-attractor filtration $(A_r)_{r=0}^m$ such that $M_r = A_r \cap (A_{r-1})^*_{\mathcal{T}_0}$, for all $r \in [\![1, m]\!]$. For every $r \in [\![0, m]\!]$ let V_r and V_r^* be closed sets with $A_r = \operatorname{Inv}_{\mathcal{T}_0}(V_r) \subset \operatorname{Int}_{X_0}(V_r)$ and $(A_r)^*_{\mathcal{T}_0} = \operatorname{Inv}_{\mathcal{T}_0^-}(V_r^*) \subset \operatorname{Int}_{X_0}(V_r^*)$. Let $r \in [\![1, m]\!]$ be arbitrary. Since M_r is $\mathcal{T}_0$-invariant and $M_r \subset V_r \cap V_{r-1}^*$, we see that

$$\begin{aligned} M_r \subset \operatorname{Inv}_{\mathcal{T}_0}(V_r \cap V_{r-1}^*) &\subset \operatorname{Inv}_{\mathcal{T}_0}(V_r) \cap \operatorname{Inv}_{\mathcal{T}_0}(V_{r-1}^*) \\ &= \operatorname{Inv}_{\mathcal{T}_0}(V_r) \cap \operatorname{Inv}_{\mathcal{T}_0^-}(V_{r-1}^*) = A_r \cap (A_{r-1})^*_{\mathcal{T}_0} = M_r \end{aligned}$$

so

$$\begin{aligned} M_r = \operatorname{Inv}_{\mathcal{T}_0}(V_r \cap V_{r-1}^*) &= \operatorname{Inv}_{\mathcal{T}_0}(V_r) \cap \operatorname{Inv}_{\mathcal{T}_0^-}(V_{r-1}^*) \\ &\subset \operatorname{Int}_{X_0}(V_r) \cap \operatorname{Int}_{X_0}(V_{r-1}^*) \subset \operatorname{Int}_{X_0}(V_r \cap V_{r-1}^*). \end{aligned} \tag{4.23}$$

For $r \in [\![0, m]\!]$ and $\varepsilon \in]0, \varepsilon_0]$ define

$$A_r^\varepsilon = \operatorname{Inv}_{\mathcal{T}_\varepsilon}([V_r]_{\varepsilon,\eta}), \quad \tilde{A}_r^\varepsilon = \operatorname{Inv}_{\mathcal{T}_\varepsilon^-}([V_r^*]_{\varepsilon,\eta}).$$

By Theorem 4.10 there is an $\varepsilon_1 \in]0, \varepsilon_0]$ such that, for all $\varepsilon \in]0, \varepsilon_1]$, the sequence $(A_r^\varepsilon)_{r=0}^m$ is a $\mathcal{T}_\varepsilon$-attractor filtration and $(\tilde{A}_r^\varepsilon)_{r=0}^m$ is its dual $\mathcal{T}_\varepsilon$-repeller filtration. It follows that, for all $\varepsilon \in]0, \varepsilon_1]$, the sequence $(\tilde{M}_r^\varepsilon)_{r=1}^m$ is a $\mathcal{T}_\varepsilon$-Morse decomposition, where

$$\tilde{M}_r^\varepsilon = A_r^\varepsilon \cap \tilde{A}_{r-1}^\varepsilon, \quad r \in [\![1, m]\!].$$

Proceeding as in the proof of formula (4.23) we see that

$$\begin{aligned} M_r^\varepsilon &= \operatorname{Inv}_{\mathcal{T}_\varepsilon}([V_r]_{\varepsilon,\eta} \cap [V_{r-1}^*]_{\varepsilon,\eta}) = \operatorname{Inv}_{\mathcal{T}_\varepsilon}([V_r]_{\varepsilon,\eta}) \cap \operatorname{Inv}_{\mathcal{T}_\varepsilon^-}([V_{r-1}^*]_{\varepsilon,\eta}) \\ &\subset \operatorname{Int}_{Z_\varepsilon}([V_r]_{\varepsilon,\eta}) \cap \operatorname{Int}_{Z_\varepsilon}([V_{r-1}^*]_{\varepsilon,\eta}) \subset \operatorname{Int}_{Z_\varepsilon}([V_r]_{\varepsilon,\eta} \cap [V_{r-1}^*]_{\varepsilon,\eta}). \end{aligned}$$

Notice that, for all subsets B and C of X_0 and all $\beta \in]0, \infty[$ and $\varepsilon \in]0, \varepsilon_0]$ we have

$$[B]_{\varepsilon,\beta} \cap [C]_{\varepsilon,\beta} = [B \cap C]_{\varepsilon,\beta}\,. \tag{4.24}$$

Now (4.22), (4.23), (4.24) and Corollary 4.7 imply that there is an $\tilde{\varepsilon} \in]0, \varepsilon_1]$ such that for all $r \in [\![1, m]\!]$ and for all $\varepsilon \in]0, \tilde{\varepsilon}]$,

$$\tilde{M}_r^\varepsilon = \operatorname{Inv}_{\mathcal{T}_\varepsilon}([V_r]_{\varepsilon,\eta} \cap [V_{r-1}^*]_{\varepsilon,\eta}) = \operatorname{Inv}_{\mathcal{T}_\varepsilon}([V_r \cap V_{r-1}^*]_{\varepsilon,\eta}) = \operatorname{Inv}_{\mathcal{T}_\varepsilon}([W_r]_{\varepsilon,\eta}) = M_r^\varepsilon.$$

The theorem is proved. ∎

We will now prove the main result of this section.

THEOREM 4.12. *Suppose $\mathcal{T}_\varepsilon \xrightarrow{\text{sg}} \mathcal{T}_0$, where, for each $\varepsilon \in [0, \varepsilon_0]$, $\mathcal{T}_\varepsilon \subset \mathcal{C}_\varepsilon$ is compact, translation and cut-and-glue invariant. Suppose $(M_i)_{i \in P}$ is a $\prec$-ordered $\mathcal{T}_0$-Morse decomposition. For each $i \in P$, let V_i be closed in X_0 such that*

$$M_i = \operatorname{Inv}_{\mathcal{T}_0}(V_i) \subset \operatorname{Int}_{X_0}(V_i). \tag{4.25}$$

Moreover, for every $I \in I(\prec)$, let V_I be closed in X_0 such that

$$M(I) = \mathrm{Inv}_{\mathcal{T}_0}(V_I) \subset \mathrm{Int}_{X_0}(V_I).$$

(In view of Proposition 2.13 such sets V_i, $i \in P$ and V_I, $I \in I(\prec)$, always exist.) For $\varepsilon \in \,]0, \varepsilon_0]$ and $i \in P$ set $M_i(\varepsilon) := \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_i]_{\varepsilon,\eta})$. Then there is an $\tilde{\varepsilon} \in \,]0, \varepsilon_0]$ such that for every $\varepsilon \in \,]0, \tilde{\varepsilon}]$ the family $(M_i(\varepsilon))_{i \in P}$ is a $\prec$-ordered $\mathcal{T}_\varepsilon$-Morse decomposition. Moreover, for every $I \in I(\prec)$,

$$M_I(\varepsilon) := \bigcup_{(i,j) \in I \times I} \mathrm{CS}_{\mathcal{T}_\varepsilon}(M_i(\varepsilon), M_j(\varepsilon)) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_I]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_I]_{\varepsilon,\eta}).$$

To prove this theorem we need the following lemma.

LEMMA 4.13. *Under the assumptions of the theorem there is an $\varepsilon' \in \,]0, \varepsilon_0]$ such that for every $\varepsilon \in \,]0, \varepsilon']$ the family $(M_i(\varepsilon))_{i \in P}$ is a $\prec$-ordered $\mathcal{T}_\varepsilon$-Morse decomposition. Moreover, for every $I \in A(\prec)$*

$$M_I(\varepsilon) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_I]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_I]_{\varepsilon,\eta}) \tag{4.26}$$

and

$$M_{P \setminus I}(\varepsilon) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_{P \setminus I}]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_{P \setminus I}]_{\varepsilon,\eta}). \tag{4.27}$$

Proof. We will consider two cases.

Case 1. We first suppose that $\prec$ is a total order. In view of Remark 2.5 we may assume that $P = [\![1, m]\!]$ for some $m \in \mathbb{N}$ and $\prec$ is $<$. It follows that $(M_i)_{i \in P}$ is a $\mathcal{T}$-Morse decomposition of the second kind. Therefore, Theorem 4.11 implies that there exists a $\tilde{\varepsilon}_0$ such that for every $\varepsilon \in \,]0, \tilde{\varepsilon}_0]$, $(M_i(\varepsilon))_{i=1}^m$ is a $\mathcal{T}_\varepsilon$-Morse decomposition.

For $r \in [\![0, m]\!]$ let A_r, W_r and W_r^* be as in the proof of case 1 of Lemma 3.4, with (X, d) replaced by (X_0, d_0) and $\mathcal{T}$ replaced by $\mathcal{T}_0$. As in that proof we see that

$$M_r = \mathrm{Inv}_{\mathcal{T}_0}(W_r \cap W_{r-1}^*) \subset \mathrm{Int}_{X_0}(W_r \cap W_{r-1}^*). \tag{4.28}$$

For each $r \in [\![0, m]\!]$ and $\varepsilon \in \,]0, \varepsilon_0]$, define $A_r^\varepsilon = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r]_{\varepsilon,\eta})$ and $\widetilde{A}_r^\varepsilon = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r^*]_{\varepsilon,\eta})$. Theorem 4.10 implies that there exists an $\varepsilon_1 \in \,]0, \varepsilon_0]$ with $\varepsilon_1 \leq \tilde{\varepsilon}_0$ such that, for every $\varepsilon \in \,]0, \varepsilon_1]$ and $r \in [\![0, m]\!]$, $A_r^\varepsilon \subset \mathrm{Int}_{Z_\varepsilon}([W_r]_{\varepsilon,\eta})$, $\widetilde{A}_r^\varepsilon \subset \mathrm{Int}_{Z_\varepsilon}([W_r^*]_{\varepsilon,\eta})$, $(A_r^\varepsilon)_{r=0}^m$ is a $\mathcal{T}_\varepsilon$-attractor filtration and $(\widetilde{A}_r^\varepsilon)_{r=0}^m$ is its dual $\mathcal{T}_\varepsilon$-repeller filtration.

For all $r \in [\![1, m]\!]$ and for all $\varepsilon \in \,]0, \varepsilon_1]$ define $\widetilde{M}_r^\varepsilon = A_r^\varepsilon \cap \widetilde{A}_{r-1}^\varepsilon$. It follows that the set $\widetilde{M}_r^\varepsilon$ is $\mathcal{T}_\varepsilon$-invariant and

$$\widetilde{M}_r^\varepsilon \subset [W_r]_{\varepsilon,\eta} \cap [W_{r-1}^*]_{\varepsilon,\eta} = [W_r \cap W_{r-1}^*]_{\varepsilon,\eta}\,.$$

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Hence

$$\begin{aligned}\widetilde{M}_r^\varepsilon &\subset \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r \cap W_{r-1}^*]_{\varepsilon,\eta}) \subset \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r]_{\varepsilon,\eta}) \cap \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_{r-1}^*]_{\varepsilon,\eta}) \\ &= \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r]_{\varepsilon,\eta}) \cap \mathrm{Inv}_{\mathcal{T}_\varepsilon^-}([W_{r-1}^*]_{\varepsilon,\eta}) = A_r^\varepsilon \cap (A_{r-1}^\varepsilon)^*_{\mathcal{T}_\varepsilon} = \widetilde{M}_r^\varepsilon\end{aligned}$$

and so

$$\begin{aligned}\widetilde{M}_r^\varepsilon &= \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r \cap W_{r-1}^*]_{\varepsilon,\eta}) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r]_{\varepsilon,\eta}) \cap \mathrm{Inv}_{\mathcal{T}_\varepsilon^-}([W_{r-1}^*]_{\varepsilon,\eta}) \\ &\subset \mathrm{Int}_{Z_\varepsilon}([W_r]_{\varepsilon,\eta}) \cap \mathrm{Int}_{Z_\varepsilon}([W_{r-1}^*]_{\varepsilon,\eta}) = \mathrm{Int}_{Z_\varepsilon}([W_r \cap W_{r-1}^*]_{\varepsilon,\eta}).\end{aligned} \tag{4.29}$$

Now, formulas (4.25) and (4.28) and Corollary 4.6 imply that there exists a $\varepsilon' \in]0, \varepsilon_1]$ such that for all $r \in [\![1, m]\!]$

$$\mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r \cap W_{r-1}^*]_{\varepsilon,\eta}) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_r]_{\varepsilon,\eta}), \text{ for all } \varepsilon \in]0, \varepsilon'].$$

Thus, for all $r \in [\![1, m]\!]$ and for all $\varepsilon \in]0, \varepsilon']$, $\widetilde{M}_r^\varepsilon = M_r(\varepsilon)$. Let $\varepsilon \in]0, \varepsilon']$ and $I \in A(<)$ be arbitrary. Then there is an $r \in [\![0, m]\!]$ such that $I = [\![1, r]\!]$. Now Proposition 3.9 in [3] implies that

$$\begin{aligned}\mathrm{Inv}_{\mathcal{T}_\varepsilon}([W_r]_{\varepsilon,\eta}) = A_r^\varepsilon &= \{\, x \mid \exists \sigma \in \mathcal{T}_\varepsilon \text{ with } \sigma(0) = x \text{ and } \alpha(\sigma) \subset \textstyle\bigcup_{i=1}^r M_i(\varepsilon) \,\} \\ &= M_{[\![1,r]\!]}(\varepsilon).\end{aligned}$$

Since $(A_r^\varepsilon, \widetilde{A}_r^\varepsilon)$ and $(M_{[\![1,r]\!]}(\varepsilon), M_{[\![r+1,m]\!]}(\varepsilon))$ are $\mathcal{T}_\varepsilon$-attractor-repeller pairs, it follows that $\widetilde{A}_r^\varepsilon = M_{[\![r+1,m]\!]}(\varepsilon)$ and this completes the proof of the first case.

Case 2. Now suppose that $\prec$ is an arbitrary strict partial order on P. It follows from case 1 that for each total order $\prec^*$ extending $\prec$, $(M_i(\varepsilon))_{i\in P}$ is a $\prec^*$-ordered $\mathcal{T}_\varepsilon$-Morse decomposition for all $\varepsilon > 0$ small enough. Since there is a finite number of such extensions, it also follows that there exists an $\tilde{\varepsilon}_0 \in]0, \varepsilon_0]$ such that, for every extension $\prec^*$ of $\prec$ and for all $\varepsilon \in]0, \tilde{\varepsilon}_0]$, $(M_i(\varepsilon))_{i\in P}$ is a $\prec^*$-ordered $\mathcal{T}_\varepsilon$-Morse decomposition.

Let $\varepsilon \in]0, \tilde{\varepsilon}_0]$ be arbitrary. It follows that the sets $M_i(\varepsilon)$, $i \in P$, are closed, $\mathcal{T}_\varepsilon$-invariant and pairwise disjoint. Moreover, for each $\sigma \in \mathcal{T}_\varepsilon$ either

(1) $\sigma(\mathbb{R}) \subset M_k(\varepsilon)$ for some $k \in P$

or else

(2) there are $i, j \in P$ such that $i \neq j$, $\alpha(\sigma) \subset M_j(\varepsilon)$ and $\omega(\sigma) \subset M_i(\varepsilon)$.

Assume the second alternative. We shall prove that $i \prec j$. Indeed, define

$$J := \{\, k \in P \mid k = j \text{ or } k \prec j \,\}.$$

It is clear that $j \in J$. Moreover, $J \in A(\prec)$. By Proposition 2.2, there is a total order $\prec^*$ extending $\prec$ such that $J \in A(\prec) \cap A(\prec^*)$. We also have that $(M_i(\varepsilon))_{i\in P}$ is a $\prec^*$-ordered $\mathcal{T}_\varepsilon$-Morse decomposition. Thus $i \prec^* j$ and so $i \in J$. This implies, as $i \neq j$, that $i \prec j$. This concludes the proof of our claim. Hence, we have proved that $(M_i(\varepsilon))_{i\in P}$ is a $\prec$-ordered $\mathcal{T}_\varepsilon$-Morse decomposition for all $\varepsilon \in]0, \tilde{\varepsilon}_0]$.

Let $I \in A(\prec)$. By Proposition 2.2, there is a total order $\prec^*$ extending $\prec$ such that $I \in A(\prec) \cap A(\prec^*)$.

It follows from case 1 that

$$M_I(\varepsilon) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_I]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_I]_{\varepsilon,\eta}), \quad \varepsilon \in \left]0, \varepsilon'\right]$$

and

$$M_{P\setminus I}(\varepsilon) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_{P\setminus I}]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_{P\setminus I}]_{\varepsilon,\eta}), \quad \varepsilon \in \left]0, \varepsilon'\right].$$

This completes the proof the lemma. ∎

Proof of Theorem 4.12. Let ε' be as in Lemma 4.13 and $K \in A(\prec)$ be arbitrary. Using the notation of Proposition 2.12 we will now show that

(4.30) the assumptions of Theorem 4.12 hold with P replaced by K and $\mathcal{T}_\varepsilon$, $\varepsilon \in \left]0, \varepsilon_0\right]$, replaced by $\mathcal{T}_\varepsilon^K$, $\varepsilon \in \left]0, \varepsilon_0\right]$.

First we claim that $\mathcal{T}_\varepsilon^K \stackrel{\mathrm{sg}}{\to} \mathcal{T}_0^K$. Indeed, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\left]0, \varepsilon_0\right]$ with $\varepsilon_n \to 0$ as $n \to \infty$ and $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence such that $\sigma_n \in \mathcal{T}_{\varepsilon_n}^K$ for all $n \in \mathbb{N}$. Since $\mathcal{T}_\varepsilon \stackrel{\mathrm{sg}}{\to} \mathcal{T}_0$, there is a subsequence $(\sigma_{n_m})_{m\in\mathbb{N}}$ and a $\sigma \in \mathcal{T}_0$ such that

$$\Gamma_{\varepsilon_{n_m}}\left(\sigma_{n_m}(t), (\sigma(t), \theta_{\varepsilon_{n_m}})\right) \to 0 \text{ as } m \to \infty \tag{4.31}$$

uniformly on compact subsets of $\mathbb{R}$. We only need to show that $\sigma \in \mathcal{T}_0^K$. It follows from Lemma 4.13 that for all ε_{n_m} with m large enough

$$M_K(\varepsilon_{n_m}) = \mathrm{Inv}_{\mathcal{T}_{\varepsilon_{n_m}}}([V_K]_{\varepsilon_{n_m},\eta}) \subset \mathrm{Int}_{Z_{\varepsilon_{n_m}}}([V_K]_{\varepsilon_{n_m},\eta}).$$

Since $\sigma_{n_m}(t) \in M_K(\varepsilon_{n_m})$ for all $t \in \mathbb{R}$ and for all $m \in \mathbb{N}$, it follows that, for $m \in \mathbb{N}$ large enough, $\sigma_{n_m}(t) \in [V_K]_{\varepsilon_{n_m},\eta}$ for $t \in \mathbb{R}$. In particular, we have

$$\phi_{n_m}(t) \in V_K \text{ for all } t \in \mathbb{R} \tag{4.32}$$

where $\sigma_{n_m}(t) = (\phi_{n_m}(t), \psi_{n_m}(t)) \in X_0 \times Y_{\varepsilon_{n_m}}$. Formula (4.31) implies that for all $t \in \mathbb{R}$

$$d_0(\phi_{n_m}(t), \sigma(t)) \to 0 \text{ and } d_{\varepsilon_{n_m}}(\psi_{n_m}(t), \theta_{\varepsilon_{n_m}}) \to 0 \text{ as } m \to \infty.$$

This together with formula (4.32) implies that $\sigma(t) \in V_K$ for each $t \in \mathbb{R}$, i.e., $\sigma(\mathbb{R}) \subset M(K)$. The proof of our claim is complete.

Since for each $i \in K$ we have $M_i \subset M(K)$ and $\mathcal{T}_0^K \subset \mathcal{T}_0$, it follows that

$$M_i \subset \mathrm{Inv}_{\mathcal{T}_0^K}(V_i) \subset \mathrm{Inv}_{\mathcal{T}_0}(V_i) = M_i$$

so

$$M_i = \mathrm{Inv}_{\mathcal{T}_0^K}(V_i) \subset \mathrm{Int}_{X_0}(V_i).$$

Let $I \in I(\prec)$, $I \subset K$, be arbitrary. It follows that $M(I) \subset M(K)$, $M(K \setminus I) \subset M(K)$ and so

$$M(I) \subset \mathrm{Inv}_{\mathcal{T}_0^K}(V_I) \subset \mathrm{Inv}_{\mathcal{T}_0}(V_I) = M(I),$$

$$M(K \setminus I) \subset \mathrm{Inv}_{\mathcal{T}_0^K}(V_{K\setminus I}) \subset \mathrm{Inv}_{\mathcal{T}_0}(V_{K\setminus I}) = M(K \setminus I).$$

Thus

$$M(I) = \mathrm{Inv}_{\mathcal{T}_0^K}(V_I) \subset \mathrm{Int}_{X_0}(V_I)$$

and

$$M(K \setminus I) = \mathrm{Inv}_{\mathcal{T}_0^K}(V_{K\setminus I}) \subset \mathrm{Int}_{X_0}(V_{K\setminus I}).$$

Now Proposition 2.12 implies that, indeed, (4.30) holds.

Now let $J \in I(\prec)$ be arbitrary. By Proposition 2.1 there are I and $K \in A(\prec)$ with $I \subset K$ and $J = K \setminus I$. Therefore, (4.30) and Lemma 4.13 imply that

$$M_J(\varepsilon) = M_{K\setminus I}(\varepsilon) = \mathrm{Inv}_{\mathcal{T}_\varepsilon^K}([V_J]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_J]_{\varepsilon,\eta}), \text{ for all } \varepsilon \in]0, \tilde{\varepsilon}_J],$$

where $\tilde{\varepsilon}_0(J) \in]0, \varepsilon']$ is chosen small enough. We claim that

$$\mathrm{Inv}_{\mathcal{T}_\varepsilon^K}([V_J]_{\varepsilon,\eta}) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_J]_{\varepsilon,\eta}) \text{ for all } \varepsilon > 0 \text{ small enough.} \tag{4.33}$$

Suppose that (4.33) is not true. Since $\mathrm{Inv}_{\mathcal{T}_\varepsilon^K}([V_J]_{\varepsilon,\eta}) \subset \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_J]_{\varepsilon,\eta})$ for all $\varepsilon \in]0, \varepsilon_0]$, it follows that there exist a sequence $(\varepsilon_n)_n$ in $]0, \varepsilon']$ with $\varepsilon_n \to 0$ as $n \to \infty$ and a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, $\sigma_n \in \mathcal{T}_{\varepsilon_n}$, $\sigma_n(\mathbb{R}) \subset [V_J]_{\varepsilon_n,\eta}$ and $\sigma_n(\mathbb{R}) \not\subset M_{\varepsilon_n}(K) = \mathrm{Inv}_{\mathcal{T}_{\varepsilon_n}}([V_K]_{\varepsilon_n,\eta})$. Hence, by the translation invariance of $\mathcal{T}_\varepsilon$, $\varepsilon \in]0, \varepsilon_0]$, we may assume that $\sigma_n(0) \notin [V_K]_{\varepsilon_n,\eta}$, Taking subsequences if necessary we may assume that there is a $\sigma \in \mathcal{T}_0$ such that

$$\Gamma_{\varepsilon_n}(\sigma_n(t), (\sigma(t), \theta_{\varepsilon_n})) \to 0, \tag{4.34}$$

uniformly on compact subsets in $\mathbb{R}$.

Letting $t = 0$ in (4.34), it follows that

$$d_0(\phi_n(0), \sigma(0)) \to 0 \text{ and } d_{\varepsilon_n}(\psi_n(0), \theta_{\varepsilon_n}) \to 0 \text{ as } n \to \infty. \tag{4.35}$$

Formula (4.35) implies that

$$d_{\varepsilon_n}(\phi_n(0), \theta_{\varepsilon_n}) < \eta \text{ for all } n \text{ large enough.} \tag{4.36}$$

Since $\sigma_n(0) \notin [V_K]_{\varepsilon_n,\eta}$, formula (4.36) implies that

$$\phi_n(0) \notin V_K \text{ for all } n \text{ large enough.} \tag{4.37}$$

We claim that $\sigma(\mathbb{R}) \subset M(J)$. Indeed, formula (4.34) implies that for each $t \in \mathbb{R}$, $d_0(\phi_n(t), \sigma(t)) \to 0$ as $n \to \infty$. Since $\sigma_n(\mathbb{R}) \subset [V_J]_{\varepsilon_n,\eta}$ for all $n \in \mathbb{N}$, it follows that

$\phi_n(t) \in V_J$ for all $n \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Since V_J is a closed set in X_0, it follows that $\sigma(\mathbb{R}) \subset V_J$ and so $\sigma(\mathbb{R}) \subset \mathrm{Inv}_{\mathcal{T}_0}(V_J) = M(J)$. This completes the proof of our claim. Since $M(J) \subset M(K) \subset \mathrm{Int}_{X_0}(V_K)$, our claim implies that, for all n large enough, we have $\sigma_n(0) \in \mathrm{Int}_X(V_K)$ which contradicts (4.37). Thus, formula (4.33) holds. Therefore, for each $J \in I(\prec)$, there exists an $\varepsilon_J \in \,]0, \tilde{\varepsilon}_J]$ such that

$$M_J(\varepsilon) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_J]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_J]_{\varepsilon,\eta}), \text{ for all } \varepsilon \in \,]0, \varepsilon_J]. \tag{4.38}$$

Since the set $I(\prec)$ is finite, formula (4.38) implies that there is a $\tilde{\varepsilon} \in \,]0, \varepsilon_0]$ such that, for all $\varepsilon \in \,]0, \tilde{\varepsilon}]$ and for all $J \in I(\prec)$,

$$M_J(\varepsilon) = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_J]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_J]_{\varepsilon,\eta}).$$

The theorem is proved. ▮

Specializing to the semiflow case we obtain the following results.

Corollary 4.14. *Assume the following hypotheses:*

(1) η is a positive number, $(\pi_\varepsilon)_{\varepsilon \in]0,\varepsilon_0]}$ is a family of local semiflows that converges singularly to the local semiflow π_0 and N is a singularly strongly admissible set with respect to η and $(\pi_\varepsilon)_{\varepsilon \in [0,\varepsilon_0]}$.

$(M_i)_{i \in P}$ is a $\prec$-ordered Morse decomposition of $S_0 := \mathrm{Inv}_{\pi_0}(N)$ relative to π_0.

(2) For each $i \in P$, $V_i \subset N$ is closed in X_0 and such that

$$M_i = \mathrm{Inv}_{\pi_0}(V_i) \subset \mathrm{Int}_{X_0}(V_i). \tag{4.39}$$

Moreover, for every $I \in I(\prec)$, $V_I \subset N$ is closed in X_0 and such that

$$M_{\pi_0,S_0}(I) = \mathrm{Inv}_{\pi_0}(V_I) \subset \mathrm{Int}_{X_0}(V_I).$$

For $\varepsilon \in \,]0, \varepsilon_0]$ and $i \in P$ set $M_i(\varepsilon) := \mathrm{Inv}_{\pi_\varepsilon}([V_i]_{\varepsilon,\eta})$. Then there is an $\tilde{\varepsilon} \in \,]0, \varepsilon_0]$ such that for every $\varepsilon \in \,]0, \tilde{\varepsilon}]$ the family $(M_i(\varepsilon))_{i \in P}$ is a $\prec$-ordered Morse decomposition of $S_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}([N]_{\varepsilon,\eta})$ relative to π_ε. Furthermore, for every $I \in I(\prec)$,

$$M_I(\varepsilon) := M_{\pi_\varepsilon,S_\varepsilon} = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_I]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_I]_{\varepsilon,\eta}).$$

Proof. For each $\varepsilon \in \,]0, \varepsilon_0]$, define $\mathcal{T}_\varepsilon := \mathcal{T}_{\pi_\varepsilon}$ be the set of all full solutions of π_ε lying in $S_\varepsilon = \mathrm{Inv}_{\pi_\varepsilon}([N]_{\varepsilon,\eta})$ and $\mathcal{T}_0 := \mathcal{T}_{\pi_0}$ be the set of all full solutions of π_0 lying in $S_0 = \mathrm{Inv}_{\pi_0}(N)$. Then $\mathcal{T}_\varepsilon \overset{\mathrm{sg}}{\to} \mathcal{T}_0$ by Proposition 4.3. Now Theorem 4.12 implies the assertion. ▮

Corollary 4.15. *Assume hypothesis (1) of Corollary 4.14. Moreover, suppose that $S_0 \subset \mathrm{Int}_X(N)$. Then there are sets V_i, $i \in P$ and sets V_I, $I \in I(\prec)$, such that hypothesis (2) of Corollary 4.14 (and, consequently, its assertion) holds.*

Proof. The proof is identical to the proof of Corollary 3.5. ▮

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We will now apply Corollaries 4.14 and 4.15 to a thin domain problem considered in [12] and [2]. We assume the reader's familiarity with [2] and only recall some of the relevant notations and definitions.

Let M and N be positive integers. Write (x, y) for a generic point of $\mathbb{R}^M \times \mathbb{R}^N$. Let Ω be an arbitrary nonempty bounded domain in $\mathbb{R}^M \times \mathbb{R}^N$ with Lipschitz boundary and let $\varepsilon > 0$ be arbitrary. Define the symmetric bilinear form

$$a_\epsilon \colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

by

$$a_\epsilon(u, v) := \int_\Omega \left(\nabla_x u \cdot \nabla_x v + \frac{1}{\epsilon^2} \nabla_y u \cdot \nabla_y v \right) dx\, dy.$$

and let b be the scalar product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$. Let $A_\varepsilon \colon D(A_\varepsilon) \subset H^1(\Omega) \to L^2(\Omega)$ be the linear operator generated by the pair (a_ε, b). We define on $H^1(\Omega)$ the scalar product

$$(u, v)_\varepsilon := a_\varepsilon(u, v) + b(u, v), \quad u, v \in H^1(\Omega)$$

and the corresponding norm

$$|u|_\epsilon := \left(a_\epsilon(u, u) + |u|^2_{L^2(\Omega)} \right)^{1/2}, \quad u \in H^1(\Omega)$$

which is equivalent to the usual norm on $H^1(\Omega)$.

We also define the "limit" space $H^1_s(\Omega)$ by

$$H^1_s(\Omega) = \{\, u \in H^1(\Omega) \mid \nabla_y u = 0 \,\}.$$

Note that $H^1_s(\Omega)$ is a closed linear subspace of $H^1(\Omega)$ so $H^1_s(\Omega)$ is a Hilbert space under the usual scalar product of $H^1(\Omega)$.

Furthermore, define the space $L^2_s(\Omega)$ to be the closure of the set $H^1_s(\Omega)$ in $L^2(\Omega)$. It follows that $L^2_s(\Omega)$ is a Hilbert space under the scalar product of $L^2(\Omega)$.

Now let $a_0 \colon H^1_s(\Omega) \times H^1_s(\Omega) \to \mathbb{R}$ be the "limit" bilinear form defined by

$$a_0(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx\, dy = \int_\Omega \nabla_x u \cdot \nabla_x v \, dx\, dy.$$

Finally, let b_0 be the restriction of the scalar product b to $L^2_s(\Omega) \times L^2_s(\Omega)$. Denote by A_0 the operator generated by the pair (a_0, b_0).

Now let $\varepsilon_0 \in \,]0, 1]$ be arbitrary and $(f_\varepsilon)_{\varepsilon \in [0, \varepsilon_0]}$ be a family satisfying hypothesis (A1) introduced in Definition 2.6 in [2]. For $\varepsilon \in \,]0, \varepsilon_0]$ let π_ε be the local semiflow on $H^1(\Omega)$ generated by the solutions of the evolution equation

$$\dot{u} = A_\varepsilon u + f_\varepsilon(u).$$

Moreover, let π_0 be the local semiflow on $H^1_s(\Omega)$ generated by the solutions of the evolution equation

$$\dot{u} = A_0 u + f_0(u).$$

We now have the following singular convergence result.

PROPOSITION 4.16. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers convergent to zero. Moreover let $t \in [0, \infty[$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0, \infty[$ converging to t. Finally, let $u_0 \in H^1_s(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ such that*

$$|u_n - u_0|_{\varepsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Assume that $u_0\pi_0 t$ is defined. Then, for all $n \in \mathbb{N}$ large enough, $u_n\pi_n t_n$ is defined and

$$\left|u_n\pi_{\varepsilon_n}t_n - u_0\pi_0 t\right|_{\varepsilon_n} \to 0 \quad \text{as } n \to \infty.$$

Proof. We choose an open ball $Y \subset H^1(\Omega)$ such that $u_0\pi_0\,[0,t] \subset Y$. Then we modify the family $(f_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ as in Proposition 2.9 of [2] to obtain the modified family $(f'_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ satisfying the stronger hypothesis (A2). Let $(\pi'_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$ be the corresponding family of modified semiflows. These modified semiflows are global and coincide on Y with the original local semiflows π_ε, $\varepsilon \in [0, \varepsilon_0]$. Now Corollary 2.15 in [2] shows that

$$|u_n\pi'_{\varepsilon_n}s_n - u_0\pi'_0 s|_{\varepsilon_n} \to 0 \tag{4.40}$$

for every sequence $(s_n)_n$ with $s_n \in [0, t_n]$, $n \in \mathbb{N}$ such that $s_n \to s \in [0, t]$. In particular, for all $n \in \mathbb{N}$ large enough, $u_n\pi'_n\,[0, t_n] \subset Y$ so, in particular, $u_n\pi_n t_n$ is defined and $u_n\pi_n t_n = u_n\pi'_n t_n$. Now formula (4.40) concludes the proof. ∎

For all $\varepsilon \in\,]0, \varepsilon_0]$ set $\theta_\varepsilon := 0 \in H^1(\Omega)$ and let $Q_\varepsilon \colon H^1(\Omega) \to H^1(\Omega)$ be the orthogonal projector onto $H^1_s(\Omega)$ with respect to the scalar product $(\cdot,\cdot)_\varepsilon$. Let $X_0 := H^1_s(\Omega)$ be endowed with the usual norm of $H^1(\Omega)$ and d_0 be the corresponding metric on X_0. Moreover, let $Y_\varepsilon := (I - Q_\varepsilon)(H^1(\Omega))$ be endowed with the norm $|\cdot|_\varepsilon$ and let d_ε be the corresponding metric on Y_ε. Set $Z_\varepsilon := X_0 \times Y_\varepsilon \cong H^1(\Omega)$ and note that the norm

$$\|(u, v)\|_\varepsilon := \max\{|u|_{H^1(\Omega)}, |v|_\varepsilon\}, \quad (u, v) \in X_0 \times Y_\varepsilon,$$

is equivalent to the norm $|\cdot|_\varepsilon$ on $H^1(\Omega)$ with constants independent of $\varepsilon \in\,]0, \varepsilon_0]$. Let Γ_ε be the metric on Z_ε generated by the norm $\|\cdot\|_\varepsilon$.

The remarks just made imply that, for every $\varepsilon \in\,]0, \varepsilon_0]$, π_ε is a local semiflow on Z_ε and π_0 is a local semiflow on X_0, while Proposition 4.16 just says that $(\pi_\varepsilon)_{\varepsilon\in]0,\varepsilon_0]}$ singularly converges to π_0.

Now an application of Lemma 2.21 in [2] shows that whenever $\eta > 0$ and N is closed and bounded in X_0 then N is singularly admissible with respect to η and the family $(\pi_\varepsilon)_{\varepsilon\in[0,\varepsilon_0]}$. Therefore we finally obtain the following result:

THEOREM 4.17. *Let η be a positive number and $N \subset H^1_s(\Omega)$ be closed and bounded. Suppose that $(M_i)_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S_0 := \mathrm{Inv}_{\pi_0}(N)$ relative to π_0. For each $i \in P$, let $V_i \subset N$ be closed in X_0 and such that*

$$M_i = \mathrm{Inv}_{\pi_0}(V_i) \subset \mathrm{Int}_{X_0}(V_i).$$

Moreover, for every $I \in I(\prec)$, let $V_I \subset N$ be closed in X_0 and such that

$$M_{\pi_0,S_0}(I) = \mathrm{Inv}_{\pi_0}(V_I) \subset \mathrm{Int}_{X_0}(V_I).$$

For $\varepsilon \in]0,\varepsilon_0]$ and $i \in P$ set $M_i(\varepsilon) := \mathrm{Inv}_{\pi_\varepsilon}([V_i]_{\varepsilon,\eta})$. Then there is an $\tilde{\varepsilon} \in]0,\varepsilon_0]$ such that for every $\varepsilon \in]0,\tilde{\varepsilon}]$ the family $(M_i(\varepsilon))_{i\in P}$ is a $\prec$-ordered Morse decomposition of $S_\varepsilon := \mathrm{Inv}_{\pi_\varepsilon}([N]_{\varepsilon,\eta})$ relative to π_ε. Furthermore, for every $I \in I(\prec)$,

$$M_I(\varepsilon) := M_{\pi_\varepsilon,S_\varepsilon} = \mathrm{Inv}_{\mathcal{T}_\varepsilon}([V_I]_{\varepsilon,\eta}) \subset \mathrm{Int}_{Z_\varepsilon}([V_I]_{\varepsilon,\eta}).$$

Theorem 4.17 also holds for the more general case of reaction-diffusion equations on curved squeezed domains considered in [13]. Furthermore, an analogous result can be proved for damped wave equations on squeezed domains considered in [5] and [4]. The formulation of this result is left to the interested reader.

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