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Euler Obstruction, polar multiplicities and equisingularity of map germs in $\mathcal{O}(n, p)$, $n < p$.

V. H. Jorge Peréz

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil
E-mail: vhjperez@icmc.usp.br

Marcelo José Saia *

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil
E-mail: mjsaia@icmc.usp.br

We investigate the relationship between the local Euler obstruction for nonsingular varieties and the Whitney equisingularidade of a one parameter deformation of a corank one finitely determined holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $n < p$. We also study how to determine the minimal number of invariants to garantee the Whitney equisingularity of such a family. According to a result of Gaffney, these are the 0-stable invariants and all polar multiplicities which appear in the stable types of a stable deformation of the germ. First we describe all stable types which appear when $n < p$. Then we show that the number of polar multiplicities necessary can be reduced to a half and show that the local Euler obstruction is an invariant for the Whitney equisingularity. May, 2003 ICMC-USP

1. INTRODUCTION

The local Euler obstruction for nonsingular varieties, introduced by R. MacPherson in a purely obstructional way is an invariant that plays an important role in his affirmative response to a conjecture of Deligne and Grothendieck on the existence of Chern class for singular complex algebraic varieties (see [12],[20]). Gonzales-Sprinberg gave in [8] a purely algebraic interpretation of the local Euler obstruction. Lê and Teissier proved in [13] a formula for the multiplicity of the local polar varieties, and with the aid of Gonzales-Sprinberg's interpretation of the local Euler obstruction, they showed that the local Euler obstruction is an alternate sum of the multiplicities of the local polar varieties.

On the other side, Gaffney in [6] showed that the Whitney equisingularity, hence the topological triviality, of a 1-parameter family of map germs is controlled by the multiplic-

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ities of the local polar varieties and the zero stable invariants of all the stable types which appear in the source and in the target.

Gaffney in [6] and [3] uses this result to analyze mapping from the plane to plane and the plane to space. The first named author in [9] and [10] applies the results of Gaffney for mapping from 3-space to 3-space and 3-space to 4-space. More recently Gaffney's approach has been used by Vohra [23], to study map germs from n-space ($n \geq 3$) to the plane. The case $n = p$ is investigated in [11] and as a consequence, it is also shown how to compute the Euler obstruction, in terms of the polar multiplicities, for all stable types.

In this paper we deal with the case of map germs from n-space to p-space with $n < p$. According to Gaffney's result, for a family $f_t : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ to be Whitney equisingular it is needed the constancy of $2(p-k(p-n+1)+\ell)$ invariants for each partition $\mathcal{P} = (r_1, ..., r_\ell)$ of all k satisfying $p - k(p - n + 1) + \ell \geq 0$. We reduce this number in the case of corank 1 germs. We do this by finding relations among the invariants and using the fact that these are upper semi-continuous.

We apply these results to obtain explicit and algebraic formulae for the Euler obstruction in the stable type of mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$, with $n < p$. As a consequence we show that the local Euler obstruction is an invariant for the Whitney equisingularity.

2. BASIC DEFINITIONS AND RESULTS

We follow the notation used by Gaffney in [6] and denote by $\mathcal{O}(n, p)$ the set of origin preserving germs of holomorphic mappings from $\mathbb{C}^n$ to $\mathbb{C}^p$, $\mathcal{O}_e(n, p)$ denotes the set of germs at the origin but not necessarily origin preserving. We denote by $\mathcal{R}$ the group of diffeomorphisms of the source $(\mathbb{C}^n, 0)$, and by $\mathcal{L}$ the group of diffeomorphisms of the target $(\mathbb{C}^p, 0)$. The action of the product $\mathcal{A} := \mathcal{R} \times \mathcal{L}$ leads to $\mathcal{A}$-equivalence of map germs: $f,\ g \in \mathcal{O}(n, p)$ are $\mathcal{A}$-equivalent if they are equivalent by smooth coordinate changes at source and target. Similarly the action of the semi direct product $\mathcal{K} := (\mathcal{R}.\mathcal{C})$ gives rise to $\mathcal{K}$-equivalence of map germs.

A germ is said to be k-$\mathcal{A}$-determined if any $g \in \mathcal{O}(n, p)$ with the same k-jet as f, i.e. $j^k g = j^k f$, is $\mathcal{A}$-equivalent to f. The germ f is said to be finitely $\mathcal{A}$-determined if it is k-$\mathcal{A}$-determined for some k.

A map-germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ **is stable** if, up families of (bianalytic) diffeomorphisms in source and target, every deformation is trivial. That is, if f_t is a 1-parameter family with $f_0 = f$, then there should exist 1-parameter families φ_t and ψ_t of diffeomorphisms of source and target such that and $\psi_t \circ f \circ \varphi_t^{-1} = f_t$. **Stable type** is the $\mathcal{A}$-equivalence class of stable germs.

Our interest is primarily in corank one $\mathcal{A}$-finitely determined map-germs $f \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $n < p$, $\Sigma(f)$ denotes the critical set of f (if $n < p$, $\Sigma(f) = \mathbb{C}^n$). We denote by F a versal unfolding of such an f.

We say that a **stable type** $\mathcal{Q}$ appears in F if for any representative $F = (id, f_u(x))$ of F, there exists a point $(s, y) \in \mathbb{C}^s \times \mathbb{C}^p$ such that the germ $f_u : \mathbb{C}^n, S \to \mathbb{C}^p, y$ is a stable germ of type $\mathcal{Q}$ where $S = f^{-1}(y) \cap \Sigma(f_u)$. The points (s, y) and (s, x) with $x \in S$ are called points of stable type $\mathcal{Q}$ in the target and in the source, respectively.

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If f is stable, we denote the set of points in $\mathbb{C}^s \times \mathbb{C}^p$ of type $\mathcal{Q}$ by $\mathcal{Q}(f)$ and the set $\mathcal{Q}_S(f) = f^{-1}(\mathcal{Q}(f)) - \mathcal{Q}_\Sigma(f)$, where $\mathcal{Q}_\Sigma(f)$ denotes $f^{-1}(\mathcal{Q}(f)) \cap \Sigma(f)$.

If f is finitely determined, we denote by $\overline{\mathcal{Q}(f)} = (\{0\} \times \mathbb{C}^p) \cap \overline{\mathcal{Q}(F)}$ and $\overline{\mathcal{Q}_S(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_S(F)}$, $\overline{\mathcal{Q}_\Sigma(f)} = (\{0\} \times \mathbb{C}^n) \cap \overline{\mathcal{Q}_\Sigma(F)}$, here the bar over a set means the closure of this set.

We say that $\mathcal{Q}$ is a **zero-dimensional stable type** for the pair (n, p) if $\mathcal{Q}(f)$ has dimension 0 where f is a representative of the stable type $\mathcal{Q}$.

We observe that the set $\overline{\mathcal{Q}(F)} = \cap F(j^{(p+1)} F^{-1}(\overline{\mathcal{A}z_i}))$ is closed and analytic, where z_i is the $p+1$ jet of the stable type $\mathcal{Q}$ and $\mathcal{A}z_i$ is the $\mathcal{A}$-orbit of z_i.

A finitely determined germ f has **discrete stable type** if there exist a versal unfolding of f in which only a finite number of stable types occur. If the numbers (n, p) are in Mather's "nice dimensions" (which is our focus here) or on the boundary thereof, then every finitely determined germ $f \in \mathcal{O}(n, p)$ has discrete stable type.

2.1. Colength and multiplicities of ideals

The colength of a given ideal I in a complex analytic ring R, is defined as $\dim_{\mathbb{C}}(R/I)$; it may or not be finite. The multiplicity of an ideal I is an integer invariant denoted by $e(I)$ that is defined whenever I has finite colength. If $R =< R, m >$ is local and Cohen-Macaulay, if I is m-primary and a complete intersection, then the multiplicity of I is just its colength.

2.2. Finite maps and degree

A smooth map germ $f : (X, x) \to (Y, y)$ is said to be **finite** if the dimension of its local algebra is finite, i.e. if the number $m(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{(X,x)}}{f^*(m_y)\mathcal{O}_{(X,x)}} < \infty$. Note that for f to be finite, it is necessary that $\dim X \leq \dim Y$. In the context of complex analytic geometry, we have the following important result for finite maps.

Let $f : (X, x) \to (Y, y)$ be an analytic map of analytic spaces of the same dimension, such that $f(X)$ is Zariski-dense in Y. Suppose $f(x)$ is a smooth point of Y, and $\{x\}$ a component of the fiber $f^{-1}(f(x))$. For open neighborhoods $U \subset X$ of x, and $V \subset Y$ and a closed analytic subset $B \subset Y$ such that:

(i) $V \setminus B$ is connected;
(ii) $f(U) \subset V$, $f|_U$ is proper, $f^{-1}(f(x)) = \{x\}$; and
(iii) $f|_{U \setminus f^{-1}(B)}$ smooth,

then the number of pre-images in U, counted with multiplicity, of any point $y \in V \backslash B$, is called the **degree** of f at x, denoted $\deg(f)$, for a proof see 3.12 in [19].

In particular if $X \subset (\mathbb{C}^n, 0)$ is an analytic space germ defined as the zero set of germs $g_1, ..., g_t$ with $\dim_0(X) = d$, and if $\mathcal{O}_X$ is Cohen-Macaulay, then we can often use a projection $\pi : (\mathbb{C}^n, 0) \to (\mathbb{C}^d, 0)$ such that $\deg(\pi|_X)$ is the colength of the ideal $(\pi_1, ..., \pi_d)$ in $\mathcal{O}_X$, i.e. $\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\pi_1, ..., \pi_d, g_1, ..., g_t)}$.

2.3. Unfoldings

A 1-parameter unfolding F of f is a good unfolding if there exist neighborhoods U and W of the origin in $\mathbb{C} \times \mathbb{C}^n$ and in $\mathbb{C} \times \mathbb{C}^p$, respectively, such that $F^{-1}(W) = U$, F maps $(U \cap \Sigma(f) \backslash \mathbb{C} \times 0)$ to $W \backslash \mathbb{C} \times 0$, and if $(t_0, y_0) \in W \backslash \mathbb{C} \times 0$, with $S := F^{-1}(t_0, y_0) \cap \Sigma(F)$, then the germ $f_{t_0} : (\mathbb{C}^n, S) \to (\mathbb{C}^p, y_0)$ is a stable germ. We say that a good unfolding F of f is **excellent** if f is a finitely determined germ of discrete stable type, and the 0-stable invariants are constant in F, if $n = p$, then an additional requirement for excellence is that the degree of f be constant in the unfolding.

2.4. Whitney equisingularity and stratification of maps

A **Whitney stratification** of a given space is a stratification such that for any pair of strata S, S', with $S' \subset S$, the big stratum $\overline{S}$ is Whitney regular along S'. The local topological type remains constant along each stratum of a Whitney stratification of a given space. Note that a Whitney stratification always exists in the local complex analytic space, see section 1 of [5]. We say a given space X is **Whitney equisingular** along Y if there is a Whitney stratification of X with Y as a stratum.

Recall that if $F : \mathbb{C}^n \to \mathbb{C}^p$ is a morphism, and $A \subset \mathbb{C}^n$, $A' \subset \mathbb{C}^p$ subsets such that $F(A) \subset A'$, then a **stratification** of $F : A \to A'$ is a pair $(\mathcal{A}, \mathcal{A}')$ of stratifications of A and A' respectively, such that F maps strata submersively to strata. A given stratification $(\mathcal{A}, \mathcal{A}')$ of F is a regular stratification if $\mathcal{A}$, $\mathcal{A}'$ satisfy the Whitney regularity conditions, and all pairs of incident strata in the source satisfy Thom's A_f condition: *Let U be an open subset of some affine space, $f : U \to \mathbb{C}$ be an analytic function and M be a submanifold of U. Thom's A_f condition is satisfied between $U - \Sigma(f)$ and M if, whenever $p_i \in U - \Sigma(f)$, $p_i \to p \in M$, and $T_{p_i}V(f - f(p_i)) \to \mathcal{T}$, then $T_pM \subseteq \mathcal{T}$.*

If $F = f_t$ is an unfolding with parameter axis T, then a regular stratification $(\mathcal{A}, \mathcal{A}')$ of F is said to be a **Whitney equisingular** along T if T is a stratum of $\mathcal{A}$ and of $\mathcal{A}'$, ($\mathcal{A}$ and $\mathcal{A}'$ are Whitney equisingular along T. We also say in this case that F is a Whitney equisingular map.

The polar multiplicities of the polar varieties (defined by Teissier in [22]) of the stable types are the invariants needed to show the Whitney equisingularity of unfoldings.

Definition 2.1. Suppose $f : (X, 0) \to (S, 0)$ is a smooth flat map which fibers at every point of $X - sing(X)$, where X is a d-dimensional anlytic complex variety. Let $p : \mathbb{C}^n \to \mathbb{C}^{d-k+1}$ be a linear projection such that ker $p = D_{d-k+1}$ where D_{d-k+1} is a linear subspace of $(\mathbb{C}^n, 0)$ of dimension k, with $0 \le k \le d-1$. For $x \in X - sing(X)$, the fiber $X(f(x))$ is non-singular at x contained in $\{f(x)\} \times \mathbb{C}^n$ and one denotes by $\pi_x : X(f(x)) \to \mathbb{C}^{d-k+1}$ the restriction of p to $X(f(x))$. Let $P_k(f, p)$ be the closure of points $x \in X - sing(X)$ such that $x \in \Sigma(\pi_x)$, one calls the closed analytic subspace $P_k(f, p)$ of X, the relative polar variety of codimension k associate to D_{d-k+1}. If f is the constant map, we denote it by $P_k(X)$, called absolute polar variety.

The key invariant of $P_k(f, p)$ is its polar multiplicity which we denote by $m_k(X, f)$, if f is the constant map, we denote it by $m_0(P_k(X))$ or $m_k(X)$.

Gaffney in [6] page 195, defines a new invariant as following: Take a versal unfolding $F : \mathbb{C}^n \times \mathbb{C}^s \to \mathbb{C}^s \times \mathbb{C}^{n+1}$ of f. Specify a stable singularity type or stratum $\mathcal{D}(f)$ in source or target such that dim $\overline{\mathcal{D}(f)} \ge 1$. Select D_1 a linear subspace of $(\mathbb{C}^{n+1}, 0)$ of dimension 1 and form $P_d(\mathcal{D}(F))$ the polar variety on $\mathcal{D}(F)$ with the projection $(p, \pi_s) : \mathbb{C}^n \times \mathbb{C}^s \to$

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$\mathbb{C} \times \mathbb{C}^s$ where $d = \dim(\overline{\mathcal{D}(F)}) - s$. The d-th stable multiplicity of f of type $\mathcal{D}(f)$, denoted $m_d(\mathcal{D}(f))$, is the multiplicity of $m_s\mathcal{O}_{\overline{\mathcal{D}(F)},(0,0)}$ in $\mathcal{O}_{\overline{\mathcal{D}(F)},(0,0)}$.

Using the polar multiplicities of the stable types and Thom's A_f, condition, Gaffney showed the following:

Theorem 2.1. *[6] pp. 206-207 Suppose that $F : \mathbb{C} \times \mathbb{C}^n, (0,0) \to \mathbb{C} \times \mathbb{C}^p, (0,0)$ is an excellent unfolding of a finitely determined germ $f \in \mathcal{O}(n,p)$. Also suppose that the polar invariants of all the stable types defined in the discriminant $\Delta(f)$, in $\Sigma(f)$ and in $f^{-1}(\Delta(f)) - \Sigma(f)$ are constant at the origin for f_t. Then the unfolding is Whitney equisingular.*

The theorem also implies that such unfolding is topologicaly trivial; for the proof of this result Gaffney uses Thom's second isotopy lemma for complex analytic mappings, see [6] p. 204.

The theorem remains valid if we replace the term "an excellent unfolding" in the hypothesis by "a 1-parameter unfolding which, when stratified by stable types and by the parameter axis T, has only the parameter axis T as 1-dimensional stratum at the origin". see [23]

We remember that in the case $n < p$, our subject in this article, the set $f^{-1}(\Delta(f)) - \Sigma(f)$ is empty.

The following results are a key tool in finding relations among our invariants.

Theorem 2.2. (Lê-Greuel, [14],[17]) *Let X_1 be a complete intersection with isolated singularity at $0 \in \mathbb{C}^n$ (an ICIS). Let X be an ICIS defined in X_1 by $f_k = 0$, and let $f_1, ..., f_{k-1}$ be the generators of the ideal that defines X_1 at 0 in $\mathbb{C}^n$. Then*

$$\mu(X_1, 0) + \mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(f_1, ..., f_{k-1}, J(f_1, ..., f_k))}$$

Remark 2. 1. For a zero-dimensional ICIS we use the simpler formula: Let $f : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ be a germ such that $X = f^{-1}(0)$ is an ICIS. Then $\mu(X, 0) = \delta(f) - 1$, where $\delta(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{f^*(m_n)\mathcal{O}_n}$, see [16] page 78.

Other elementary result that we appeal is: Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a finitely determined germ. Then $f : (\mathbb{C}^n, 0) \to f(\mathbb{C}^n) \subset (\mathbb{C}^{n+1}, 0)$ is bimeromorphic; see[4] page 138.

3. EQUISINGULARITY OF MAP GERMS IN $\mathcal{O}(N, P)$

3.1. The stable types in $\mathcal{O}(n,p)$, $n < p$

As highlighted in the introduction, our aim is to minimize the number of invariants defined in the stable types of f whose constancy in the family f_t implies the family is Whitney equisingular (therefore topologically trivial).

The strategy is to apply the Theorem 2.1 and the techniques used by Gaffney in [6], that is, stratify the source and the target by the stable types and establish relations among the invariants on the strata. As these invariants are upper semi-continuous, the relations will allow us to reduce the number of invariants required in Gaffney's theorem.

The main purpose of this subsection is to give a full description of the stratification of the source and the target by the stable types. For this we first give the following preliminary definition. Given a continuous mapping $f : X \to Y$ on analytic spaces, we define the k^{tk} multiple point space of f as

$$D^k(f) = \text{closure}\{(x_1, x_2, ..., x_k) \in X^k : f(x_1) = ... = f(x_k) \,\text{for}\, x_i \neq x_j, i \neq j\}.$$

If $g : \mathbb{C}^n \to \mathbb{C}$ is a function then we define $V_i^k(g) : \mathbb{C}^{n+k-1} \to \mathbb{C}$ to be

$$\begin{vmatrix} 1 & z_1 & \cdots & z_1^{i-1} & g(x,z_1) & z_1^{i+1} & \cdots & z_1^{k-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_k & \cdots & z_k^{i-1} & g(x,z_k) & z_k^{i+1} & \cdots & z_k^{k-1} \end{vmatrix} \Big/ \begin{vmatrix} 1 & z_1 & \cdots & z_1^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_k & \cdots & z_k^{k-1} \end{vmatrix}$$

Suppose $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, with $p \geq n$, is of corank 1 and is given in the form $f(x_1, ..., x_{n-1}, z) = (x_1, ..., x_{n-1}, h_1(x, z), ..., h_{p-n+1}(x, z))$.

Theorem 3.1. *([18]) $D^k(f)$ is defined in $\mathbb{C}^{n+k-1}$ by the ideal $\mathcal{I}^k(f)$ generated by $V_i^k(h_j(x, z))$ for all $i = 1, ..., k-1$ and $j = 1, ..., p-n+1$.*

In what follows we will take coordinates on $\mathbb{C}^{n+k-1} = \mathbb{C}^{n-1} \times \mathbb{C}^k$ to be $(x, z) = (x_1, ..., x_{n-1}, z_1, ..., z_k)$.

Example 3.1. For a corank 1 map-germ $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $D^2(f)$ is defined by the ideal generated by the system $\left\{ \frac{h_i(x,z_1) - h_i(x,z_2)}{z_1 - z_2},\ i = 1, ..., p-n+1 \right\}$.

The main result of [18] is the theorem 2.14 where a description of $D^k(f)$ is obtained for a finitely $\mathcal{A}$-determined corank 1 map germ. There, it is shown that the multiple point spaces of f are ICIS. More precisely, f is finitely-determined if and only if for each k, with $p - k(p-n) \geq 0$, $D^k(f)$ is ICIS of dimension $p - k(p-n)$ or empty, and for those k with $p - k(p-n) < 0$, $D^k(f) = \{0\}$. Furthermore, f is stable if and only if the spaces are non-singular or empty.

Definition 3.1. Let $\mathcal{P} = (r_1, r_2, ..., r_m)$ be a partition of k, i.e., $r_1 + r_2 + ... + r_m = k$, with $k = 1, ..., p$. Let $\mathcal{I}(\mathcal{P})$ be the ideal in $\mathcal{O}_{n-1+k}$ generated by the $k-m$ elements $y_i - y_{i+1}$ for $r_1 + r_2 + ... + r_{j-1} + 1 \leq i \leq r_1 + r_2 + ... + r_j - 1$, $1 \leq j \leq m$, and let $\Delta(\mathcal{P}) = V(\mathcal{I}(\mathcal{P}))$.

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If $\mathcal{P}, \mathcal{R}$ are two partitions of k, we say $\mathcal{P} < \mathcal{R}$ if $\mathcal{I}(\mathcal{P}) \subset \mathcal{I}(\mathcal{R})$. We define a generic point of $\Delta(\mathcal{P})$ for any partition $\mathcal{R}$ of k with $\mathcal{P} < \mathcal{R}$.

Define

$$\mathcal{I}^k(f,\mathcal{P}) = \mathcal{I}^k(f) + \mathcal{I}(\mathcal{P}), \text{ and } \quad D^k(f,\mathcal{P}) = V(\mathcal{I}^k(f,\mathcal{P})),$$

equipped with the sheaf structure $\mathcal{O}_{n-1+k}/\mathcal{I}^k(f,\mathcal{P})$.

For any k satisfying $p - k(p-n+1) + \ell \geq 0$, we consider a partition $\mathcal{P} = (r_1, ..., r_m)$ of k and define projections $\pi_i(\mathcal{P}) : \mathbb{C}^{n-1+k} \to \mathbb{C}^n$, for $1 \leq i \leq m$, by $\pi_i(\mathcal{P})(x, z_1, ..., z_k) = (x, z_{r_1+...+r_{i-1}+1})$.

The geometric significance of $D^k(f,\mathcal{P})$ was given by Marar and Mond:

Lemma 3.1. *([18], Lemma 2.7, p. 559) Let $\mathcal{P} = (r_1, ..., r_m)$ be a partition of k, with $p - k(p-n+1) + \ell \geq 0$; at a generic point (x,z) of $\Delta(\mathcal{P})$ we have:*

$$\mathcal{I}_k(f,\mathcal{P}) = \mathcal{I}(\mathcal{P}) + \left\{ \frac{\partial^s f_j}{\partial z^s} \circ \pi_i(\mathcal{P}) \mid j = 1, ..., p-n+1, 1 \leq s \leq r_i - 1, 1 \leq i \leq m \right\} +$$

$$\{f_j \circ \pi_1(\mathcal{P}) - f_j \circ \pi_i(\mathcal{P}) \mid j = 1, ...p - n + 1, 2 \leq i \leq m\} \text{ in } \mathcal{O}_{n-1+k}, (x, z).$$

In the corollary 2.15 of [18] page 562 it is shown the following result. If f is finitely determined then for each partition $\mathcal{P} = (r_1, ..., r_\ell)$ of k satisfying $p - k(p-n+1) + \ell \geq 0$, the germ of $D^k(f,\mathcal{P})$ at 0 is either an ICIS of dimension $p - k(p-n+1) + \ell$, or is empty. Moreover, those $D^\ell(f,\mathcal{P})$ for $\mathcal{P}$ not satisfying the inequality consist at most of the single point 0.

With the definitions above, for any finitely determined germ $f \in \mathcal{O}(n,p)$ we denote by $D_1^\ell(f,\mathcal{P})$, the projections of $D^\ell(f,\mathcal{P})$, to the (x, y, z_1)-space. Therefore we conclude that the stratification in the source and target are as follows.

In the source: The regular part of the critical points set $\Sigma(f) = \mathbb{C}^n$ and the regular part of the multiple points set $D_1^\ell(f)$, for each partition $\mathcal{P}$ of k with $p - k(p-n+1) + \ell \geq 0$.

In the target: The regular part of the image of the multiple points set:
$f(D^\ell(f,\mathcal{P}))$ for each partition $\mathcal{P}$ of k with $p - k(p-n+1) + \ell \geq 0$.

Remark 3. 1. In the case that the number k satisfies the equality $p-k(p-n+1)+\ell = 0$, for each partition $\mathcal{P}$ of k, the set $f(D^\ell(f,\mathcal{P}))$ is 0-dimensional and is called a 0-stable type of the germ. We shall denote this set by $\mathcal{Q}(f,\mathcal{P})$.

4. POLAR VARIETIES IN THE TARGET

To show how the polar multiplicities of the stable types in the target are related we describe the polar varieties of the set $f(D^\ell(f,\mathcal{P}))$ for each partition $\mathcal{P}$ of k with $p - k(p-n+1) + \ell \geq 0$.

For a map germ $f \in \mathcal{O}(n,p)$ with corank 1, each analytic set $f(D^\ell(f,\mathcal{P}))$ is of dimension $d = p - k(p-n+1) + \ell$, the polar varieties of codimension j of these sets, with $0 \leq j \leq p - k(p-n+1) + \ell$, are obtained in the following way:

We choose generic projections $p : \mathbb{C}^{n-1} \times \mathbb{C}^\ell \to \mathbb{C}^p$ and $p_{d-j+1} : \mathbb{C}^p \to \mathbb{C}^{d-j+1}$, each polar variety is defined as

$$P_j(f(D^\ell(f,\mathcal{P}))) = \overline{\Sigma(p_{n-m-j+1}|f \circ p(D_1^\ell(f,\mathcal{P})^0)}$$

with $0 \leq j \leq d$.

We see in [22] that each polar variety is of codimension j in $\mathbb{C}^n$. The polar invariants associate to each polar variety are the multiplicities $m_j(P_j(f \circ p(D^\ell(f,\mathcal{P})))),\ 0 \leq j \leq d$ for each partition $\mathcal{P}$ of all k with $p - k(p-n+1) + \ell \geq 0$.

To compute the multiplicites $m_j(P_j(f \circ p(D^\ell(f,\mathcal{P}))))$, we need to consider the sets

$$\overline{\Sigma(p_{n-m-j+1}|f \circ p(D^\ell(f,\mathcal{P})^0)},$$

however it is better to work with the sets

$$X_j(\mathcal{P}) = \overline{\Sigma(p_{n-m-j+1} \circ f \circ p|D_1^\ell(f,\mathcal{P}))},$$

which are nothing more than

$$X_j(\mathcal{P}) = V\left(\mathcal{I}^\ell(f,\mathcal{P}), J(p_{n-m-j+1} \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})\right).$$

The advantages to work with these sets is that they are in the source and the equations that define the associate polar varieties are computable. Our strategy is to compute the polar invariants for these sets, choosing generic projections, and considering the fact that f is bimeromorphic, we compute the polar invariants in the target.

Let $\mathcal{P} = (r_1, \cdots, r_\ell)$ be a partition of $m \leq n$ with $r_1 \geq r_2 \geq \cdots r_\ell \geq 1$. Define $N(\mathcal{P})$ to be the order of the sub group of S_ℓ which fixes $\mathcal{P}$. Here S_ℓ acts on $\mathbb{R}^\ell$ by permuting the coordinates, for example, if $\mathcal{P} = (4,4,4,2,2,2,1,1)$ we have $N(\mathcal{P}) = (3!)^2.2!$, we remark that if $\mathcal{P} \neq (r_i)$, then $N(\mathcal{P}) \neq 1$.

4.1. Relations among the invariants in the target

The other strata in the target are the regular part of $f \circ p(D^\ell(f,\mathcal{P}))$ for each partition $\mathcal{P} = (r_1, ..., r_\ell)$ of k with $p - k(p-n+1) + \ell \geq 0$, if there is no confusion, we shall denote $f \circ p(D^\ell(f,\mathcal{P}))$ by $f(D^\ell(f,\mathcal{P}))$. These sets have dimensions $d = p - k(p-n+1) + \ell$ and their polar multiplicities are $m_i(f(D^\ell(f,\mathcal{P}))),\ 0 \leq i \leq d$.

To compute $m_d(f(D^\ell(f,\mathcal{P})))$, we take a versal unfolding $F : \mathbb{C}^n \times \mathbb{C}^s \to \mathbb{C}^p \times \mathbb{C}^s$ of f and consider $(p,\pi_s) : \mathbb{C}^n \times \mathbb{C}^s \to \mathbb{C}^n \times \mathbb{C}^s$ a projection in the source such that $F \circ (p,\pi_s) : D^\ell(F,\mathcal{P}) \subset \mathbb{C}^{n-1-\ell} \times \mathbb{C}^s \to F(D^\ell(F,\mathcal{P})) \subset \mathbb{C}^p \times \mathbb{C}^s$. We know that $F \circ (p,\pi_s)|D^\ell(F,\mathcal{P})^0$ is a ℓ-fold cover of $F(D^\ell(F,\mathcal{P}))^0$.

Choose $p_1 : \mathbb{C}^p \to \mathbb{C}$, a generic linear projection for $(F(D^\ell(F,\mathcal{P})),(p_1,\pi_s))$. To work directly with

$$P_d(F(D^\ell(F,\mathcal{P})),(p_1,\pi_s),D_1 \times \mathbb{C}^s) = \overline{\Sigma((p_1,\pi_s)|F(D^\ell(F,\mathcal{P}))^0)}$$

with $D_1 \times \mathbb{C}^s$ is the kernel of (p_1, π_s) we must work with

$$\overline{\Sigma((p_1, \pi_s)|F(D^2(F))^0)}.$$

However, it is much easier to work with $\Sigma((p_1, \pi_s) \circ F \circ (p, \pi_s)|D^\ell(F, \mathcal{P}))$ which is

$$V(\mathcal{I}^\ell(F, \mathcal{P}), J_x((p_1, \pi_s) \circ F \circ (p, \pi_s), \mathcal{I}^\ell(F, \mathcal{P}))),$$

where $\mathcal{I}^\ell(F, \mathcal{P})$ is the ideal that defines $D^\ell(F,)$, we include the singular set of $D^\ell(F, \mathcal{P})$ in the critical set of $(p_1, \pi_s) \circ F \circ (p, \pi_s)|D^\ell(F)$. This set has two advantages. It is in the source, and its equations are computable. We shall extract an invariant from it which will be simply related to $m_d(f(D^\ell(f, \mathcal{P})))$ and in fact, will control it.

The variety V has dimension s, since u is a generic parameter value, the degree of $\pi_s|V(\mathcal{I}^\ell(F), J_x((p_1, \pi_s) \circ F \circ (p, \pi_s), \mathcal{I}^\ell(F, \mathcal{P})))$ is just the colength, denoted by $e_{D^\ell(f,\mathcal{P})}$, of m_s in the local ring of the source at $(0, 0)$:

$$e_{D^\ell(f,\mathcal{P})} := \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f, \mathcal{P}), J_x((p_1 \circ f \circ p, \mathcal{I}^\ell(f)))} \tag{4.1}$$

where $\mathcal{I}^\ell(f, \mathcal{P})$ defines $D^\ell(f, \mathcal{P})$.

We then have the following relation between $e_{D^\ell(f,\mathcal{P})}$ and other invariants.

THEOREM 4.1. *Let $f \in \mathcal{O}(n, p)$ be a finitely determined germ of kernel rank 1. Then*

$$e_{D^\ell(f,\mathcal{P})} = N(\mathcal{P}) m_d(f(D^\ell(f, \mathcal{P}))) + \Sigma_{\mathcal{P}} c_{\mathcal{P}} \sharp \mathcal{Q}(f, \mathcal{P})$$

for all partition $\mathcal{P}$ of k satisfying $p - k(p - n + 1) + \ell \geq 0$ and $c_{\mathcal{P}} \in \mathbb{Z}^+$

Proof The components of the variety

$$V(\mathcal{I}^\ell(F, \mathcal{P}), J_x((p_1, \pi_s) \circ F \circ (p, \pi_s), \mathcal{I}^\ell(F, \mathcal{P})))$$

are the closure of the set $F^{-1}(P_d(D^\ell(F, \mathcal{P}))$ and the set $F^{-1}(\mathcal{Q}(f, \mathcal{P}))$. These have dimension at least s.

For a generic projection p_1 and f, the dimension of V is s. As the multiplicity of V is the sum of the multiplicities of these components, it is enough to calculate the contribution of the degree of π_s restricted to each component, where π_s is the projection of $\mathbb{C}^p \times \mathbb{C}^s$ to $\mathbb{C}^s$.

We choose neighbourhoods U_1 of 0 in $\mathbb{C}^s$ and U_2 of 0 in $\mathbb{C}^p \times \mathbb{C}^s$ such that at each point in U_1, π_s has $e_{D^\ell(f,\mathcal{P})}$ pre-images in $V \cap U_2$ counting multiplicity. If $u \in \mathbb{C}^s$ is a generic parameter close to the origin we have

$$\begin{aligned} e_{D^\ell(f,\mathcal{P})} &= \sum_{x \in S} \dim_{\mathbb{C}} \tfrac{\mathcal{O}_{s+n-1+\ell,x}}{(m_s, \mathcal{I}^\ell(F,\mathcal{P}), J_x((p_1,\pi_s) \circ F \circ (p,\pi_s), \mathcal{I}^\ell(F,\mathcal{P})))} \\ &= \sum_{x \in S} \dim_{\mathbb{C}} \tfrac{\mathcal{O}_{n-1+\ell,x}}{(\mathcal{I}^\ell(f_s,\mathcal{P}), J_x(p_1 \circ f_s \circ p, \mathcal{I}^\ell(f_s,\mathcal{P})))} \end{aligned}$$

where $S = \pi_s^{-1}(0) \cap V$.

We can compute this last number using normal forms. Hence the singularities of type $\mathcal{Q}(f,\mathcal{P})$ for each partition $\mathcal{P}$ of k with $p-k(p-n+1)+\ell=0$ contributes with $c_{\mathcal{P}} \in \mathbb{Z}^+$. A similar argument shows that $F^{-1}(P_d(F(D^\ell(F,\mathcal{P}))))$ contributes with $N(\mathcal{P})m_d(f(D^\ell(f,\mathcal{P})))$. Since F restricted to each type of each component is bimeromorphic and finite, and the generic point of each component is reduced, we have

$$\deg(\pi_{|V}) = N(\mathcal{P})m_d(f(D^\ell(f,\mathcal{P}))) + \Sigma_{\mathcal{P}} c_{\mathcal{P}} \sharp \mathcal{Q}(f,\mathcal{P})$$

Proposition 4.1. *Let $\mathcal{P}$ be a partition of k with $p-k(p-n+1)\ell \geq 0$, then*

$$m_j(P_j(f(D^\ell(f,\mathcal{P})))) = \frac{1}{N(\mathcal{P})}\deg((p_{d-j}\circ f\circ p)|X_j(\mathcal{P}))$$

Proof For each $j=0,...,d$, we have $X_j(\mathcal{P}) \subset D_1^\ell(f,\mathcal{P})$, let $\mathbf{y}=(x,z_1,z_2,\ldots,z_\ell) \in X_j(\mathcal{P})$ and $\sigma \in S_\ell$, then we have

$$\mathbf{y}_\sigma = (x, z_{\sigma(1)}, z_{\sigma(2)}, ..., z_{\sigma(\ell)}) \in X_j(\mathcal{P})$$

if and only if $r_{\sigma(k)} = r_i$ for all $k,i=1,...,\ell$. There exist $N(\mathcal{P})$ of these Σ, such that the points $\mathbf{y}$, $\mathbf{y}_\sigma$ are different, but the corresponding sets $\{z_1,z_2,...,z_\ell\}$ are equal, i.e., the germ f in $\{(x,z_1),(x,z_2),...,(x,z_\ell)\}$ has an ordinary ℓ-multiple point in $X_j(\mathcal{P})$, we observe that each of these points point gives $N(\mathcal{P})$ points in $(p_{d-j}\circ f\circ p)^{-1}(z)$, and these point are all computable in $X_j(\mathcal{P})$.

Remark 4. 1. Since the projections p_{d-j} and p are generic and the germ f is finitely determined, the variety $X_j(\mathcal{P})$ is an ICIS, therefore it is Cohen Macaulay, then

$$\deg(K_{d-j}|X_j(\mathcal{P})) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(K_{d-j},\mathcal{I}^\ell(f,\mathcal{P}),J(K_{d-j+1},\mathcal{I}^\ell(f,\mathcal{P}))}$$

with $K_{d-j} = p_{d-j}\circ f$

Theorem 4.2. *Let $f \in \mathcal{O}(n,p)$, with $n<p$, be a finitely determined germ of corank 1. Then*

$$\begin{aligned}\Sigma_{i=0}^{d-1}(-1)^i N(\mathcal{P})m_i(f(D^\ell(f,\mathcal{P}))) \;&=\; (-1)^d\mu(D^\ell(f,\mathcal{P}))+1+\\ &\quad (-1)^{d+1}dim_{\mathbb{C}}\tfrac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}),J(p_1\circ f,\mathcal{I}^\ell(f,\mathcal{P})))}.\end{aligned}$$

Proof Choose a generic projection $p_d:\mathbb{C}^n\to\mathbb{C}^d$ such that the degree of $p_d|f(D^\ell(f,\mathcal{P}))$ is equal to the multiplicity of $f(D^\ell(f,\mathcal{P}))$ at the origin and a generic linear projection $p_{d-1}|P_1(f(D^\ell(f,\mathcal{P}):\mathbb{C}^n\to\mathbb{C}^{d-1}$ such that its degree is the multiplicity of the polar variety $P_1(f(D^\ell(f,\mathcal{P})))$, denoted by $m_1(f(D^\ell(f,\mathcal{P})))$.

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Let $X_1 = V(p_{d-1} \circ f, \mathcal{I}^\ell(f,\mathcal{P}))$, and $X = V(p_d \circ f, \mathcal{I}^\ell(f,\mathcal{P}))$. As these varieties are I.C.I.S., we apply the Theorem of Lê-Greuel and obtain

$$\mu(X_1) + \mu(X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(p_{d-1} \circ f, \mathcal{I}^\ell(f,\mathcal{P}), J\,(p_d \circ f, \mathcal{I}^\ell(f,\mathcal{P})))}$$

Since the germ f is bimeromorphic and the projections are generic, we have

$$\deg((p_{d-1} \circ f)|X_1(\mathcal{P})) = \deg(p_{d-1}|P_1(f(D^\ell(f,\mathcal{P}))))$$

from the Proposition 4.1 and the Remark 4.1, it follows that

$$\mu(X_1) + \mu(X) = N(\mathcal{P}) m_1(f(D^\ell(f,\mathcal{P}))) \tag{4.2}$$

Now we choose a generic projection $p_{d-2} : \mathbb{C}^n \to \mathbb{C}^{d-2}$ such that the degree of $p_{d-2}|P_2(f(D^\ell(f,\mathcal{P})))$ is $m_2(f(D^\ell(f,\mathcal{P})))$ and call $X_2 = V(p_{d-2} \circ f, \mathcal{I}^\ell(f,\mathcal{P}))$ hence

$$\mu(X_1) + \mu(X_2) = dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(p_{d-2} \circ f, \mathcal{I}^\ell(f,\mathcal{P}), J(p_{d-1} \circ f, \mathcal{I}^\ell(f,\mathcal{P})))} \tag{4.3}$$

Again, as f is bimeromorphic and the projection p_{d-2} is generic, we obtain

$$\deg((p_{d-2} \circ f)|X_2(\mathcal{P})) = \deg(p_{d-2}|P_2(f(D^\ell(f,\mathcal{P}))))$$

applying the Proposition 4.1 to the equality (4.3), it follows from the Remark 4.1 that

$$\mu(X_1) + \mu(X_2) = N(\mathcal{P}) m_2(f(D^\ell(f,\mathcal{P}))).$$

This equality and the equality (4.2) gives us

$$N(\mathcal{P}) m_1(f(D^\ell(f,\mathcal{P}))) - \mu(X) + \mu(X_2) = N(\mathcal{P}) m_2(f(D^\ell(f,\mathcal{P})))$$

Now, for all s with $3 < s < n-m-2$ we choose generic projections p_{d-s} in a successive way and construct the sets X_s and X_{s-1} analogously than above to obtain the sets

$$X_{d-1} = V(p_1 \circ f, \mathcal{I}^\ell(f,\mathcal{P})), \qquad X_d = D^\ell(f,\mathcal{P})$$

and the equality

$$\Sigma_{i=1}^{d-1}(-1)^i N(\mathcal{P}) m_i(f(D^\ell(f,\mathcal{P}))) - \mu(X_{n-m}) + \mu(X) = dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f \circ p, \mathcal{I}^\ell(f,\mathcal{P})))},$$

therefore $\mu(X) = \deg(p_d \circ f, \mathcal{I}^\ell(f,\mathcal{P})) - 1$.

Since $f \circ p : D^\ell(f,\mathcal{P}) \to f(D^\ell(f,\mathcal{P}))$ is finite and bimeromorphic we obtain $\deg(p_d|f(D^\ell(f,\mathcal{P}))) = \deg(p_d \circ f, \mathcal{I}^\ell(f,\mathcal{P}))$ and $\deg(p_d \circ f, \mathcal{I}^\ell(f,\mathcal{P})) = N(\mathcal{P}) m_0(f(D^\ell(f,\mathcal{P})))$.

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4.2. Relations among the invariants of the stable types in the source

In this section we stablish relations among the invariants of the stable types in the source. The strata in this case are the regular part of the critical points set $\Sigma(f) = \mathbb{C}^n$ and the regular part of the multiple points set $D_1^\ell(f)$, for each partition $\mathcal{P}$ of k with $p-k(p-n+1)+\ell \geq 0$. The situation is less difficult than in the case of the target because, in the case of corank 1 germs, all these sets are ICIS.

We know from [21] that the absolute polar multiplicities of a hypersurface X with isolated singularity are related to the Milnor numbers $\mu^{(k)}$ of the plane sections ($\mu^{(k)}(X) = \mu(X \cap H^k)$) by the following equalities

$$m_k(X) = \mu^{(k+1)}(X) + \mu^{(k)}(X),$$

for $0 \leq k \leq d-1$, where $d = \dim(X)$. This result is also valid for ICIS (see [14], [7]). The absolute polar multiplicities are defined when the dimension of X is ≥ 1. The multiplicity $m_d(X)$ cannot be defined directly like the other m_k, $0 \leq k \leq d-1$, because the singularities of $p_1|X$ are isolated points. However, Gaffney [7] defines this multiplicity for spaces that are ICIS as follows.

Definition 4.1. The d-th polar multiplicity of $(X^d, 0)$ (X^d is ICIS of dimension d), denoted by $m_d(X^d)$, is defined by

$$m_d(X^d) = \dim_{\mathbb{C}} \frac{\mathcal{O}_X}{J(p_1, f)}$$

where $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-d}, 0)$, $f^{-1}(0) = X^d$ and $p_1 : \mathbb{C}^n \to \mathbb{C}$ is a generic linear projection.

Remark 4. 2. As $V(p_1, f)$ is ICIS, then by Lê-Greuel theorem, we have

$$m_d(X^d) = \mu(X^d) + \mu(X^d \cap p_1^{-1}(0)).$$

When $f \in \mathcal{O}(n,p)$ is finitely determined and of corank 1, $n < p$ the multiple points sets $D^\ell(f, \mathcal{P})$ for all partitions P of k, with $p-k(p-n+1)+\ell \geq 0$ are ICIS. Therefore we can apply the definition 4.1 and all the properties above to obtain the following.

Proposition 4.2. *Let $f \in \mathcal{O}(n,p)$ be a finitely determined germ of corank 1. For each partition P of k, with $p-k(p-n+1)+\ell \geq 0$ we have*

$$\Sigma_{i=0}^{d-1}(-1)^i m_i(D^\ell(f,\mathcal{P})) = (-1)^d \mu(D^\ell(f,\mathcal{P})) + 1 + (-1)^{d+1} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}^\ell(f,\mathcal{P})))}.$$

$$\Sigma_{i=0}^{d-1}(-1)^i m_i(D_1^\ell(f,\mathcal{P})) = (-1)^d \mu(D_1^\ell(f,\mathcal{P})) + 1 + (-1)^{d+1} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n}}{(\mathcal{I}_1^\ell(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}_1^\ell(f,\mathcal{P})))}.$$

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Therefore we can now deduce our main theorem. From the results given in the subsections 4.1 and 4.2, we reduce the number of invariants in Gaffney's theorem 2.1 for half in the corank 1 case.

THEOREM 4.3. *Suppose that $f \in \mathcal{O}(n,p)$ with $n < p$ is a finitely determined germ of corank* 1. *If $F = (t, f_t)$ is a good 1−parameter unfolding of f, then F is Whitney equisingular along $T = \mathbb{C} \times \{0\}$ if, and only if, $m_{2i-1}(D_1^\ell(f_t,\mathcal{P}))$ and $m_{2i-1}(f_t(D_1^\ell(f_t,\mathcal{P})))$ are constant for t close to the origin.*

5. FORMULA FOR THE EULER OBSTRUCTION IN STABLE TYPES

The local Euler obstruction for nonsingular varieties, introduced by R. MacPherson in [20], in a purely obstructional way is a topological invariant that is also associated to the polar multiplicities. The local Euler obstruction plays an important role in his affirmative response to a conjecture of Deligne and Grothendieck on the existence of Chern class for singular complex algebraic varieties (see [12],[20].)

Definitions equivalent to MacPherson's have been given by several authors. We recall here the one given in [2], see also [1], pp. 38–42.

Let X be an d-dimensional analytic complex subvariety of an m-dimensional manifold M. We consider the Nash transform $\tilde{X}$ and the restriction $\nu : \tilde{X} \to X$, where $\nu : G \to \mathbb{P}^m$ denotes the Grassmann bundle over $\mathbb{P}^m$ whose fibre over x is the Grassmann manifold $G(d,m)$ of d-linear subspaces in $T_x\mathbb{P}^m$.

Let us consider a Whitney stratification $\{V_\alpha\}$ of M such that X is a union of strata. Let v be a radial vector field with an isolated singularity at $x \in V_\alpha$. Let B be a ball centered at x, small enough to be transversal to every stratum V_β with $V_\alpha \subset V_\beta$, and such that x is the unique zero of v inside B. In the proposition 9.1 of [2] it is shown that, using the Whitney conditions it is possible to prove that there is a canonical lifting $\tilde{v}$ of $v_{|\partial B\cap X}$ as a section of $\tilde{E}_{|\nu^{-1}(\partial B\cap X)}$. The obstruction to the extension of $\tilde{v}$, on $\nu^{-1}(B\cap X)$, as a non-zero section of $\tilde{E}$, evaluated on the corresponding fundamental class, is an integer, called the Euler obstruction and denoted by $Eu_x(X)$.

In [13], Lê and Teissier proved a formula for the multiplicity of the local polar varieties, and, with the aid of Gonzales-Sprinberg's purely algebraic interpretation of the local Euler obstruction, they showed that the local Euler obstruction is an alternate sum of the multiplicity of the local polar variety.

THEOREM 5.1. *[13] Let X be an analytic space of dimension d reduced at $0 \in \mathbb{C}^{n+1}$. Then*

$$Eu_0(X) = \Sigma_{i=0}^{d-1}(-1)^{d-i-1}m_i(X),$$

where $Eu_0(X)$ denotes the Euler obstruction of X at 0 and $m_i(X)$ is the polar multiplicity of the polar varieties $P_i(X)$.

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With this theorem and the theorems above we formulate the formula for Euler obstruction for the stable types in the source and the target.

THEOREM 5.2. *Let $f \in \mathcal{O}(n,p)$, $n < p$ be a finitely determined germ of corank 1. Then*

$$N(\mathcal{P})Eu_0(f(D^\ell(f,\mathcal{P}))) = (-1)^d\mu(D^\ell(f,\mathcal{P})) + 1 + (-1)^{d+1} dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}^\ell(f,\mathcal{P})))}.$$

$$Eu_0(D^\ell(f,\mathcal{P})) = (-1)^d\mu(D^\ell(f,\mathcal{P})) + 1 + (-1)^{d+1} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-1+\ell}}{(\mathcal{I}^\ell(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}^\ell(f,\mathcal{P})))}.$$

$$Eu_0(D_1^\ell(f,\mathcal{P})) = (-1)^d\mu(D_1^\ell(f,\mathcal{P})) + 1 + (-1)^{d+1} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n}}{(\mathcal{I}_1^\ell(f,\mathcal{P}), J(p_1 \circ f, \mathcal{I}_1^\ell(f,\mathcal{P})))}.$$

Remark 5. 1. For a germ f of corank 1, consider the partition $\mathcal{P} = (1)$, then we have $D^1(f,(1)) = \mathbb{C}^n$, therefore $\mu(D^1(f,(1))) = 0$ and

$$dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(\mathcal{I}^1(f,(1)), J(p_1 \circ f, \mathcal{I}^1(f,(1))))} = 0$$

for a linear projection $p_1 = x$ therefore $Eu_0(f(\mathbb{C}^n)) = 1$ for all f.

In particular if $f(x,y) = (x, y^2, xy)$ we have that the Euler obstruction of the Cross-cap is 1.

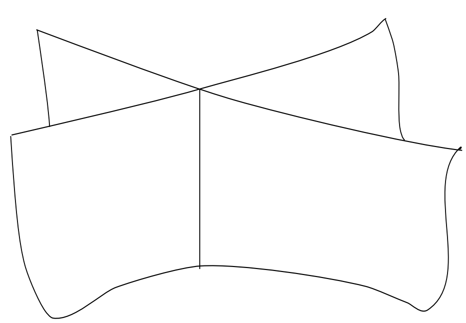

FIG. 1. Cross-Cap

We observe that Gonzalez-Sprinberg in [8], p. 28, uses the usual method to compute the local Euler obstruction of this example, and our method is much simpler to do this computation.

As a consequence of Gaffney's theorem 2.1 and the Theorem 5.1 we obtain in the next theorem that the local Euler obstruction is an invariant for the Whitney equisingularity.

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THEOREM 5.3. *Suppose that $f \in \mathcal{O}(n,p)$, with $n < p$, is a finitely determined germ of corank* 1 *and $F = (t, f_t)$ is a good 1−parameter unfolding. If F is Whitney equisingular along $T = \mathbb{C} \times \{0\}$, then $Eu_0(D_1^\ell(f_t, \mathcal{P}))$, $Eu_0(f_t \circ p(D^\ell(f_t, \mathcal{P})))$, are constant for t close to the origin, for all partition $\mathcal{P}$ of k satisfying $p - k(p - n + 1) + \ell \geq 0$.*

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