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Rank–1 Codimension one Singularities of Positive Quadratic Differential Forms

V. Guíñez*

Universidad de Santiago de Chile, Facultad de Ciencia, Casilla 307, Correo 2, Santiago, Chile
E-mail: vguinez@usach.cl

Carlos Gutierrez†

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP, Brazil
E-mail: gutp@icmc.usp.br

We complete the study of first order structural stability at singular points of positive quadratic differencial forms on two manifolds. For this, we consider the generic 1–parameter bifurcation of a $D_{23}-$singular point.This situation consists in having, before the bifurcation, two locally stable singular points (one of type D_2 and the other of type D_3) which collapse at the $D_{23}-$singular point when the bifurcation parameter is reached, and afterwards disappear. (See Figure 2.) In local $(x, y)-$coordinates, such a point appears at the origin of a planar differential equation of the form $a(x, y)dy^2+2b(x, y)dxdy+c(x, y)dx^2$, with $(b^2 - ac)(x, y) \geq 0$, such that: (1) the first jet of the map (a, b, c) at the origin is $T_1(a, b, c)(0, 0) = (y, 0, -y)$ and (2) $\frac{\partial^2 b}{\partial x^2} \neq 0$. May, 2003 ICMC-USP

Key Words: One–dimensional foliation, quadratic differential form, singular point, bifurcation.

1. INTRODUCTION

We associate to each positive quadratic differential form ω defined on a connected, oriented, two–dimensional manifold M two transversal, one–dimensional foliations $f_1(\omega)$ and $f_2(\omega)$ with common set of singular points. Structural stability under small perturbations of the positive quadratic differential form as well as the genericity of the stability property are studied in [20] and [21]. The set $S(M)$ consisting of all structurally stable positive $C^\infty-$quadratic differential forms is characterized by conditions on the singular points and compact leaves, as well as by the asymptotic behavior of the non–compact leaves, especially the singular sepatrices. The above conditions on the singular points generate three types of locally stable singular points, namely the types D_1, D_2, and D_3. This paper focuses on

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generic 1–parameter bifurcations at singular points of smooth positive quadratric differential forms. In order to generate codimension one singular points, we weaken in the mildest way the conditions on a singular point for being locally stable. Thus we obtain two types of non–locally stable singular points, namely the types D_{12} and D_{23}. The former, which is a transition between the types D_1 and D_2, is studied in [22]. This paper deals with the latter, that is the type D_{23}, and completes the study of first order structural stability at singular points of positive quadratic differencial forms.

We first recall standard definitions, and next give an overview of the main results of [20], [21] and [22].

Let M be a $C^\infty-$compact, connected, oriented, two–dimensional manifold. A C^r-quadratic differential form on M is an element of the form $\omega = \sum_{k=1}^{n} \phi_k \psi_k$, where ϕ_k and ψ_k are C^r 1–forms on M. Therefore for each point p in M, we have that $\omega(p)$ is a quadratic form on the tangent space T_pM. We say ω is *positive* if for every point p in M, the subset $\omega(p)^{-1}(0)$ of T_pM is either the union of two transversal lines or all T_pM. In the former case, p is called a *regular point* of ω. In the latter case, p will be called a *singular point* of ω. If p is a regular point of such an ω, we let $L_1(\omega)(p)$ and $L_2(\omega)(p)$ denote the lines which make up $\omega(p)^{-1}(0)$ and are characterized as follows. Let C be a positively oriented circle around the origin of T_pM. Then $q \in C \cap L_1(\omega)(p)$ (resp. $C \cap L_2(\omega)(p)$) if there exists a small arc $[q_1, q_2]$ on C containing q such that $\omega(p)$ is positive (resp. negative) on $[q_1, q[$ and negative (resp. positive) on $]q, q_2]$. Thus associated to each positive C^r-quadratic differential form ω on M there is a triplet $\{f_1(\omega),\ f_2(\omega),\ Sing(\omega)\}$, is denoted by $C(\omega)$, and is called the *configuration of* ω, where $Sing(\omega)$ is the set of singular points of ω, and $f_1(\omega)$ and $f_2(\omega)$ are the two transversal, C^r one–dimensional foliations defined on $M - Sing(\omega)$ whose tangent lines at each regular point p are $L_1(\omega)(p)$ and $L_2(\omega)(p)$, respectively.

Recall that two positive C^r-quadratic differential forms ω_1 and ω_2 are called **equivalent** if there exists a homeomorphism h of M such that $h(C(\omega_1)) = C(\omega_2)$. In other words, h maps $Sing(\omega_1)$ onto $Sing(\omega_2)$, and maps leaves of $f_1(\omega_1)$ (resp. $f_2(\omega_1)$) onto leaves of $f_1(\omega_2)$ (resp. $f_2(\omega_2)$). Also, a positive C^r-quadratic differential form ω will be called **structurally stable** if any positive $C^\infty-$quadratic differential form sufficiently C^1-close to ω is equivalent to ω.

In what follows, we will denote the set of all positive $C^\infty-$quadratic differential forms defined on M by $\mathcal{F}(M)$, and will denote this set endowed with the C^k-topology by $\mathcal{F}^k(M)$. Remark that $\mathcal{F}^k(M)$ is a subspace of the Banach space $\widetilde{\mathcal{F}^k}(M)$, consisting of all the $C^\infty-$quadratic differential forms (not necessarily positive) defined on M, which is also endowed with the C^k-topology.

The study of the global features of the configurations associated to structurally stable positive quadratic differential forms was begun in [20]. Combining the results of [20] and [21], a complete characterization of the structurally stable positive $C^\infty-$quadratic differential forms with the C^2-topology is obtained. In order to review this characterization, we next recall some definitions. (See [22].)

Let p be a singular point of a positive $C^\infty-$quadratic differential form ω. We will say p is an **s–singular point** of ω if there exists a local chart $(x, y) : (U, p) \to (\mathbb{R}^2, (0, 0))$ such that if $(x, y)^*(\omega) = a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2$, then the Jacobian matrix of the

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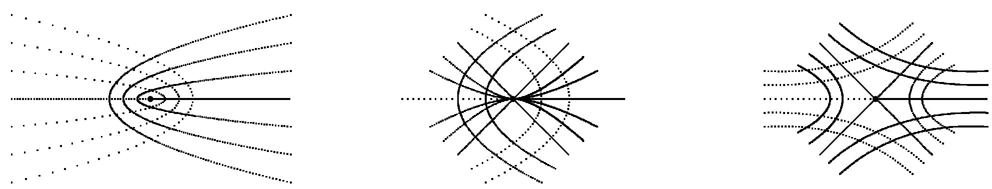

Type D_1 Type D_2 Type D_3

Figure 1

map $g = (a, 2b, c)$ at $(0, 0)$ is of the type

$$\begin{pmatrix} 0 & a_2 \\ 1 & 2b_2 \\ 0 & c_2 \end{pmatrix},$$

with $a_2^2 + c_2^2$ positive, and one of the following inequalities holds:

$D_1)$ $b_2^2 - a_2(1 + c_2) < 0$;
$D_2)$ $1 + c_2 < 0$ or $(-1 < c_2 \leq 0$ and $b_2^2 - a_2(1 + c_2) > 0)$;
$D_3)$ $c_2 > 0$ or $(c_2 = 0$ and $a_2 < 0)$.

In each case, the corresponding local configuration of ω at the point p is shown in Figure 1.

The index i of D_i, with $i = 1, 2, 3$, denotes the number of **singular separatrices of** p, that is the leaves which tend to the singular point p and separate regions of different patterns of approach to p. We say that a singular separatrix is a **singular connection** if it is a separatrix of two different singular points or twice a separatrix of the same singular point.

Finally, we say that a compact leaf c of the foliation $f_i(\omega)$ is **hyperbolic** if the Poincaré first return map P associated to a transversal line to c at a point $q \in c$ verifies $P'(q) \neq 1$.

We let $S_i(M)$, with $i = 1, 2, 3, 4$, denote the set consisting of the positive C^∞−quadratic differential forms ω which satisfy the following corresponding i−condition.

1) All singular points of ω are s–singular points.

2) All compact leaves of the foliation $f_i(\omega)$, with $i = 1, 2$, are hyperbolic.

3) The $f_i(\omega)$, with $i = 1, 2$, has no singular connections.

4) The limit set of every leaf of $f_i(\omega)$, with $i = 1, 2$, is the union of singular points, compact leaves, and singular connections.

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Thus a form ω in $\mathcal{F}(M)$ is structurally stable if and only if $\omega \in S(M) = \cap_{i=1}^{4} S_i(M)$. Further, the set $S(M)$ is dense in $\mathcal{F}^2(M)$.

To obtain codimension one singular points, we must weaken in the mildest way the conditions which define the s–singular points. For this, we must recall further definitions. Let p be a singular point of a positive $C^\infty-$quadratic differential form ω, and let $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ be a local chart. Assume $(x,y)^*(\omega) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2$. By the **matrix of ω at p in the chart** (x,y) will be meant the Jacobian matrix of the map $(a, 2b, c)$ at the origin. The **separatrix polynomial of ω at p in the chart** (x,y) is the homogeneous degree 3 polynomial

$$S(\omega,p)(x,y) = Da_{(0,0)}(x,y)y^2 + 2\,Db_{(0,0)}(x,y)xy + Dc_{(0,0)}(x,y)x^2\,.$$

Finally, we will say that p is a **rank–k** singular point of ω, with $k = 0, 1, 2$, if the matrix of ω at p in the chart (x,y) has rank–k.

Generically, an s–singular point is a rank–2 singular point with separatrix polynomial which has only simple roots. If we maintain the rank–2 condition on the singular point and admit a root of multiplicity two for the separatrix polynomial, we obtain the following.

Definition 1.1. A singular point p of a $\omega \in \mathcal{F}(M)$ is called a $\mathbf{D_{12}}$**−singular point** if there exists a local chart $(x,y) : U \to \mathbb{R}^2$ such that the matrix of the ω at p is

$$\begin{pmatrix} 0 & a_2 \\ 1 & 2b_2 \\ 0 & c_2 \end{pmatrix},$$

with

$$(1+c_2)a_2 \neq 0 \quad \text{and} \quad b_2^2 - a_2(1+c_2) = 0.$$

This singular point is studied in [22]. We again recall definitions to describe the principal results.

Let $S_1(b_1)$ be the set consisting of the $\omega \in \cap_{i=2}^{4} S_i(M)$ such that all of the singular points of ω are s–singularities except one, which is a $D_{12}-$singular point. Let $\Phi(M)$ be the set of $C^\infty-$maps $\xi : [0,1] \to \mathcal{F}^2(M)$ with the C^1-uniform topology, and let $\tilde{\Phi}(M)$ be the subset of $\xi \in \Phi(M)$ such that $\xi(0) \in S(M)$ and there exists $t < 1$ for which $\xi(t) \notin S(M)$. For such a ξ, we let b_0 be the first bifurcation parameter, or in other words $b_0 = \inf\{t \in [0,1]/\xi(t) \notin S(M)\}$. Consider the set $\tilde{\Phi}_1(M)$ of $\xi \in \tilde{\Phi}(M)$ such that $\xi(b_0) \in \cap_{i=2}^{4} S_i(M) - S_1(M)$ and $\xi(b_0)$ has a non–locally stable rank–2 singular point.

In [22], it is shown that the set $\mathcal{M}_1$ consisting of the 1–parameter families in $\tilde{\Phi}_1(M)$ which transversally cross $S_1(b_1)$ is open and dense, and that its elements are stable. This means that for any $\xi : [0,1] \to \mathcal{F}(M)$ in $\mathcal{M}_1$, there exist $t_0 > b_0$ and a neighborhood $\mathcal{V} \subset \Phi(M)$ such that for each $\eta \in \mathcal{V}$, there is an increasing homeomorphism $\mu : [0,1] \to [0,1]$ for which $\eta(t)$ is equivalent to $\xi(\mu(t))$, for all $t \in [0,t_0]$.

Here while we maintain the condition on the roots of the separatrix polynomial at the singular point for being simple, we weaken the rank–2 condition on the singular point to

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that of rank–1. Further, we impose a generic condition on the second jet of the quadratic differential form at the point. The resulting singular point is the following.

Definition 1.2. A singular point p of $\omega \in \mathcal{F}(M)$ will be called a $\mathbf{D_{23}}$**−singular point** if there exists a local chart $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ such that if $(x,y)^*(\omega) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2$, then $\dfrac{\partial^2 b}{\partial x^2}(0,0) \neq 0$ and the matrix of ω at p is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $S_1(b_2)$ be the set consisting of the $\omega \in \cap_{i=2}^{4} S_i(M)$ such that all of the singular points of ω are s–singularities except one, which is a $D_{23}-$singular point. We show that the set $S_1(b_2)$ is the intersection of $\mathcal{F}(M)$ with a codimension one immersed submanifold of $\tilde{\mathcal{F}}^2(M)$, is open in $\mathcal{F}(M) - S_1(M)$, and is dense in the subset of all the positive smooth quadratic differential forms whose rank–2 singular points are locally stable and which have an isolated rank–1 singular point. Further, we prove that $S_1(b_2)$ separates locally a connected component of $S(M)$ from other positive smooth quadratic differential forms with different configurations, as well as that every $\omega \in S_1(b_2)$ is structurally stable along $\mathcal{F}(M) - S_1(M)$.

We consider the set $\tilde{\Phi}_2(M)$ consisting of the $\xi \in \tilde{\Phi}(M) - \tilde{\Phi}_1(M)$ such that $\xi(b_0) \in \cap_{i=2}^{4} S_i(M) - S_1(M)$ and $\xi(b_0)$ has an isolated rank–1 singular point. We show that the set $\mathcal{M}_2$ consisting of all the 1–parameter families in $\tilde{\Phi}_2(M)$ which transversally cross $S_1(b_2)$ is open, and that its elements are stable. We also show that any 1–parameter family ξ in $\tilde{\Phi}_2(M)$ can be arbitrarily approximated by a family that may be either in $\mathcal{M}_1 \cup \mathcal{M}_2$ or outside of $\tilde{\Phi}_1(M) \cup \tilde{\Phi}_2(M)$.

For $\xi \in \mathcal{M}_2$ and t near $b_0(\xi)$, the local configuration of $\xi(t)$ around the corresponding $D_{23}-$singular point is shown in Figure 2.

The local and global behaviours of quadratic differential forms have been studied in many papers and in diverse contexts. We include various references related to the present work.

2. GENERIC SINGULARITIES

Next proposition gives a characterization of the D_1, D_2 and D_3 singularities better suited to our needs.

Let $\omega \in \mathcal{F}(M)$. Given a chart $(x,y) : U \to \mathbb{R}^2$, if $(x,y)^*(\omega) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2$, consider the maps $g : (x,y)(U) \subset \mathbb{R}^2 \to \mathbb{R}^3$ and $\Delta, H_1, H_2 : (x,y)(U) \subset \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

1) $g = (a, 2b, c)$.

2) $\Delta(x,y)$ is the discriminant of the homogeneous degree three polynomial $Da_{(x,y)}(u,v)v^2 + 2Db_{(x,y)}(u,v)uv + Dc_{(x,y)}(u,v)u^2$.

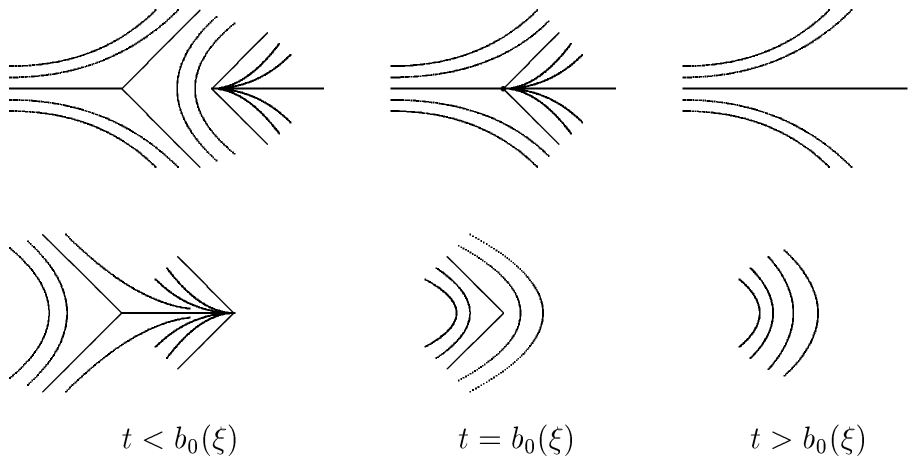

Figure 2

3) $H_1(x, y)$ and $H_2(x, y)$ are the determinant of the Jacobian matrix of the maps $g_1 = (a, b)$ and $g_2 = (b, c)$ at the point (x, y), respectively.

Remark 2. 1. Let p be a singular point of $\omega \in \mathcal{F}(M)$, and let $(x, y) : (U, p) \to (\mathbb{R}^2, (0, 0))$ be a local chart. Consider the maps g, Δ, H_1, and H_2 associated to this chart defined above. Then:

a) The singular point p is of rank–2 if and only if the map H_1 or H_2 does not vanish at the origin.
b) The singular point p is of rank–1 if and only if both maps H_1 and H_2 vanish at the origin, and the Jacobian matrix of the map g does not vanish at the origin.
c) $H_1(0, 0) > 0$ (resp. $H_1(0, 0) < 0$) implies $H_2(0, 0) \geq 0$ (resp. $H_2(0, 0) \leq 0$).

Proposition 2.1. *Let p be a rank–2 singular point of $\omega \in \mathcal{F}(M)$, and let $(x, y) : (U, p) \to (\mathbb{R}^2, (0, 0))$ be a local chart. If $\Delta(0, 0) \neq 0$, then*

1)p is of type D_1 if and only if $\Delta(0, 0) > 0$.
2)p is of type D_2 if and only if $\Delta(0, 0) < 0$ and $H_1(0, 0) + H_2(0, 0) < 0$.
3)p is of type D_3 if and only if $\Delta(0, 0) < 0$ and $H_1(0, 0) + H_2(0, 0) > 0$.

Proof. Since the roots of the equation

$$Da_{(x,y)}(u, v)v^2 + 2Db_{(x,y)}(u, v)uv + Dc_{(x,y)}(u, v)u^2 = 0$$

correspond to the possible directions of asymptotic convergence to the singular point for the leaves of the foliations, the sign of $\Delta(0, 0)$ is invariant by coordinate changes. On the other

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hand, $H_1(0,0)+H_2(0,0)$ is negative (resp. positive) if and only if the Poincaré index of the singular point is $\frac{1}{2}$ (resp. $-\frac{1}{2}$). Therefore, the sign of $H_1(0,0)+H_2(0,0)$ is also invariant by coordinate changes. The Proposition results from Remark 2.6 and [21, Theorem B]. ▮

3. $\mathbf{D_{23}}$−SINGULAR POINTS

In this section, we show certain facts about D_{23}−singular points used to establish our main results.

First we characterize the D_{23}−singular points, which we give in the next Remark.

Remark 3. 1. Let $\omega \in \mathcal{F}(M)$, and let p be a singular point of ω. Then the following three properties are equivalent:

a) The point p is a D_{23}−singular point.

b) There exists a local chart $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ such that if $(x,y)^*(\omega) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2$, then matrix of ω at p is

$$\begin{pmatrix} 0 & 1 \\ 0 & b_2 \\ 0 & c_2 \end{pmatrix},$$

and the following condition is satisfied

$$c_2 \left(c_2 \frac{\partial^2 b}{\partial x^2}(0,0) \; - \; b_2 \frac{\partial^2 c}{\partial x^2}(0,0) \right) \neq 0 \, .$$

c) There exists a local chart $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ such that if $(x,y)^*(\omega) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2$, then the matrix of ω at p is

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix},$$

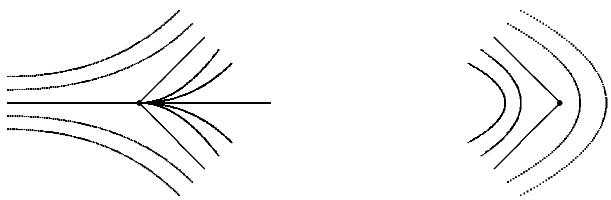
and the following condition is satisfied

$$\frac{\partial^2 b}{\partial x^2}(0,0) \; + \; \frac{\partial^2 c}{\partial x^2}(0,0) \neq 0 \, .$$

The proof is straightforward, hence omitted.

We now recall a Proposition that gives the phase portrait around a D_{23}− singular point, which appears in [21, Section 11.2.1 Case A].

PROPOSITION 3.1. *Let $\omega \in \mathcal{F}(M)$, and let p be a D_{23}−singular point of ω. Then the local phase portraits of the foliations associated to ω around p are homeomorphic to the ones shown in Figure 3.*



$f_1(\omega)$ $f_2(\omega)$

Figure 3

Proof. Consider a local chart $(x, y) : (U, p) \to (\mathbb{R}^2, (0, 0))$ such that if $(x, y)^*(\omega) = a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2$, then $\dfrac{\partial^2 b}{\partial x^2}(0, 0) \neq 0$ and the matrix of ω at p is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

We may suppose

$$\begin{aligned} a(x, y) &= y + a_{20}x^2 + P_1(x, y)\,, \\ b(x, y) &= b_{20}x^2 + P_2(x, y)\,, \\ c(x, y) &= -y + c_{20}x^2 + P_3(x, y)\,, \end{aligned} \tag{1}$$

where $T_2(P_i, (0, 0)) = 0$ with $i = 1, 2, 3$, and where we let $T_2(P_i, (0, 0))$ denote the order 2 Taylor polynomial of the map P_i at the origin.

Indeed, if

$$\begin{aligned} T_2((a, b, c), (0, 0)) = (1, 0, -1)y + (a_{20}, b_{20}, c_{20})x^2 \\ (a_{11}, b_{11}, c_{11})xy + (a_{02}, b_{02}, c_{02})y^2\,, \end{aligned}$$

we set

$$(x, y) = (s, t) + (\alpha_{20}, \beta_{20})s^2 + (\alpha_{11}, \beta_{11})st + (\alpha_{02}, \beta_{02})t^2\,,$$

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with

$$\begin{aligned} a_{11} + 3\beta_{11} &= 0\,, \\ a_{02} + 5\beta_{02} &= 0\,, \\ b_{11} - \alpha_{11} + 2\beta_{20} &= 0\,, \\ b_{02} - 2\alpha_{02} + \beta_{11} &= 0\,, \\ c_{11} - 4\alpha_{20} - \beta_{11} &= 0\,, \\ c_{02} - 2\alpha_{11} - \beta_{02} &= 0\,, \end{aligned}$$

and we obtain $(s,t)^*(\omega)$ in the desired form.

Thus we suppose ω as in (1).

In order to study the configuration of ω around the origin, we consider the blowing-up

$$(x,y) = (u, uv)\,.$$

Then $(u,v)^*(\omega) = u\omega_1$ where $g_{\omega_1} = (u^2A_1, 2uB_1, C_1)$, with

$$\begin{aligned} &A_1(u,v) = v + a_{20}u + u^2N_1(u,v)\,, \\ &B_1(u,v) = b_{20}u + 2a_{20}uv + 2v^2 + u^2(N_2(u,v) + vN_1(u,v))\,, \\ &C_1(u,v) = c_{20}u - v + 2b_{20})uv + a_{20}uv^2 + v^3 + \\ &\qquad u^2(N_3(u,v) + 2vN_2(u,v) + v^2N_1(u,v))\,, \end{aligned}$$

and where $P_i(u,uv) = u^3N_i(u,v)$, with $i = 1,2,3$.

Thus the singular points of ω_1 over the line $u = 0$ are the origin and the points $(0,1)$ and $(0,-1)$. Since $h_{\omega_1} = u^2H$, with $H(0,v) = 4v^2$, and $B_1(0,v) = 2v^2$, we conclude that the points $(0,1)$ and $(0,-1)$ are regular points for the vector field

$$Y_1(\omega_1) = (uA_1, -B_1 - H^{\frac{1}{2}})\,,$$

and that they are singular points for

$$Y_2(\omega_1) = (uA_1, -B_1 + H^{\frac{1}{2}})$$

as well as hyperbolic saddles for

$$Z_2(\omega_1) = (u(B_1 + H^{\frac{1}{2}}), -C_1)\,.$$

To study the origin, we consider the blowing-up

$$(x,y) = (u, u^2v)\,.$$

Then $(u,v)^*(\omega) = u^2\omega_2$ where $g_{\omega_2} = (u^4 A_2, 2u^2 B_2, C_2)$, with

$$\begin{aligned}
A_2(u,v) &= v + a_{20} + uM_1(u,v)\,,\\
B_2(u,v) &= b_{20} + 2a_{20})uv + 2uv^2 + u(M_2(u,v) + 2uvM_1(u,v))\,,\\
C_2(u,v) &= c_{20} - v + 4b_{20})uv + 4a_{20}u^2v^2 + 4u^2v^3 +\\
&\qquad u(M_3(u,v) + 4uvM_2(u,v) + 4u^2v^2M_1(u,v))\,,
\end{aligned}$$

and where $P_i(u, u^2v) = u^3 M_i(u,v)$, with $i = 1,2,3$.

The unique singular point of ω_2 over the line $u = 0$ is the point $(0, c_{20})$. Since $h_{\omega_2} = u^4 H$, with $H(0,c_{20}) = b_{20}^2$, and $B(0,c_{20}) = b_{20}$, we conclude that this point is a regular point for

$$Y_1(\omega_2) = (u^2 A_2, -B_2 - H^{\frac{1}{2}})\,,$$

and that it is a singular point for

$$Y_2(\omega_2) = (u^2 A_2, -B_2 + H^{\frac{1}{2}})$$

as well as a saddle-node for

$$Z_2(\omega_2) = (u^2(B_2 + H^{\frac{1}{2}}), -C_2)\,.$$

To complete our analysis we consider the blowing-up

$$(x,y) = (uv, uv^2)\,.$$

Then $(u,v)^*(\omega) = uv^2\omega_3$ where $g_{\omega_3} = (u^2 A_3, 2uvB_3, v^2 C_3)$, with

$$\begin{aligned}
A_3(u,v) &= -1 + c_{20}u + 4b_{20}uv + 4v^2 + 4a_{20}uv^2 +\\
&\qquad u^2v(L_3(u,v) + 4vL_2(u,v) + 4v^2L_1(u,v))\,,\\
B_3(u,v) &= -1 + c_{20}u + 3b_{20}uv + 2v^2 + 2a_{20}uv^2 +\\
&\qquad u^2v(L_3(u,v) + 3vL_2(u,v) + 2v^2L_1(u,v))\,,\\
C_3(u,v) &= -1 + c_{20}u + 2b_{20}uv + v^2 + a_{20}uv^2 +\\
&\qquad u^2v(L_3(u,v) + 2vL_2(u,v) + v^2L_1(u,v))\,,
\end{aligned}$$

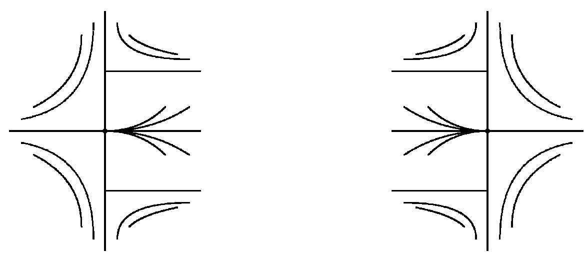
and where $P_i(uv, uv^2) = u^3v^3 L_i(u,v)$, with $i = 1,2,3$.

Since $h_{\omega_3} = u^2v^4H$, with $H(0,0) = 1$, and

$$Y_i(\omega_3) = (u^2 A_3, uv(-B_3 + (-1)^i vH^{\frac{1}{2}})\,,$$

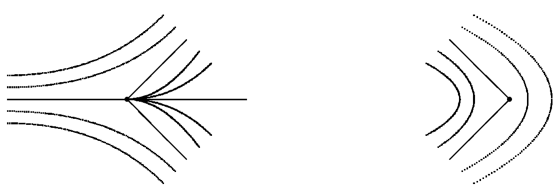
we conclude that the origin is a saddle singular point for Y_i, with $i = 1,2$.

In the case $b_{20} > 0$, the configuration of ω_1 (resp. ω) around the line $u = 0$ (resp. the origin) is shown in Figure 4 (resp. Figure 5). In the case $b_{20} < 0$ the configuration



$f_1(\omega_1)$ $f_2(\omega_1)$

Figure 4



$f_1(\omega)$ $f_2(\omega)$

Figure 5

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of ω_1 (resp. ω) around the line $u = 0$ (resp. the origin) is shown in Figure 4 (resp. Figure 5) with the foliations interchanged. The proof of the Proposition is now complete. ▮

PROPOSITION 3.2. *Let $\omega \in \mathcal{F}(M)$, and let p be a $D_{23}-$singular point of ω. Then there exist a local chart $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$, neighborhoods $\mathcal{N}$ of ω in $\mathcal{F}^1(M)$ and $U_0 \subset U$ of p in M, and a smooth map $p : \mathcal{N} \to U_0$ such that for every $\tilde{\omega} \in \mathcal{N}$, the following two properties are satisfied.*

a)$\det D(c_{\tilde{\omega}}, 2b_{\tilde{\omega}})(p(\tilde{\omega})) = 0$*;*

b)$c_{\tilde{\omega}}(p(\tilde{\omega}) = 0$

where $a_{\tilde{\omega}}, b_{\tilde{\omega}}$ and $c_{\tilde{\omega}}$ are the components of of the map $g_{\tilde{\omega}}$ asociated to $\tilde{\omega}$ in the chart (x,y). (See Section 2.)

Proof. Consider a local chart $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ as in (1) and the map $S : \tilde{\mathcal{F}}(M) \times (x,y)(U) \to \mathbb{R}^2$ defined by

$$S(\tilde{\omega}, q) = (c_{\tilde{\omega}}(q), \det D(2b_{\tilde{\omega}}, c_{\tilde{\omega}})(q))\,.$$

Thus S is smooth, $S(\omega,(0,0)) = (0,0)$, and

$$D_2 S(\omega,(0,0)) = \begin{pmatrix} 0 & -1 \\ -4b_{20} & 0 \end{pmatrix}.$$

According to the Implicit Function Theorem, there exist neighborhoods $\tilde{\mathcal{N}}$ of ω in $\tilde{\mathcal{F}}^1(M)$ and V of the origin in $(x,y)(U)$, and a smooth map $s : \tilde{\mathcal{N}} \to V$ such that $s(\omega) = (0,0)$ and $S(\tilde{\omega}, s(\tilde{\omega})) = (0,0)$ for all $\tilde{\omega} \in \tilde{\mathcal{N}}$. The neighborhoods $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{F}(M)$ and $U_0 = (x,y)^{-1}(V)$, as well as the map $p : \mathcal{N} \to U_0$ defined by $p(\tilde{\omega}) = (x,y)^{-1}(s(\tilde{\omega}))$ satisfy the Proposition. ▮

PROPOSITION 3.3. *Let $\omega \in \mathcal{F}(M)$, and let p be a $D_{23}-$singular point of ω. Then there exist neighborhoods $\mathcal{N}$ of ω in $\mathcal{F}^1(M)$ and U of p in M, and a smooth map $f : \mathcal{N} \to \mathbb{R}$ such that for every $\tilde{\omega} \in \mathcal{N}$, the following three properties are satified.*

a)$f(\tilde{\omega}) = 0$ *if and only if* $\tilde{\omega}$ *has a unique singular point in U, which is a $D_{23}-$singular point.*

b)$f(\tilde{\omega}) < 0$ *if and only if* $\tilde{\omega}$ *has only two singular points in U, one is a D_3-singular point and the other a D_2-singular point.*

c)$f(\tilde{\omega}) > 0$ *if and only if* $\tilde{\omega}$ *has no singular points in U.*

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Proof. Let $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ be a local chart such that if $(x,y)^*(\omega) = a(x,y)dy^2 + 2b(x,y)dxdy + c(x,y)dx^2$, then $\dfrac{\partial^2 b}{\partial x^2}(0,0) \neq 0$ and the matrix of ω at p is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Without loss of generality, assume that $b_{20} = \dfrac{\partial^2 b}{\partial x^2}(0,0) > 0$. Consider a neighborhood $U_0 \subset (x,y)(U)$ of the origin and a square $R_\delta = I_\delta \times I_\delta$ where $I_\delta = [-\delta,\delta]$, with $\delta > 0$ such that $R_\delta \subset U_0$. Taking δ suficiently small if necessary, we asuume the following three properties:

1) $\Delta(x,y) \neq 0$ in R_δ, where $\Delta(x,y)$ is the discriminant of the homogeneous degree three polynomial $Da_{(x,y)}(u,v)v^2 + 2Db_{(x,y)}(u,v)uv + Dc_{(x,y)}(u,v)u^2$.

2) There exists a smooth map $h = h(\omega) : I_\delta \to \mathbb{R}$ such that $c(x,y) = -(y-h(x))M_3(x,y)$, with $M_3(0,0) = 1$.

3) The curves $y = h(x)$ and $H_2(x,y) = 0$, where $H_2(x,y)$ is the determinant of the Jacobian matrix of the map $(2b,c)$ at (x,y), have the point $(x,y)(p(\omega)) = (0,0)$ as the unique common point in R_δ; furthermore, the intersection is transversal.

Let $\mathcal{N}$ be a small neighborhood of ω in $\mathcal{F}^{\,1}(M)$ such that for every $\tilde\omega \in \mathcal{N}$, with $(x,y)^*(\tilde\omega) = a_{\tilde\omega}(x,y)dy^2 + 2b_{\tilde\omega}(x,y)dxdy + c_{\tilde\omega}(x,y)dx^2$, the properties 1), 2) and 3) hold. Observe that $p(\tilde\omega)$ is the point of the curve $y = h(\tilde\omega)(x)$ for which $H_2(\tilde\omega)(x,y) = 0$.

We next show that the map $f : \mathcal{N} \to \mathbb{R}$ given by $f(\tilde\omega) = b_{\tilde\omega}((x,y)(p(\tilde\omega))$ satisfies our Proposition.

In effect, the map f is smooth. We set $m(x) = b_{\tilde\omega}(x, h(\tilde\omega)(x))$.

Then

$$m'(x) = (\frac{\partial c_{\tilde\omega}}{\partial y}(x, h(\tilde\omega)(x)))^{-1} H_2(\tilde\omega)(x, h(\tilde\omega)(x))\,.$$

Since b_{20} is positive, the map $H_2(\tilde\omega)(x,h(\tilde\omega)(x))$ is positive (resp. negative) for $-\delta < x < 0$ (resp. $0 < x < \delta$). This implies that $m(x)$ decreases strictly in $]-\delta, x(\tilde\omega)[$ and increases strictly in $]x(\tilde\omega), \delta[$. Assertions a), b) and c) now follow from Proposition 2.1. ∎

Let $S_1(b_2)$ be the set consisting of $\omega \in \cap_{i=2}^4 S_i(M)$ such that all of the singular points of ω are s-singularities except for one which is a $D_{23}-$singular point. The following will be proved in Section 4.

Theorem 3.1. *Let $\omega_0 \in S_1(b_2)$, and let $p_0 \in Sing(\omega_0)$ be the corresponding D_{23}-singular point. Consider the neighborhoods $\mathcal{N}$ of ω_0 in $\mathcal{F}^{\,1}(M)$ and V of p_0 in M, and the $C^\infty-$map $f : \mathcal{N} \to \mathbb{R}$ of Proposition 3.3. If $\mathcal{N}$ is sufficiently small, then the following three properties are satisfied.*

1. There exist a neighborhood $\tilde{\mathcal{N}}$ of ω_0 in $\tilde{\mathcal{F}}^{\,l}(M)$ and a $C^\infty-$map $F : \tilde{\mathcal{N}} \to \mathbb{R}$ such that 0 is a regular value of F, and F restricted to $\mathcal{N}$ is f.

2. We have $f^{-1}(0) = S_1(b_2) \cap \mathcal{N}$.
3. Every $\omega \in S_1(b_2) \cap \mathcal{N}$ *is equivalent to* ω_0.

Let $\xi \in \tilde{\Phi}_2(M)$, with $\xi(b_0) \in S_1(b_2)$, and let p_0 be the corresponding $D_{23}-$ singular point. In Theorem 3.1, set $\omega_0 = \xi(b_0)$. For $\delta > 0$ small, we can define a $C^\infty-$map $f :]b_0 - \delta, b_0 + \delta[\to \mathbb{R}$ by

$$f(t) = f(\xi(t)) .$$

Then we define $\mathcal{M}_2$ as the set consisting of the $\xi \in \tilde{\Phi}_2(M)$ such that $\xi(b_0) \in S_1(b_2)$ and $f'(b_0) \neq 0$. Our main theorem can be stated as follows.

THEOREM 3.2. *The set* $\mathcal{M}_2$ *is open and every* $\xi \in \mathcal{M}_2$ *is stable. Moreover any* $\xi \in \tilde{\Phi}_2(M)$ *can be arbitrarily approximated by either a family in* $\mathcal{M}_1 \cup \mathcal{M}_2$ *or a family outside* $\tilde{\Phi}_2(M)$.

Section 5 is devoted to proving Theorem 3.2.

4. PROOF OF THEOREM 3.1

This section is devoted to the proof of Theorem 3.1 for which we first prove the next two results. We will use standard arguments on 1–parameter families. Consider the Banach space $\tilde{\mathcal{F}}^1(M)$.

PROPOSITION 4.1. *Let* $\omega \in S_1(b_2)$. *There exist a neighborhood* $\tilde{\mathcal{N}}$ *of* ω *in* $\tilde{\mathcal{F}}^1(M)$ *and a* $C^\infty-$*map* $F : \tilde{\mathcal{N}} \to \mathbb{R}$ *such that 0 is a regular value of* F *and*

$$F^{-1}(0) \cap \mathcal{F}(M) = S_1(b_2) \cap \mathcal{N} ,$$

where $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{F}(M)$.

Proof. Let p_0 be the $D_{23}-$singular point of ω. Let $(x, y) : (U, p_0) \to (\mathbb{R}^2, (0, 0))$ be a chart where U is a neighborhood of p_0 in M such that $Sing(\omega) \cap U = \{p_0\}$. If $(x, y)^*(\omega) = a(x, y)dy^2 + 2b(x, y)dxdy + c(x, y)dx^2$, we may suppose that

$$D(a, b, c)(0, 0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$$

and that

$$\frac{\partial^2 b}{\partial x^2}(0, 0) \neq 0 .$$

According to Proposition 3.2 and its proof, there exist a neighborhood $\tilde{\mathcal{N}} \subset \tilde{\mathcal{F}}^1(M)$ of ω and a smooth map $p : \tilde{\mathcal{N}} \to U$ such that for every $\tilde{\omega} \in \tilde{\mathcal{N}}$, the curves $\det D(b_{\tilde{\omega}}, c_{\tilde{\omega}}) = 0$

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and $c_{\tilde{\omega}} = 0$ are transversal over $(x,y)(U)$, and they meet at a single point $(x,y)(p(\tilde{\omega}))$. Moreover, if $\alpha \in \mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{F}(M)$ and $b_{\tilde{\omega}}((x,y)(p(\tilde{\omega}))) = 0$, then $Sing(\alpha) \cap U = \{p(\tilde{\omega})\}$. Let $F : \tilde{\mathcal{N}} \to \mathbb{R}$ be the map given by $F(\tilde{\omega}) = b_{\tilde{\omega}}((x,y)(p(\tilde{\omega})))$. Therefore, F is smooth. To show that 0 is a regular value of F, it suffices to prove that $\frac{\partial F}{\partial W}(\omega) = B((x,y)(p(\omega)))$, for all $W \in T_\omega \tilde{\mathcal{F}}(M) = \tilde{\mathcal{F}}(M)$ such that $(x,y)^*(W) = A(x,y)dy^2 + 2B(x,y)dxdy + C(x,y)dx^2$. Indeed,

$$\begin{aligned}
\frac{\partial F}{\partial W}(\omega) &= \lim_{s\to 0} \frac{F(\omega + sW) - F(\omega)}{s} \\
&= \lim_{s\to 0} \frac{(b+sB)((x,y)(p(\omega+sW)))}{s} \\
&= \lim_{s\to 0} \frac{b((x,y)(p(\omega+sW)))}{s} + B((x,y)(p(\omega))) \\
&= B((x,y)(p(\omega))),
\end{aligned}$$

with the last equality by definition of $p(\omega + sW)$.

The fact that $F^{-1}(0) \cap \mathcal{F}(M) = S_1(b_2) \cap \mathcal{N}$ for $\mathcal{N} = \tilde{\mathcal{N}} \cap \mathcal{F}(M)$ sufficiently small follows from Proposition 3.3, which completes the proof. ▮

Proposition 4.2. *The set $S_1(b_2)$ is open in $\mathcal{F}^{\,1}(M) - S_1(M)$.*

Proof. The proof follows directly from Proposition 3.3. ▮

Proof of Theorem 3.1. To complete the proof of Theorem 3.1, it reamins to prove that every $\omega_0 \in S_1(b_2)$ is structurally stable relative to $\mathcal{F}(M) - S_1(M)$. For this, it suffices to show that any $\omega \in S_1(b_2)$ which is C^1-close to ω_0 has the same decomposition into canonical regions as ω_0 (compare [25, pp. 210–214], [18]). For an outline about canonical regions, compare [25], [18, pp. 72–83], [22, pp. 642–643].

Let $\omega_0 \in S_1(b_2)$, and let p_0 be its corresponding $D_{23}-$singular point. We assume that p_0 has three $f_1(\omega_0)-$singular separatrices, say σ_0^1, σ_1^1 and σ_2^1, and that σ_1^1 and σ_2^1 bound the parabolic sector. Therefore, p_0 has only two $f_2(\omega_0)-$singular separatrices, say σ_1^2 and σ_2^2, and no parabolic sector. (See Figure 3.) The $\omega - f_k$-limit of σ_i^k obtained by orienting the separatrices in such a way that p_0 is the $\alpha - f_k$-limit of σ_i^k will be called A_i^k.

Let V^1 be the union of the oriented $f_1(\omega_0)-$leaves which converge to the point p_0, except for σ_0^1 if $p_0 \neq A_0^1$. Then V^1 is the union of the closure of parallel $f_1(\omega_0)-$canonical regions, say $\tau_0^1, \cdots, \tau_m^1$, with σ_1^1 (resp. σ_2^1) at the boundary of τ_0^1 (resp. of τ_m^1).

Let τ_{m+1}^1 be the parallel $f_1(\omega_0)-$canonical region consisting of oriented leaves going from A_1^1 to A_0^1 and bounded by σ_0^1 and σ_1^1. Let τ_{m+2}^1 be the parallel $f_1(\omega_0)$–canonical region consisting of oriented leaves going from A_2^1 to A_0^1 and bounded by σ_0^1 and σ_2^1. Let U^1 be the union of the closures of the τ_{m+1}^1 and τ_{m+2}^1 above.

Concerning the foliation $f_2(\omega_0)$, let τ_1^2 and τ_2^2 be the parallel canonical regions consisting of oriented leaves going from A_1^2 to A_2^2 which have $\sigma_1^2 \cup \sigma_2^2$ as common boundary.

If $\mathcal{N}$ is a neighborhood of ω_0 sufficiently small, then such a decomposition is the same for every $\omega \in S_1(b_2) \cap \mathcal{N}$, and the conclusion follows. ▮

5. PROOF OF THEOREM 3.2

The openness of $\mathcal{M}_2$ in $\tilde{\Phi}_2(M)$ follows from previous results and standard arguments on 1–parameter families. Subsection 5.1 contains the proof of the stability of the 1–parameter families of $\mathcal{M}_2$. Subsection 5.2 is devoted to the proof of the genericity of these families.

5.1. Stability

Let $\xi \in \mathcal{M}_2$, and let p_0 be the corresponding $D_{23}-$singular point of $\xi(b_0)$. As in the proof of Theorem 3.1, we assume that p_0 has three $f_1(\xi(b_0))-$singular separatrices, say σ_0^1, σ_1^1 and σ_2^1, and that σ_1^1 and σ_2^1 bound the parabolic sector. Therefore, p_0 has only two $f_2(\xi(b_0))-$singular separatrices, say σ_1^2 and σ_2^2, and no parabolic sector. Againt, the $\omega - f_k$-limit of σ_i^k obtained by orienting the separatrices in such a way that p_0 is the $\alpha - f_k$-limit of σ_i^k will be called A_i^k.

Let V^1 be the union of the oriented $f_1(\xi(b_0))-$leaves which converge to the point p_0, except for σ_0^1 if $p_0 \neq A_0^1$. Then V^1 is the union of the closure of parallel $f_1(\xi(b_0))-$canonical regions, say $\tau_0^1, \cdots, \tau_m^1$, with σ_1^1 (resp. σ_2^1) at the boundary of τ_0^1 (resp. of τ_m^1).

Let τ_{m+1}^1 be the parallel $f_1(\xi(b_0))-$canonical region consisting of oriented leaves going from A_1^1 to A_0^1 and bounded by σ_0^1 and σ_1^1. Let τ_{m+2}^1 be the parallel $f_1(\xi(b_0))$–canonical region consisting of oriented leaves going from A_2^1 to A_0^1 and bounded by σ_0^1 and σ_2^1. Let U^1 be the union of the closures of the τ_{m+1}^1 and τ_{m+2}^1 above.

Concerning the foliation $f_2(\xi(b_0))$, let τ_1^2 and τ_2^2 be the parallel canonical regions consisting of oriented leaves going from A_1^2 to A_2^2 which have $\sigma_1^2 \cup \sigma_2^2$ as common boundary.

Let $\mathcal{N}$ be a neighborhood of $\xi(b_0)$ as in the proof of Theorem 3.1, and let $\delta > 0$ be such that $\xi(t) \in \mathcal{N}$ for all $t \in]b_0 - \delta, b_0 + \delta[$. Assume also that $f(t) = f(\xi(t)) \neq 0$ for all $t \in]b_0, b_0 + \delta[$. We use the following notation.

- We denote the corresponding $C^\infty-$continuation of A_i^k as critical element of $f_i(\xi(t))$ by $A_i^k(t)$.
- If $f(t) = f(\xi(t)) < 0$, there are two singular points of $\xi(t)$ which collapse in p_0 for $t = b_0$. One is a D_3-singular point, denoted $p_3(t)$, and the other is a D_2-singular point, denoted $p_2(t)$.
- Also, if $f(t) = f(\xi(t)) < 0$, we let $\sigma_i^k(t, 3)$ (resp. $\sigma_i^k(t, 2)$) denote the $f_k(\xi(t))$–singular separatrix which goes from $p_3(t)$ (resp. $p_2(t)$) to $A_i^k(t)$.

We now prove the stability in the two following cases: $p_0 \neq A_i^1 \neq A_j^1$ for $i \neq j$, and $p_0 = A_0^1$ and $A_i^1 \neq A_j^1$ for $i \neq j$. In any other case, both the situation and the proof are similar to those in the preceding cases.

Case A) Assume that $p_0 \neq A_i^1 \neq A_j^1$ for $i \neq j$. We orient the leaves of the canonical regions τ_j^1, with $j = 0, 1, \cdots, m, m+1, m+2$ in such a way that the $w - f_1(\xi(b_0))-$limit of the leaves of τ_j^1 is the point p_0 for $j = 0, 1, \cdots, m$, while it is A_0^1 for $j = m+1, m+2$.

Consider $t \in]b_0, b_0 + \delta[$.

When $f(t) < 0$, the canonical regions τ_{m+1}^1 and τ_{m+2}^1 (resp. τ_j^1, with $j = 0, 1, \cdots, m$) have well–defined corresponding $C^\infty-$continuations $\tau_{m+1}^1(t)$ and $\tau_{m+2}^1(t)$ (resp. $\tau_j^1(t)$, with $j = 0, 1, \cdots, m$) as $f_1(\xi(t))-$canonical regions, though now with the separatrices of $p_3(t)$

(resp. $p_2(t)$, for $j = 0$ and $j = m$) on its boundary. There also appears a new parallel $f_1(\xi(t))-$canonical region $\tau^1_{m+3}(t)$ consisting of oriented leaves going from $A^1_1(t)$ to $A^1_2(t)$ and bounded by $f_1(\xi(t))-$singular separatrices of $p_3(t)$ and $p_2(t)$. Concerning the other foliation, the $f_2(\xi(b_0))-$canonical regions τ^2_1 and τ^2_2 have well–defined corresponding $C^\infty-$continuations $\tau^2_1(t)$ and $\tau^2_2(t)$ as $f_2(\xi(t))-$canonical regions. There now appear two new $f_2(\xi(t))-$canonical regions: one, say $\tau^2_3(t)$ and $\tau^2_4(t)$. The first consists of oriented leaves going from $A^2_1(t)$ to $p_2(t)$, which is bounded by the $f_2(\xi(t))-$singular separatrices $\sigma^2_1(t,2)$ and $\sigma^2_1(t,3)$. The other consists of oriented leaves going from $p-2(t)$ to $A^2_2(t)$, which is bounded by the $f_2(\xi(t))-$singular separatrices $\sigma^2_2(t,2)$ and $\sigma^2_2(t,3)$.

When $f(t) > 0$, the canonical regions τ^1_{m+1} and τ^1_0 (resp. τ^1_{m+2} and τ^1_m) generate a unique $f_1(\xi(t)-$canonical region $\tau^1_0(t)$ (resp. $\tau^1_m(t)$) consisting of oriented leaves going from $A^1_0(t)$ to $A^1_1(t)$ (resp. from $A^1_0(t)$ to $A^1_2(t)$). The remaining canonical regions τ^1_j, with $j = 1, \cdots, m-1$, also have well–defined corresponding $C^\infty-$continuations $\tau^1_j(t)$, with $j = 1, \cdots, m-1$, though now the oriented leaves have $A^1_0(t)$ as their $w - f_1(\xi(t))-$limits. Concerning the other foliation, the $f_2(\xi(b_0))-$canonical regions τ^2_1 and τ^2_2 generate a unique $f_2(\xi(t))-$canonical region, denoted $\tau^2_{1,2}$, which consists of oriented leaves going from $A^2_2(t)$ to $A^2_1(t)$.

Case B) Assume that $p_0 = A^1_0$ and $A^1_i \neq A^1_j$ for $i \neq j$. In this case, the leaves of the $f_1(\xi(b_0))-$canonical regions τ^1_{m+1} (resp. τ^1_{m+2}) go from A^1_1 (resp. A^1_2) to p_0. Since in Case A), when $f(t) < 0$ the canonical regions τ^1_{m+1} and τ^1_{m+2} (resp. τ^1_j, with $j = 0, 1, \cdots, m$) have well–defined corresponding $C^\infty-$continuation $\tau^1_{m+1}(t)$ and $\tau^1_{m+2}(t)$ (resp. $\tau^1_j(t)$, with $j = 0, 1, \cdots, m$) as $f_1(\xi(t))-$canonical regions, though now with the separatrices of $p_3(t)$ (resp. $p_2(t)$, for $j = 0$ and $j = m$) on their boundary. In this case there also appears a new parallel $f_1(\xi(t))-$canonical region, denoted $\tau^1_{m+3}(t)$, consisting of oriented leaves going from $A^1_1(t)$ to $A^1_2(t)$ and bounded by $f_1(\xi(t))-$singular separatrices of $p_3(t)$ and $p_2(t)$.

When $f(t) > 0$, the canonical regions τ^1_{m+1} and τ^1_0 (resp. τ^1_{m+2} and τ^1_m) generate a unique $f_1(\xi(t)-$canonical region, denoted $\tau^1_0(t)$ (resp. $\tau^1_m(t)$) consisting of oriented leaves going from $A^1_1(t)$ (resp. $A^1_2(t)$) to the unique hyperbolic compact leaf $\gamma^k(t)$ that appears close to the loop σ^1_0 . The remaining canonical regions τ^1_j, with $j = 1, \cdots, m-1$, also have well–defined corresponding $C^\infty-$continuation $\tau^1_j(t)$, with $j = 1, \cdots, m-1$, though now the oriented leaves have $\gamma^k(t)$ as their $w - f_1(\xi(t))-$limits.

The situation for the other foliation is similar to the one in Case A).

Finally, since any $\tilde{\xi}$ sufficiently C^1-close to ξ is in $\mathcal{M}_2$, and since it has a decomposition into canonical regions which is similar to that of ξ, the result follows from standard arguments on 1–parameter families. ▌

5.2. Genericity

Definition 5.1. Let p be a rank–1 singular point of $\omega \in \mathcal{F}(M)$, and let (u, v) be coordinates in the tangent space T_pM. The point p is called a **1–simple singular point** if the separatrix polynomial $Dw_p(u, v)(u, v)$ has only simple roots.

Remark 5. 1. The following are two characterizations of 1–simple singular points.

a) Let p be a rank-1 singular point of $\omega \in \mathcal{F}(M)$. Let $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ be a local chart such that the matrix of ω at p is

$$\begin{pmatrix} 0 & a_2 \\ 0 & b_2 \\ 0 & c_2 \end{pmatrix}.$$

Then p is 1–simple if and only if the following inequalities hold

$$c_2 \neq 0 \quad \text{and} \quad b_2^2 - a_2\, c_2 > 0\,.$$

b) Let p be a singular point of $\omega \in \mathcal{F}(M)$. Then p is 1-simple if and only if there exists a local chart $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ such that the matrix of ω at p is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

We classify the rank–1 singular point in the following.

Lemma 5.1. *Let p be a rank–1 singular point of $\omega \in \mathcal{F}(M)$. There exists a local chart $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ such that the matrix of ω at p is one of the following types*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof. Clearly, there exists a local chart $(x,y) : (U,p) \to (\mathbb{R}^2,(0,0))$ such that the matrix of ω at p is of form

$$\begin{pmatrix} 0 & a_2 \\ 0 & b_2 \\ 0 & c_2 \end{pmatrix},$$

with $(a_2,b_2,c_2) \neq (0,0,0)$ and $b_2^2 - a_2c_2 \geq 0$.

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear isomorphism, and let $(u,v) = A \circ (x,y)$, with $A^{-1}(u,v) = (\alpha u + \beta v, \delta v)$. If $(u,v)^*(\omega) = \tilde{a}(u,v)dv^2 + 2\tilde{b}(u,v)dudv + \tilde{c}(u,v)du^2$, then

$$(*) \quad \begin{cases} \tilde{a}_1 &= 0 \\ \tilde{a}_2 &= \delta(\delta^2 a_2 + \beta\delta b_2 + \beta^2 c_2) \\ \tilde{b}_1 &= 0 \\ \tilde{b}_2 &= \alpha\delta(\delta b_2 + 2\beta c_2) \\ \tilde{c}_1 &= 0 \\ \tilde{c}_2 &= \alpha^2\delta c_2 \end{cases}$$

where $\tilde{d}_1 = \frac{\partial \tilde{d}}{\partial u}(0,0)$ and $\tilde{d}_2 = \frac{\partial \tilde{d}}{\partial v}(0,0)$, for $d = a, b, c$.

If $b_2 = c_2 = 0$, then $\tilde{b}_2 = \tilde{c}_2 = 0$. Hence there exists δ such that $\tilde{a}_2 = 1$, thus obtaining the third matrix. If $c_2 = 0$ and $b_2 \neq 0$, then $\tilde{c}_2 = 0$. Taking $\delta = 1$, $\beta = -a_2 b_2^{-1}$ and $\alpha = b_2^{-1}$, we obtain the second matrix.

Now if $c_2 \neq 0$ taking $\alpha = 1$, $\delta = -c_2^{-1}$ and $\beta = b_2 2^{-1} c_2^{-2}$, we obtain $\tilde{c}_2 = -1$ and $\tilde{b}_2 = 0$. Thus assume $c_2 = -1$ and $b_2 = 0$ in (*). So taking $\beta = 0$ and $\delta = \alpha^{-2}$, we obtain $\tilde{c}_2 = -1$, $\tilde{b}_2 = 0$, and $\tilde{a}_2 = \alpha^{-4} a_2$. If $a_2 = 0$, then we obtain the fourth matrix. If $a_2 \neq 0$, setting $\alpha = a_2^{\frac{1}{4}}$, then we obtain the first matrix. ∎

There are ways of perturbing a 1–parameter family of positive C^∞−quadratic differential forms that maintain both the set of singular points and their rank, which we give in the next Lemma.

Lemma 5.2. *Let $\xi : I \to \mathcal{F}(M)$ be a smooth 1–parameter family of positive C^∞–quadratic differential forms. Let $(x,y) : U \subset M \to \mathbb{R}^2$ be a local chart. Assume that $(x,y)^*(\xi(t)) = a(x,y,t)dy^2 + 2b(x,y,t)dxdy + c(x,y,t)dx^2$, for all $t \in I$. Given an open set V such that $\overline{V} \subset U$, an open interval J such that $\overline{J} \subset int(I)$, and a smooth non–negative map $\phi : \mathbb{R}^3 \to \mathbb{R}$ which vanishes outside $(x,y)(V) \times J$, consider the maps defined as follows:*

$$\begin{aligned}
(a_1, b_1, c_1)(x,y,t) &= (a,b,c)(x,y,t) + \phi(x,y,t)\,(2b, c-a, -2b)(x,y,t)\,,\\
(a_2, b_2, c_2)(x,y,t) &= (a,b,c)(x,y,t) + \phi(x,y,t)\,(0,0,-a)(x,y,t)\,,\\
(a_3, b_3, c_3)(x,y,t) &= (a,b,c)(x,y,t) + \phi(x,y,t)\,(-a,0,0)(x,y,t)\,.
\end{aligned}$$

Then the maps $\xi_i : I \to \mathcal{F}(M)$, with $i = 1,2,3$, given by

$$\xi_i(t)(p) = \begin{cases} \xi(t)(p)\,, & \text{if} \quad p \notin U \\ \theta_i(t)(p)\,, & \text{if} \quad p \in U \end{cases}$$

where for all $t \in I$ and all $p \in U$,

$$\begin{aligned}
\theta_i(t)(p) &= (x,y)_*(a_i((x,y)(p),t)dy^2 + 2b_i((x,y)(p),t)dxdy + \\
&\quad c_i((x,y)(p),t)dx^2)
\end{aligned}$$

are well defined and smooth. Moreover for all $t \in I$, and $i = 1,2,3$, we have that $Sing(\xi_i(t)) = Sing(\xi(t))$, and that p is a singular point of rank-k of $\xi_i(t)$ if and only if p is a singular point of rank-k of $\xi(t)$.

Proof.
Since

$$\begin{aligned}
b_1^2 - a_1 c_1 &= b^2 - ac + \phi^2 (a-c)^2\,,\\
b_2^2 - a_2 c_2 &= b^2 - ac + \phi a^2\,, \quad \text{and}\\
b_3^2 - a_3 c_3 &= b^2 - ac + \phi c^2\,,
\end{aligned}$$

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we have that the maps ξ_i are well defined. These equalities also imply that $Sing(\xi_i(t)) = Sing(\xi(t))$, for all $t \in I$ and $i = 1, 2, 3$.

Finally, let $p \in Sing(\xi(t)) \cap U$ with $(x, y)(p) = q$. Using the notations of Section 2, we have

$$(H_1(\xi_1(t)) + H_2(\xi_1(t)))(q) = (1 + 4\phi(x_0, y_0, t)^2)(H_1(\xi(t)) + H_2(\xi(t)))(q)\,,$$

$$(H_1(\xi_2(t)) + H_2(\xi_2(t)))(q) = (((1 + \phi(x_0, y_0, t))H_1(\xi(t))) + H_2(\xi(t)))(q)\,,$$

$$(H_1(\xi_3(t)) + H_2(\xi_3(t)))(q) = (H_1(\xi(t)) + ((1 + \phi(x_0, y_0, t))H_2(\xi(t))))(q)\,,$$

and the conclusion follows. ▮

The following Remark give some local expressions of a 1–parameter family around a $D_{23}-$singular point.

Remark 5. 2. Let $\xi(t)$ be a 1–parameter family passing through a $D_{23}-$singular point p_0 at $t = 0$.

1) Let $(x, y) : (U, p_0) \to (\mathbb{R}^2, (0, 0))$ be a local chart such that $(x, y)^*(\xi(t)) = a(x, y, t)dy^2$ $+2b(x, y, t)dxdy + c(x, y, t)dx^2$ where

$$\begin{aligned} a(x,y,t) &= y + a_{20}x^2 + P_1(x,y) + t[\alpha_0 + R_1(x,y,t)]\,, \\ b(x,y,t) &= b_2 y + b_{20}x^2 + P_2(x,y) + t[\beta_0 + R_2(x,y,t)]\,, \\ c(x,y,t) &= c_2 y + c_{20}x^2 + P_3(x,y) + t[\gamma_0 + R_3(x,y,t)]\,, \end{aligned}$$

and where $T_2(P_i(0,0)) = 0$ and $T_1(R_i(0,0,0)) = 0$, with $i = 1, 2, 3$. We let $T_2(P_i(0,0))$ (resp. $T_1(R_i(0,0,0))$) denote the order 2 (resp. order 1) Taylor polynomial of the map P_i (resp. R_i) at the origin.

Since p_0 is a $D_{23}-$singular point at $t = 0$, we have that $c_2(c_2 b_{20} - b_2 c_{20}) \neq 0$ and that the map $p(t)$ of Proposition 3.2 has the form

$$p(t) = (q_1(t), t\left[-\frac{\gamma_0}{c_2} + q_2(t)\right])\,,$$

with $(q_1, q_2)(0) = 0$. Moreover, the map $f(t)$ is

$$f(t) = b(p(t), t) = t\left[\beta_0 - \frac{b_2}{c_2}\gamma_0\right] + g(t)\,,$$

with $g(0) = 0$ and

$$f'(0) = \beta_0 - \frac{b_2}{c_2}\gamma_0 = \frac{1}{c_2}\left(\frac{\partial c}{\partial y}\frac{\partial b}{\partial t} - \frac{\partial b}{\partial y}\frac{\partial c}{\partial t}\right)(0,0,0)\,.$$

2) In the case $b_2 = 1 = -c_2$, we may suppose $a_{20}\, c_{20}\, b_{20} \neq 0$. Then the maps a, b, c in a small neighborhood V of the origin became

$$\begin{aligned} a(x,y,t) &= [y + B(x,t) + A(x,t)]M_1(x,y,t)\,, \\ b(x,y,t) &= [y + B(x,t)]M_2(x,y,t)\,, \\ c(x,y,t) &= -[y + B(x,t) + C(x,t)]M_3(x,y,t)\,, \end{aligned}$$

where $(M_1, M_2, M_3)(0,0,0) = (1,1,1)$. In addition,

$$\begin{aligned} A(x,t) &= -(b_{20} - a_{20})[x^2 + 2a_1(t)x + a_2(t)]A_1(x,t)\,, \\ B(x,t) &= b_{20}[x^2 + 2b_1(t)x + b_2(t)]B_1(x,t)\,, \\ C(x,t) &= -(b_{20} + c_{20})[x^2 + 2c_1(t)x + c_2(t)]C_1(x,t)\,, \end{aligned}$$

where $(A_1, B_1, C_1)(0,0) = (1,1,1)$, $a_1(0) = a_2(0) = c_1(0) = c_2(0) = 0$, $a_2'(0) = \beta_0 - \alpha_0$, and $c_2'(0) = \beta_0 + \gamma_0$. Further, since the quadratic differential forms are positive, we necessarily have $A(x,t)C(x,t) \geq 0$.

Then the condition for p_0 to be a $D_{23}-$singular point is $b_{20} + c_{20} \neq 0$. In addition, $f'(0) = \beta_0 + \gamma_0$, and therefore $f'(0) \neq 0$ is equivalent to $c_2'(0) \neq 0$.

Lemma 5.3. *Let $\xi : I \to \mathcal{F}(M)$ be a smooth 1–parameter family of positive $C^\infty-$quadratic differential forms in $\tilde{\Phi}_2(M)$. Let p_0 be an isolated rank–1 singular point of $\xi(b_0)$, where $b_0 = b_0(\xi)$. Then there exists a smooth 1–parameter family $\tilde{\xi} : I \to \mathcal{F}(M)$ in $\tilde{\Phi}_1(M) \cup \tilde{\Phi}_2(M)$ which is arbitrarily close to ξ , and is such that either $\tilde{\xi} \in \mathcal{M}_1$ or $b_0(\tilde{\xi}) = b_0$ and p_0 is a 1-simple singular point of $\tilde{\xi}(b_0)$.*

Proof. Let $\mathcal{V}$ be a neighborhood of ξ. According to Lemma 5.1, there exists a local chart $(x,y) : (U, p_0) \to (\mathbb{R}^2, (0,0))$ such that the matrix of $\xi(b_0)$ at p_0 is either

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let V and W be open neighborhoods of p_0 such that $\overline{W} \subset V \subset \overline{V} \subset U$ and $\mathrm{Sing}(\xi(b_0)) \cap V = \{p_0\}$. Given $\delta > 0$ small, consider the open intervals $J_1 =]b_0 - \delta, b_0 + \delta[$ and $J_2 =]b_0 - 2\delta, b_0 + 2\delta[$, and a smooth map $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\phi^{-1}(1) = \overline{W} \times \bar{J}_1$ and $\phi^{-1}(0) = \mathbb{R}^3 - (V \times J_2)$.

Suppose that $(x,y)^*(\xi(t)) = a(x,y,t)dy^2 + 2b(x,y,t)dxdy + c(x,y,t)dx^2$, and consider $\varepsilon > 0$ small. For the three types of matrices above, the corresponding maps $a_\varepsilon, b_\varepsilon, c_\varepsilon$ are given by

$$\begin{aligned} a_\varepsilon(x,y,t) &= a(x,y,t) + 2\varepsilon\phi(x,y,t)b(x,y,t)\,, \\ b_\varepsilon(x,y,t) &= b(x,y,t) + \varepsilon\phi(x,y,t)(c(x,y,t) - a(x,y,t))\,, \\ c_\varepsilon(x,y,t) &= c(x,y,t) - 2\varepsilon\phi(x,y,t)b(x,y,t)\,, \end{aligned}$$

or

$$\begin{aligned} a_\varepsilon(x,y,t) &= a(x,y,t)\,, \\ b_\varepsilon(x,y,t) &= b(x,y,t)\,, \\ c_\varepsilon(x,y,t) &= c(x,y,t) - \varepsilon\phi(x,y,t)a(x,y,t)\,, \end{aligned}$$

or

$$\begin{aligned} a_\varepsilon(x,y,t) &= a(x,y,t) - \varepsilon\phi(x,y,t)c(x,y,t)\,, \\ b_\varepsilon(x,y,t) &= b(x,y,t)\,, \\ c_\varepsilon(x,y,t) &= c(x,y,t)\,. \end{aligned}$$

Let $\tilde{\xi} : I \to \mathcal{F}(M)$ be given by

$$\tilde{\xi}(t)(p) = \begin{cases} \xi(t)(p)\,, & \text{if} \quad p \notin U \\ \theta(t)(p)\,, & \text{if} \quad p \in U \end{cases} \tag{2}$$

where for all $t \in I$ and all $p \in U$, we have

$$\begin{aligned} \theta(t)(p) \; = \; & (x,y)_*(a_\varepsilon((x,y)(p),t)dy^2 + 2b_\varepsilon((x,y)(p),t)dxdy + \\ & c_\varepsilon((x,y)(p),t)dx^2)\,. \end{aligned}$$

According to Lemma 5.2, for all $t \in I$, we have that $Sing(\tilde{\xi}(t)) = Sing(\xi(t))$ and that q is a rank-k singular point of $\tilde{\xi}(t)$ if and only if q is a rank-k singular point of $\xi(t)$. Next fix $\varepsilon > 0$ small such that $\tilde{\xi} \in \mathcal{V}$. If there exist $t_0 < b_0$ and $q \in \mathrm{Sing}(\tilde{\xi}(t_0))$, which is not an s–singular point of $\tilde{\xi}(t_0)$, then there is an element of $\mathcal{M}_1$ in $\mathcal{V}$ because q is a rank-2 singular point of $\tilde{\xi}(t_0)$. (See [22, Theoerem 2.13].) Otherwise, $b_0(\tilde{\xi}) = b_0$ and p_0 is a 1-simple singular point of $\tilde{\xi}(b_0)$. ∎

Lemma 5.4. *Let $\xi : I \to \mathcal{F}(M)$ be a smooth 1–parameter family of positive $C^\infty-$quadratic differential forms in $\tilde{\Phi}_2(M)$. Let p_0 be an isolated 1–simple singular point of $\xi(b_0)$, where $b_0 = b_0(\xi)$. Then there exists a smooth 1–parameter family $\tilde{\xi} : I \to \mathcal{F}(M)$ in $\tilde{\Phi}_2(M)$ which is arbitrarily close to ξ, with $b_0(\tilde{\xi}) = b_0$, and such that p_0 is a $D_{23}-$singular point of $\tilde{\xi}(b_0)$.*

Proof. To simplify the notation, we suppose $b_0 = 0$. Let $\mathcal{V}$ be a neighborhood of ξ. We consider a local chart $(x,y) : (U,p_0) \to (\mathbb{R}^2,(0,0))$ such that $(x,y)^*(\xi(t)) = a(x,y,t)dy^2 + 2b(x,y,t)dxdy + c(x,y,t)dx^2$, with

$$\begin{aligned} a(x,y,t) &= y + a_{20}x^2 + P_1(x,y) + t\,A(x,y,t)\,, \\ b(x,y,t) &= y + b_{20}x^2 + P_2(x,y) + t\,B(x,y,t)\,, \\ c(x,y,t) &= -y + c_{20}x^2 + P_3(x,y) + t\,C(x,y,t)\,, \end{aligned}$$

where $T_2(P_i,(0,0)) = 0$, with $i = 1,2,3$. We let $T_2(P_i,(0,0))$ denote the order 2 Taylor polynomial of the map P_i at the origin.

Recall that by Remark 3.1, the point p_0 is a $D_{23}-$singular point of $\xi(b_0)$ if and only if $b_{20} + c_{20} \neq 0$.

Let V and W be open neighborhoods of p_0 such that $\overline{W} \subset V \subset \overline{V} \subset U$ and $\mathrm{Sing}(\xi(b_0)) \cap V = \{p_0\}$. Given $\delta > 0$ small, consider the open intervals $J_1 =]-\delta,\delta[$ and $J_2 =]-2\delta, 2\delta[$, as well as a smooth map $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\phi^{-1}(1) = \overline{W} \times \bar{J_1}$ and $\phi^{-1}(0) = \mathbb{R}^3 - (V \times J_2)$. We may further assume that for all $(x,y,t) \in V \times J_2$,

$$\begin{aligned} a(x,y,t) &= [y + A(x,t)]M_1(x,y,t)\,, \\ b(x,y,t) &= [y + B(x,t)]M_2(x,y,t)\,, \\ c(x,y,t) &= -[y + C(x,t)]M_3(x,y,t)\,, \end{aligned}$$

with $(M_1, M_2, M_3)(0,0,0) = (1,1,1)$.
Then

$$b^2 - ac \;=\; y^2(M_2^2 + M_1M_3) + y(2BM_2^2 + (A+C)M_1M_3) + B^2M_2^2 + ACM_1M_3\,.$$

Since $b^2 - ac \geq 0$, the discriminant with respect to y

$$\Delta = (2BM_2^2 + (A+C)M_1M_3)^2 - 4(B^2M_2^2 + ACM_1M_3)(M_2^2 + M_1M_3)\,,$$

must be less than or equal to zero at any point $(x,y,t) \in V \times J_2$.

We may suppose that $a_{20}\, b_{20}\, c_{20} \neq 0$ and that $a_{20} \neq c_{20}$. If any of these two conditions were not verified, for $\varepsilon \neq 0$ small, we consider the maps

$$(\tilde{a}, \tilde{b}, \tilde{c}) = (a,b,c) + \varepsilon x^2 \phi (M_1, M_2, M_3)\,.$$

Set $Y = y + \varepsilon x^2 \phi$. Then

$$\tilde{b}^2 - \tilde{a}\tilde{c} = Y^2(M_2^2 + M_1M_3) + Y(2BM_2^2 + (A+C)M_1M_3) + B^2M_2^2 + ACM_1M_3\,,$$

which has discriminant, with respect to $Y = y + \varepsilon x^2\phi$, equal to the discriminant Δ of $b^2 - ac$, with respect y. Therefore, as in (2), the maps $\tilde{a}, \tilde{b}, \tilde{c}$ define a 1–parameter family $\tilde{\xi}$ of positive $C^\infty-$quadratic differential forms which is arbitrarily close to ξ and verifies the required conditions.

Hence we assume that for our initial maps a, b, c, we have that $a_{20}\, b_{20}\, c_{20} \neq 0$ and that $a_{20} \neq c_{20}$.

If $b_{20} + c_{20} = 0$, for $\varepsilon > 0$ small, we consider the maps

$$\begin{aligned} a_\varepsilon(x,y,t) &= a(x,y,t)\,, \\ b_\varepsilon(x,y,t) &= b(x,y,t)\,, \\ c_\varepsilon(x,y,t) &= c(x,y,t) \;-\; \varepsilon\, \phi(x,y,t)\, a(x,y,t)\,. \end{aligned}$$

By Lemma 5.2, these maps define a smooth 1–parameter family $\tilde{\xi}$ which is arbitrarily close to ξ and such that $b_0(\tilde{\xi}) = b_0$. Moreover, the point p_0 is a $D_{23}-$singular point of $\tilde{\xi}(b_0)$ because

$$\left(\frac{\partial c_\varepsilon}{\partial y}\frac{\partial^2 b_\varepsilon}{\partial x^2} - \frac{\partial b_\varepsilon}{\partial y}\frac{\partial^2 c_\varepsilon}{\partial x^2}\right)(0,0,0) = \varepsilon\,(a_{20} - c_{20}) \neq 0\,,$$

which completes the proof. ∎

Lemma 5.5. *Let $\xi : I \to \mathcal{F}(M)$ be a family in $\tilde{\Phi}_2(M)$. Let p_0 be a $D_{23}-$ singular point of $\xi(b_0)$, where $b_0 = b_0(\xi)$. Then there exists a smooth 1–parameter family $\tilde{\xi} : I \to \mathcal{F}(M)$ in $\tilde{\Phi}_2(M)$ which is arbitrarily close to ξ such that $\tilde{\xi}(\tilde{b}_0) \in S_1(b_2)$, if $b_0(\tilde{\xi}) = \tilde{b}_0$.*

Proof. By Proposition 3.3, there exist a small neighborhood U of p_0 in M, and a small open interval J containing b_0 such that $Sing(\xi(b_0)) \cap U = \{p_0\}$ and that, for all $t \in J$, with $t < b_0$, the following four properties are satisfied:

(a) $Sing(\xi(t)) \cap U$ vanishes or $Sing(\xi(t)) \cap U = \{p_1(t), p_2(t)\}$, where $p_1(t)$ (resp. $p_2(t)$) is a D_2 (resp. D_3) singular point, and $\lim_{t\to b_0^-} p_1(t) = \lim_{t\to b_0^-} p_2(t) = p_0$.

(b) The compact leaves of $\xi(t)$ do not meet U.

(c) There is no singular separatrix of $\xi(t)$ which enters U and afterwards leaves U.

(d) For $k = 1, 2$, there exists $\alpha > 0$ such that if $p(t) \in Sing(\xi(t)) \cap U$ and $q(t)$ is the first point where a singular separatrix of $f_k(\xi(t))$ of the point $p(t)$ intersects ∂U, then any external singular separatrix (that is, a separatrix of a singular point not in U) of $f_k(\xi(\beta))$ does not intersect the closed ball with centre $q(t)$ and radius α, for all $\beta \in J$.

Let $(x, y) : (U, p_0) \to (\mathbb{R}^2, (0, 0))$ be a local chart with

$$(x,y)^*(\xi(t)) = a(x,y,t)dy^2 + 2b(x,y,t)dxdy + c(x,y,t)dx^2\,, \text{ for all } t \in I\,.$$

Let $\varepsilon > 0$. First consider the rectangles $R_k = [-kr, kr] \times [-kr, kr]$, with $k = 1, 2$ and $0 < r < 1$ such that $R_2 \subset (x, y)(U)$, and a smooth map $\phi_1 : \mathbb{R}^2 \to [0,\ 1]$, with $\|\phi_1\|_1 < (2/r)$, such that $\phi_1^{-1}(1) = R_1$ and $\phi_1^{-1}(0) = \mathbb{R}^2 - int(R_2)$.

Next consider $0 < \delta < r$, with $[b_0 - 8\delta, b_0 + 8\delta] \subset J$, and a smooth increasing map $m : I \to I$ such that:

$$\begin{aligned} m(t) &= t\,, \quad \text{for} \quad |t - b_0| \geq \delta\,,\\ m(t) &> t\,, \quad \text{for} \quad |t - b_0| < \delta\,,\\ \|m - 1_I\|_1 &< \frac{\varepsilon\delta}{4M}\,, \end{aligned}$$

where $M = \max\{\|a/R_2 \times J\|_2, \|b/R_2 \times J\|_2, \|c/R_2 \times J\|_2\}$.

And finally, consider a smooth map $\phi_2 : \mathbb{R} \to [0,\ 1]$, with $\|\phi_2\|_1 < \frac{2}{2\delta} = \frac{1}{\delta}$, such that $\phi_2^{-1}(1) = [b_0 - 2\delta, b_0 + 2\delta]$ and $\phi_2^{-1}(0) = \mathbb{R} -]b_0 - 4\delta, b_0 + 4\delta[$.

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Now let

$$\begin{aligned}\tilde{a}(x,y,t) &= a(x,y,\phi(x,y,t)m(t)+(1-\phi(x,y,t))t)\,,\\ \tilde{b}(x,y,t) &= b(x,y,\phi(x,y,t)m(t)+(1-\phi(x,y,t))t)\,,\\ \tilde{c}(x,y,t) &= c(x,y,\phi(x,y,t)m(t)+(1-\phi(x,y,t))t)\,,\end{aligned}$$

where $\phi:\mathbb{R}^3\to[0,\,1]$ is given by $\phi(x,y,t)=\phi_1(x,y)\phi_2(t)$.

Then $\max\{\|\,\tilde{a}-a\,\|_1,\|\,\tilde{b}-b\,\|_1,\|\,\tilde{c}-c\,\|_1\}\;<\;\varepsilon$.

Let $\tilde{\xi}:I\to\mathcal{F}(M)$ be given by

$$\tilde{\xi}(t)(p)=\begin{cases}\xi(t)(p)\,, & \text{if}\quad p\notin U\\ \theta(t)(p)\,, & \text{if}\quad p\in U\end{cases}$$

where for all $t\in I$ and all $p\in U$,

$$\begin{aligned}\theta(t)(p) \;=\; & (x,y)_*(\tilde{a}((x,y)(p),t)dy^2+2\tilde{b}((x,y)(p),t)dxdy+\\ & \tilde{c}((x,y)(p),t)dx^2)\,.\end{aligned}$$

Let $\tilde{b}_0$ be the number in $[b_0-\delta,b_0+\delta]$ such that $m(\tilde{b}_0)=b_0$. Then:

1. We have that $\tilde{b}_0<b_0$.
2. For $t<b_0-\delta$, we have $\tilde{\xi}(t)=\xi(t)\in S(M)$.
3. For $b_0-\delta<t<\tilde{b}_0$ and $(x,y)\in\mathbb{R}^2$, we have

$$t\;\le\;\phi(x,y,t)m(t)+(1-\phi(x,y,t))t\;\le\;m(t)\;\le\;b_0\,.$$

Therefore, by the condition imposed on U and J, we have that $\tilde{\xi}(t)\in S(M)$ for all $t<\tilde{b}_0$, that $\tilde{\xi}(\tilde{b}_0)\in\cap_{i=2}^4 S_i(M)$, and that all singular points of $\tilde{\xi}(\tilde{b}_0)$, except for the point p_0, are s-singular points. The proof is now complete. ▮

Theorem 5.1. *Any $\xi\in\tilde{\Phi}_2(M)$ can be arbitrarily approximated by either a family in $\mathcal{M}_1\cup\mathcal{M}_2$ or a family outside $\tilde{\Phi}_1(M)\cup\tilde{\Phi}_2(M)$.*

Proof. According to the preceding three lemmas, it suffices to consider the case for which ξ is any smooth 1–parameter family in $\tilde{\Phi}_2(M)$, with $\xi(b_0)\in S_1(b_2)$.

Let $\xi:I\to\mathcal{F}(M)$ be one of these families, and let p_0 be the corresponding $D_{23}-$singular point of $\xi(b_0)$. For simplicity, we assume that $b_0=0$. Let U be a neighborhood of p_0 such that $Sing(\xi(0))\cap U=\{p_0\}$. Consider a local chart $(x,y):(U,p_0)\to(\mathbb{R}^2,(0,0))$ such that $(x,y)^*(\xi(t))=a(x,y,t)dy^2+2b(x,y,t)dxdy+c(x,y,t)dx^2$, with

$$\begin{aligned}a(x,y,t) &= y+a_{20}x^2+P_1(x,y)+t\,[\alpha_0+R_1(x,y,t)]\,,\\ b(x,y,t) &= y+b_{20}x^2+P_2(x,y)+t\,[\beta_0+R_2(x,y,t)]\,,\\ c(x,y,t) &= -y+c_{20}x^2+P_3(x,y)+t\,[\gamma_0+R_3(x,y,t)]\,,\end{aligned}$$

where $T_2(P_i(0,0)) = 0$ and $T_1(R_i(0,0,0)) = 0$, with $i = 1,2,3$. We let $T_2(P_i(0,0))$ (resp. $T_1(R_i(0,0,0))$) denote the order 2 (resp. order 1) Taylor polynomial of the map P_i (resp. R_i) at the origin. Since p_0 is a $D_{23}-$singular point at $t = 0$, we have $b_{20} + c_{20} \neq 0$. We may further assume $0 \neq b_{20} \neq a_{20}$.

Let V and W be open neighborhoods of p_0 such that $\overline{W} \subset V \subset \overline{V} \subset U$. Given $\delta > 0$ small, consider the open intervals $J_1 =]-\delta, \delta[$ and $J_2 =]-2\delta, 2\delta[$, and a smooth map $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\phi^{-1}(1) = \overline{W} \times \bar{J}_1$ and $\phi^{-1}(0) = \mathbb{R}^3 - (V \times J_2)$. We may further assume that for all $(x,y,t) \in V \times J_2$,

$$\begin{aligned} a(x,y,t) &= [y + B(x,t) - \lambda A(x,t)] M_1(x,y,t)\,, \\ b(x,y,t) &= [y + B(x,t)] M_2(x,y,t)\,, \\ c(x,y,t) &= -[y + B(x,t) - \mu C(x,t)] M_3(x,y,t)\,, \end{aligned}$$

with the conditions $(M_1, M_2, M_3)(0,0,0) = (1,1,1)$, $\lambda = b_{20} - a_{20}$, $\mu = b_{20} + c_{20}$, and $\lambda\,\mu\, A(x,t)C(x,t) \geq 0$. Then

$$b^2 - ac = [y+B]^2(M_2^2 + M_1 M_3) - [y+B](\lambda A + \mu C) M_1 M_3 + \lambda\mu A C M_1 M_3\,.$$

Since $b^2 - ac \geq 0$, the discriminant with respect to y

$$\Delta = (\lambda A - \mu C)^2 M_1^2 M_3^2 - \lambda\,\mu\, A C M_1 M_3 M_2^2\,,$$

must be less than or equal to zero at any point $(x,y,t) \in V \times J_2$.

In addition, reducing V, if necessary, we may suppose that

$$\begin{aligned} A(x,t) &= [x^2 + 2a_1(t)x + a_2(t)] A_1(x,t) \quad \text{and} \\ C(x,t) &= [x^2 + 2c_1(t)x + c_2(t)] C_1(x,t)\,, \end{aligned}$$

with the conditions $(A_1, B_1, C_1)(0,0) = (1,1,1)$, $a_1(0) = a_2(0) = c_1(0) = c_2(0) = 0$, $\lambda\, a_2'(0) = \beta_0 - \alpha_0$, and $\mu\, c_2'(0) = \beta_0 + \gamma_0$.

Recall that by Remark 5.2, the family $\xi \in \mathcal{M}_2$ if and only if $c_2'(0) \neq 0$.

So we suppose $c_2'(0) = 0$.

We essentially have only two cases:

a) $c_2(t) - c_1(t)^2 < 0$, for either all $t \neq 0$, or all $t > 0$, or all $t < 0$.
b) $c_2(t) - c_1(t)^2 \geq 0$, for all $t \neq 0$ small.

In Case a), we necessarily have $(a_1(t), a_2(t)) = (c_1(t), c_2(t))$, for all $t \neq 0$ small and $\lambda\,\mu > 0$. We set $D(x,t) = x^2 + 2a_1(t)x + a_2(t)$. Then

$$\Delta = D^2[(\lambda A_1 - \mu C_1)^2 M_1^2 M_3^2 - \lambda\mu A_1 C_1 M_1 M_3 M_2^2] \leq 0\,.$$

Setting

$$(\tilde{a}, \tilde{b}, \tilde{c}) = (a,b,c) - \varepsilon\phi t(\lambda A_1 M_1, 0, -\mu C_1 M_3)\,,$$

we have that $\tilde{b}^2 - \tilde{a}\tilde{c} \geq 0$ because the corresponding discriminant is

$$\tilde{\Delta} = (D + \varepsilon t\phi)^2 \Delta \leq 0\,.$$

Now for $|\, t \,|$ small, the corresponding map $\tilde{c}_2$ verifies $\tilde{c}_2(t) = c_2(t) + \varepsilon t$. Therefore, $c_2'(0) \neq 0$ and the corresponding family $\tilde{\xi} \in \mathcal{M}_2$.

In Case b), we necessarily have that $a_2(t) - a_1(t)^2 \geq 0$, for all $t \neq 0$ small, and hence we have no singular points for these t.
We set

$$(\tilde{a}, \tilde{b}, \tilde{c}) = (a, b, c)(x, y, t) + \varepsilon\phi(-M_1, 0, M_3)\,,$$

with ε such that $\varepsilon\lambda > 0$. Then the corresponding discriminant is

$$\tilde{\Delta} = (\lambda A - \mu C)^2 M_1^2 M_3^2 - (\lambda A + \varepsilon\phi)(\mu C + \varepsilon\phi) M_1 M_3 M_2^2 \leq 0$$

and the corresponding family $\tilde{\xi}$ has no singular points for $|\, t \,| < \delta$. Therefore, the family $\tilde{\xi}$ can be arbitrarily approximated by either a family in $\mathcal{M}_1 \cup \mathcal{M}_2$ or a family outside $\tilde{\Phi}_1(M) \cup \tilde{\Phi}_2(M)$, which completes the proof. ▌

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