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Bilipschitz triviality of polynomial maps

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We give in this paper a condition on $\theta : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, in terms of the Newton filtration, for the deformation $f_t = f + t\theta$ to be bilipschitz trivial, where $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ is a non-degenerate polynomial map. May, 2003 ICMC-USP

1. INTRODUCTION

In the sixties S. Lojasiewicz obtained some metric properties of singular sets which indicated a new route to singularity theory. Twenty years later, T. Mostowski consolidated the indicatrizes of S. Lojasiewicz by presenting the following result:

The Mostowski's theorem: The family of all complex algebraic sets of complexity bounded by some number k has a finite number of bilipschitz equivalence classes (see [5]).

In [6], A. Parusinski presents a version of the Mostowiski's theorem for real singular sets; thereafter the study of the singular sets from the metric viewpoint (metric theory of singularities) gained substancial interest.

One can see the importance of Mostowski-Parusinski theorem by considering the following Whitney example:

$$X_t = \{(x, y) \in \mathbb{R}^2 \ / \ xy(x + y)(x - ty) = 0\}, \ 0 < t < 1.$$

Any two members X_{t_1}, X_{t_2} of this family, with $t_1 \neq t_2$, are not differentiable equivalents. However, they are bilipschitz equivalents.

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In [4] Henry and Parusinski gave the following example:

$$f_t(x,y) = x^3 + y^6 - 3t^2xy^4$$

which satisfies the following: for any $t \neq s \in (0, \frac{1}{2})$ there is no bilipschitz map $\varphi : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0$ such that $f_t = f_s \circ \varphi$. This turns the metric study of maps more interesting.

Motivated by the above result, Fernandes and Ruas [3] obtained estimates to the bilipschitz determinacy of weighted homogeneous analytic function-germs.

In this paper we use the concept of non-degeneracy with respect to some Newton polyhedron. This concept is used to solve localized equations using controlled vector fields and obtain bilipschitz triviality. We obtain estimates for the filtration of a map-germ $\theta : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ in such way that the deformation $f_t = f + t\theta$, of a non-degenerated map-germ $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, is bilipzchitz trivial. This result complements that given in [3].

2. THE NEWTON FILTRATION

In this section we define the Newton polyhedron and the Newton filtration associated to a matrix A of rational numbers (see [11], [10]).

Let $A = (a_i^j), i = 1, ..., n$ and $j = 1, ..., m$ be a matrix of non-negative rational numbers. We denote the rows and the columns of A respectively by:

$$a_i = (a_i^1, ..., a_i^m),\ i = 1, ..., n;$$

$$a^j = (a_1^j, ..., a_n^j),\ j = 1, ..., m.$$

The support, the Newton polyhedron and the Newton boundary of A are denoted respectively by:

$$\begin{aligned} Supp(A) &= \{a^j,\ j = 1, ..., m\}; \\ \Gamma_+(A) &= \text{ Convex hull in } \mathbb{R}^n \text{ of the set } Supp(A) + \mathbb{R}^n_+; \\ \Gamma(A) &= \text{ Union of all compact faces of } \Gamma_+(A). \end{aligned}$$

We call A a *matrix of vertices* if $supp(A) =$ "set of the vertices of $\Gamma_+(A)$".

We say that A is a *Newton matrix* if A is a matrix of vertices and $\Gamma_+(A)$ is a Newton polyhedron in the sence of [2], that is, the intersection of $\Gamma_+(A)$ with each coordinate axis is not empty. If $p \in \mathbb{R}_+$, we denote $p_i^j = pa_i^j$ and $p^j = pa^j$.

Hereafter we fix a Newton matrix A.

Definition 2.1. The control function $\rho(x)$ of the Newton polyhedron $\Gamma_+(A)$ is defined as

$$\rho(x) = \left(\sum_{j=1}^{m} x^{2p^j} \right)^{\frac{1}{2p}} = \left(\sum_{j=1}^{m} x_1^{2p_1^j} x_2^{2p_2^j} \cdots x_n^{2p_n^j} \right)^{\frac{1}{2p}}.$$

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We can choose p large enough such that the numbers $2p_i^j$ are integers and ρ^{2p} is a polynomial.

As A is a Newton matrix, we can consider $A = (a_i^j)$ with $a^j = (0, ..., 0, a_i^j, 0, ..., 0)$ and $a_i^j > 0,\ \forall\ j = 1, ..., n$, that is, the first $n \times n$ block of A is a diagonal matrix. For example for $n = 2$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 1 \end{pmatrix}.$$

Definition 2.2. Let A be a Newton matrix and d a non-negative rational number. The matrix dA is a Newton matrix. We define and denote:

$$\Gamma_+(\rho^d) := \Gamma_+(dA);$$

$$\Gamma(\rho^d) := \Gamma(dA).$$

Definition 2.3. Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be a real analytic function-germ defined by

$$f(x) = \sum_\nu c_\nu x^\nu, \text{ where } \nu = (\nu_1, ..., \nu_n).$$

We say that f is an A-form of degree d if $\nu \in \Gamma(\rho^d)$ for all $c_\nu \neq 0$ in the taylor series of f.

Definition 2.4. Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be a real analytic function-germ defined by

$$f(x) = H_d(x) + \cdots + H_l(x) + \cdots$$

where H_i are A-forms of degree i. We say that 0 is an $\Gamma_+(\rho^d)$-isolated point of f if, for each compact face γ of $\Gamma(\rho^d)$, the equation

$$f_\gamma(x) = 0$$

has not solution in $(\mathbb{R} - \{0\})^n$.

Lemma 2.1 ([10], p. 525). *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be a real analytic function-germ defined by*

$$f(x) = H_d(x) + \cdots H_l(x) + \cdots .$$

If f admits 0 as an $\Gamma_+(\rho^d)$-isolated point, then there exists a positive real number c_2 such that

$$\|f(x)\| \geq c_2 \rho(x)^d$$

in a neighbourhood of the origin in $\mathbb{R}^n$.

Let $\Gamma_+(A)$ be a Newton polyhedron. We denote by $\mathbb{R}^{n^*}$ the dual space of $\mathbb{R}^n$ and for each $a = (a_1, ..., a_n) \in \mathbb{R}^n_+$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^{n^*}_+$ we denote by

$$\langle a, \alpha\rangle = a_1\alpha_1 + \cdots + a_n\alpha_n,$$
$$\ell(\alpha) = \min\{\langle a, \alpha\rangle \ / \ a \in \Gamma_+(A)\},$$
$$\gamma(\alpha) = \{a \in \Gamma_+(A) \ / \ \langle a, \alpha\rangle = \ell(\alpha)\}.$$

The vector α is called a primitive integer vector if α is the vector with minimum length in $C(\alpha) \cap (\mathbb{Z}^n_+ - \{0\})$, where $C(\alpha)$ is the half ray emanating from 0 and passing through α.

Let $\lambda : \mathbb{R}, 0 \to \mathbb{R}, 0$ a real analytic funtion. We can write

$$\lambda(t) = a_k t^k + a_{k+1} t^{k+1} + \cdots, \text{ with } a_k \neq 0.$$

We say that $\lambda(t)$ is equivalent to t^k and denote by $\lambda(t) \sim t^k$.

Lemma 2.2. *Let* $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ *be an analytic function-germ. If supp*$(f) \in \Gamma_+(\rho^d)$ *there exists* $c_1 > 0$ *such that, in a neighbourhood of the origin,*

$$\|f(x)\| \leq c_1 \rho(x)^d,$$

that is, $\frac{f(x)}{\rho(x)^d}$ *is bounded.*

Proof. It is suficient to prove that for all $x^a \in \Gamma_+(\rho^d)$ there exists a real number $c > 0$ such that

$$\rho^d \geq c\|x^a\|$$

in a neighbourhood of the origin. Let us make this by contradiction. Suppose that for all $c > 0$, $\rho^d < c\|x^a\|$. Then, 0 is in the hull of the set $X := \{(x, c) \ / \ \rho(x)^d < c\|x^a\|\}$.

As X is semi-analytic, by the curve selection lemma, there exists an analytic curve $\gamma : (0, \epsilon] \to X$, with $\gamma(0) = 0$, $\gamma(t) = (\lambda_1(t), ..., \lambda_n(t), \lambda_{n+1}(t))$ and

$$\lambda_1(t) \sim t^{\alpha_1}, ..., \lambda_n(t) \sim t^{\alpha_n} \text{ e } \lambda_{n+1}(t) \sim t^{\beta}.$$

Then, $\rho^d(\gamma(t)) < t^\beta\|\lambda(t)^a\|$, with $\lambda(t) = (\lambda_1(t), ..., \lambda_n(t))$, give us

$$\left(\sum_j t^{2p\alpha_1 a_1^j} \cdots t^{2p\alpha_n a_n^j}\right)^{\frac{d}{2p}} < t^\beta . t^{\alpha_1 a_1} \cdots t^{\alpha_n a_n}$$
$$< t^{\alpha_1 a_1} \cdots t^{\alpha_n a_n}$$

therefore

$$\left(\sum_j t^{2p\langle\alpha,a^j\rangle}\right)^{\frac{d}{2p}} < t^{\langle\alpha,a\rangle}$$

and we obtain that

$$\langle\alpha,a\rangle < \inf_j\{\langle\alpha,da^j\rangle\}$$

what it contradicts the fact that $\gamma(\alpha)$ is a face of the polihedron $\Gamma_+(\rho^d)$ and therefore contain some da^j as a vertex. ▮

Definition 2.5. Let $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be a real analytic function-germ and $v^k = (v_1^k, ..., v_n^k)$, $k = 1, ..., r$, the primitive vectors of the compact faces γ (of dimension $n-1$) of $\Gamma(A)$. We define the Newton filtration of f with respect to $\Gamma_+(A)$ by

$$fil(f) := \inf\{\varphi(\alpha) \;/\; \alpha \in Supp(f)\}$$

where

$$\varphi(\alpha) = \min_{k=1}^{r}\left\{\frac{M}{\ell(v^k)}\langle\alpha,v^k\rangle\right\}$$

and $M = \text{l.c.m.}\{\ell(v^k)\}$.

3. BILIPSCHITZ TRIVIALITY

The aim of this section is to give a sufficient condition for the bilipschitz triviality of families of map-germs.

Definition 3.1. Let $\lambda \in \mathbb{R}$ be a positive number. A mapping $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^p$ is called λ-Lipschitz if it satisfies:

$$\|\phi(x) - \phi(y)\| \leq \lambda\|x - y\|, \;\forall x, y \in U.$$

Let $f_t : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ $t \in I$ (an interval in $\mathbb{R}$) be a smooth family of smooth map-germs, that is, there exist a neighbourhood U of 0 in $\mathbb{R}^n$ and a smooth map $F : U \times I \to \mathbb{R}^p$ such that $F(0,t) = 0$ and $f_t(x) = F(x,t)$, $\forall t \in I, \forall x \in U$.

Definition 3.2. We call f_t bilipschitz trivial when there exist $t_0 \in I$, $\lambda > 0$ and $\varphi : U \times I \to U \times I$ of type $\varphi(x,t) = (\phi_t(x), t)$, with $\phi_t(x)$ λ-Lipschitz and its inverse $\frac{1}{\lambda}$-Lipschitz, such that $f_t \circ \phi_t = f_{t_0}$ for all $t \in I$.

We fix a Newton filtration associated to $\Gamma_+(A)$. We define the numbers

$$R = \max_j \max_i\left\{\frac{M}{\ell(v^j)}v_i^j\right\} \text{ and } r = \min_j \min_i\left\{\frac{M}{\ell(v^j)}v_i^j\right\}$$

where $M = \text{l.c.m.}\{\ell(v^j)\}$ and $v^j = (v_1^j, ..., v_n^j)$ are the associated primitive vectors of the compact faces (of dimension $n-1$) of $\Gamma(A)$.

Let $h : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ be a function-germ such that $supp(h) \in \Gamma_+(\rho^d)$, it follows from Lemma 2.2 that $\frac{h(x)}{\rho(x)^d}$ is bounded. In fact, if the filtration of h is sufficiently higher that the filtration of ρ^d, it is possible to prove that $Grad\left(\frac{h}{\rho^d}\right)$ is bounded, hence $\frac{h}{\rho^d}$ is Lipschitz. Therefore we have the following result

PROPOSITION 3.1. *If $fil(h) \geq d.M + R$ then $\frac{h(x)}{\rho(x)^d}$ is Lipschitz.*

Proof. Let $f(x) = \frac{h(x)}{\rho(x)^d}$, then

$$Grad(f(x)) = \frac{1}{\rho(x)^{2d}}(\rho^d.Grad(h) - h.Grad(\rho^d)),$$

$$\begin{aligned} fil(\rho^d.Grad(h) - h.Grad(\rho^d)) &\geq fil(h) + fil(\rho^d) - R \\ &\geq d.M + fil(\rho^d) \\ &= fil(\rho^{2d}). \end{aligned}$$

It follows by Lemma 2.2 that $Grad(f)$ is bounded, therefore f is Lipschitz. ∎

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be an analytic map-germ and let $N^*_{\mathcal{R}}f := \sum_I M_I^{2\alpha_I}$ where M_I denotes a $p \times p$ minor of df, $I = \{i_1, ..., i_p\}$, $\alpha_I = \alpha/s_I$, with $s_I := fil(M_I)$ and $\alpha :=$ l.c.m.$\{s_I\}$.

Let suppose that $N^*_{\mathcal{R}}f = H_D + \cdots + H_L$, where H_i are A-forms of degree i, with 0 an $\Gamma_+(\rho^D)$-isolated point of $N^*_{\mathcal{R}}f$.

It follows by Lemmas 2.2 and 2.1 that

$$N^*_{\mathcal{R}}f \leq \|H_D\| + \cdots + \|H_L\| \leq c_D\rho^D + \cdots + c_L\rho^L \leq (c_D + \cdots + c_L)\rho^D$$

and

$$N^*_{\mathcal{R}}f \geq c\rho^D.$$

Hence, there exist $c_1, c_2 > 0$ such that

$$c_1\rho^D \leq N^*_{\mathcal{R}}f \leq c_2\rho^D.$$

Suppose that $f_t = f + t\theta$ is a deformation of f with $fil(\theta_i) > fil(f_i)$. Then, defining $N^*_{\mathcal{R}}f_t := \sum_I M_{t_I}^{2\alpha_I}$ we have

$$N^*_{\mathcal{R}}f_t = N^*_{\mathcal{R}}f + t\Theta, \text{ with } fil(\Theta) > fil(N^*_{\mathcal{R}}f).$$

Therefore $N^*_{\mathcal{R}}f \leq N^*_{\mathcal{R}}f_t + \|\Theta\|, \forall\, 0 \leq t \leq 1.$

Now, there exists $c_1 > 0$ such that $c_1\rho^D \leq N^*_{\mathcal{R}} f \leq N^*_{\mathcal{R}} f_t + \|\Theta\|$ and since $fil(\Theta) > fil(N^*_{\mathcal{R}} f)$ we obtain $\lim_{x\to 0} \Theta/\rho^D = 0$. Hence

$$c_1'\rho^D \leq N^*_{\mathcal{R}} f_t,$$

Now,

$$N^*_{\mathcal{R}} f_t \leq N^*_{\mathcal{R}} f + \|\Theta\| \leq c_2\rho^D + \|\Theta\| \leq (c_2 + c_3)\rho^D.$$

From this, we obtain the following result.

Lemma 3.1. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a polynomial map-germ. Suppose that $N^*_{\mathcal{R}} f := \sum_I M_I^{2\alpha_I} = H_D + \cdots + H_L$ admits 0 as an $\Gamma_+(\rho^D)$-isolated point. Then, if $f_t = f + t\theta$ is a deformation of f with $fil(\theta_i) > fil(f_i)$, there exist $c_1, c_2 > 0$ such that*

$$c_1\rho^D \leq N^*_{\mathcal{R}} f_t \leq c_2\rho^D.$$

Theorem 3.1. *Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ be a polynomial map-germ. Suppose that $N^*_{\mathcal{R}} f := \sum_I M_I^{2\alpha_I} = H_D + \cdots + H_L$ admits 0 as an $\Gamma_+(\rho^D)$-isolated point. If $f_t = f + t\theta$ is a deformation of f with $fil(\theta_i) \geq fil(f_i) + R - r$, then f_t is bilipschitz trivial.*

Proof. Let be M_{t_I} a $p \times p$ minor of $df_t, I = (i_1, ..., i_p) \subset (1, ..., n)$. Then, there exists a vector field W_I associated to M_{t_I}, such that $\frac{\partial f_t}{\partial t} M_{t_I} = df(W_I)$, where

$$W_I = \sum_{i=1}^{n} w_i \frac{\partial}{\partial x_i}, \text{ with } \begin{cases} w_i = 0, \text{ if } i \notin I \\ w_{i_m} = \sum_{j=1}^{p} N_{ji_m} (\frac{\partial f_t}{\partial t})_j, \text{ if } i_m \in I \end{cases}$$

and N_{ji_m} is the $(p-1) \times (p-1)$ minor cofactor of $\frac{\partial f_j}{\partial x_{i_m}}$ in df. (See [7] for more details).

Let $W_{\mathcal{R}} := \sum_I M_I^{2\alpha_I - 1} W_I$. We have $N^*_{\mathcal{R}} f_t \cdot \frac{\partial f_t}{\partial t} = df_t(W_{\mathcal{R}})$ and

$$\begin{aligned} fil(W_{\mathcal{R}}) &= \min\{fil(M_I^{2\alpha_I - 1}) + fil(W_I)\} \\ &\geq \min\{2\alpha - fil(M_I) + fil(N_{ji_m}) + fil(\theta_j)\} \\ &\geq \min\{2\alpha - fil(M_I) + fil(M_I) - fil(\frac{\partial f_j}{\partial x_{i_m}}) + fil(\theta_j)\} \\ &\geq \min\{2\alpha - (fil(f_j) - r) + fil(\theta_j)\} \\ &\geq 2\alpha + R. \end{aligned}$$

Let $V : \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}, 0$ be the vector field

$$V(x) = \frac{W_{\mathcal{R}}}{N^*_{\mathcal{R}} f_t}.$$

It follows from Lemma 3.1 and Proposition 3.1 that V is Lipschitz.

The equation $\frac{\partial f_t}{\partial t}(x,t) = (df_t)_x(V(x,t))$ implies the bilipschitz triviality of the family f_t in a neighbourhood of $t=0$.

As the same argument holds in a neighbourhood of $t = t_0, \forall t_0 \in [0,1]$, the proof is completed. ▮

4. EXAMPLES

Example 1

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be $f(x,y) = (xy, x^{2b+2} - y^{2b} - x^{2r}y^{2s})$, where $r+s=b$, $br+(b+1)s < (b+1)b$, $r+2s = b+1$ and $r > s$.

Then, the 2×2 minor of df is

$$M = -2((b+1)x^{2b+2} + by^{2b} + (r-s)x^{2r}y^{2s})$$

Let A be the Newton matrix

$$A = \begin{pmatrix} \frac{1}{b} & 0 & \frac{r}{(b+1)b} \\ 0 & \frac{1}{b+1} & \frac{s}{(b+1)b} \end{pmatrix}$$

The control is

$$\rho(x,y) = (x^{2b+2} + y^{2b} + x^{2r}y^{2s})^{\frac{1}{2b(b+1)}}$$

Observe that $M = H_{2b(b+1)}$ where $H_{2b(b+1)} = -2((b+1)x^{2b+2} + by^{2b} + (r-s)x^{2r}y^{2s})$ is an A-form of degree $2b(b+1)$.

We have that $v^1 = (1,1)$, $v^2 = (1,2)$, and $\ell(v_1) = 2b$, $\ell(v_2) = 2b+2$. Therefore $m.m.c\{\ell(v_1), \ell(v_2)\} = 2b(b+1)$ and

$$\begin{aligned} fil(xy) &= \min\{(b+1)\langle(1,1),(1,1)\rangle, b\langle(1,2),(1,1)\rangle\} = 2b+2, \\ fil(y^{2b}) &= \min\{(b+1)\langle(1,1),(0,2b)\rangle, b\langle(1,2),(0,2b)\rangle\} = 2b(b+1), \\ fil(x^{2r}y^{2s}) &= \min\{(b+1)\langle(1,1),(2r,2s)\rangle, b\langle(1,2),(2r,2s)\rangle\} = 2b(b+1), \\ fil(x^{2b+2}) &= \min\{(b+1)\langle(1,1),(2b+2,0)\rangle, b\langle(1,2),(2b+2,0)\rangle\} = 2b(b+1). \end{aligned}$$

Hence, $fil(xy) = 2b+2$, $fil(x^{2b+2} - y^{2b} + x^{2r}y^{2s}) = 2b(b+1)$, and we have $R = 2b$, $r = b$.

Now, let be $\theta_1 = x^{b-1}y$ and $\theta_2 = x^{2b+1}y$. Then,

$$fil(\theta_1) = b(b+1) \geq fil(f_1) + R - r = 3b+2$$

$$fil(\theta_2) = b(2b+3) = fil(f_2) + R - r = b(2b+3)$$

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Therefore, the family $f_t(x,y) = f(x,y) + t(x^{b-1}y, x^{2b+1}y)$ is bilipschitz trivial.

The Briançon-Speder example [1]

Let $f : \mathbb{K}^3, 0 \to \mathbb{K}, 0$ defined by $f(x,y,z) = x^{15} + xy^7 + z^5$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

When $\mathbb{K} = \mathbb{C}$, the family $F(x,y,z,t) = x^{15} + xy^7 + z^5 + ty^6z$ is topological trivial, since the Milnor number $\mu(f_t)$ is constant for all t. Briançon and Speder showed in [1] that the variety $F^{-1}(0)$ in $\mathbb{C}^4$ is not equisingular along the parameter space $0 \times \mathbb{C}$ at 0. A complete description of all equisingular deformations of f is given in [9]. The variety $F^{-1}(0)$, defined by $F(x,y,z,t) = f(x,y,z) + tx^ay^bz^c$ is equisingular along the parameter space at 0 if, and only if, the monomial $x^ay^bz^c$ is in the Newton polyhedron $\Gamma_+(\{(15,0,0),(0,8,0),(0,0,5),(1,7,0)\})$.

We consider here the analogous question for the real family $F : \mathbb{R}^3 \times \mathbb{R}, 0 \to \mathbb{R}, 0$; $F(x,y,z,t) = f(x,y,z) + tx^ay^bz^c$.

Fernandes and Ruas in [3] showed that if $a+2b+3c \geq 17$ then the family is equisingular along of the parameter space at 0.

Let

$$A = \begin{pmatrix} 14 & 0 & 0 & 1 \\ 0 & 7 & 0 & 6 \\ 0 & 0 & 4 & 0 \end{pmatrix}.$$

Then $N^*_{\mathcal{R}}f := M_1^{2\alpha_1} + M_2^{2\alpha_2} + M_3^{2\alpha_3}$ admits 0 as an $\Gamma_+(\rho^{84})$-isolated point.

The primitive vectors of the compact faces are $v^1 = (6,13,21)$, $v^2 = (12,12,21)$, and $\ell(v^1) = 84$, $\ell(v^2) = 28$.

In this case, since l.c.m.$\{84, 28\} = 84$, we have

$$\begin{aligned} \varphi(x^ay^bz^c) &:= \min\{\langle (a,b,c),(6,13,21)\rangle, 3\langle (a,b,c),(4,4,7)\rangle\} \\ &= \min\{6a+13b+21c, 12a+12b+21c\}. \end{aligned}$$

Hence, $\varphi(x^{15}) = 90$, $\varphi(xy^7) = 96$ and $\varphi(z^5) = 105$. Therefore $fil(f) = 90$.

Observe that $R = 21$ and $r = 6$. Then, for f_t to be bilipschitz trivial it is sufficient that

$$fil(x^ay^bz^c) \geq fil(f) + R - r,$$

i.e.

$$\min\{6a+13b+21c, 12a+12b+21c\} \geq 90 + 21 - 6 = 105.$$

Therefore the variety $F^{-1}(0)$ is equisingular along of the parameter space $0 \times \mathbb{R}$ at 0.

Remark: In the work [3] of Fernandes and Ruas cannot detect whether the deformations by the monomials $xy^3z^3, x^3y^2z^3, x^2yz^4, xz^5, z^5, x^7z^3$ are equisingular. From the Theorem 3.1 we have that deformations by this monomials are Whitney equisingular. It is valuable to observe that the family $f_t(x,y,z) = f(x,y,z) + tx^{17}$ is bilipschitz trivial (c.f. [3]), however here $fil(x^{17}) = 102$.

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