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Annihilating Properties of Complex Spherical Convolution Operators

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Spherical convolution operators are a useful tool in the construction of approximations to continuous functions on the sphere. This paper studies some annihilating properties of operators generated by complex spherical convolution which can be of use in approximation problems. The word complex refers to the fact that the convolution involves complex functions defined on the unit sphere Ω_{2q} of $\mathbb{C}^q$. The main tool in the study is a Funk-Hecke type formula for spherical harmonics on Ω_{2q}, an elementary proof of which is included here.

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1. INTRODUCTION

Let q be an integer at least 2 and Ω_{2q} the unit sphere in $\mathbb{C}^q$. Let σ_q be the unique positive Borel measure on Ω_{2q}, so normalized that

$$\int_{\Omega_{2q}} d\sigma_q = \frac{2\pi^q}{(q-1)!}. \tag{1.1}$$

This measure is invariant with respect to $\mathcal{O}(2q)$, where $\mathcal{O}(2q)$ denotes the group of isometries of $\mathbb{C}^q$ that fix the origin. This means that

$$\int_{\rho(B)} d\sigma_q = \int_B d\sigma_q, \quad \rho \in \mathcal{O}(2q). \tag{1.2}$$

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whenever B is a Borel subset of Ω_{2q}. Integrability with respect to this measure can be resumed like this: a complex $d\sigma_q$-measurable function f is *p-integrable* ($1 \leq p < \infty$) if

$$\|f\|_p := \left(\int_{\Omega_{2q}} |f(z)|^p d\sigma_q(z)\right)^{1/p} < \infty. \tag{1.3}$$

Equation (1.3) defines a norm in the space $L^p(\Omega_{2q})$, the class of all p-integrable functions on Ω_{2q}. The space $L^2(\Omega_{2q})$ is then a Hilbert space with inner product given by

$$\langle f, g\rangle_2 := \int_{\Omega_{2q}} f(z)\overline{g(z)} d\sigma_q(z), \quad f, g \in L^2(\Omega_{2q}). \tag{1.4}$$

The purpose of this paper is to study some annihilating properties of complex spherical convolution operators over $L^p(\Omega_{2q})$, that is, operators of the form

$$f \in L^p(\Omega_{2q}) \longmapsto \int_{\Omega_{2q}} K(\langle z, \cdot\rangle) f(z) d\sigma_q(z), \tag{1.5}$$

where $\langle\cdot,\cdot\rangle$ is the usual inner product of $\mathbb{C}^q$ and K is a fixed function.

The study was motivated by problems of the following nature: given a family $\{\Phi_\lambda\}_\lambda$ of linear functionals, all generated by linear integral transformations of convolution-type, to determine whether or not $\cap_\lambda \ker \phi_\lambda$ is trivial. One possible application of results of this type is in the search for fundamental sets in $C(\Omega_{2q})$ or $L^p(\Omega_{2q})$ ([11]). Another instance has geometric flavor, since the results have to do with those of determining properties of an object from information concerning its lower-dimensional sections and integrals over those sections ([17]).

A Funk-Hecke type formula for spherical harmonics on Ω_{2q} is the main tool required in our analysis. Standard references in the literature do not quote or prove such formula but we like to believe it is folkloric. The first goal in the paper is then to provide an elementary proof for it.

Section 2 outlines some basic material about analysis on the sphere of $\mathbb{C}^q$ while Section 3 deals with the Funk-Hecke formula. In Section 4 we formalize the notion of complex spherical convolution and derive some of its annihilating properties. Section 5 specializes the results of Section 4 for some particular choices of the function K.

2. ANALYSIS ON THE SPHERE

In this section we introduce basic material about analysis on Ω_{2q}. We refer the reader to [5, 6, 7, 8, 9, 10, 14] for detailed information and deeper results on this topic.

The closed unit disk in $\mathbb{C}$ will be written as $B[0,1]$ while $B(0,1)$ will stand for its interior. The unitary space of polynomials in the independent variables z and $\overline{z}$ of $\mathbb{C}^q$ will be denoted by Π. Hence, if p is an element of Π, there are nonnegative integers m e n such that

$$p(z) = \sum_{|\mu|\leq m} \sum_{|\nu|\leq n} a_{\mu,\nu} z^\mu \overline{z}^\nu, \quad z \in \mathbb{C}^q, \quad a_{\mu,\nu} \in \mathbb{C}, \quad \mu, \nu \in \mathbb{Z}_+^q. \tag{2.1}$$

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The reader is advised that z is then used as both, a single complex variable and a q-tuple of complex variables.

At least two subspaces of Π will be of use to us. The symbol $\Pi^h_{m,n}$ will stand for the subspace of Π formed by polynomials that are homogeneous of degree m on z and of degree n on $\overline{z}$. Thus,

$$p(\lambda z) = \lambda^m \overline{\lambda}^n p(z), \quad \lambda \in \mathbb{C}, \quad p \in \Pi^h_{m,n}. \tag{2.2}$$

We will write $\mathcal{H}^h_{m,n}(\mathbb{C}^q)$ or simply $\mathcal{H}^h_{m,n}$, depending on the context, to denote the subspace of $\Pi^h_{m,n}$ composed of polynomials that are in the kernel of the complex laplacian. A *complex spherical harmonic* is then the restriction to Ω_{2q} of an element of $\cup_{m,n=0}^{\infty}\mathcal{H}^h_{m,n}$. We will write $\mathcal{H}_{m,n}(\Omega_{2q})$ or simply $\mathcal{H}_{m,n}$, to denote the space of complex spherical harmonics that are restrictions of elements of $\mathcal{H}^h_{m,n}$. The dimension of $\mathcal{H}_{m,n}(\Omega_{2q})$ will be denoted by $D_q(m,n)$ or simply $D(m,n)$.

In many places in the paper, the set

$$\{Y^j_{m,n} : j = 1, \ldots, D(m,n)\} \tag{2.3}$$

will denote an orthonormal basis of $\mathcal{H}_{m,n}$ with respect to the inner product of $L^2(\Omega_{2q})$. The spherical harmonic expansion of a function f in $L^2(\Omega_{2q})$ is then given by

$$f \sim \sum_{m,n\in\mathbb{N}} \sum_{j=1}^{D(m,n)} a^j_{m,n}(f) Y^j_{m,n} \tag{2.4}$$

so that

$$a^j_{m,n}(f) = \langle f, Y^j_{m,n}\rangle_2, \quad j = 1, \ldots, D(m,n), \quad m,n \in \mathbb{N}. \tag{2.5}$$

In $B[0,1]$, we will consider the normalized Lebesgue measure

$$d\nu_q(z) := \frac{q-1}{\pi}\left(1 - x^2 - y^2\right)^{q-2} dx\, dy, \quad z = x + iy \in B[0,1]. \tag{2.6}$$

The class of complex functions that are p-integrable in $B[0,1]$ with respect to ν_q will be denoted by $L^{p,q}(B[0,1])$. In particular, the norm in this space is given by

$$\|f\|_{p,q} := \left(\int_{B[0,1]} |f(z)|^p d\nu_q(z)\right)^{1/p}, \quad f \in L^{p,q}(B[0,1]). \tag{2.7}$$

Next, we introduce some classes of disk polynomials. The *disk polynomial* of degree m in z and degree n in $\overline{z}$ associated to the integer $q-2$ is the polynomial $P^{q-2}_{m,n}$ given by

$$P^{q-2}_{m,n}(z) = \begin{cases} P_n^{(q-2,m-n)}(2z\overline{z}-1)z^{m-n} & \text{if } m \geq n \\ P_m^{(q-2,n-m)}(2z\overline{z}-1)\overline{z}^{n-m} & \text{otherwise,} \end{cases} \tag{2.8}$$

where $P_k^{(q-2,m-n)}$ is the usual Jacobi polynomial of degree k associated to the pair of numbers $(q-2,m-n)$. Here, the Jacobi polynomials are normalized so that $P_k^{(q-2,m-n)}(1)=1$, $k=0,\ldots$. It is not difficult to see that $\{P_{m,n}^{q-2} : 0 \le m,n < \infty\}$ is a complete orthogonal subset of $L^2(B[0,1],d\nu_q)$.

We close this section quoting some properties of disk polynomials to be used in the paper. Some of them follow from the definitions given above while others can be found in the references mentioned at the beginning of this section.

Lemma 2.1 *The following properties hold:*
i) $P_{m,n}^{q-2}(e^{i\varphi}z) = e^{i(m-n)\varphi}P_{m,n}^{q-2}(z)$, $\varphi \in [0,2\pi)$, $z \in B[0,1]$;
ii) $|P_{m,n}^{q-2}(z)| \le 1$, $z \in B[0,1]$;
iii) $P_{m,n}^{q-2}(\overline{z}) = \overline{P_{m,n}^{q-2}(z)} = P_{n,m}^{q-2}(z)$, $z \in B[0,1]$.

Lemma 2.2 *If* $\xi,\zeta \in \Omega_{2q}$ *then*

$$P_{m,n}^{q-2}(\langle \xi,\zeta\rangle) = \frac{2\pi^q}{(q-1)!D(m,n)} \sum_{j=1}^{D(m,n)} Y_{m,n}^j(\xi)\overline{Y_{m,n}^j(\zeta)}. \tag{2.9}$$

3. THE FUNK-HECKE FORMULA

The classical Funk-Hecke formula is a transformation formula for spherical harmonics having many applications in analysis and harmonic analysis on the sphere ([1,3,13]). Its most common form is as follows: if $f : [-1,1] \mapsto \mathbb{R}$ is continuous and Y is a standard spherical harmonic of degree k in m variables then

$$\int_{S^{m-1}} f(x\cdot y)Y(y)d\tau_{m-1}(y) = \lambda_k^{m-1}(f)Y(y), \quad x \in S^{m-1}, \tag{3.1}$$

where

$$\lambda_k^{m-1}(f) := |S^{m-2}| \int_{-1}^{1} f(t)P_k^{(m-2)/2}(t)(1-t^2)^{(m-3)/2}dt. \tag{3.2}$$

Here, $\cdot$ stands for the usual inner product in $\mathbb{R}^m$, S^{m-1} denotes de unit sphere in $\mathbb{R}^m$, τ_{m-1} is the surface measure on S^{m-1}, $|S^{m-2}|$ is the surface area of S^{m-2} and $P_k^{(m-2)/2}$ is the normalized Gegenbauer polynomial of degree k associated to $(m-2)/2$.

The formula for complex harmonics has a similar structure. Since we are dealing with both complex variables and complex functions, the formula has conjugate cousins. Next, we list all technical facts needed in the proof of the formula, postponing the formula itself and its conjugates to the end of the section.

Lemma 3.1. *(Stone-Weierstrass Theorem) Every complex continuous function on a compact subset B of $\mathbb{C}$ can be uniformly approximated on B by a polynomial in the complex variables z and $\overline{z}$.*

Lemma 3.2. *Let w be an element of Ω_{2q}. If K is a function in $L^{1,q}(B[0,1])$ then the mapping $z \in \Omega_{2q} \mapsto K(\langle z, w\rangle)$ is in $L^1(\Omega_{2q})$ and*

$$\begin{aligned}\int_{\Omega_{2q}} K(\langle z, w\rangle) d\sigma_q(z) &= \frac{2\pi^q}{(q-1)!}\int_{B[0,1]} K(z) d\nu_q(z) \\ &= \frac{2\pi^{q-1}}{(q-2)!}\int_0^{2\pi}\int_0^1 K(re^{i\theta})(1-r^2)^{q-2} r\, dr\, d\theta.\end{aligned}$$

Proof. This follows from a result of Forelli ([3]). The form presented here can be found in [14, p. 15]. ∎

Lemma 3.3 *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of $L^{1,q}(B[0,1])$ with uniform limit f. Then f is an element of $L^{1,q}(B[0,1])$ and*

$$\int_{B[0,1]} f(z) d\nu_q(z) = \lim_{n\to\infty}\int_{B[0,1]} f_n(z) d\nu_q(z). \tag{3.3}$$

Proof. The proof is standard and will be omitted. ∎

There is a version of Lemma 3.3 for functions in $L^1(\Omega_{2q})$. The statement and proof of such version are left to the reader who is advised that both, the stated and unstated versions will be used ahead.

Lemma 3.4 below is an adapted to our purposes version of Lusin's Theorem ([2]).

Lemma 3.4 *If $K : B[0,1] \mapsto \mathbb{C}$ is a ν_q-integrable function then there exists a sequence $\{K_n\}_{n=0}^{\infty}$ of complex continuous functions on $B[0,1]$ such that $|K_n| \leq \sup_{z\in B[0,1]} |K(z)|$, $n = 0, \ldots$ and $\lim_{n\to\infty} K_n(z) = K(z)$ a.e..*

The Funk-Hecke formula for complex spherical harmonics is as follows.

Theorem 3.5 *Let w be in Ω_{2q}, Y an element of $\mathcal{H}_{m,n}$, and K an element of $L^{1,q}(B[0,1])$. Then the mapping $z \in \Omega_{2q} \longmapsto K(\langle z, w\rangle)\overline{Y(z)}$ is in $L^1(\Omega_{2q})$ and*

$$\int_{\Omega_{2q}} K(\langle z, w\rangle)\overline{Y(z)} d\sigma_q(z) = \lambda_{m,n}^{q-2}(K)\overline{Y(w)}, \tag{3.4}$$

in which

$$\lambda_{m,n}^{q-2}(K) := \frac{2\pi^q}{(q-1)!} \int_{B[0,1]} K(z)\overline{P_{m,n}^{q-2}(z)}d\nu_q(z). \tag{3.5}$$

Proof. First we analyze the case in which K is the disk polynomial $P_{k,l}^{q-2}$. Certainly, $P_{k,l}^{q-2}(\langle\cdot,w\rangle)\overline{Y(\cdot)}$ is σ_q-integrable on Ω_{2q}. Decomposing Y in the form

$$Y = \sum_{j=1}^{D(m,n)} a_j Y_{m,n}^j, \quad a_j \in \mathbb{C}, \tag{3.6}$$

and using the Addition Formula we obtain

$$\int_{\Omega_{2q}} P_{k,l}^{q-2}(\langle z,w\rangle)\overline{Y(z)}d\sigma_q(z) = \frac{2\pi^q}{(q-1)!D(k,l)}\,\delta_{mk}\delta_{nl} \sum_{i=1}^{D(k,l)} \left(\sum_{j=1}^{D(m,n)} \overline{a_j}\delta_{ij}\right)\overline{Y_{k,l}^i(w)}. \tag{3.7}$$

On the other hand, the use of Lemma 3.2 and the Addition Formula once again lead to

$$\begin{aligned}
\lambda_{m,n}^{q-2}(P_{k,l}^{q-2}) &= \frac{2\pi^q}{(q-1)!}\int_{B[0,1]} P_{k,l}^{q-2}(z)\overline{P_{m,n}^{q-2}(z)}d\nu_q(z)\\
&= \int_{\Omega_{2q}} P_{k,l}^{q-2}(\langle z,w\rangle)\overline{P_{m,n}^{q-2}(\langle z,w\rangle)}d\sigma_q(z)\\
&= \frac{2\pi^q}{(q-1)!D(k,l)}\,\frac{2\pi^q}{(q-1)!D(m,n)}\delta_{mk}\delta_{nl} \sum_{i=1}^{D(k,l)} \overline{Y_{k,l}^i(w)} \left(\sum_{j=1}^{D(m,n)} Y_{m,n}^j(w)\,\delta_{ij}\right).
\end{aligned}$$

It is now clear that (3.4) holds when $(k,l) \neq (m,n)$. If $(k,l) = (m,n)$, we use the Addition Formula to simplify the above relations to obtain

$$\lambda_{m,n}^{q-2}(P_{k,l}^{q-2})\overline{Y(w)} = \frac{2\pi^q}{(q-1)!D(m,n)}\overline{Y(w)} = \int_{\Omega_{2q}} P_{m,n}^{q-2}(\langle z,w\rangle)\overline{Y(z)}d\sigma_q(z), \tag{3.8}$$

thus showing that (3.4) holds in all cases. This gives the desired formula for K being a disk polynomial. The set $\{P_{k,l}^{q-2} : k,l = 0,\ldots\}$ being a basis for Π and the linear functional

$$K \in L^{1,q}(B[0,1]) \longmapsto \lambda_{m,n}^{q-2}(K) \in \mathbb{C} \tag{3.9}$$

being continuous allows us to use a linear combination argument to conclude that the formula holds for elements of Π. Next, if K is a continuous function on $B[0,1]$, we can use Lemma 3.1 to select a sequence $\{p_j\}_{j=0}^{\infty}$ from Π such that $\lim_{j\to\infty} p_j = K$ uniformly on

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$B[0,1]$. By Lemma 3.3, we deduce that

$$\begin{aligned}\int_{\Omega_{2q}} K(\langle z,w\rangle)\overline{Y(z)}d\sigma_q(z) &= \lim_{j\to\infty}\int_{\Omega_{2q}} p_j(\langle z,w\rangle)\overline{Y(z)}d\sigma_q(z)\\ &= \lim_{j\to\infty}\lambda^{q-2}_{m,n}(p_j)\overline{Y(w)}\\ &= \lambda^{q-2}_{m,n}(K)\overline{Y(w)}.\end{aligned}$$

Finally, if K is ν_q-integrable, we select a sequence $\{K_j\}_{j=0}^{\infty}$ of continuous functions as described in Lemma 3.4. Hence,

$$\lim_{j\to\infty} K_j(\langle\cdot,w\rangle) = K(\langle\cdot,w\rangle) \quad a.e, \tag{3.10}$$

and the Dominated Convergence Theorem yields

$$\begin{aligned}\int_{\Omega_{2q}} K(\langle z,w\rangle)\overline{Y(z)}d\sigma_q(z) &= \lim_{j\to\infty}\int_{\Omega_{2q}} K_j(\langle z,w\rangle)\overline{Y(z)}d\sigma_q(z)\\ &= \lim_{j\to\infty}\lambda^{q-2}_{m,n}(K_j)\overline{Y(w)}\\ &= \lambda^{q-2}_{m,n}(K)\overline{Y(w)}.\end{aligned}$$

This concludes the proof of the theorem. ∎

Corollary 3.6 *Under the conditions given in Theorem 3.5, the following formulae hold:*

$$\int_{\Omega_{2q}} K(\langle z,w\rangle)Y(z)d\sigma_q(z) = \lambda^{q-2}_{n,m}(K)Y(w), \tag{3.11}$$

$$\int_{\Omega_{2q}} K(\langle w,z\rangle)Y(z)d\sigma_q(z) = \lambda^{q-2}_{m,n}(K)Y(w), \tag{3.12}$$

$$\int_{\Omega_{2q}} K(\langle w,z\rangle)\overline{Y(z)}d\sigma_q(z) = \lambda^{q-2}_{n,m}(K)\overline{Y(w)}. \tag{3.13}$$

Proof. Formula (3.11) is a re-writing of that in the statement of the theorem. Formula (3.12) follows from (3.11) and from the fact that $\lambda^{q-2}_{n,m}(K) = \lambda^{q-2}_{m,n}(\widetilde{K})$, where $\widetilde{K}(z) := K(\overline{z})$, $z \in B[0,1]$. Formula (3.13) is a variation of the others. ∎

4. COMPLEX SPHERICAL CONVOLUTION OPERATORS

In this section we first define the spherical convolution of a kernel K in $L^{p,q}(B[0,1])$ and an appropriated measure and analyze one of its annihilating properties. The symbol $\mathcal{M}$

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will denote the set of all regular complex Borel measures over Ω_{2q}. The formal expansion of a kernel $K \in L^{1,q}(B[0,1])$ in terms of disk polynomials is given by

$$K(z) \sim \sum_{m,n\in\mathbb{N}} \frac{(q-1)!}{2\pi^q} \lambda_{m,n}^{q-2}(K) P_{m,n}^{q-2}(z), \quad z \in B[0,1]. \tag{4.1}$$

When $K(\langle\cdot,\cdot\rangle)$ is integrable with respect to the product $\sigma_q \times \mu$, $\mu \in \mathcal{M}$, we can define the *convolution of* K *and* μ as the function $\Psi_K : \mathcal{M} \mapsto L^1(\Omega_{2q})$ given by

$$\Psi_K(\mu)(z) = \int_{\Omega_{2q}} K(\langle z, w\rangle) d\mu(w), \quad z \in \Omega_{2q}. \tag{4.2}$$

Theorem 4.1 is an abstract form of the type of property we are looking for.

Theorem 4.1 *Let μ be an element of $\mathcal{M}$ and K an element of $L^{1,q}(B[0,1])$. If $\Psi_K(\mu) = 0$ then either $\mu = 0$ or there exists a pair (m,n) and $Y \in \mathcal{H}_{m,n}$ such that $\lambda_{n,m}^{q-2}(K) = 0 \neq \int_{\Omega_{2q}} Y(w) d\mu(w)$.*

Proof. Due to the hypotheses, the function $z \in \Omega_{2q} \mapsto \int_{\Omega_{2q}} K(\langle z,w\rangle) d\mu(w)$ is σ_q-integrable. An application of Fubini's Theorem ([2, p. 223]) with a help of Corollary 3.6 reveals that

$$\begin{aligned}\int_{\Omega_{2q}} \Psi_K(\mu)(z) Y_{m,n}(z) d\sigma_q(z) &= \int_{\Omega_{2q}} \left(\int_{\Omega_{2q}} K(\langle z,w\rangle) Y_{m,n}(z) d\sigma_q(z) \right) d\mu(w) \\ &= \lambda_{n,m}^{q-2}(K) \int_{\Omega_{2q}} Y_{m,n}(w) d\mu(w), \quad m,n \in \mathbb{N}.\end{aligned}$$

If $\int_{\Omega_{2q}} Y(w) d\mu(w) = 0$, $Y \in \mathcal{H}_{m,n}$ whenever $\lambda_{n,m}^{q-2}(K) = 0$ then the above calculation reveals that the linear functional

$$f \longmapsto \int_{\Omega_{2q}} f(z) d\mu(z) \tag{4.3}$$

annihilates $\cup_{m,n\in\mathbb{N}} \mathcal{H}_{m,n}$. This set being fundamental in $C(\Omega_{2q})$ ([9]) would force the measure μ to be zero. ∎

Next, we go from measures to functions. Precisely, we consider the convolution $\Psi_K(f)$ of a kernel $K \in L^{1,q}(B[0,1])$ and a function f as introduced in (1.5):

$$\Psi_K(f)(w) := \int_{\Omega_{2q}} K(\langle z,w\rangle) f(z) d\sigma_q(z), \quad w \in \Omega_{2q}. \tag{4.4}$$

Hölder's inequality and Tonelli's theorem ([2]) imply that

$$\|\Psi_K(f)\|_p \leq \lambda_{0,0}^{q-2}(|K|) \|f\|_p, \quad f \in L^p(\Omega_{2q}). \tag{4.5}$$

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Thus, Ψ_K maps $L^p(\Omega_{2q})$ into itself and has norm $\lambda_{0,0}^{q-2}(|K|)$. The Funk-Hecke formula implies that

$$\Psi_K(Y) = \lambda_{n,m}^{q-2}(K)Y, \quad Y \in \mathcal{H}_{m,n}. \tag{4.6}$$

Fubini's theorem ([2]) shows that the following commuting property holds

$$\int_{\Omega_{2q}} \Psi_K(f)(z)g(z)d\sigma_q(z) = \int_{\Omega_{2q}} \Psi_{\widetilde{K}}(g)(z)f(z)d\sigma_q(z), \quad f, g \in L^p(\Omega_{2q}), \tag{4.7}$$

with $\widetilde{K}$ as defined at the end of the previous section. As a consequence, if $f \in L^2(\Omega_{2q})$ has an expansion as in (2.4) then $\Psi_K(f)$ has a spherical harmonic expansion in the form

$$\Psi_K(f) \sim \sum_{m,n\in\mathbb{N}} \sum_{j=1}^{D(m,n)} \lambda_{n,m}^{q-2}(K)a_{m,n}^j(f)Y_{m,n}^j. \tag{4.8}$$

Indeed,

$$a_{m,n}^j(\Psi_K(f)) = \left\langle \Psi_K(f), Y_{m,n}^j \right\rangle_2 = \lambda_{n,m}^{q-2}(K)\left\langle f, Y_{m,n}^j \right\rangle_2 = \lambda_{n,m}^{q-2}(K)a_{m,n}^j(f). \tag{4.9}$$

The following result collects the annihilating properties of Ψ_K. It is a consequence of Formulas (4.8) and (4.9).

Theorem 4.2 *Let K be an element of $L^{1,p}(B[0,1])$. The following assertions hold:*
i) If $\lambda_{m,n}^{q-2}(K) = 0$ for some (m,n) then $\mathcal{H}_{n,m} \subset \ker \Psi_K$;
ii) If $f \in \ker \Psi_K$ and $\lambda_{n,m}^{q-2}(K) \neq 0$ for some (m,n) then $a_{m,n}^{q-2}(f) = 0$;
iii) If $\lambda_{m,n}^{q-2}(K) \neq 0$, $m,n \in \mathbb{N}$ then $\ker \Psi_K = \{0\}$.

Remarks i) Theorem 4.2-iii) takes an interesting form when one is dealing with zonal functions only. Indeed, assuming the assumption in iii) and that a function f satisfies $f \circ A = f$, whenever $A \in \{A \in \mathcal{O}(2q) : A(\zeta) = \zeta\}$, for some $\zeta \in \Omega_{2q}$, the condition $\Psi_K(f)(w_0) = 0$ for some $w_0 \in \Omega_{2q}$ forces the function f to be zero, as we now explain. For each $w \in \Omega_{2q}$, select $A_w \in \mathcal{O}(2q)$ such that $A_w(w_0) = w$ and $A_w(\zeta) = \zeta$. It follows that,

$$\begin{aligned}
\Psi_K(f)(w) &= \int_{\Omega_{2q}} K(\langle A_w(z), w\rangle)f(A_w(z))d\sigma_q(A_w(z)) \\
&= \Psi_K(f)(A_w^{-1}(w)) \\
&= \Psi_K(f)(w_0) = 0, \quad w \in \Omega_{2q}.
\end{aligned}$$

Thus, $\Psi_K(f) = 0$ and Theorem 4.2-ii) implies that $f = 0$.
ii) The properties stated in Theorem 4.2 may be re-stated as annihilating properties of the family of linear functionals $\{\Psi_{K,w}\}_{w\in\Omega_{2q}}$, where $\Psi_{K,w}(f) = \Psi_K(f)(w)$. Thus, the results in the theorem belong to the class of problems that served as motivation to us, as briefly explained in the introduction.

5. CONVOLUTION USING SPECIAL KERNELS

In this section, we specialize the results of Section 4, by letting K to be the characteristic function of a proper subset of Ω_{2q}. Such case encompasses operators corresponding to integration over sub-spheres, spherical caps, spherical collars, etc.

The first case begins with the choice of a pole $w \in \Omega_{2q}$ and $\gamma \in B[0,1]$. The intersection of Ω_{2q} with the hyperplane $\langle z, w\rangle = \gamma$, will be termed a *sub-sphere of* Ω_{2q} *with pole* w *and radius* $(1-\gamma\overline{\gamma})^{1/2}$ and will be denoted by $\Omega_w^{\gamma,q}$. The terminology is justified since the hyperplane is orthogonal to w and contains γw.

Every element z of Ω_{2q} has a polar representation in the form

$$z = te^{i\varphi}w + \sqrt{1-t^2}\, z', \quad 0 \le t \le 1, \quad \varphi \in [0, 2\pi), \quad z' \in \Omega_w^{0,q} := \Omega_{2q-2}. \tag{5.1}$$

When $z \in \Omega_w^{\gamma,q}$ the representation takes the form

$$z = \gamma w + \sqrt{1-\gamma\overline{\gamma}}\, z', \quad z' \in \Omega_{2q-2}, \quad \langle z, w\rangle = \gamma. \tag{5.2}$$

The restriction ν_w^γ of σ_q to $\Omega_w^{\gamma,q}$ makes $\Omega_w^{\gamma,q}$ into a measurable space and equation (5.2) yields the following relation between surface elements:

$$d\nu_w^\gamma(z) = (1-\gamma\overline{\gamma})^{(2q-3)/2} d\sigma_{q-1}(z'). \tag{5.3}$$

The following technical result will be needed in the proof of Theorem 5.2. It relies on the above representations.

Lemma 5.1 *Let m and n nonnegative integers. For each pair (k,l) in the set $\{0,\dots,m\} \times \{0,\dots,n\}$, let $\{Z_{k,l}^j : j = 1,\dots,D_{q-1}(k,l)\}$ be an orthonormal basis of $\mathcal{H}_{k,l}(\Omega_{2q-2})$. Then there exist positive constants $c_{q,m,n}^{k,l}$ and associated functions*

$$B_{m,n}^{k,l}(te^{i\varphi}) := c_{q,m,n}^{k,l}(1-t^2)^{(k+l)/2} P_{m-k,n-l}^{q-2+k+l}(te^{i\varphi}) \tag{5.4}$$

such that the set

$$\left\{ B_{m,n}^{k,l}(te^{i\varphi}) Z_{k,l}^j(z') : j = 1,\dots,D_{q-1}(k,l) \right\} \tag{5.5}$$

constitutes an orthonormal basis of $\mathcal{H}_{m,n}(\Omega_{2q})$.

Proof. See [9,10]. ∎

Theorem 5.2 *Let $\gamma \in B[0,1]$ and $w \in \Omega_{2q}$. Then*

$$\int_{\Omega_w^{\gamma,q}} Y(z)d\nu_w^\gamma(z) = \frac{2\pi^{q-1}}{(q-2)!}(1-\gamma\overline{\gamma})^{(2q-3)/2} P_{m,n}^{q-2}(\gamma) Y(w), \quad Y \in \mathcal{H}_{m,n}. \tag{5.6}$$

Proof. Let $Y \in \mathcal{H}_{m,n}$. Using Lemma 5.1, we may write

$$Y(z) = \sum_{k=0}^{m}\sum_{l=0}^{n}\sum_{j=1}^{D_{q-1}(k,l)} a_j^{k,l} B_{m,n}^{k,l}(\gamma)\, Z_{k,l}^{j}(z'), \quad a_j^{k,l} \in \mathbb{C}, \quad z \in \Omega_w^{\gamma,q}, \tag{5.7}$$

whence

$$\int_{\Omega_w^{\gamma,q}} Y(z) d\nu_w^{\gamma}(z) = \sum_{k=0}^{m}\sum_{l=0}^{n}\sum_{j=1}^{D_{q-1}(k,l)} a_j^{k,l}\, B_{m,n}^{k,l}(\gamma)(1-\gamma\overline{\gamma})^{(2q-3)/2} \int_{\Omega_{2q-2}} Z_{k,l}^{j}(z') d\sigma_{q-1}(z').$$

By the orthogonality of the spherical harmonics, the expression above reduces to

$$\begin{aligned}\int_{\Omega_w^{\gamma,q}} Y(z) d\nu_w^{\gamma}(z) &= a_1^{0,0} B_{m,n}^{0,0}(\gamma)\, Z_{0,0}^{1} \frac{2\pi^{q-1}}{(q-2)!}(1-\gamma\overline{\gamma})^{(2q-3)/2} \\ &= a_1^{0,0} c_{q,m,n}^{0,0}\, Z_{0,0}^{1} P_{m,n}^{q-2}(\gamma) \frac{2\pi^{q-1}}{(q-2)!}(1-\gamma\overline{\gamma})^{(2q-3)/2} \\ &= Y(w) P_{m,n}^{q-2}(\gamma) \frac{2\pi^{q-1}}{(q-2)!}(1-\gamma\overline{\gamma})^{(2q-3)2},\end{aligned}$$

where we have used the fact that $B_{m,n}^{k,l}(1) = 0$, $(k,l) \neq (0,0)$. ∎

The reader is advised that an alternative proof of Theorem 5.2 can be given using Corollary 3.6. The calculations are similar to others already used in the paper.

Definition 5.3 Let $\gamma \in B(0,1)$ and $w \in \Omega_{2q}$. The *average* of $f \in L^2(\Omega_{2q})$ over $\Omega_w^{\gamma,q}$ is the complex number

$$\Phi^{(\gamma)}(f)(w) := \frac{(q-2)!}{2\pi^{q-1}(1-\gamma\overline{\gamma})^{(2q-3)/2}} \int_{\Omega_w^{\gamma,q}} f(z) d\nu_w^{\gamma}(z). \tag{5.8}$$

Theorem 5.4 *Let $\gamma \in B(0,1)$ and $f, g \in L^2(\Omega_{2q})$. Then*

$$\int_{\Omega^{2q}} \Phi^{(\gamma)}(f)(z) g(z) d\sigma_q(z) = \int_{\Omega^{2q}} \Phi^{(\overline{\gamma})}(g)(z) f(z) d\sigma_q(z). \tag{5.9}$$

Proof. First observe that

$$\Phi^{(\gamma)}(f)(z) = \frac{(q-2)!}{2\pi^{q-1}(1-\gamma\overline{\gamma})^{(2q-3)/2}}\, \Psi_{K_\gamma}(f)(z), \tag{5.10}$$

where

$$K_\gamma(z) = \begin{cases} 1 & \text{if } z = \gamma \\ 0 & \text{if } z \neq \gamma. \end{cases} \tag{5.11}$$

¿From (4.7) it follows that

$$\begin{aligned}\int_{\Omega^{2q}} \Phi^{(\gamma)}(f)(z)g(z)d\sigma_q(z) &= \frac{(q-2)!}{2\pi^{q-1}(1-\gamma\overline{\gamma})^{(2q-3)/2}} \int_{\Omega_{2q}} \Psi_{K_\gamma}(f)(z)g(z)d\sigma_q(z) \\ &= \frac{(q-2)!}{2\pi^{q-1}(1-\gamma\overline{\gamma})^{(2q-3)/2}} \int_{\Omega_{2q}} \Psi_{\widetilde{K_\gamma}}(g)(z)f(z)d\sigma_q(z) \\ &= \int_{\Omega_{2q}} \Phi^{(\overline{\gamma})}(g)(z)f(z)d\sigma_q(z),\end{aligned}$$

completing the proof. ∎

Since

$$\int_{\Omega_{2q}} \Phi^{(\gamma)}(f)(z)d\sigma_q(z) = \int_{\Omega_{2q}} \Phi^{(\overline{\gamma})}(1)(z)f(z)d\sigma_q(z) = \int_{\Omega_{2q}} f(z)d\sigma_q(z), \tag{5.12}$$

an obvious necessary condition for a function f to be in $\ker \Phi^{(\gamma)}$ is that $a^1_{0,0}(f) = 0$. In particular, $\ker \Phi^{(\gamma)} \subseteq \ker \Psi_1$.

Theorem 5.5 below describes additional properties of $\ker \Phi^{(\gamma)}$. The following notation is needed:

$$\mathcal{A} := \left\{z \in B(0,1) : P^{q-2}_{m,n}(z) = 0, \text{ for some}\,(m,n)\right\}. \tag{5.13}$$

Theorem 5.5 *Let $\gamma \in B(0,1)$. The following assertions hold:*
i) If $\gamma \in \mathcal{A}$ then $\ker \Phi^{(\gamma)} \neq \{0\}$;
ii) If $\gamma \notin \mathcal{A}$ then $\ker \Phi^{(\gamma)} = \{0\}$.

Proof. If $\gamma \in \mathcal{A}$, we can use Theorem 5.2 to write

$$\Phi^{(\gamma)}(Y) = P^{q-2}_{m,n}(\gamma)Y, \quad Y \in \mathcal{H}_{m,n}. \tag{5.14}$$

It follows that $\mathcal{H}_{m,n} \subset \ker \Phi^{(\gamma)}$, for at least one pair (m,n). Next, assume that $\gamma \notin \mathcal{A}$ and let $f \in \ker \Phi^{(\gamma)}$. Due to Theorems 5.4 and 5.2 we have that

$$0 = \int_{\Omega_{2q}} \Phi^{(\overline{\gamma})}(Y)(z)f(z)d\sigma_q(z) = P^{q-2}_{m,n}(\overline{\gamma}) \int_{\Omega_{2q}} f(z)Y(z)d\sigma_q(z), \quad Y \in \mathcal{H}_{m,n}. \tag{5.15}$$

Hence,

$$\int_{\Omega_{2q}} f(z)Y(z)d\sigma_q(z) = 0, \quad Y \in \cup_{m,n=0}^{\infty}\mathcal{H}_{m,n}, \tag{5.16}$$

that is, $f = 0$. ∎

The above result has the following geometric consequence.

Corollary 5.6 *Let $\gamma \notin \mathcal{A}$. If the average of $f \in L^2(\Omega_{2q})$ over every $\Omega_w^{\gamma,q}$ is zero then f vanishes.*

We end our discussion on the operator $\Phi^{(\gamma)}$ by showing that $\mathcal{A}$ is dense in $B[0,1]$. Due to Lemma 2.1-i,iii) it suffices to show that

$$\mathcal{A} \cap [0,1] = \left\{ r \in [0,1] : P_{m\wedge n}^{(q-2,|m-n|)}(2r^2-1) = 0, \text{ for some } (m,n) \right\} \tag{5.17}$$

is dense in $[0,1]$. To this end, we will show that

$$\mathcal{A}' := \left\{ r \in (0,1) : P_m^{(q-2,0)}(2r^2-1) = 0, \text{ for some } m \right\} \tag{5.18}$$

is dense in $[0,1]$ or, equivalently, that

$$\mathcal{A}'' := \left\{ \theta \in (0,\pi) : P_m^{(q-2,0)}(\cos\theta) = 0, \text{ for some } m \right\} \tag{5.19}$$

is dense in $[0,\pi]$. We will require a comparison theorem of Sturm's type as described in [16, p. 19]. We fix $a, b \in (0,\pi)$, $a < b$, and define

$$f_m(\theta) = \left(\frac{2m+q-1}{2}\right)^2 + \frac{1-4(q-2)^2}{16\sin^2\theta/2} + \frac{1}{16\cos^2\theta/2} \tag{5.20}$$

and $f(\theta) = \pi^2/(b-a)^2$. It is very easy to see that $f_m(\theta) \geq f(\theta)$, $\theta \in (0,\pi)$, $m \geq m_0$, for some m_0. On the other hand,

$$Y_m(\theta) = \left(\sin\frac{\theta}{2}\right)^{(2q-3)/2} \left(\cos\frac{\theta}{2}\right)^{1/2} P_m^{(q-2,0)}(\cos\theta) \tag{5.21}$$

is a solution of the differential equation $y''(\theta) + f_m(\theta)y(\theta) = 0$ ([16, p. 66]) while $Y(\theta) = \sin(\theta-a)/(b-a)$ is a solution of $y''(\theta) + f(\theta)\,y(\theta) = 0$. Since $Y(a) = Y(b) = 0$, Sturm's theorem reveals that Y_m has a root in (a,b) whenever $m \geq m_0$. It is now clear that $\mathcal{A}''$ is dense in $[0,\pi]$.

Next, we investigate linear operators corresponding to integration over caps of Ω_{2q}.

Definition 5.7 Let $\gamma \in B(0,1)$ and $w \in \Omega_{2q}$. The *average of $f \in L^2(\Omega_{2q})$ over the cap of* $\Omega_w^{\gamma,q}$ is the number

$$\Psi^{(\gamma)}(f)(w) := \frac{q-1}{\pi(1-\gamma\overline{\gamma})^{q-1}} \int_0^{2\pi} \int_{|\gamma|}^1 \Phi^{(z)}(f)(w)(1-r^2)^{q-2} r\, dr\, d\theta, \quad z = r\,e^{i\theta}. \tag{5.22}$$

Despite normalization, $\Psi^{(\gamma)}(f)$ corresponds to integration of f over a cap sitting over the sub-sphere $\Omega_w^{\gamma,q}$. Due to Theorem 5.4, the following property holds:

$$\int_{\Omega_{2q}} \Psi^{(\gamma)}(f)(z)g(z)d\sigma_q(z) = \int_{\Omega_{2q}} \Psi^{(\gamma)}(g)(z)f(z)d\sigma_q(z), \quad f, g \in L^2(\Omega_{2q}). \tag{5.23}$$

The first equality in (5.14) and Lemma 2.1-i) imply that

$$\begin{aligned}\Psi^{(\gamma)}(Y) &= \frac{(q-1)Y}{\pi(1-\gamma\overline{\gamma})^{q-1}} \int_0^{2\pi}\int_{|\gamma|}^1 P_{m,n}^{q-2}(re^{i\theta})(1-r^2)^{q-2} r\,dr\,d\theta \\ &= \frac{(q-1)Y}{\pi(1-\gamma\overline{\gamma})^{q-1}} \int_0^{2\pi}\int_{|\gamma|}^1 e^{i(m-n)\theta} P_{m,n}^{q-2}(r)(1-r^2)^{q-2} r\,dr\,d\theta, \quad Y \in \mathcal{H}_{m,n}.\end{aligned}$$

We call $I_{m,n}$ the last double integral above. Since $I_{m,n} = 0$ when $m \neq n$, we assume $m = n$. The integral reduces to

$$\begin{aligned}I_{m,m} &= 2\pi \int_{|\gamma|}^1 P_m^{(q-2,0)}(2r^2-1)(1-r^2)^{q-2} r\,dr \\ &= \frac{\pi}{2^{q-1}} \int_{2\gamma\overline{\gamma}-1}^1 P_m^{(q-2,0)}(r)(1-r)^{q-2}dr.\end{aligned}$$

It is very easy to see that $I_{0,0} = \pi(1-\gamma\overline{\gamma})^{q-1}(q-1)^{-1}$. We handle the other cases by using the following formula of Rodrigues type ([1, p. 173])

$$2m\int_0^x P_m^{(q-2,0)}(t)(1-t)^{q-2}dt = P_{m-1}^{(q-1,1)}(0) - (1-x)^{q-1}(1+x)P_{m-1}^{(q-1,1)}(x), \quad x \geq 0. \tag{5.24}$$

In both cases, $2\gamma\overline{\gamma}-1 \geq 0$ and $2\gamma\overline{\gamma}-1 < 0$, we arrive at

$$\begin{aligned}I_{m,m} &= \frac{\pi}{2^q m}\left(\int_0^1 P_m^{(q-2,0)}(r)(1-r)^{q-2}dr - \int_0^{2\gamma\overline{\gamma}-1} P_m^{(q-2,0)}(r)(1-r)^{q-2}dr\right) \\ &= \frac{\pi}{m}(1-\gamma\overline{\gamma})^{q-1}\gamma\overline{\gamma}P_{m-1}^{(q-1,1)}(2\gamma\overline{\gamma}-1) \\ &= \frac{\pi}{m}(1-\gamma\overline{\gamma})^{q-1}\overline{\gamma}P_{m,m-1}^{q-1}(\gamma).\end{aligned}$$

Concluding,

$$\Psi^{(\gamma)}(Y) = \begin{cases} 0 & m \neq n \\ Y & m = n = 0 \\ m^{-1}(q-1)\overline{\gamma}P_{m,m-1}^{q-1}(\gamma)Y & m = n > 0. \end{cases} \tag{5.25}$$

In what follows,

$$\mathcal{B} := \{z \in B(0,1) : P_{m,m-1}^{q-1}(z) = 0, \text{ for some } m > 0\}. \tag{5.26}$$

Theorem 5.8 *Let $\gamma \in B(0,1)$. The following assertions hold:*
i) $\cup_{m\neq n}\mathcal{H}_{m,n} \subset \ker \Psi^{(\gamma)}$*;*
ii) If $\gamma \notin \mathcal{B}$ *then* $\ker \Psi^{(\gamma)} \subset (\cup_{m=0}^{\infty}\mathcal{H}_{m,m})^{\perp}$.

As our final example, consider the difference operator $\Psi^{(\gamma,\delta)} := \Psi^{(\gamma)} - \Psi^{(\delta)}$, $\gamma,\delta \in B(0,1)$. After normalization, it corresponds to integration over a spherical collar of Ω_{2q}. For $Y \in \mathcal{H}_{m,n}$, it is easy to see that

$$\Psi^{(\gamma,\delta)}(Y) = \begin{cases} 0 & m \neq n;\ m=n=0 \\ m^{-1}(q-1)\left[\overline{\gamma}P_{m,m-1}^{q-1}(\gamma) - \overline{\delta}P_{m,m-1}^{q-1}(\delta)\right]Y & m=n>0. \end{cases} \tag{5.27}$$

Thus, we have the following properties.

Theorem 5.9 *Let $\gamma,\delta \in B(0,1)$. The following assertions hold:*
i) $\mathcal{H}_{0,0} \cup (\cup_{m\neq n}\mathcal{H}_{m,n}) \subset \ker \Psi^{(\gamma,\delta)}$*;*
ii) If $(\gamma,\delta) \notin \{(z,w) : \overline{z}P_{m,m-1}^{q-1}(z) - \overline{w}P_{m,m-1}^{q-1}(w) = 0\}$ *then* $\ker \Psi^{(\gamma,\delta)} \subset (\cup_{m=1}^{\infty}\mathcal{H}_{m,m})^{\perp}$.

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