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On Families of Square Matrices

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In this paper we consider smooth families of linear maps (matrices) between equidimensional vector spaces, classified up to parametrised automorphisms (changes of bases) in source and target, and change of parameter. We classify all simple such families, that is those that do not involve moduli. This can be viewed as a non-linear version of the classical theory of linear systems of matrices and has applications to the study of vector bundle maps and the dependency set of n vector fields on an n-dimensional manifold. October, 2002
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1. INTRODUCTION

In this paper we classify families of square matrices (linear maps between equidimensional spaces) up to a natural equivalence. Let K denote the real or complex field, and V and W equidimensional K-vector spaces. If $\hom(V, W)$ is the space of linear mappings $V \to W$ we study smooth map germs $K^r, 0 \to \hom(V, W)$ up to an equivalence which is the natural analogue of that given by the standard action of the product of general linear groups $GL(V) \times GL(W)$ on $\hom(V, W)$ (corresponding to basis change). So we allow parametrised families of such changes of coordinates, and smooth changes of coordinates in the source of the germ; the result is referred to as $\mathcal{G}$-equivalence. We obtain a list of all the corresponding simple mappings (that is those with no adjacent moduli). This work is parallel to that in [3] where we dealt with symmetric matrices. The approach we take is self contained, but we show that our problem can be viewed as as special case of Damons work on sections of varieties. Indeed in [8] Damon considers sections of singular matrices.

One obvious motivation for this work is the following. Let ξ and η be n-dimensional vector bundles over smooth manifolds X and Y and let $\phi : \xi \to \eta$ be a vector bundle

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homomorphism. Then because the bundles are locally trivial, for each point $x \in X$ there is a germ $f : X, x \to \hom(V, W)$ where V and W are the fibres of ξ and η at x and its image in Y, respectively. If the mapping on the base space is an immersion at x then the germ f is defined precisely up to the equivalence given above.

Another motivation arises in the theory of dynamical systems. Consider n general vector (or direction) fields in n-space. It is natural to consider points where these fields are dependent, given by the vanishing of the determinant of the $n \times n$ matrix associated to the n fields. (When $n = 2$ such pairs are studied by several authors, see for example [10] and [13].) At each point of n-space we can form a matrix by writing down the column vectors at that point corresponding to the fields. In this way we get, at each point of K^n, a germ of the above sort. The equivalence above clearly preserves the discriminant. Similarly one can consider n-tuples of 1-forms, determining, at most points, n hyperplanes in general position. Those points where these hyperplanes are not in general position correspond again to a discriminant. So, aside from their intrinsic interest, the results in this paper yield information about the singularities of such n-tuples of vector fields and differential forms. More generally our results can be thought of as the nonlinear version of the classical theory of linear systems of matrices. They also provide information on the deformations of determinantal hypersurfaces.

A version of this paper has been in existence since 1998. Some results relating the Tjurina numbers of the discriminants and the $\mathcal{G}$-codimension of the singularities above have already been obtained; see [4].

For background in singularity theory and notation, see [1] and [14]. The organisation and layout of the paper are as follows. In §2 we give some basic definitions, in §3 the tangent space to the orbits of this action are computed, and we give some determinacy results. In §4 we determine the pairs (r, n) for which there exist simple germs $K^r, 0 \to M_n$, where M_n is the space of $n \times n$ matrices with entries in K, and prove the classification theorem that follows. In §5 we begin a discussion of the associated geometry. The paper finishes with an appendix where an alternative approach to part of the classification is outlined.

Before giving the list we note that using a "splitting" result in Proposition 4.1, we need only consider matrices vanishing at the origin, i.e. at 0 the matrix has rank 0. All other simple singularities are direct sums of an identity matrix with ones listed below. We shall refer to the rank of the matrix at 0 as the linear rank of the germ, to distinguish it from the rank of its derivative. The discriminant is the set of points in the source where the corresponding matrix is singular. The *criminant* is defined in the body of the paper.

THEOREM 1.1. *The $\mathcal{G}$-simple germs of linear rank 0 at the origin are those that appear in the following list.*
(1) *When $r = 1$ all finitely-$\mathcal{G}$-determined germs are simple and $\mathcal{G}$-equivalent to a germ of the form $\operatorname{diag}(x^{m_1}, x^{m_2}, \dots, x^{m_n})$ where $m_1 \le m_2 \le \dots \le m_n$. This germ has $\mathcal{G}$-codimension $\sum_{i=1}^{n}(2(n-i)+1)m_i - 1$. Its criminant is empty.*
(2) *When the corank of the derivative of the germ is 0 we have a normal form $A : K^{n^2} \times K^s \to M_n$ given by $A(x, z)_{ij} = x_{ij}$ which is $\mathcal{G}$-simple, and has $\mathcal{G}$-codimension 0. (The z's are redundant variables.) Its criminant is smooth.*
(3) *When the corank of the derivative of the germ is 1 we have two cases:*

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(i) *a normal form* $A: K^{n^2-1} \times K^s \to M_n$ *given by*

$$(\sum_{i=2}^{n} x_{ii} + f(z))E_{11} + \sum_{(ij)\neq(11)} x_{ij}E_{ij}$$

where E_{ij} *is the matrix with a* 1 *in the* (ij) *place and zeros elsewhere, and* $f: K^s, 0 \to K$ *is one of Arnold's* $\mathcal{R}$*-simple germs. The* $\mathcal{G}$*-codimension of* A *coincides with the Tjurina number of* f*.*

(ii) *A normal form* $A: K^{n^2-1} \times K^s \to M_n$ *given by*

$$(\sum_{i=2}^{n-1} x_{ii} + f(x_{nn}, z))E_{11} + \sum_{(ij)\neq(11)} x_{ij}E_{ij}$$

where $f: K \times K^s, 0 \to K$ *is one of Arnold's* $\mathcal{R}_\delta$ *simple germs of singularities of functions on manifolds with boundary. Here* $\mathcal{R}_\delta$ *is the subgroup of the right-group of diffeomorphisms* $K^{s+1}, 0 \to K^{s+1}, 0$*, changes of* (x_{nn}, z) *variables, preserving the boundary* $x_{nn} = 0$*. However the full group acting is the semi-direct product of this and the* $\mathcal{C}$*-group, denoted by* $\mathcal{K}_\delta$*. The simple singularities of the two equivalences coincide, but the former is a better known equivalence. The* $\mathcal{G}$*-codimension of* A *coincides with that* $\mathcal{K}_\delta-$ *or* $\mathcal{R}_\delta$*-codimension of* f*.*
In both cases the criminant is smooth.
(4) *When the corank of the derivative of the germ is* 2 *then* $n = 3, r = 7$ *and we have the following* 4 *cases (in all cases the criminant is smooth) in Table 1.*
Table 1. $K^7 \to M(3,3)$

Normal form	*Codimension*
$\begin{pmatrix} -x_{33} & x_{12} & x_{13} \\ x_{21} & -x_{33} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	2
$\begin{pmatrix} -x_{22} & x_{21}+x_{33} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	2
$\begin{pmatrix} -x_{22} & x_{12} & x_{13} \\ -x_{33}+x_{12}^k & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \quad k \geq 2$	$2k$
$\begin{pmatrix} -x_{22}-x_{33}\pm x_{13}^k & x_{12} & x_{13} \\ -x_{32}+x_{13}^l & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \quad k=2 \text{ or } 2 \leq l \leq 3$	$k+l$

(5) *When* $n = 2$ *the simple germs that are not covered in (1)-(3) are given in Table 2 and Table 3 below, together with the* $\mathcal{K}$*-type of their discriminant and criminants and their* $\mathcal{G}$*-codimensions.*

Table 2. $K^2 \to M(2,2)$

Normal form	*Discriminant*	*Criminant*	*Codimension*
$(x, y^k, \pm y^l, x), 1 \leq k \leq l$	A_{k+l-1}	$2A_0,\ k = l$ $A_{l-k-1},\ k < l$	$2k + l - 1$
$(x, y, x^2 \pm y^k, 0), 2 \leq k$	D_{k+2}	$A_0 + A_{k-3}$	$k + 3$
$(x, x^2 \pm y^k, y, 0), 2 \leq k$	D_{k+2}	$A_0 \| A_{k-1}$	$k + 3$
(x, y, y^3, x^2)	E_6	A_0	7
(x, y^3, y, x^2)	E_6	(t^3, t^4, t^5)	7
(x, y, xy^2, x^2)	E_7	A_1	8
(x, xy^2, y, x^2)	E_7	$A_0 \| A_2$	8
(x, y, y^4, x^2)	E_8	A_2	9
(x, y^4, y, x^2)	E_8	(t^3, t^5, t^7)	9
$(x, 0, 0, y^2 \pm x^k), 2 \leq k$	D_{k+2}	$A_0 + A_{k-1}$	$k + 4$
$(x, 0, 0, xy + y^k), 3 \leq k$	D_{2k}	$A_0 + A_1$	$3k$
$(x, y^k, \pm y^l, xy), 3 \leq k \leq l$	D_{k+l+1}	$A_1,\ k = l$ $3A_0,\ k + 1 = l$ $A_0 + A_{l-k-2},\ k + 1 < l$	$2k + l + 1$
$(x, \pm y^l, y^k, xy), 3 \leq k < l$	D_{k+l+1}	$A_0 \| A_{l-k}$	$2k + l + 1$
(x, y^2, y^2, x^2)	E_6	A_2	8
$(x, y^2, 0, x^2 + y^3)$	E_7	$2A_0$	9
$(x, 0, y^2, x^2 + y^3)$	E_7	$A_0 \| A_2$	9
$(x, 0, 0, x^2 + y^3)$	E_7	$A_0 + A_2$	10
(x, y^2, y^3, x^2)	E_8	A_0	10
(x, y^3, y^2, x^2)	E_8	(t^3, t^4, t^5)	10

Table 3. $K^3 \to M(2,2)$

Normal form	*Discriminant*	*Criminant*	*Codimension*
$(x, z^k, \pm z^l, y), 1 \leq k \leq l$	A_{k+l-1}	$A_{k-1} + A_{l-1}$	$k + l - 1$
$(x, -y, y + z^k, x)$	A_{2k-1}	$2A_0$	$2k - 1$
$(x, y, z^2 \pm y^k, x), 2 \leq k$	D_{k+2}	$A_0 + D_{k+1}$	$k + 2$
$(x, y, y^2, x + z^2)$	E_6	$A_0 + D_5$	6
$(x, y, y^2 + z^3, x)$	E_7	$A_0 + E_6$	7
$(x, y, yz, x + z^k), 2 \leq k$	D_{2k+1}	$A_0 + A_{2k}$	$2k + 1$
$(x, y, yz + z^k, x), 3 \leq k$	D_{2k}	$A_0 + A_{2k-1}$	$2k$

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(6) *When $n = 3$ the simple germs that are not covered in (1)-(5) are given in Table 4 below, together with the $\mathcal{K}$-type of their discriminant and their $\mathcal{G}$-codimensions.*

Table 4. $K^2 \to M(3,3)$

Normal form	*Discriminant*	*Criminant*	*Codimension*
$\begin{pmatrix} x & y^k & 0 \\ \pm y^l & x & 0 \\ 0 & 0 & y \end{pmatrix}, 1 \leq k \leq l$	D_{k+l+2}	$3A_0,\ l = k$ $A_0 + A_{l-k-1},\ l \neq k$	$2k + l + 4$
$\begin{pmatrix} x & y & 0 \\ 0 & x & y \\ y^{k+1} & 0 & x \end{pmatrix}, 1 \leq k$	$J_{p+1,0},\ k = 3p$ $E_{6(p+1)},\ k = 3p + 1$ $E_{6(p+1)+2},\ k = 3p + 2$	D_4 E_{6p} E_{6p+2}	$2k + 7$
$\begin{pmatrix} x & y & 0 \\ 0 & x & y \\ xy^k & 0 & x \end{pmatrix}, 1 \leq k$	$J_{p+1,0},\ k = 2p$ $E_{6(p+1)+1},\ k = 2p + 1$	$J_{p,0}$ E_{6p+1}	$2k + 8$
$\begin{pmatrix} x & y & 0 \\ y^2 & x & 0 \\ 0 & 0 & x \end{pmatrix}$	E_7	$2A_0$	11
$\begin{pmatrix} x & y & 0 \\ 0 & x & y^2 \\ y^2 & 0 & x \end{pmatrix}$	E_8	A_0	12

Remark 1.1. (1) Over $\mathbb{C}$ all of the signs $\pm$ in Theorem 1.1 can be set to $+$. The second germ in Table 3 is equivalent over $\mathbb{C}$ to the first one in the table for $l = k$.
(2) In the third column of Tables 2-4, the singularities of the criminant are computed in the complex case (in the real case some of them do not appear). We explain the notation in §5 below.
(3) In the case of corank 1 germs, (3) in the Theorem 1.1, it is perhaps more instructive to view the image of the map A in a slightly different way. So in case (i) consider $F : M_n \times K^s, 0 \to K, 0$ defined by $F(A, z) = trace A + f(z)$; then we can think of the given normal form as coming from the natural projection of $F^{-1}(0)$ to M_n. Similarly for (ii) we have $F : M_n \times K^s, 0 \to K, 0$ given by $F(A, z) = \sum_{i=1}^{n-1} a_{ii} + f(a_{nn}, z)$, and project $F^{-1}(0)$ to M_n. This is the alternative approach to the classification we sketch out in the Appendix.

2. BASIC RESULTS AND DEFINITIONS

We start with a few basic facts. Let $V,\ W$ be two vector spaces of dimension n over the field K which is $\mathbb{R}$ or $\mathbb{C}$. Then $\hom(V,W)$ denotes the space of linear maps $V \to W$ and there is a natural action of the product $GL(V)\times GL(W)$ on this space, where $GL(U)$ is the group of linear automorphisms of the vector space U. We need a few basic results from linear algebra which we collect here for convenience. A good general reference is [12].

Proposition 2.1. (1) *There are natural isomorphisms* $\hom(V,W) \equiv V^* \otimes W$ *and* $\hom(V,W) \equiv \hom(W^*,V^*)$. *With respect to bases and their dual bases the second isomorphism is given by taking the transpose of matrices.*
(2) *There is a natural isomorphism* $\hom(V,W)^* \equiv \hom(W,V)$. *This isomorphism is given as follows: if $\theta \in hom(W,V)$ then define $\bar{\theta} : hom(V,W) \to K$ by $\bar{\theta}(\alpha) = trace(\alpha\circ\theta)$. This trace map gives a non-degenerate bilinear form* $\hom(V,W)\times\hom(W,V)\to K$.

A crucial part of the work will be done at the level of 1-jets, where the problems remain linear (indeed reduce to the classical theory of linear systems of matrices). To carry this work out we shall need the following basic results.

In what follows U,T are finite dimensional vector spaces, and H is a subgroup of $GL(T)$. We say that $\alpha_1,\alpha_2 \in \hom(U,T)$ are ($GL(U)\times H$-)equivalent if for some $\beta\in GL(U), \gamma\in H$ we have $\alpha_2 = \gamma\circ\alpha\circ\beta$. If $\alpha\in\hom(U,T)$ we write α^* for the corresponding element of $\hom(T^*,U^*)$. If S is a subspace of U we write $S^\perp$ for $\{\phi\in U^* : \phi(S)=0\}$.

Lemma 2.1. (1) *The maps α_1,α_2 are equivalent if and only if some $\beta\in H$ takes* $\operatorname{im}\alpha_1$ *to* $\operatorname{im}\alpha_2$.
(2) *We have* $(\operatorname{im}\alpha)^\perp = \ker\alpha^*, (\ker\alpha)^\perp = \operatorname{im}\alpha^*$.
(3) *The maps α_1,α_2 are equivalent if and only if some $\beta\in H$ the map β^* takes* $\ker\alpha_1^*$ *to* $\ker\alpha_2^*$.

So classifying maps of rank r in $\hom(U,T)$ up to $GL(V)\times H$-equivalence is equivalent to classifying r-dimensional subspaces of T up to H-equivalence, or $(\dim T - r)$-dimensional subspaces of T^* up to H-equivalence. This provides a useful duality we can exploit.

Here we are interested in the case when $T=\hom(V,W)$ and $H = GL(V)\times GL(W)$ acting as a subgroup of $GL(\hom(V,W))$ in the obvious way. We need to classify k-dimensional subspaces of $\hom(V,W)$ up to H-equivalence. But this classifies n^2-k-dimensional subspaces of $\hom(V,W)^*\equiv\hom(W,V)$ up to $GL(W)\times GL(V)$-equivalence (where $n=\dim V$). The practical output from this discussion is the following. Here we take $V=W=K^n$, $GL(V)=GL_n$ the space of invertible $n\times n$ matrices, $\hom(V,W)=M_n$ the space of $n\times n$ matrices.

Proposition 2.2. *Let $W_i, i\in I$ be a listing of the subspaces of M_n of dimension k up to GL_n^2 equivalence. Then the subspaces $W_i^\perp, i\in I$, where $W^\perp = \{A\in M_n : trace(AB) = 0$ for all $B\in W\}$, is a listing of the subspaces of dimension n^2-k in M_n.*

The space of singular linear maps Δ is itself an extremely singular set in $\hom(V,W)$. The following gives two related good desingularisations $\tilde{\Delta}_j,\ j=1,\ 2$, of Δ. (The projective space associated with a vector space V is denoted by PV.) We also describe the dual of Δ, that is the closure of the set of tangent hyperplanes at smooth points of the projective variety determined by Δ. This shows that although singular the set Δ is not very curved. Its dual indeed is smooth and has dimension $2n-2$ rather than the maximum possible dimension n^2-1.

Proposition 2.3. (1) *The sets*

$$\tilde{\Delta}_1 = \{(A,v) \in \hom(V,W) \times PV : Av = 0\}$$

$$\tilde{\Delta}_2 = \{(A,\psi) \in \hom(V,W) \times PW^* : \psi \circ A = 0\}$$

are smooth of dimension n^2-1 and the projections

$$\pi_1 : \tilde{\Delta}_1 \subset \hom(V,W) \times PV \to \Delta \subset \hom(V,W)$$

$$\pi_2 : \tilde{\Delta}_2 \subset \hom(V,W) \times PW^* \to \Delta \subset \hom(V,W)$$

are isomorphisms over the set of maps of corank 1.
(2) *If we replace $V,\ W$ by $W^*,\ V^*$ respectively then using the natural isomorphisms above we see that $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ are interchanged.*
(3) *There is a natural map $V \times W^* \to \hom(V,W)^*$, $(v,w^*) \mapsto (\alpha \mapsto w^*(\alpha(v))$. This gives a well defined map $\Theta : PV \times PW^* \to P(hom(V,W)^*)$ with Θ a smooth embedding and its image the dual of $\Delta \subset P(\hom(V,W))$.*

Proof: Part (2) is clear, so it is enough to establish (1) in one case. Identifying V and W with K^n and replacing $\hom(V,W)$ by the space of $n \times n$ matrices with entries in K, consider the equations $\sum a_{ij}x_j = 0,\ 1 \leq i \leq n$, and their derivatives with respect to $a_{jk}, 1 \leq j \leq n$. We have the required rank unless $x_k = 0$. Since not all of the x_is are zero the result follows.
(3) The fact that Θ is a smooth embedding is not difficult to establish. We actually show that the dual of the image of Θ is Δ; the result then follows by duality. But any non-zero $\alpha \in \hom(V,W)$ determines a hyperplane in $P(\hom(V,W)^*)$. This plane meets the image of Θ in the set $\{(v,w^*) \in P(V) \times P(W^*) : w^*\alpha(v) = 0\}$, this is singular, that is the plane is tangent to Θ, precisely when $\alpha \in \Delta$. □

This concludes the general introductory material. We now move to the relevant definitions. As motivation we use one of the examples from the introduction. So consider an n-tuple of differential 1-forms in $\mathbb{R}^n$

$$\alpha_i = a_{i1}(x)dx_1 + \ldots + a_{in}(x)dx_n, 1 \leq i \leq n,$$

where the a_{ij} are germs of smooth functions. Clearly we can associate to each such n-tuple the family of matrices $(a_{ij}(x))$ which we can think of as a map $A : \mathbb{R}^n, 0 \to M(n, \mathbb{R})$. The discriminant of the n-tuple, which consists of points where the forms are dependent, is given by $\det(A) = 0$. Given the matrix A above, and two other $n \times n$ matrices X and Y, whose entries are smooth functions in x, and which are invertible at the origin, we can consider the matrix valued function XAY. Clearly its determinant vanishes at precisely the same points as that of A. Similarly it is not hard to show that any smooth change of coordinates in the source of A, via a diffeomorphism ϕ, takes the discriminant of A to that of $A \circ \phi$. This motivates the following definitions.

First some notation. Let $\mathcal{R}$ the group of diffeomorphisms $K^r, 0 \to K^r, 0$. Let $\mathcal{H}$ denote the set of germs of smooth mappings $K^r, 0 \to GL(V) \times GL(W)$, and M the set of germs $A : K^r, 0 \to \hom(V, W)$. We can think of such a germ as an r-parameter family of matrices. The set $\mathcal{H}$ can be given a group structure inherited from the product group in the target.

Definition 2.1. If $A,\ B : K^r, 0 \to \hom(V, W)$ are smooth map germs we say that they are $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ equivalent if and only if for some $(\phi, (X, Y)) \in \mathcal{R} \times \mathcal{H}$ we have $B = X(A \circ \phi)Y$. Giving $\mathcal{G}$ the usual semi-direct product group structure, the map sending A to $X^{-1}(A \circ \phi^{-1})Y$ gives a group action on the space M. (Generally we omit the inverses.)

Relating this back to one of our examples we have the following.

Proposition 2.4. *Given a n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of differential 1-forms we denote the corresponding element in M by $A(\alpha)$. If $\phi : K^n, 0 \to K^n, 0$ is the germ of a diffeomorphism taking α to $\beta = (\beta_1, \ldots, \beta_n)$, then $A(\alpha)$ and $A(\beta)$ are $\mathcal{G}$-equivalent.*

Proof: Denoting the differential of ϕ by $d\phi$ the new n-tuple has corresponding matrix $A(\beta)$ with

$$A(\beta) = (A(\alpha) \circ \phi)(d\phi)$$

and clearly $A(\alpha)$ and $A(\beta)$ are $\mathcal{G}$-equivalent elements of M. (Note that multiplying the 1-forms by non-zero functions leaves the hyperplane fields unchanged, and this also yields $\mathcal{G}$-equivalent n-tuples.) $\square$

An element of M can also be considered as a map $K^r, 0 \to K^N$, where we identify $\hom(V, W)$ with the $n \times n$ matrices, and $N = n^2$.

Lemma 2.2. *The group $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ acts on the space of mappings $K^r, 0 \to K^N \equiv \hom(V, W)$ as a subgroup of the corresponding contact group $\mathcal{K}$ for germs $K^r, 0 \to K^N, 0$(see* [14]*).*

Proof: The action of the group $\mathcal{R}$ in both cases clearly coincides. The group $\mathcal{C}$ is the set of mappings $K^r, 0 \to GL(n^2, K)$. There is a natural homomorphism $\theta : GL(V) \times GL(W) \to GL(\hom(V, W)) = GL(V^* \otimes W)$, or in more concrete terms $GL_n \times GL_n \to GL_{n^2}$, with $det(\theta(X, Y)) = det(X)^n det(Y)^n$. Moreover the map θ gives a group homomorphism $\mathcal{H} \to \mathcal{C}$, and the actions correspond. The result then follows. $\square$

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For example consider the case $n = 2$; here the matrices $(X, Y) \in \mathcal{H}$ act on a mapping $K^r, 0 \to K^4, 0$ in the same way as the matrix $Z \in \mathcal{C}$, where

$$X = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}, \; Y = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}, \; Z = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_3 & \alpha_2\beta_1 & \alpha_2\beta_3 \\ \alpha_1\beta_2 & \alpha_1\beta_4 & \alpha_2\beta_2 & \alpha_2\beta_4 \\ \alpha_3\beta_1 & \alpha_3\beta_3 & \alpha_4\beta_1 & \alpha_4\beta_3 \\ \alpha_3\beta_2 & \alpha_3\beta_4 & \alpha_4\beta_2 & \alpha_4\beta_4 \end{pmatrix}.$$

Remark 2. 1. (1) It is not hard to see that $\mathcal{G}$ is one of Damon's geometric subgroups of $\mathcal{K}$, and as a consequence of results of Damon we can use all of the standard techniques of singularity theory (for example those concerning determinacy and unfoldings) to investigate these singularities. See [7] and [6].
(2) Note that the definitions all make perfect sense in the case when the matrices are not square. Note also that if $\dim V$ or $\dim W$ is 1 then $\mathcal{G}$-equivalence simply reduces to ordinary $\mathcal{K}$-equivalence.

As mentioned above one of our main interests in elements of M is with their associated discriminants. We next discuss some further structure preserved by this equivalence relation.

Definition 2.2. The discriminant of an element $A \in M$ is the set-germ $D(A), 0 = \{x \in K^r : \det A(x) = 0\}, 0$.

Proposition 2.5. *If A, $B \in \mathcal{S}$ are $\mathcal{G}$-equivalent then there is a germ of a diffeomorphism $\phi : K^r, 0 \to K^r, 0$ preserving discriminants, i.e. taking $D(A)$ to $D(B)$.*

Proof: This is as in the discussion at the beginning of this section. □

Remark 2. 2. (1) So given n differential 1-forms in K^n, if we are only interested in the locus of points where they are dependent, then it is enough to consider the family of matrices constructed above up to this notion of equivalence.
(2) The above proposition provides an important tool for detecting moduli. If the discriminant of a matrix has a modulus that can be realized by varying the coefficients of the matrix, then, by this proposition, the matrix is also not simple.

Proposition 2.6. *If $n \geq 2$ and $r \leq 4$ the complement of the set of map-germs $K^r, 0 \to M_n$ whose discriminant has an isolated singularity is of infinite codimension. Conversely if $r \geq 5$ and $n \geq 2$ the set of map germs whose discriminant has a non-isolated singularity is open.*

Proof: Consider, in M_n, the singular matrices Δ; these are of codimension 1. If $n \geq 2$ the singular set of Δ consists of those matrices of corank 2. This is of codimension 4.

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If $r \leq 4$ then it is easy to prove that for all A off a set of map-germs $K^r, 0 \to M_n$ of infinite codimension, A meets the singular set in isolated points, and the discriminant has an isolated singularity. Conversely if $r \geq 5$ the set of mappings for which the discriminant has a non-isolated singular point is clearly open. □

Now suppose given a smooth map germ $A : K^r, 0 \to M_n$. We have defined a desingularisation $\pi : \tilde{\Delta}_1 \to M_n$. We define the criminant to be the corresponding fibre product of A and π.

Definition 2.3. The full criminant of A, $FC(A)$, is defined to be the set

$$\{(x, v) \in K^r \times PV : A(x)v = 0\}.$$

The dual full criminant of A, $FC^*(A)$, is the set

$$\{(x, \psi) \in K^r \times PW^* : \psi \circ A(x) = 0\}.$$

We will shortly see that we can suppose we are dealing with germs of linear rank 0, so the above are set-germs along $0 \times PV$.

In the first case this is the family of solution sets of the family of linear equations provided by A. If the maps A and π are transverse then $FC(A)$ is smooth. While the above definition works well in many situations, it sometimes requires modifying as the fibre at the origin may have higher dimension than that of nearby fibres.

Definition 2.4. The criminant of A, $C(A)$ is defined to be the closure of the set

$$\{(x, v) \in K^r \times PV : A(x)v = 0, \text{ and rank } A = n - 1\}.$$

Similarly for the dual criminant.

The full criminants, criminants and their duals are clearly $\mathcal{G}$-invariants in a natural sense.

Definition 2.5. Given any map-germ $A : K^r, 0 \to \hom(V, W)$ there is a corresponding germ $A^* : K^r, 0 \to \hom(W^*, V^*)$ obtained using the canonical isomorphism given above. We shall say that the germs A and A^* are *dual*.

Remark 2. 3. (1) If A is given as a matrix then A^* is obtained by taking the transpose of A.
(2) It is not difficult to see which germs in our list are self dual (i.e. which A are equivalent to A^*), and to pair up the remainder; dual germs are adjacent in the above lists. Note that dual germs have the same discriminants, degrees of determinacy and $\mathcal{G}$-codimension.
(3) In particular dual map germs may not be equivalent. Indeed we see in the table above that, for example, the germs $(x, y, x^2 \pm y^k, 0)$ and $(x, x^2 \pm y^k, y, 0), 2 \leq k$ are dual but distinct because their criminants are. However the criminant of any germ A is naturally isomorphic to the dual criminant of A^*.

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3. TANGENT SPACES AND DETERMINACY

We seek to classify the simple mappings, that is those germs whose orbits (in all sufficiently large jet spaces) have neighbourhoods containing only finitely many orbits. The first task is to determine the tangent space for the action of this group. Given a family of matrices $A \in M$, we write $A_{x(i)}$ for the matrix $\frac{\partial A}{\partial x_i}$. We shall also write $\mathcal{O}_r$ for the ring of smooth functions $K^r, 0 \to K$, and $\mathcal{M}_r$ for its maximal ideal (of functions which vanish at the origin). The set M can be identified with $\mathcal{O}_r^N$, $N = n^2$, which is an $\mathcal{O}_r$-module, and the group $\mathcal{G}$ as a subgroup of the corresponding contact group $\mathcal{K}$. So the tangent space will be viewed as an $\mathcal{O}_r$-submodule of $\mathcal{O}_r^N$.

PROPOSITION 3.1. (1) *The $\mathcal{R}_e$-tangent space to the orbit of the element $A \in M$ is the $\mathcal{O}_r$-module spanned by the $A_{x(i)}$.*
(2) *Let $R_i(A)$ (resp. $C_j(A)$) denotes the i^{th} row (resp. j^{th} column) of A. Then the extended tangent space to the orbit of A under the subgroup $\mathcal{H}$ of $\mathcal{C}$ is the $\mathcal{O}_r$-module spanned by the set of matrices R_{il} (resp. C_{jm}) with l^{th} row (resp. m^{th} column) $R_i(A)$ (resp. $C_j(A)$) and with zeros elsewhere, for $1 \le l, m \le n$ and $1 \le i, j \le n$, that is*

$$T_e\mathcal{G}.A = \mathcal{O}_r\{A_{x(i)}, R_{il}, C_{jm}\}.$$

Proof: The vectors emerging from the action of the $\mathcal{R}$ group are obtained as usual. Now consider the action of the group $\mathcal{H}$. We consider the action, first on the left, and then on the right of $I + t\alpha E_{ij}$ on the matrix A for t small and $\alpha \in \mathcal{O}_r$. Computing the tangent vector of the resulting paths in M at $t = 0$ yields, for $1 \le i,\ j \le n$, the required result. □

We next characterize finitely $\mathcal{G}$-determined germs in the complex case. Recall that the action of $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ on $M(n,\mathbb{C})$ has $n+1$ orbits, with representatives diagonal matrices with s 1's and $n-s$ 0's. These orbits determine a stratification of $M(n,\mathbb{C})$. (Here $GL(n,\mathbb{C}), M(n,\mathbb{C})$ are simply GL_n, M_n with $K = \mathbb{C}$.)

PROPOSITION 3.2. *An element $A : \mathbb{C}^r, 0 \to m(n,\mathbb{C})$ is finitely $\mathcal{G}$-determined if and only A is transverse to the orbit stratification of $M(n,\mathbb{C})$ off $0 \in \mathbb{C}^r$. In particular there are no points x in a neighbourhood of the origin at which the matrix $A(x)$ has corank s where $r \le s^2$.*

Proof: We follow the arguments of Gaffney (see [14]). We say that A is $\mathcal{G}$-stable if $T_e\mathcal{G}.A = M$. If A is to be $\mathcal{G}$-stable then working modulo $\mathcal{M}_r.M$ we see that the vectors $A_{x(i)}(0)$ must be transverse to the $GL(n,\mathbb{C}) \times GL(n,\mathbb{C})$ orbit of $A(0)$, and the converse follows by Nakayama's lemma.

We now need to sheafify the above constructions to obtain the result. Choose a neighbourhood U of $0 \in \mathbb{C}^r$, and a neighbourhood V of $A(0)$ in $M(n,\mathbb{C})$, with $A : U \to V$. We consider the sheaf $M(U)$ of germs $U \to M(n,\mathbb{C})$. Let $\mathcal{O}(U)$ be the sheaf of functions on

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U and $\mathcal{O}(U,V)$ the sheaf of mappings from U to V. We have an $\mathcal{O}(U)$-homomorphism $tA : M(U) \to \mathcal{O}(U,V)$, and an $\mathcal{O}(U)$-submodule, $\mathcal{N}$, generated by the R_{il} and C_{jm}. We now set

$$\mathcal{T}_{\mathcal{G}}(A) = \mathcal{O}(U,V)/\{tA(M(U)) + \mathcal{N}\}.$$

This is a coherent sheaf, and by the Nullstellensatz for coherent sheaves its stalk at 0 is of finite dimension over $\mathbb{C}$ if and only if there is a neighbourhood $U' \subset U$ such that $\mathcal{T}_{\mathcal{G}}(A)$ vanishes on $U' - \{0\}$. But the stalk at any point x can be identified with the extended $\mathcal{G}$-tangent space, and the first result follows.

Finally note that the set of matrices of corank s has codimension s^2 in $M(n,\mathbb{C})$. Suppose given a map-germ $A : \mathbb{C}^r, 0 \to M(n,\mathbb{C})$ with $A(0) = (0)$. If it is transverse to the orbit stratification, the inverse image of the set of corank s matrices is, in a punctured neighbourhood of the origin, a smooth manifold of dimension $r - s^2$, so is empty if $r \leq s^2$. □

It is natural to ask how common $\mathcal{G}$-finitely-determined germs are. A standard result in singularity theory asserts that almost all mappings are finitely $\mathcal{K}$-determined. A small variant on the usual proofs (see for example [14]) of this result yields the following.

Proposition 3.3. *For all r and n the set of germs $A : K^r, 0 \to M_n$ which are not of finite $\mathcal{G}$-codimension form a set of infinite codimension in M.*

Proof: We first complexify. It is enough to show that for all germs $A : \mathbb{C}^r, 0 \to M(n,\mathbb{C}) = \mathbb{C}^N$ of polynomial maps of degree say m we can find a polynomial map $B : \mathbb{C}^r, 0 \to M(n,\mathbb{C})$ of degree $m+1$ with $j^m B = A$ and B finitely $\mathcal{G}$-determined. Let H^{m+1} be the space of homogeneous polynomial maps $\mathbb{C}^r, 0 \to M(n,\mathbb{C})$ of degree $m+1$ and consider the map $G : \mathbb{C}^r \times H^{m+1} \to M(n,\mathbb{C})$ given by $G(x,C) = A(x) + C(x)$. It is not difficult to see that G is a submersion when restricted to $(\mathbb{C}^r - \{0\}) \times H^{m+1}$, and in particular $G : (\mathbb{C}^r - \{0\}) \times H^{m+1} \to M(n,\mathbb{C})$ is transverse to the discriminant Δ (the set of singular matrices). Now the Thom transversality lemma shows that for almost all $C \in H^{m+1}$ we have $G(-,C) : (\mathbb{C}^r - \{0\} \to M(n,\mathbb{C})$ transverse to Δ. We take $B = A + C$; it is finitely $\mathcal{G}$-determined by the geometric criterion given in Proposition 3.2. The real case follows in the usual way. □

We need to relate our equivalence relation with Damon's notion of $\mathcal{K}_V$-equivalence, where V is a variety in $\mathbb{C}^n$; see [9]. Here we consider V to be the set Δ of matrices in $M(n,\mathbb{C})$ with zero determinant. The key is the following result about vector fields.

Proposition 3.4. *The module of holomorphic vector fields tangent to the discriminant Δ, denoted by $Derlog(\Delta)$, coincides with the module $V\mathcal{H}$ of vector fields on M_n determined by the group $\mathcal{H}$. More precisely $Derlog(\Delta)$ is the $\mathcal{O}_N$-module of vector fields generated by the R_{il}, C_{jm}. (Note that since $\hom(V,W)$ is a vector space a vector field is simply a smooth map $\hom(V,W) \to \hom(V,W)$.)*

Proof: Clearly $V\mathcal{H} \subset Derlog(\Delta)$, since $\mathcal{H}$ preserves Δ. Conversely let $\theta = \sum \alpha_{ij}\partial/\partial a_{ij}$ be a vector field tangent to Δ; then $\theta(\det(a_{ij})) = \alpha \det(a_{ij})$ for some analytic α. However

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the Euler field lies in both modules, so subtracting some multiple of this field from θ we are considering $\theta(\det(a_{ij})) = 0 = \sum \alpha_{ij}(-1)^{i+j}M_{ij}$ where M_{ij} is the $(i,j)^{th}$ minor of A. However it is well known that the set of relations on the M_{ij} are generated by the linear ones obtained from the identity $A.adj(A) = (\det A).I$. So there are essentially n^2-1 of them arising from $\sum(-1)^{i+j}a_{ij}M_{ik} = \delta_{jk}\det A$. It easily follows that $\sum a_{ij}\partial/\partial a_{ik}$ for $1 \leq j,k \leq n$ generate $Derlog(\Delta)$. But these are our generators for $V\mathcal{H}$. □

Corollary 3.1. *Over $\mathbb{C}$ the notions of $\mathcal{K}_\Delta$ and $\mathcal{G}$-equivalence coincide for finitely determined germs.*

Proof: Clearly the group action of $\mathcal{H}$ preserves the set Δ, so $\mathcal{G} \subset \mathcal{R} \times \mathcal{C}_\Delta = \mathcal{K}_\Delta$. Since the tangent spaces coincide germs are finitely determined with respect to one equivalence if and only if with respect to the other. Given such a germ A work in a sufficient jet space. Clearly $J^k\mathcal{G} \subset J^k\mathcal{K}_\Delta$, so we have an inclusion of orbits $J^k\mathcal{G}.A \subset J^k\mathcal{K}_\Delta.A$. We need to establish the reverse inclusion. To do this we apply Mather's lemma, stated below in 4.3 (see also [14]); the group $J^k\mathcal{G}$ is connected, the dimension of the tangent space to its orbits at points of $J^k\mathcal{K}_\Delta$ is constant, and we have an inclusion, indeed an equality of tangent spaces. So $J^k\mathcal{K}_\Delta.A$ is contained in a single orbit, namely $J^k\mathcal{G}.A$. □

The result above means that, for example, the geometric characterization of finite determinacy given above can be deduced from work of Damon. However a priori the calculation of the vector fields tangent to the space Δ of singular matrices looks rather complicated, whereas the tangent space to the action of $\mathcal{H}$ seems more amenable and natural.

Finally, we have the following property of the criminant.

Proposition 3.5. *If $A : \mathbb{C}^r, 0 \to M(n,\mathbb{C})$ is finitely-$\mathcal{G}$-determined then $C(A)$ is the germ of a variety (along $0 \times \mathbb{C}P^{n-1}$) of dimension $r-1$, and is smooth off $0 \times \mathbb{C}P^{n-1}$.*

Proof: If A is finitely $\mathcal{G}$-determined we know that A is transverse, on a punctured neighbourhood of $0 \in \mathbb{C}^r$, to the stratification of $M(n,\mathbb{C})$. In particular the closure of the set $\{x : rank\ A(x) = n-1\}$ is the whole of $D(A)$. It follows that A and π are transverse at all points away from 0 and that $C(A)$ is smooth off $0 \times \mathbb{C}P^{n-1}$, and of the right dimension, $r-1$. □

4. SIMPLICITY AND THE CLASSIFICATION

We start with a splitting result. Given two map germs $A_j : K^{r_j}, 0 \to M_{n_j}$, $j = 1,2$, let $A_1 \oplus A_2 : K^{r_1+r_2}, 0 \to M_{n_1+n_2}$ denote the mapping given by

$$(x,y) \mapsto \begin{pmatrix} A_1(x) & 0 \\ 0 & A_2(y) \end{pmatrix}.$$

Proposition 4.1. *Suppose given a germ $A : K^r, 0 \to M_n$ with linear rank s at the origin. Then A is $\mathcal{G}$-equivalent to a germ of the form $I_s \oplus B$ where $B : K^r, 0 \to M_{n-s}$*

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has rank 0 *at* 0*, and* I_s *is the* $s \times s$ *identity matrix. The* $\mathcal{G}$*-codimension of* A *is equal to that of* B*, their unfoldings are naturally isomorphic and their discriminants coincide. The simplicity of one is equivalent to the simplicity of the other. We say that* A *is a suspension of* B*.*

Proof: This is simply row and column reduction of matrices but with parameters. □

This means from now on that we need only consider germs whose linear rank is zero; that is we may suppose from now on that all germs vanish at $0 \in K^r$.

We shall use the determinacy methods from [6] and complete transversal methods from [5]; this is simply a formulation of the usual inductive approach to classification by degree. The result we require is the following, deducible from [5].

Proposition 4.2. *Let* A *be a* k*-jet in* J^kM*, write* $\mathcal{G}_1$ *for the subgroup of* $\mathcal{G}$ *of elements with 1-jet the identity, and* $H^{k+1}M$ *for the homogeneous mappings in* M *of degree* $k+1$*. Suppose that* $T_AJ^{k+1}\mathcal{G}_1.A \cap H^{k+1}M + T = H^{k+1}M$ *for some subspace* T *of* $H^{k+1}M$*. Then every* $(k+1)$*-jet with* k*-jet* A *is* $J^{k+1}\mathcal{G}$*-equivalent to a jet of the form* $A+B$*, with* $B \in T$*. Such a* T *is called a complete transversal.*

This result provides a powerful tool but sometimes more direct (and brutal) calculation is required using Mathers lemma, mentioned above, which we state for completeness.

Proposition 4.3. *Let* G *be a Lie group acting on a smooth manifold* M *and let* N *be a connected submanifold of* M*. Then if for all* $x \in N$ *we have (i)* $T_xN \subset T_xG.x$ *and (ii)* $\dim T_xG.x$ *is constant for* $x \in X$*, then* N *is contained in a single orbit of* G*.*

As is usual a key issue is to get the induction started, that is identify the relevant 1-jets. Again as is frequently the case this reduces to a classical linear problem, and we seek conditions on r, n and other invariants which need to hold for there to be any simple germs.

The next result gives a criterion for the existence of moduli.

Proposition 4.4. *Let* G *be an algebraic group acting on a* K*-affine space* L*. Let* S *be a smooth subvariety of* G *with the property that the set* $\{s \in S : T_sS \subset T_sG.s\}$ *is a proper subvariety of* S*. Then given any* $s \in S$*, any neighbourhood* U *of* $s \in S$ *contains infinitely many* G*-orbits.*

Proof: Suppose on the contrary that some open set U contains finitely many orbits. Then the intersection of some orbit with S contains an open set, a contradiction. □

We give a brief description of 1-jets and the action we are considering on the 1-jet space. For this discussion it is more convenient to write K^r as U. Then a 1-jet is simply a linear map $\alpha : U \to \hom(V,W)$, that is an element of $\hom(U, \hom(V,W))$. The natural equivalence is via the group $GL(U) \times GL(V) \times GL(W)$. As we have seen if we are looking at germs of rank s we need to classify subspaces of $\hom(V,W)$ of dimension s up to $GL(V) \times GL(W)$-equivalence.

PROPOSITION 4.5. (1) *There are no simple germs if* $n \geq 3$ *and* $3 \leq r \leq n^2 - 3$.
(2) *If a simple germ has 1-jet* $\alpha \in \hom(U, \hom(V, W))$ *then either:*
(i) $n = 1, 2$ *or* $r = 1$*;*
(ii) *for* $n \geq 3$ *rank* $\alpha = 1, 2$ *or corank* $\alpha = 0, 1, 2$.

Proof: (1) As remarked the action of $J^1\mathcal{G}$ reduces to the action of $GL(U) \times GL(V) \times GL(W)$ on $\hom(U, \hom(V, W))$. The dimension of the group is $r^2 + 2n^2$, but considering the action of multiples of the identity of each of the three components one can see that the dimension of any orbit is at most $r^2 + 2n^2 - 2$. Now we clearly have moduli if the dimension of each orbit is less than the dimension of the space, so we have moduli if $r^2 - rn^2 + 2n^2 - 1 \leq 0$. One can check that for $n \geq 3$ this inequality holds if and only if $3 \leq r \leq n^2 - 3$. When $n = 1,\ 2$ it never holds.
(2) Suppose that we are considering germs of rank s so we consider s-dimensional subspaces of $\hom(V, W)$. This Grassmannian $Gr(s, n^2)$ has dimension $s(n^2 - s)$. We consider the action of GL_n^2 on this Grassmannian. The dimension of the group is $2n^2$ and of the space $s(n^2 - s)$. However by considering the effect of the action of multiples of the identities in the GL_n we see that the dimension of each orbit is at most $2n^2 - 2$ so we must have moduli at the 1-jet level if $sn^2 \geq r^2 + 2n^2 - 1$. This occurs if $n \geq 3$ and $3 \leq s \leq n^2 - 3$. So for $n \geq 3$ we need $s = 0, 1, 2$ or $s = n^2 - 2, n^2 - 1, n^2$ for $\mathcal{G}$-simplicity. This means that for $n \geq 3$ we have rank $\alpha \leq 2$ or corank $\alpha \leq 2$. There are no conditions when $n = 1,\ 2$. Note that if the rank is 0 then we can consider the corresponding space of homogeneous 2-jets. These form a space of dimension $n^2 r(r+1)/2$, with a group acting of effective dimension $2n^2 + r^2 - 2$. It is not difficult to see we have moduli unless $r = 1$ or $n = 1$. $\square$

We have seen that classifying subspaces of M_n of dimension $1, 2$, yields a classification of subspaces of codimension $1, 2$. Our first task is to list these.

PROPOSITION 4.6. (1) *The subspaces of dimension* 1 *are spanned by the matrices* $I_t^*, 1 \leq t \leq n$ *which have* 1*'s in the first* t *diagonal entries and zeros elsewhere.*
(2) *If* $n = 2$ *the subspace of dimension* 2*, the so called pencils, have normal forms (using the vector notation of the main theorem):* $(x, 0, 0, y)$, $(x, -y, y, x)^*$, $(x, y, 0, x)$, $(x, 0, y, 0)$, $(x, y, 0, 0)$. *Over* $\mathbb{C}$ *the pencil marked by an asterisk is not needed; similarly in (3) below. The open orbits are provided by the first (two in the* $\mathbb{R}$ *case) pencil(s).*
(3) *If* $n = 3$ *the pencils have the normal forms, and are labelled as follows:* $x(E_{11} + E_{33}) + y(E_{22} + E_{33}), (Ii)$*;* $x(E_{11} + E_{22}) + y(E_{12} - E_{21} + E_{33})^*, (Iii)$*;* $x(E_{11} + E_{22}) + y(E_{12} + E_{33}), (IIi)$*;* $x(E_{11} + E_{22}) + yE_{33}, (IIii)$*;* $xI + y(E_{12} + E_{23}), (IIIi)$*;* $xI + yE_{12}, (IIIii)$*; (these 6 are the non-singular pencils).* $x(E_{12} + E_{21}) + y(E_{13} + E_{31})$*;* $x(E_{11} + E_{22}) + y(E_{13} + E_{21})$*;* $x(E_{11} + E_{22}) + y(E_{12} + E_{31})$*;* $x(E_{12} + E_{21}) + yE_{13}$*;* $x(E_{12} + E_{21}) + yE_{31}$ *(the irreducible singular pencils).* $A \oplus 0$*, where* A *is one of the pencils in (2) considered as* 3×3 *matrices in the obvious way.*
(4) *There are no simple pencils if* $n \geq 4$.

Proof: This is a by-product of a classical set of results, due to Weierstrass and Kronecker, see [11]. However most references consider pairs, not pencils, and usually the complex case only, so clean lists are not easy to find. So it may be more informative to briefly describe an ad-hoc approach. If the pencil contains a matrix of maximal rank (classically referred to as the nonsingular case) then one reduces it to the identity and can act on a second matrix by conjugation. One then uses the Jordan normal form (or rational normal form in the real case) and cleans up using the fact that one can make a linear change of (x, y) variables. For the singular pencils (those not containing a matrix of maximal rank) choose one of maximal rank (2 or 1), and reduce the second to normal form, using the fact that no combination can have rank greater than the first. The final part (4) follows because given a pencil, say $xA + yB$, the binary form $\det(xA + yB)$ is, up to the obvious action of GL_4, an invariant of our equivalence. This form is of degree n and so has moduli if $n \geq 4$ (the cross ratio invariant can clearly be realized by varying A and B). ∎

Remark 4. 1. (1) There is a geometrical interpretation for these orbits. Consider the case $n = 2$. Projectivising the set of matrices M_2 determines a 3-dimensional projective space PM_2, and the singular matrices form a quadric $\{(a : b : c : d) : ad - bc = 0\}$ in this space. The 1-jet $xA + yB$ is a line through A and B (i.e. a pencil of matrices), and the position of this line in relation to the quadric yields the various orbits. In the 5 given cases the line respectively: meets the quadric in two real points, in two imaginary points, in a single point (is tangent to the quadric), and lies on the quadric in the last two cases (providing examples of each of the two rulings of the cone).
(2) Similarly in the case $n = 3$ the singular matrices form a cubic hypersurface in projective 8-space PM_3. One can distinguish the nonsingular pencils by considering the number of times the line meets this cubic and the rank of the matrix at those points, that is the strata it meets. For the irreducible singular pencils (lying on Δ) one can distinguish the various types by considering the ideal generated by 2×2 minors and whether there is a constant vector v with $vA = 0$ or $Av = 0$ for all x, y.
(3) It is useful to list the adjacencies for these orbits, at least in the nonsingular case, that is those orbits lying in the closure of others. It is straightforward to show that it is as follows, where an arrow means that the source contains the target in its closure: $Ii, Iii \to IIi; IIi \to IIii; IIi \to IIIi; IIIi, IIii \to IIIii$. Note that adjacency is transitive.

We now deal with the cases of germs of corank $0, 1$.

Proposition 4.7. (1) *If corank $\alpha = 0$ then the 1-jet is a submersion. Any germ $A \in M$ with such a 1-jet is 1-$\mathcal{G}$-determined, indeed $\mathcal{G}$-stable in the natural sense. A normal form is obtained by placing a different co-ordinate function in each of the n^2 places.*
(2) *If corank $\alpha = 1$ then $r \geq n^2 - 1$ and we label the first $n^2 - 1$ of the x-variables x_{ij}, $1 \leq i, j \leq n$, $(i, j) \neq (1, 1)$. Then α is equivalent to a 1-jet of the form*

$$(\sum_{i=2}^{t} x_{ii})E_{11} + \sum_{(ij)\neq(11)} x_{ij}E_{ij}$$

where $1 \leq t \leq n$ (the case $t = 1$ above is interpreted as the first sum being absent).

Proof: The proof of (1) is straightforward. For (2), we can proceed directly, re-arrange coordinates and write the 1-jet as $f(x)E_{11}+\sum_{(ij)\neq(11)} x_{ij}E_{ij}$, where x consists of the n^2-1 variables x_{ij}. Further linear changes of coordinates together with the GL_n^2 action reduces f to the desired form. An alternative approach is as that described previously; namely we know that the $I_t^*, 1 \leq t \leq n$ span the 1-dimensional subspaces of M_n up to our equivalence. The dual spaces of codimension 1 are given by considering $\{X \in M_n : trace(XI_t^*) = 0\}$, and the result follows. □

Proposition 4.8. *The classification in case (ii) in Proposition 4.7 reduces to the classification of function germs* $K^s \times M_m, (0,0) \to K$ *where* $m = n - t$, $s = r - n^2 + 1$. *The relevant group of equivalences is the semi-direct product of the group of diffeomorphisms preserving* $\Lambda = K^s \times \Delta$ *and the usual group* $\mathcal{C}$ *for function germs, denoted* $\mathcal{K}_\Lambda$.

Simple germs occur only when $t = n$ *or* $t = n - 1$. *These reduce respectively to the classification of functions, functions on a manifold with boundary, with the* $\mathcal{C}$ *group present.*

Proof: We consider the tangent space to the orbit of the given germ. Let z denote the missing variables in the 1-jet. A short calculation shows that a complete transversal is obtained by adding a general term $h(X, z)$ in the $(1,1)$ entry, where X denotes the variables x_{ij} with $t+1 \leq i, j \leq n$.

We can make any change of co-ordinates in the z variables. On the other hand if we wish to preserve the form of the complete transversal the change of co-ordinates in the X variables must be retrievable from the action of the group $\mathcal{H}$. This means that if $\mathcal{H}_t$ is the subgroup of $\mathcal{H}$ corresponding to those pairs of matrices of the form $(I \oplus A, I \oplus B)$, with I the identity $t \times t$ matrix and A, B general $m \times m$ matrices, then we can change co-ordinates on the X variables by the action $X \mapsto AXB$, where A, $B : K^s \times M_m \to GL(m, K)$. One can check that this is a change of co-ordinates. It follows that if two functions $h_1(X, z)$ and $h_2(X, z)$ are equivalent as above then the corresponding map-germs are $\mathcal{G}$-equivalent. On the other hand one can check that the normal spaces for the $\mathcal{G}$-action for the map-germs and the $\mathcal{K}_\Delta$-action for the corresponding function germs are isomorphic. In other words we have identified a subgroup of $\mathcal{G}$, and in any jet space the dimension of the orbits of $J^k\mathcal{G}$ and $J^k(\mathcal{K}_\Delta)$ coincide. So the orbits also coincide as in 3.1.

We finally seek conditions for simple germs. To do this we work in the space of 2-jets, and assume that $h(0, z)$ is not degenerate (the other cases follow by adjacency). Then we can change coordinates in z and write $h = \sum_{i=1}^s \pm z_i^2 + g(X)$. Consider the linear space T of 2-jets of this form; we are classifying up to $J^2(\mathcal{K}_\Delta)$-equivalence. It follows from 4.4 that there will be moduli unless at a generic point of T the tangent space to the orbit contains T. But T has dimension $m^2(m^2+1)/2$ and $J^2(\mathcal{K}_\Delta)$ can yield at most $2m^2$ vectors in T. So we can have simple germs only if $2m^2 \geq m^2(m^2+1)/2$, if and only if $m = 0$ or $m = 1$, that is when $t = n$ or $t = n - 1$.

When $t = n$ the classification reduces to the $\mathcal{K}$-classification of functions, and when $t = n-1$ to that of functions on a manifold with boundary, with the group the semi-direct product of the group of diffeomorphisms preserving the boundary and $\mathcal{C}$. The simple singularities in both cases were determined by Arnold; see [1] and [2]. □

We turn now to the cases when rank $\alpha = 1, 2$.

Proposition 4.9. *If rank $\alpha = 1$ then simple singularities with 1-jet $x_1 I_t^*$ can occur only if either $n = 1$, $r = 1$ or $(r, n) = (2, 2)$.*

Proof: Consider the case $t = n$, and the subspace T of the space of 2-jets of the form $x_1 I_n^* + \sum f_{ij}(x_2, \ldots, x_r) E_{ij}$ where f_{ij} is a general quadratic and the summation is over all $1 \leq i, j \leq n$. We apply 4.4 to this transversal, so there will be no simple jets unless the set of jets in T for which the tangent space to the orbit contains the tangent space to the transversal is open. So we need $n^2 r(r-1)/2$ tangent vectors from the orbit space.

From the $J^2\mathcal{R}$-tangent space we obtain $(3r^2 - 5r + 2)/2$ relevant vectors. The relevant elements from the jet-group $J^2\mathcal{H}$ contribute a subspace of dimension $\leq n^2 - 1$ (when a row is moved one column has to move to eliminate the term x_1). So we will have moduli if $(3r^2 - 5r + 2)/2 + n^2 - 1 < n^2 r(r-1)/2$, which is the case when $r \geq 3$ and $n \geq 2$.

When $r = 2$ and $n \geq 3$ we consider a complete 2-transversal. The discriminant of the matrix has a non-simple singularity and the modulus can be retrieved from the matrix, so we have no simples in this case. Therefore we have simple germs only when $n = 1$, $r = 1$, and $(n, r) = (2, 2)$.

When $t < n$ we use the fact that any finitely determined matrix with 1-jet $x_1 I_t^*$ is adjacent to a finitely determined matrix with 1-jet $x_1 I_n^*$. So simple germs occur only in the above cases. □

Proposition 4.10. *When $n = 3$ simple singularities with a 1-jet of rank 2 can only occur when $r = 2$.*

Proof: When $n = 3$, we consider $A = xE_{11} + yE_{22} + (x+y)E_{33},\ x(E_{11} + E_{22}) + y(E_{12} - E_{21} + E_{33})$; these are the least degenerate 1-jets. In the first case suppose that there are a further m-variables, which we denote by z. We consider a subspace T of the set of 2-jets, those of the form $A + Q(z)$ where $Q(z)$ is a matrix with entries which are quadratic in the z's. We again apply 4.4, and see that we have moduli unless $m = 0$. The second case is similar. □

Remark 4. 2. It follows from these results that the remaining cases we need to consider are as follows:

A. The case $r = 1$.
B. The case $n = 2,\ rank(\alpha) = 1, 2$.
C. The case $n = 3,\ rank(\alpha) = 2,\ r = 2$.
D. The case $n = 3,\ corank(\alpha) = 2,\ r = 7$.

Case A, $r = 1$**.**

Proposition 4.11. *When $r = 1$ all finitely-$\mathcal{G}$-determined germs are simple and $\mathcal{G}$-equivalent to a germ of the form* $\mathrm{diag}(x^{m_1}, x^{m_2}, \ldots, x^{m_n})$ *where $m_1 \leq m_2 \leq \ldots \leq m_n$. This germ has $\mathcal{G}$-codimension* $\displaystyle\sum_{i=1}^{n} (2(n-i)+1)m_i - 1$.

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Proof: This is straightforward using the complete transversal method. If the first non-zero jet is the k^{th} then it is of the form $x^k A$ where A is a constant matrix. If the rank of A is s we reduce to $x^k I_s^* \oplus 0$. It is not hard to see that a complete transversal at the next stage is of the form $x^k I_s^* \oplus x^{k+1} B$, where B is a general $(n-s) \times (n-s)$-matrix. The result follows by induction. ∎

Case B, $n = 2$**.**

Proposition 4.12. *When $n = 2$ the simple germs that have not been covered previously occur when $r = 2$ or $r = 3$ and are as shown in Table 2 and 2 of Theorem 1.1.*

Proof:
Since $n = 2$ we shall write the matrix A as a vector (a, b, c, d). The $\mathcal{G}$-tangent space is spanned by

$$(a_i, b_i, c_i, d_i), 1 \leq i \leq r, (a, b, 0, 0), (0, 0, a, b),$$
$$(c, d, 0, 0), (0, 0, c, d), (a, 0, c, 0), (0, a, 0, c), (b, 0, d, 0), (0, b, 0, d).$$

As normal forms for 1-jets (over $\mathbb{R}$) we have:

$$\begin{aligned} rank(\alpha) = 2 &: \ (x,0,0,y), (x,-y,y,x), (x,y,0,x), (x,y,0,0), (x,0,y,0), \\ rank(\alpha) = 1 &: \ (x,0,0,x), (x,0,0,0) \\ rank(\alpha) = 0 &: \ (0,0,0,0). \end{aligned}$$

We now work our way through the various alternatives, starting with the case $r = 2$.
• *The 1-jets* $(x, 0, 0, y)$ *and* $(x, -y, y, x)$ are 1-determined. In fact $(x, 0, 0, y)$ is equivalent to (x, y, y, x), which is a more convenient form for our list.
• *The 1-jet* $(x, y, 0, 0)$. A complete 2-transversal is given by $A = (x, y, ax^2 + by^2, cx^2 + dy^2)$.
The orbits in this transversal are:

$$(x, y, x^2 \pm y^2, 0), (x, y, x^2, 0), (x, y, 0, x^2), (x, y, 0, 0).$$

The first germ is 2-determined. A complete k-transversal of a $(k-1)$-jet $(x, y, x^2, 0)$ is given by $(x, y, x^2 + ay^k, 0)$, which is equivalent to $(x, y, x^2 \pm y^k, 0)$ if $a \neq 0$. This germ is k-determined and has codimension $k + 3$.

A complete 3-transversal of $(x, y, 0, x^2)$ is given by $(x, y, ay^3 + bxy^2, x^2)$. Using Mather's Lemma, 4.3, one can show that the orbits in this transversal are given by (x, y, y^3, x^2), (x, y, xy^2, x^2), $(x, y, 0, x^2)$. The first two germs are 3-determined.

A complete 4-transversal of $(x, y, 0, x^2)$ is given by $(x, y, ay^4 + bxy^3, x^2)$. If $a \neq 0$ the orbit (x, y, y^4, x^2) is 4-determined and is simple. If $a = 0, b \neq 0$ then a complete 5-transversal of (x, y, xy^3, x^2) is given by $(x, y, xy^3 + cy^5, x^2)$. This is not simple as its discriminant has a unimodular singularity $J_2, 0$ with the modulus depending on c. Subsequently, all higher jets coming from this branch are not simples.

• *The 1-jet* $(x,0,0,x)$. A complete k-transversal of a $(k-1)$-jet $(x,0,0,x)$ is given by $(x,ay^k,by^k,x+cy^k)$. Using Mathers's Lemma one can show that there are 4 orbits in this transversal, $(x,y^k,\pm y^k,x)$, $(x,y^k,0,x)$, $(x,0,0,x)$.

It is not hard to see that $(x,y^k,\pm y^k,x)$ is $k-\mathcal{G}$-determined and has codimension $3k-1$.

A complete l-transversal of an $l-1$-jet$(x,y^k,0,x)$ is given by (x,y^k,dy^l,x), and is equivalent to $(x,y^k,\pm y^l,x)$ if $d\neq 0$. This germ is $l-\mathcal{G}$-determined and has codimension $2k+l-1$.

• The remaining 1-jets are treated similarly.

The case $n=2,\ r\geq 3$*:* There are no simple germs when $rank(\alpha)\leq 1$ (Propositions 4.5 and 4.9). The cases $rank(\alpha)\geq 3$ is treated in parts (2) and (3) of Theorem 1.1. So the only case to consider is $rank(\alpha)=2$,

• We start by showing that when $rank(\alpha)=2$, there are no simples when $r\geq 4$. We consider the two open orbits.

The 1-jet $(x,0,0,y)$*:* using the complete transversal results, one can show that any germ with 1-jet $(x,0,0,y)$ is equivalent to $(x,a(z),b(z),y)$ for some functions a,b of the remaining $s=r-2$ variables. It is not hard to see that the classification reduces to that of the pair (a,b) under a subgroup of $\mathcal{K}$ whose tangent space is the $\mathcal{O}(z)$-module generated by $(a_{z_i},b_{z_i}), 1\leq i\leq s$, $(a,0)$ and $(0,b)$. Working in the 2-jet space whose dimension is $s(s+1)$ the dimension of the group acting is s^2+1. So we have simples only when $s=1$, that is when $r=3$. In this case the finitely determined pairs are equivalent to $(z^k,\pm z^l)$, yielding $(x,z^k,\pm z^l,y)$ for some $k,l\geq 2$. The codimension of this germ is $k+l-1$.

The 1-jet $(x,-y,y,x)$*:* here a complete transversal can be taken of the form $(x,-y,y+a(z),x+b(z))$ for some functions a,b. Again calculation shows that there are no simple germs when $r\geq 4$, and when $r=3$ the finitely determined jets are equivalent to $(x,-y,y+z^k,x)$ for some $k\geq 2$. The codimension of this orbit is $2k-1$. (Note that rather surprisingly the 1-jet stems of these two families are equivalent over $\mathbb{C}$, but the first gives a doubly indexed family and the second a single indexed family over $\mathbb{R}$.)

By adjacency the remaining 1-jets of rank 2 can only yield simple germs when $r=3$. These 1-jets are the normal forms: $(x,y,0,x)$, $(x,y,0,0)$, $(x,0,y,0)$.

The 1-jet $(x,y,0,x)$, $r=3$. A complete 2-transversal can be taken on the form $(x,y,ay^2+byz+cz^2,x+dz^2)$. The orbits in the 2-jet are $(x,y,y^2\pm z^2,x)$, (x,y,z^2,x), $(x,y,yz,x+z^2)$, (x,y,yz,x), $(x,y,y^2,x+z^2)$, (x,y,y^2,x), $(x,y,0,x+z^2)$, $(x,y,0,x)$. The last two lead to non-simple germs (to see this, consider the discriminant of the germs in the 3-transversal, which has a parameter modulus). The rest are analyzed individually and yield to the finitely determined jets in Table 3.

The 1-jets $(x,y,0,0)$ *and* $(x,0,y,0)$, $r=3$. These do not lead to simple germs for $r=3$.

□

Case C, $n=3,\ rank(\alpha)=2,\ r=2$. This is similar to Case B and is omitted.

Case D, $n=3,\ corank(\alpha)=2$.

• Recall that the classification of 1-jets of corank 2 is dual to the classification of 1-jets of rank 2. We have two open orbits, we start with the pencil *Ii*: $x(E_{11}+E_{33}+y(E_{22}+E_{33})$.

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The dual 1-jet of corank 2, say β, is

$$\begin{pmatrix} -x_{33} & x_{12} & x_{13} \\ x_{21} & -x_{33} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

If we have only these 7 variables then this germ is 1-determined. Suppose now that we have an additional m variables, $z_i,\ 1 \leq i \leq m$. Then a complete 2-transversal is given by $\alpha + f(z)E_{11} + g(z)E_{22}$, where f and g are homogeneous of degree 2 in the z_i. We apply 4.4 to this affine space T. In fact since we have all vectors off the diagonal one can reduce to looking at the diagonal elements only. We have $(-x_{33} + f, -x_{33} + g, x_{33})$. It is not hard to see that the only $J^2\mathcal{G}$-tangent vectors to T we can obtain come from $(f, g, 0)$ and the $z_i(f_j, g_j, 0),\ 1 \leq i, j \leq m$. Note that the first vector is contained in the span of the following m^2. So we have $m(m+1)$ independent tangent vectors to T and only m^2 possible candidates, so we deduce that there are moduli unless $m = 0$. The other open 1-jet is similar; again we have moduli unless there are no extra variables. By adjacency it follows that we may suppose that $r = 7$ in what follows. In particular these maps are immersions. There is consequently an alternative approach to this part of the classification, which is often computationally easier; see the Appendix.

- The dual of the *IIi* pencil $x(E_{11} + E_{22}) + y(E_{12} + E_{33})$ is

$$\begin{pmatrix} -x_{22} & x_{12} & x_{13} \\ -x_{33} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

Taking the latter α as a $(k-1)$-jet a complete transversal at the level of a k-jet is $\alpha + ax_{12}^k E_{21}$. If $a \neq 0$ then we can change co-ordinates to ensure $a = 1$ and the result is k-$\mathcal{G}$-determined.

- The dual of the *IIIi* pencil $xI + y(E_{12} + E_{23})$ is

$$\begin{pmatrix} -x_{22} - x_{33} & x_{12} & x_{13} \\ -x_{32} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}.$$

Taking the latter α as a $(k-1)$-jet a complete transversal at the level of a k-jet is $\alpha + ax_{13}^k E_{11} + bx_{13}^k E_{21}$. We can scale a and b independently to be 0 or ± 1. If one is non-zero we calculate another transversal. In this way we obtain normal forms $\alpha + ax_{13}^k E_{11} + bx_{13}^l E_{21}, 2 \leq k, l$. However we cannot scale $a,\ b$ independently if $2l - 3k + 1 = 0$. The first case where moduli occur is $(k, l) = (3, 4)$, so in fact we must have $k = 2$ or $l = 2$ or 3. The germs are $\max(k, l)$-$\mathcal{G}$-determined.

- The dual of the *IIii* pencil $x(E_{11} + E_{22}) + yE_{33})$ is

$$\begin{pmatrix} -x_{22} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{pmatrix}$$

A complete 2-transversal for this 1-jet α is $\alpha+f(x_{12}, x_{21}, x_{22})$, where f is a general quadratic function. Taking this as an affine space T one can apply 4.4 to show that we have moduli. So by adjacency we need not consider the pencil labelled *IIIii*.

5. CRIMINANTS

In this section we briefly consider the geometry of the germs and in particular the criminants, and explain the notation used in the tables. In a later paper we will explain the relationships between the criminant and discriminant and explore this geometry further. In what follows we will consider the complex case.

We have seen that if $A : \mathbb{C}^r, 0 \to M_n$ is finitely $\mathcal{G}$-determined then the criminant is a variety of dimension $r-1$ and smooth off $0 \times \mathbb{C}P^{n-1}$. So in the case $r = 1$ the criminant is empty and when $r = 2$ we get a curve. Indeed in this case the discriminant is a plane curve and we can parameterize its branches and lift to the corresponding branch of the criminant.

Proposition 5.1. *Let $A : \mathbb{C}^2, 0 \to \hom(V, W)$ be finitely $\mathcal{G}$-determined.*
(1) *If the pencil of matrices corresponding to the 1-jet of A is nonsingular (so some element is nonsingular) then the multiplicity of the discriminant is $n = \dim V = \dim W$, and the number of distinct singular tangents coincides with the number of distinct eigenvalues of the pencil, with multiplicities also coinciding.*
(2) *Each component of the criminant has a limiting point on $0 \times \mathbb{C}P^{n-1}$. There is an eigenvalue associated to the corresponding component of the discriminant and the limiting point on $0 \times \mathbb{C}P^{n-1}$ lies in the kernel of this element of the pencil. In particular if the singular elements of the pencil all have corank 1 then the limiting points coincide with this set of kernel points in $\mathbb{C}P^{n-1}$.*

Proof: The first part is clear. For the second parameterize a component of the discriminant as $(x(t), y(t))$ and write $x(t) = t^k x_1(t), y(t) = t^k y_1(t)$ with $x_1(0)$ or $y_1(0) \neq 0$. Suppose the corresponding kernel vector is $U(t) = (u(t), 1)$. Now $A(0) = 0$, write its 1-jet as $xA_x + yA_y$. Then considering lowest order terms we have $(x_1(0)A_x + y_1(0)A_y)(U(0)) = 0$. Clearly $(x_1(0), y_1(0))$ corresponds to a singular element of the pencil, and $U(0)$ is in its kernel. □

We now consider some of the germs in Table 2 of Theorem 1.1. Consider for example the discriminant of the germ $A = (x, y^k, y^l, x)$; this is given by $\delta = x^2 - y^{k+l}$ which has an A_{k+l-1}-singularity at 0. Suppose that $k+l = 2p+1$ (the case when $k+l$ is even is similar, but we need to consider two branches). Then δ is parametrised by (t^{k+l}, t^2). The kernel of the corresponding matrix $A(t)$ is then spanned by $(1 : -t^{l-k})$, so if we take the affine chart $u = 1$ in $(u : v) \in \mathbb{R}P^1$, the criminant is given by $(t^{k+l}, t^2, -t^{l-k})$, which is equivalent to a plane curve with an A_{l-k-1}-singularity.

The discriminant of the germ $A = (x, y, x^2 + y^k, 0)$ is given by $\delta = y(x^2 + y^k)$ which has a D_{k+2}-singularity. When $y = 0$, the kernel of A is spanned by $(0 : 1)$, so in the affine chart $v = 1$, the corresponding branch of the criminant is parametrised by $(t, 0, 0)$, which is smooth. The other branch $x^2 + y^k = 0$ of δ is parametrised by $(t^k, -t^2)$ (when k is odd; the case k even follows in the same way), and the kernel of the corresponding matrix $A(t)$ along this branch is spanned by $(1 : t^{k-2})$. So the corresponding branch of the criminant in the chart $u = 1$ is given by $(t^k, -t^2, t^{k-2})$. This is equivalent to a plane curve with an A_{k-3}-singularity. So the whole criminant has two separate branches of type A_0 and A_{k-3}.

The germ $A = (x, x^2 + y^k, y, 0)$ has also a discriminant with a D_{k+2}-singularity, $\delta = y(x^2 + y^k)$. When $y = 0$, the corresponding branch of the criminant is parametrised by $(t, 0, -t)$ in $v = 1$ The other branch $x^2 + y^k = 0$ yields a branch (when k is odd) parametrised by $(t^k, -t^2, 0)$ in the same plane $v = 1$. So the criminant has two branches at the same point given by $\{(t, 0, 0), (t^k, -t^2, 0)\}$. This is equivalent to a plane curve with a D_{k+2}-singularity. So the criminant distinguishes the two orbits $(x, y, x^2 + y^k, 0)$ and $(x, x^2 + y^k, y, 0)$.

Generally the criminant (C) is a curve for the cases in Table 2 and Table 4 and a surface containing the projective line for the cases in Table 3. The notation A_0 means that C is smooth; $X + Y$ indicates that C has two singularities at different points on the projective line, one of type X and the other of type Y. The notation $A_p|A_q$ (Tables 2 and 3) denotes a single point of C where there are two plane curves singularities of type A_p and A_q lying in two different planes, with intersection locally the exceptional divisor/projective line.

6. APPENDIX

In this section we sketch out an alternative approach to the classification. The results we prove apply in greater generality, so in what follows $V, 0$ is the germ of a vector space, with $D, 0$ a subvariety. We use the notation employed by Damon, and have natural equivalences $\mathcal{K}_D$ for germs $f : X, 0 \to V, 0$, and ${}_D\mathcal{K}$ for germs $V, 0 \to Y, 0$, where X, Y are smooth manifolds. Both are the natural analogues of $\mathcal{K}$-equivalence, the first with elements of $\mathcal{C}$ taking $X \times D$ to D; the second with elements of $\mathcal{R}$ preserving D.

Proposition 6.1. *Let $f : X, 0 \to V, 0$ be an immersion and choose a submersion $g : V, 0 \to Y, 0$ with $imA = f^{-1}(0)$. Given two such germs $f_j, j = 1, 2$, with corresponding submersions g_j, then f_1 and f_2 are $\mathcal{K}_D$-equivalent if and only if g_1 and g_2 are ${}_D\mathcal{K}$-equivalent.*

Proof: We start by showing that f_1, f_2 are $\mathcal{C}_D$-equivalent if and only if there is a diffeomorphism $V, 0 \to V, 0$ preserving D taking imf_1 to imf_2. By definition we have $f_1(x) = \phi(x, f_2(x))$ for some $\phi \in \mathcal{C}_D$. Now since f_1 is an immersion there is an $h : V, 0 \to X, 0$ with $h \circ f_1 = id$. Define $\phi^* : V, 0 \to V, 0$ by $\phi^*(v) = \phi(h(v), v)$. Clearly $\phi^* \in \mathcal{R}_D$ and $\phi^* \circ f_1(x) = \phi(x, f_1(x)) = f_2(x)$. It follows that f_1, f_2 are $\mathcal{K}_D$-equivalent if and only if there is an element of $\mathcal{R}_D$ taking imf_1 to imf_2 if and only if it takes $g_1^{-1}(0)$ to $g_2^{-1}(0)$ if and only if g_1 and g_2 are ${}_D\mathcal{K}$-equivalent. □

Applying this to our situation, where $V = M_n$ and $D = \Delta$, we can classify the submersions $g : M_n \to K^p, 0$ rather than the immersions $A : K^r, 0 \to M_n, 0$. If r is large this is easier to deal with (we are dealing with a map *into* K^{n^2-r} rather than *from* K^r). However we get more useful and complete information if we replace M_n by $M_n \times K^m$ and Δ by

$\Lambda = \Delta \times K^m$. Indeed this extension arises naturally if we deal with the dual issue that gave rise to the splitting result we had 4.1. For if we consider a germ $f : M_n, A_0 \to K^p, 0$ with A_0 of rank r we may suppose $A_0 = I_r^*$ and then M_n is locally diffeomorphic to $M_{n-r} \times K^{2nr-r^2}$, the diffeomorphism taking Δ_n to $\Delta_{n-r} \times K^{2nr-r^2}$. So we reduce to the consideration of germs at $0 \in M_n \times K^m$. We now show that classification up to ${}_\Lambda\mathcal{K}$ equivalence includes our $\mathcal{G}$-classification. Again we give a more general formulation. In what follows D is *not* a analytic product. Naturally the germ at 0 of the set of singular matrices Δ is not a product; in particular all the tangent vector fields to Δ vanish at the origin. .

PROPOSITION 6.2. (1) *Associate to a germ $f : X, 0 \to V, 0$ the germs $\bar{f} : X, 0 \to V \times X, 0,\ x \mapsto (f(x), x)$ and $f^* : V \times X, 0 \to V, 0,\ (v, x) \mapsto v - f(x)$. Then $f_1,\ f_2$ are $\mathcal{K}_D$-equivalent if and only if $\bar{f}_1,\ \bar{f}_2$ are $\mathcal{K}_{D\times X}$-equivalent, if and only if f_1^*, f_2^* are ${}_{D\times X}\mathcal{K}$-equivalent. Indeed to minimize dimensions we can replace $\bar{f}$ by $(f(x), \pi(x))$, where $\pi : V, 0 \to Z, 0$ is a submersion, and replace $D \times X$ by $D \times Z$. If (f, π) is an immersion and $\dim X = \operatorname{rank} Df(0) + \dim Z$, that is Z has minimal dimension, then the result continues to hold with f^* replaced by a submersion whose fibre is the image of (f, π).*
(2) *Suppose given a germ of a submersion $g : V \times X, 0 \to Y, 0$. Then we can consider the germ $G : g^{-1}(0), 0 \to V, 0$ defined by projection to the first factor. If $g_i,\ i = 1, 2$ are ${}_{D\times V}\mathcal{K}$-equivalent submersions then the corresponding $G_i, i = 1, 2$ are $\mathcal{K}_D$-equivalent.*

Proof: (1) Since the $\bar{f}_i$ are immersions, the f_i^* corresponding submersions it suffices to prove the first equivalence, and indeed replace the relevant subgroups of $\mathcal{K}$ in both cases by those of $\mathcal{C}$. So suppose that the f_j are $\mathcal{C}_D$-equivalent. Then with the usual notation $f_2(x) = \phi(x, f_1(x))$, some $\phi \in \mathcal{C}$. So define $\Phi \in \mathcal{C}_{D\times X}$ by $\Phi(x, v, y) = (\phi(x, v), y)$. Then $\Phi(x, \bar{f}_1(x)) = \Phi(x, f_1(x), x) = (\phi(x, f_1(x)), x) = (f_2(x), x) = \bar{f}_2(x)$. Note that we need to prove that $\Phi(0, -, -) : V \times X, 0 \to V \times X, 0$ is a germ of a diffeomorphism. But we know that $\phi(0, -) : V, 0 \to V, 0$ is such a germ and the result follows.

Conversely if $\Phi(x, \bar{f}_1(x)) = \bar{f}_2(x)$ for some Φ then defining $\phi(x, v) = \Phi_1(x, v, x)$ (the first component of Φ) one can check that $\phi \in \mathcal{C}_D$ and $\phi(x, f_1(x)) = f_2(x)$. Again we need to show that $\phi(0, -) : V, 0 \to V, 0$ is a germ of a diffeomorphism. However $\Phi_1(X \times D \times X) \subset D$, and Φ_1 is the germ of a submersion, so $\Phi_1^{-1}(D)$ is a local product $K^m \times D$. However it is a subset of $X \times D \times X$, so coincides with this set. So since D is not analytically trivial $\Phi_1^{-1}(0)$ is $X \times 0 \times X$, and in particular $\ker D\Phi_1(0) = T_0(X \times 0 \times X)$ and the result follows. The second case follows in the same way.
(2) This is by a similar argument. Roughly speaking since the g_i are ${}_{D\times V}\mathcal{K}$-equivalent the $g_i^{-1}(0) \cap D \times V, 0$ are diffeomorphic, but these are $G_i^{-1}(D)$, and so the G_i are $\mathcal{K}_D$-equivalent.
□

Thus the classification of ${}_\Lambda\mathcal{K}$-simple germs includes the classification of $\mathcal{G}$-simple germs. Indeed given a ${}_\Lambda\mathcal{K}$-simple germ g which is a submersion then $X = g^{-1}(0), 0$ is the germ of a manifold and we can consider the composite of the inclusion of X in $M_n \times K^m$ with projection to the first factor to obtain the corresponding germ $X, 0 \to M_n$.

We briefly sketch how this alternative classification proceeds. As usual we need the ${}_\Delta\mathcal{K}$-tangent space to a map g. The $\mathcal{R}_\Delta$ part, as we know from 3.4, comes from applying the fields $\sum a_{ij}\partial/\partial a_{ik}$, $\sum a_{ij}\partial/\partial a_{kj}$ to g. Replacing M_n by $M_n \times K^m$ and Δ by $\Lambda = \Delta \times K^m$, we simply need to augment our vector fields by $\partial/\partial z_i$, $1 \leq i \leq m$, and extend the variables. In terms of the group operation one can see that $\mathcal{R}_\Lambda$ has an interpretation as the set of germs $(X, Y, \phi) : M_n \times K^m, 0 \to GL_n \times GL_n \times K^m, (-, -, 0)$ with $\phi(0, z)$ a diffeomorphism, which operates on germs $g : M_m \times K^m, 0 \to K^p, 0$ by $(X, Y, \phi).g(A, z) = g(X^{-1}(A, z)Y(A, z), \phi^{-1}(A, z))$.

We will complete the classification of ${}_\Lambda\mathcal{K}$-simple germs elsewhere. The following simply illustrates how this alternative approach works; the result itself follows from our main theorem.

Proposition 6.3. *The simple germs $M_n \times K^m, 0 \to K, 0$, up to ${}_\Lambda\mathcal{K}$-equivalence are as follows:*
(1) $g(A, z) = z_1$;
(2) $g(A, z) = \sum_{i=1}^{n} a_{ii} + h(z)$ *where* $h : K^m, 0 \to K, 0$ *is a simple function germ;*
(3) $g(A) = \sum_{i=1}^{n-1} a_{ii} + h(a_{nn}, z)$ *where* $h : K \times K^m, 0 \to K, 0$ *is a simple function germ on a manifold* $K \times K^m$ *with boundary* $0 \times K^m$.

Proof: If some $\partial g/\partial z_i \neq 0$ we get the first form. Otherwise we consider 1-jets and obtain $\sum_{i=1}^{t} a_{ii}$ as our normal form. For a 2-transversal we have $\sum_{i=1}^{t} a_{ii} + Q_1(z) + Q_2(B, z) + Q_3(B)$ where $Q_i, i = 1, 3$ are general quadratic forms in the indicated variables, B is the matrix a_{ij}, $t + 1 \leq i, j \leq n$, and Q_2 are the quadratic cross terms in B and z. If Q_1 is non-degenerate we can diagonalise and kill Q_2. A by now standard argument shows that we have moduli unless $t = n$, $n - 1$ and we quickly reduce to the normal forms in (2), (3). Note that this dual classification was really implicit in our approach to these germs. Using the same ideas we can recover all of the list of simples already discussed. □

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