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Neumann Boundary Value Problems: Continuity of Attractors Relatively to Domain Perturbations

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In this paper we obtain the continuity of attractors for semilinear parabolic problems with Neumann boundary conditions relatively to exterior perturbations of the domain. We show that, if the perturbations on the domain are such that the convergence of eigenvalues and eigenfunctions of the Neumann Laplacian is granted then, we will also have the continuity of attractors. October, 2002 ICMC-USP

Key Words: parabolic equations, attractors, lower semi-continuity

1. INTRODUCTION

In this paper we consider parabolic problems of the form

$$\begin{array}{l} u_t - \Delta u = f(u) \quad \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega_\epsilon. \end{array} \tag{1.1}$$

where Ω_ϵ, $0 \leq \epsilon \leq \epsilon_0$ are bounded Lipschitz domains in $\mathbb{R}^N$. The nonlinearity f is assumed be a $C^2(\mathbb{R}, \mathbb{R})$ function that satisfies the growth assumption

$$\begin{array}{l} |f'(u)| \leq c(|u|^{\frac{4}{N-2}} + 1), \text{ for some } c > 0, \quad N = 3, 4, \text{ or} \\ \lim\limits_{|u|\to\infty} \dfrac{|f'(u)|}{e^{\theta|u|^2}} = 0, \ \forall\, \theta > 0, \quad \text{if } N = 2 \end{array} \tag{1.2}$$

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(no growth condition is needed for $N = 1$) and the dissipativeness assumption

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < 0. \tag{1.3}$$

It has been shown in [5], Theorem 2.6, that, under the assumptions (1.2) and (1.3) the problem (1.1) has a global attractor $\mathcal{A}_\epsilon$ and that the attractors $\mathcal{A}_\epsilon$ are bounded in $L^\infty(\Omega_\epsilon)$, uniformly in ϵ. This enable us to cut the nonlinearity f in such a way that it becomes bounded with bounded derivatives up to second order without changing the attractors. After these considerations we may assume, without loss of generality, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a $C^2(\mathbb{R})$ function satisfying

$$|f'(u)| \leq c_f\ , \qquad |f''(u)| \leq \tilde{c}_f \qquad \forall u \in \mathbb{R} \tag{1.4}$$

where c_f e $\tilde{c}_f$ are positive constants.

In this paper we prove that the family of attractors $\{\mathcal{A}_\epsilon,\ 0 \leq \epsilon \leq \epsilon_0\}$ is continuous at $\epsilon = 0$ under some sort of convergence of the domains Ω_ϵ to Ω_0. This problem has been considered in [7] for Dirichlet boundary conditions. In the case of Dirichlet boundary conditions the problem is facilitated by the existence of an isometry between $H^1_0(\Omega_\epsilon)$ and $H^1(\mathbb{R}^N)$, $\forall\ \epsilon > 0$, given by the extension by zero outside Ω_ϵ. The perturbations Ω_ϵ of the domain Ω_0 are considered to be exterior perturbations and by continuity of the family of attractors we mean that; if $\tilde{\mathcal{A}}_\epsilon = \{u_{|_{\Omega_0}} : u \in \mathcal{A}_\epsilon\}$ then the family $\tilde{\mathcal{A}}_\epsilon$ is upper and lower semicontinuous, in $H^1(\Omega_0)$, at $\epsilon = 0$ and $\sup_{u\in\mathcal{A}_\epsilon} \|u\|_{H^1(\Omega_\epsilon\setminus\Omega_0)} \to 0$ as $\epsilon \to 0$.

Next we state precisely the sort of perturbations Ω_ϵ of Ω_0 considered here. Let Ω_ϵ, $0 \leq \epsilon \leq \epsilon_0$, be a family of bounded Lipschitz domains $\Omega_\epsilon \supset \Omega_0$. We require that the perturbations Ω_ϵ be such that the eigenvalues and eigenfunctions of the Neumann Laplacian in Ω_ϵ converge as $\epsilon \to 0$ to the eigenvalues and eigenfunctions of the Neumann Laplacian in Ω_0. This condition is equivalent (see Lemma 2.1) to the following:

$$\left.\begin{array}{l}\phi_\epsilon \in H^1(\Omega_\epsilon),\ 0 < \epsilon \leq \epsilon_0 \\ \|\nabla\phi_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C,\ 0 < \epsilon \leq \epsilon_0\end{array}\right\} \Rightarrow \|\phi_\epsilon\|_{L^2(\Omega_\epsilon\setminus\Omega_0)} \to 0. \tag{1.5}$$

Also from Lemma 2.1, this condition implies that $|\Omega_\epsilon\setminus\Omega_0| \to 0$ as $\epsilon \to 0$.

In Section 3 we prove that the family of attractors $\mathcal{A}_\epsilon$ and the set of equilibria $\mathcal{E}_\epsilon$ are upper semicontinuous at $\epsilon = 0$ showing $\{\mathcal{A}_\epsilon,\ 0 \leq \epsilon \leq \epsilon_0\}$ is bounded in $H^1(\Omega_0)$ and that the family of semigroups $\{T_\epsilon(t),\ t \geq 0\}$ associated to (1.1) is continuous, uniformly in compact intervals of $[0, \infty)$, in ϵ at $\epsilon = 0$; that is, if $u_0^\epsilon \in H^1(\Omega_\epsilon)$, $0 \leq \epsilon \leq \epsilon_0$ with $\|u^\epsilon - u^0\|_{H^1(\Omega_\epsilon)} + \|u^\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)} \to 0$ as $\epsilon \to 0$, then for any $0 \leq r < R < \infty$

$$\sup_{r\leq t\leq R}\{\|T_\epsilon(t)(u^\epsilon) - T_0(t)u^0\|_{H^1(\Omega_0)} + \|T_\epsilon(t)(u^\epsilon)\|_{H^1(\Omega_\epsilon\setminus\Omega_0)}\} \to 0, \quad \text{as } \epsilon \to 0.$$

Consider the problem

$$(P)_\epsilon \qquad \left\{\begin{array}{l} -\Delta u = f(u) \quad \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega_\epsilon. \end{array}\right.$$

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for $0 \leq \epsilon \leq \epsilon_0$. Assume that $(P)_0$ has exactly m distinct solutions, u_1^0,...,u_m^0 and that zero is not an eigenvalue for the operator $\Delta + f'(u_i)I$, with Neumann boundary condition, $1 \leq i \leq m$. Under these hypotheses, in Subsection 4.1, we prove that , for suitably small ϵ, P_ϵ has exactly m distinct solutions u_1^ϵ,...,u_m^ϵ and $u_j^\epsilon \to u_j^0$ as $\epsilon \to 0$, $1 \leq j \leq m$. In Subsection 4.2, we prove that the unstable manifolds of the equilibrium points u_k^ϵ are continuous at $\epsilon = 0$.

It follows from the continuity of the unstable manifolds that the attractors are lower semicontinuous at $\epsilon = 0$. This can be proved in the following way. If $u_0 \in \mathcal{A}_0$ then u_0 belongs to the unstable manifold of u_k^0 for some $1 \leq k \leq n$. Let $\delta > 0$ be such that all the unstable manifolds around u_k^ϵ are defined in a ball of radius δ, $0 \leq \epsilon \leq \epsilon_0$. If τ is such that $w_0 = T_0(-\tau, u_0) \in B_\delta(u_k^0)$, from the continuity of the unstable manifolds there is a sequence w_{ϵ_n} which converges to w_0 in $H^1(\Omega_0)$ as $n \to \infty$ and such that $\|w_{\epsilon_n}\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \to 0$ as $n \to \infty$. Now, since the family of semigroups is continuous in this sense we have that $\mathcal{A}_{\epsilon_n} \ni T_{\epsilon_n}(\tau, w_{\epsilon_n}) \to T_0(\tau, w_0) = u_0$ in $H^1(\Omega_0)$ as $n \to \infty$ and $\|T_{\epsilon_n}(\tau, w_{\epsilon_n})\|_{H^1(\Omega_{\epsilon_n} \setminus \Omega_0)} \to 0$ as $n \to \infty$. Showing the lower semicontinuity of attractors.

It is important to notice that the the continuity of attractors is a consequence of the convergence of the eigenvalues and eigenfunctions of the Laplace operator perturbed by a varying potential in the domains Ω_ϵ as $\epsilon \to 0$. Therefore we have organized the results in the following manner. Section 2 contains the *Linear Theory* and is divided into a survey of results on spectral convergence, uniform estimates on the resolvent operators and the convergence of the linear semigroups. In Section 3 we use the Linear Theory to obtain the upper semicontinuity of attractors. In Section 4 we obtain the lower semicontinuity of the set of equilibria and the continuity of unstable manifold to obtain the lower semicontinuity of attractors.

2. LINEAR THEORY

In this section we analyze the behavior of the linear parts of the operators and prove several results that will be used throughout the paper.

2.1. Spectral convergence characterization

It is very clear that the spectral behavior of the linear operators is extremely important when analyzing the continuity properties of nonlinear dynamics. We include in this section several results on the spectral behavior of the Laplace operators with Neumann boundary conditions when the domain is perturbed from the exterior. We are interested in obtaining characterizations that guarantee that the eigenvalues and eigenfunctions behave continuously.

To fix the notations we consider the eigenvalue problems

$$\begin{cases} -\Delta u + V_\epsilon u = \lambda u, & \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0, & \partial\Omega_\epsilon \end{cases}$$

where $\|V\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq C$ and $V_{\epsilon|\Omega_0} \to V_0$ weakly in $L^2(\Omega_0)$. We denote by $\{\lambda_n^\epsilon\}_{n=1}^\infty$, for $\epsilon \in [0, \epsilon_0]$, the set of eigenvalues of the operator $-\Delta + V_\epsilon$ with Neumann boundary conditions in Ω_ϵ and by $\{\phi_n^\epsilon\}_{n=1}^\epsilon$ a corresponding family of orthonormalized eigenfunctions. We will also denote $R_\epsilon = \Omega_\epsilon \setminus \Omega_0$.

We will say that the spectra behaves continuously at $\epsilon = 0$, if we have that $\lambda_n^\epsilon \to \lambda_n^0$ as $\epsilon \to 0$ and the spectral projections converge in H^1, that is, if $a \notin \{\lambda_n^0\}_{n=0}^\infty$, and $\lambda_n^0 < a < \lambda_{n+1}^0$, then if we define the projections $P_a^\epsilon(\psi_\epsilon) = \sum_{i=1}^n (\phi_i^\epsilon, \psi_\epsilon)\phi_i^\epsilon$ then

$$\|P_a^\epsilon(\psi_\epsilon) - P_a^0(\psi_\epsilon)\|_{H^1(\Omega_0)} + \|P_a^\epsilon(\psi_\epsilon)\|_{H^1(R_\epsilon)} \to 0, \quad \text{as } \epsilon \to 0$$

In order to characterize when the spectra behaves continuously we will need the following auxiliary eigenvalue problem. Denote by τ_ϵ the first eigenvalue of the following problem,

$$\begin{cases} -\Delta u = \tau u, & R_\epsilon \\ u = 0, & \partial R_\epsilon \cap \partial\Omega_0 \\ \frac{\partial u}{\partial n} = 0, & \partial R_\epsilon \setminus \partial\Omega_0 \end{cases}$$

Observe that obviously $\tau_\epsilon > 0$ and it can be characterized as follows

$$\tau_\epsilon = \min_{\substack{\phi \in H^1(\Omega_\epsilon) \\ \phi = 0, \text{ in } \Omega_0}} \frac{\displaystyle\int_{\Omega_\epsilon} |\nabla\phi|^2}{\displaystyle\int_{\Omega_\epsilon} |\phi|^2}$$

We have the following useful characterization

Lemma 2.1. *The following three statements are equivalent*
i) The spectra of $-\Delta + V_\epsilon$ *behaves continuously as* $\epsilon \to 0$.
ii) $\tau_\epsilon \to \infty$ *as* $\epsilon \to 0$.
iii) For any family of functions ψ_ϵ *with* $\|\psi_\epsilon\|_{H^1(\Omega_\epsilon)} \le C$ *then* $\|\psi_\epsilon\|_{L^2(\Omega_\epsilon\setminus\Omega_0)} \to 0$ *as* $\epsilon \to 0$.
Moreover, if any of the three statements above is true then the following also holds
iv) $|\Omega_\epsilon \setminus \Omega_0| \to 0$ *as* $\epsilon \to 0$.

Remark 2. 1. A somehow similar statement of this lemma can be found in the works of [1, 2]. We include here a proof for completeness.

Proof. That iii) implies ii) is easy since if there exists a sequence of $\epsilon \to 0$ with τ_ϵ bounded, then the eigenfunction associated to τ_ϵ will have $L^2(R_\epsilon)$ norm equal one and the $H^1(\Omega_\epsilon)$ norm bounded. Also, if iii) holds then iv) also holds, since we can always consider $\psi_\epsilon \equiv 1 \in H^1(\Omega_\epsilon)$ and $\|\psi_\epsilon\|_{L^2(\Omega_\epsilon\setminus\Omega_0)} = |\Omega_\epsilon \setminus \Omega_0|^{\frac{1}{2}} \to 0$ as $\epsilon \to 0$.

That ii) implies iv) is also easy. If it were not true then we will have a positive $\eta > 0$ and a sequence of $\epsilon \to 0$ such that $|\Omega_\epsilon \setminus \Omega_0| \ge \eta$. Let $\rho = \rho(\eta)$ be a small number such that $|\{x \in \mathbb{R}^N \setminus \Omega_0,\, dist(x, \Omega_0) \le \rho\}| \le \eta/2$. Let us construct a smooth function $\gamma(x)$ with $\gamma(x) = 0$ in Ω_0, $\gamma(x) = 1$ $x \in \mathbb{R}^N \setminus \Omega_0$ with $dist(x, \Omega_0) \ge \rho$. Then obviously $\|\nabla\gamma\|_{L^2(\Omega_\epsilon)} \le C$ and $\|\gamma\|_{L^2(\Omega_\epsilon)} \ge (\eta/2)^{\frac{1}{2}}$. This implies that τ_ϵ is bounded.

That ii) implies iii) is proved as follows. If it is not true then there will exists a sequence of $\epsilon \to 0$ and a sequence of functions ϕ_ϵ with $\|\phi_\epsilon\|_{H^1(\Omega_\epsilon)} \le C_1$ and $\|\phi_\epsilon\|_{L^2(R_\epsilon)} \ge C_2 > 0$, for some constants C_1 and C_2 independent of ϵ. We can extract a subsequence, that we denote by ϕ_ϵ again, and a function $\phi_0 \in H^1(\Omega_0)$, such that $\phi_\epsilon \to \phi_0$ in $L^2(\Omega_0)$. If we denote by E a continuous extension operator from $H^1(\Omega_0)$ to $H^1(R^N)$, which is also continuous from $L^2(\Omega_0)$ to $L^2(\Omega_\epsilon)$, by $\mathcal{T}_\epsilon$ a restriction operator to Ω_ϵ, and denote by $\tilde\phi_\epsilon = \mathcal{T}_\epsilon E \mathcal{T}_0 \phi_\epsilon$ then we have that $E\mathcal{T}_0\phi_\epsilon \to E\phi_0$ in $L^2(R^N)$ which implies that, since the measure of $\Omega_\epsilon \setminus \Omega_0$ goes to zero by iv), then $\|\tilde\phi_\epsilon\|_{L^2(R_\epsilon)} \to 0$ as $\epsilon \to 0$. Moreover $\|\tilde\phi_\epsilon\|_{H^1(\Omega_\epsilon)} \le C$. If we define the functions $\chi_\epsilon = \phi_\epsilon - \tilde\phi_\epsilon$, then $\chi_\epsilon = 0$ in Ω_0, $\|\chi_\epsilon\|_{H^1(\Omega_\epsilon)} \le C$ and $\|\chi_\epsilon\|_{L^2(R_\epsilon)} \ge \|\phi_\epsilon\|_{L^2(R_\epsilon)} - \|\tilde\phi_\epsilon\|_{L^2(R_\epsilon)}$ which is bounded away from zero for ϵ small enough. Using this function as a test function in the Raleigh quotient to obtain τ_ϵ we will deduce that τ_ϵ is bounded for this sequence of ϵ, contradicting ii).

Let us prove now that i) implies ii). If this is not the case then we will have again a sequence of ϵ approaching zero and a positive number a with $\tau_\epsilon \le a$. Choose $n \in N$ with the property that $a < \lambda_n < \lambda_{n+1}$. Denote by ϕ_ϵ the eigenfunction with $\|\phi_\epsilon\|_{L^2(R_\epsilon)} = 1$ associated to τ_ϵ. Observe then that $\|\nabla\phi_\epsilon\|^2_{L^2(R_\epsilon)} \le a$. Consider the following family of functions $\psi_i^\epsilon = \mathcal{T}_\epsilon E\phi_i^0$ for $i = 1, \ldots, n$. Then if we consider the linear subspace $[\psi_1^\epsilon, \ldots, \psi_n^\epsilon, \phi_\epsilon] \subset H^1(\Omega_\epsilon)$ we get that for ϵ small enough they are linearly independent. As a matter of fact they are almost an orthonormal system in $L^2(\Omega_\epsilon)$. By the min-max characterization of the eigenvalues (see Courant-Hilbert) we have that

$$\lambda_{N+1}^\epsilon \le \max_{\phi \in [\psi_1^\epsilon, \ldots, \psi_n^\epsilon, \phi_\epsilon]} \frac{\displaystyle\int_{\Omega_\epsilon} |\nabla\phi|^2 + \int_{\Omega_\epsilon} V_\epsilon |\phi|^2}{\displaystyle\int_{\Omega_\epsilon} |\phi|^2}$$

By direct calculation of the above quotient we obtain that $\lambda_{n+1}^\epsilon \le \lambda_n^0 + o(1)$, as $\epsilon \to 0$. This contradicts the continuity of the eigenvalues given by i).

The proof that ii) implies i) can be deduced from [2]. Notice that since we have already proved that ii) is equivalent to iii) we trivially have that hypothesis (H) from [2], page 61, is satisfied (with the notations of [2], $\bar u_\epsilon$ can be taken identically zero). In particular we can apply Theorem 2.1 of [2] which in the particular case that $\tau_\epsilon \to \infty$ implies the continuity of the spectra. Nevertheless we provide now a self contained proof that ii) implies i). Notice first that we have already proved above that ii) implies iv). In particular we have that $|\Omega_\epsilon \setminus \Omega_0| \to 0$ as $\epsilon \to 0$. Considering the test functions $\mathcal{T}_\epsilon E\phi_1^0, \ldots, \mathcal{T}_\epsilon E\phi_n^0$ to bound λ_n^ϵ we easily obtain that $\limsup_{\epsilon\to 0} \lambda_n^\epsilon \le \lambda_n^0$.

In particular for fixed n, λ_n^ϵ is bounded above by a constant independent of ϵ. Hence for any sequence of ϵ approaching zero we can extract another subsequence, that we will still denote by ϵ, with the property that $\lambda_i^\epsilon \to \kappa_i \le \lambda_i^0$, and $R_0\phi_i^\epsilon \to \xi_i$ weakly in $H^1(\Omega_0)$ and strongly in $L^2(\Omega)$. Moreover, since for fixed n, the functions ϕ_n^ϵ are bounded uniformly in ϵ in $H^1(\Omega_\epsilon)$, applying iii), which is equivalent to ii), we get that $\|\phi_n^\epsilon\|_{L^2(R_\epsilon)} \to 0$ as $\epsilon \to 0$. In particular this implies that $\delta_{nm} = \int_{\Omega_\epsilon} \phi_n^\epsilon \phi_m^\epsilon \to \int_{\Omega_0} \xi_n \xi_m$, which means that the family of functions ξ_n is an orthonormal system in $L^2(\Omega_0)$. By passing to the limit in the weak formulation of the equation it is not difficult to see that ξ_i is a weak solution of the equation $-\Delta\xi_i + V_0\xi_i = \kappa_i\xi_i$ and $\partial\xi_i/\partial n = 0$. Since $\|\xi_i\|_{L^2(\Omega_0)} = 1$ we have that κ_i is an eigenvalue and ξ_i is an eigenfunction. Since $\kappa_i \le \lambda_i^0$ we necessarily get that $\kappa_i = \lambda_i^0$ and ξ_i is an eigenfunction associated to λ_i^0. This implies the convergence of the eigenvalues and the convergence of the eigenfunctions in L^2 and weakly in H^1. To obtain the strong convergence in H^1 we just observe that

$$\|\nabla\phi_i^\epsilon\|^2_{L^2(\Omega_\epsilon)} = \lambda_i^\epsilon - \int_{\Omega_\epsilon} V_\epsilon|\phi_i^\epsilon|^2 \to \lambda_i^0 - \int_{\Omega_0} V_0|\phi_i|^2 = \|\nabla\xi_i\|^2_{L^2(\Omega_0)} \le \liminf_{\epsilon\to 0} \|\nabla\phi_i^\epsilon\|_{L^2(\Omega_0)}$$

from where we obtain that $\|\nabla\phi_i^\epsilon\|_{L^2(\Omega_0)} \to \|\nabla\xi_i\|_{L^2(\Omega_0)}$ and $\|\nabla\phi_i^\epsilon\|_{L^2(R_\epsilon)} \to 0$. This implies the strong convergence in H^1. This concludes the proof of the lemma.

2.2. Uniform bounds on the resolvent operators

LEMMA 2.2. *Consider a family of potentials $V_\epsilon \in L^\infty(\Omega_\epsilon)$ with $\|V_\epsilon\|_{L^\infty(\Omega_)} \le C$ and $V_\epsilon \to V_0$ in $L^2(\Omega_0)$. Assume that for the operator $-\Delta + V_0$, $0 \notin \sigma(-\Delta + V_0)$. Then for ϵ small enough $0 \notin \sigma(-\Delta + V_\epsilon)$ and there exists a constant C independent of ϵ such that*

$$\|(-\Delta + V_\epsilon)^{-1} g_\epsilon\|_{H^1(\Omega_\epsilon)} \le C\|g_\epsilon\|_{L^2(\Omega_\epsilon)}, \quad g_\epsilon \in L^2(\Omega_\epsilon)$$

Proof. By the continuity of the spectra given by Lemma 2.1 we have that for ϵ small enough $0 \notin \sigma(-\Delta + V_\epsilon)$. In particular, for $g_\epsilon \in L^2(\Omega_\epsilon)$ given we have a unique solution $w_\epsilon \in H^1(\Omega_\epsilon)$ of

$$\begin{cases} -\Delta w_\epsilon + V_\epsilon w_\epsilon = g_\epsilon, & \Omega_\epsilon \\ \frac{\partial w_\epsilon}{\partial n} = 0, & \partial\Omega_\epsilon \end{cases} \tag{2.1}$$

We show first that $\|w_\epsilon\|_{L^2(\Omega_\epsilon)}$ is bounded. Suppose not, then there is a subsequence, which we again denote by $\{w_\epsilon\}$, such that $\|w_\epsilon\|_{L^2(\Omega_\epsilon)} \to \infty$. Consider $\tilde{w}_\epsilon = \frac{w_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}}$, then

$$\begin{cases} -\Delta \tilde{w}_\epsilon + V_\epsilon \tilde{w}_\epsilon = \frac{g_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}}, & \Omega_\epsilon \\ \frac{\partial \tilde{w}_\epsilon}{\partial n} = 0, & \partial\Omega_\epsilon. \end{cases} \tag{2.2}$$

Multiplying this equation by $\tilde{w}_\epsilon$ and integrating by parts we obtain that

$$\int_{\Omega_\epsilon} |\nabla\tilde{w}_\epsilon|^2 + \int_{\Omega_\epsilon} V_\epsilon|\tilde{u}_\epsilon|^2 = \int_{\Omega_\epsilon} \frac{\tilde{g}_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}}\tilde{w}_\epsilon$$

from where it follows that

$$\int_{\Omega_\epsilon} |\nabla\tilde{w}_\epsilon|^2 \le C.$$

Hence we can extract a subsequence, which we again denote by $\{\tilde{w}_\epsilon\}$, such that

$$\tilde{w}_\epsilon\big|_{\Omega_0} \to \tilde{w}_0 \begin{cases} \text{strongly} - L^2(\Omega_0) \\ \text{weakly} - H^1(\Omega_0). \end{cases}$$

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Let $\xi \in H^1(\Omega_0)$ and consider $\tilde{\xi}$ the extension of ξ to B. If we multiply the equation (2.2) by $\tilde{\xi} \in H^1(\Omega_\epsilon)$ and integrating by parts we have that

$$\int_{\Omega_\epsilon} \nabla \tilde{w}_\epsilon \nabla \tilde{\xi} + \int_{\Omega_\epsilon} V_\epsilon \tilde{w}_\epsilon \tilde{\xi} = \int_{\Omega_\epsilon} \frac{g_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}} \tilde{\xi}.$$

Now, note that

$$\begin{aligned}
&\left| \int_{\Omega_\epsilon \setminus \Omega_0} \nabla \tilde{w}_\epsilon \nabla \tilde{\xi} \right| \leq \|\nabla \tilde{w}_\epsilon\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \|\nabla \tilde{\xi}\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \to 0 \text{ as } \epsilon \to 0\\
&\left| \int_{\Omega_\epsilon \setminus \Omega_0} V_\epsilon \tilde{w}_\epsilon \tilde{\xi} \right| \leq \|V_\epsilon\|_{L^\infty(\Omega_\epsilon)} \|\tilde{w}_\epsilon\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \|\tilde{\xi}\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \to 0 \text{ as } \epsilon \to 0\\
&\left| \int_{\Omega_\epsilon} \tfrac{g_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}} \tilde{\xi} \right| \leq \tfrac{\|g_\epsilon\|_{L^2(\Omega_\epsilon)}}{\|\tilde{w}_\epsilon\|_{L^2(\Omega_\epsilon)}} \|\tilde{\xi}\|_{L^2(\Omega_\epsilon)} \to 0 \text{ as } \epsilon \to 0
\end{aligned}$$

and letting $n \to \infty$ we have that

$$\int_{\Omega_0} \nabla \tilde{w}_0 \nabla \xi + \int_{\Omega_0} V_0 \tilde{w}_0 \xi = 0.$$

Thus

$$\begin{aligned}
&-\Delta \tilde{w}_0 + V_0 \tilde{w}_0 = 0, \quad \Omega_0\\
&\tfrac{\partial \tilde{w}_0}{\partial n} = 0, \quad \partial\Omega_0.
\end{aligned} \tag{2.3}$$

which implies $\tilde{w}_0 = 0$. Now, since $\|\tilde{w}_\epsilon\|_{H^1(\Omega_\epsilon)}$ is bounded, by Lemma 2.1 we have that $\|\tilde{w}_\epsilon\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \to 0$ as $n \to \infty$. Hence, necessarily $1 = \|\tilde{w}_\epsilon\|_{L^2(\Omega_\epsilon)} \to 0$ which is a contradiction. Hence, we obtain that $\|w_\epsilon\|_{L^2(\Omega_\epsilon)}$ is uniformly bounded in ϵ.

To show that $\|\nabla w_\epsilon\|_{L^2(\Omega_\epsilon)}$ is uniformly bounded in *eps* we note that V_ϵ are uniformly bounded in $L^\infty(\Omega_\epsilon)$ and that

$$\int_{\Omega_0} |\nabla w_\epsilon|^2 = -\int_{\Omega_0} V_\epsilon |w_\epsilon|^2 + \int_{\Omega_\epsilon} g_\epsilon w_\epsilon.$$

This concludes the proof of the lemma.

Remark 2. 2. With the conditions of Lemma 2.2 we can even show the convergence of the resolvent operators. As a matter of fact, if $g_\epsilon\big|_{\Omega_0} \to g_0$ weakly in $L^2(\Omega)$, then

$$\|(-\Delta + V_\epsilon)^{-1} g_\epsilon - (-\Delta + V_0)^{-1} g_0\|_{H^1(\Omega_0)} + \|(-\Delta + V_\epsilon)^{-1} g_\epsilon\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \to 0, \quad \text{as } \epsilon \to 0.$$

Notice that if we denote by $w_\epsilon = (-\Delta + V_\epsilon)^{-1} g_\epsilon \in H^1(\Omega_\epsilon)$ and $w_0 = (-\Delta + V_0)^{-1} g_0 \in H^1(\Omega_0)$ then since $\|w_\epsilon\|_{H^1(\Omega_\epsilon)}$ is uniformly bounded in ϵ, by Lemma 2.1 we have that $\|w_\epsilon\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \to 0$ as $\epsilon \to 0$.

Moreover by extracting a subsequence and using the weak formulations of the equations we can easily obtain that $w_\epsilon\big|_{\Omega_0} \to w_0$ weakly in $H^1(\Omega_0)$ and strongly in $L^2(\Omega_0)$. Now, multiplying the equations by w_ϵ and integrating by parts we have that

$$\int_{\Omega_\epsilon} |\nabla w_\epsilon|^2 = -\int_{\Omega_\epsilon} V_\epsilon w_\epsilon^2 + \int_{\Omega_\epsilon} g_\epsilon w_\epsilon \to -\int_{\Omega_0} V_0 w_0^2 + \int_{\Omega_0} g_0 w_0 = \int_{\Omega_0} |\nabla w_0|^2.$$

from where we easily obtain the strong convergence in $H^1(\Omega_0)$.

2.3. Convergence of Linear Semigroups

With the continuity of the spectra of the operators $-\Delta + V_\epsilon$ we can obtain estimates on the behavior of the linear semigroups that will be very useful for the analysis of the nonlinear dynamics.

We have the following result

Proposition 2.1. *Let $a \in R$ and $n \in N$ such that $\lambda_n^0 < a < \lambda_{n+1}^0$. Consider the spectral projections over the linear space generated by the first n eigenfunctions P_a^ϵ defined in the pervious section. Denote also by b a number such that $b < \lambda_1^0$. There exists a $\gamma < 1$ and a function $\theta(\epsilon)$ with $\theta(\epsilon) \to 0$ as $\epsilon \to 0$ such that*

$$\|e^{A_\epsilon t} u_\epsilon - e^{-A_0 t} u_{\epsilon|_{\Omega_0}}\|_{H^1(\Omega_0)} \le M\theta(\epsilon) t^{-\gamma} e^{-bt} \|u_\epsilon\|_{L^2(\Omega_\epsilon)}, \quad u_\epsilon \in L^2(\Omega_\epsilon), \quad t > 0$$

$$\|e^{A_\epsilon t}(I - P_a^\epsilon) u_\epsilon - e^{-A_0 t}(I - P_a^0) u_{\epsilon|_{\Omega_0}}\|_{H^1(\Omega_0)} \le M\theta(\epsilon) t^{-\gamma} e^{-at} \|u_\epsilon\|_{L^2(\Omega_\epsilon)}, \quad u_\epsilon \in L^2(\Omega_\epsilon), \quad t > 0$$

$$\|e^{A_\epsilon t} u_\epsilon\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \le M\theta(\epsilon) t^{-\gamma} e^{-bt} \|u_\epsilon\|_{L^2(\Omega_\epsilon)}$$

Proof. We will very much follow the lines given by [3], Proposition 2.1. Let us prove the second inequality. So let us consider n and a given, satisfying the hypothesis of the proposition. Notice that we can choose a constant M independent of ϵ such that

$$\|e^{A_\epsilon t}(I - P_a^\epsilon) u_\epsilon\|_{H^1(\Omega_\epsilon)} \le M t^{-\frac{1}{2}} e^{-at} \|u_\epsilon\|_{L^2(\Omega_\epsilon)}, \quad u_\epsilon \in L^2(\Omega_\epsilon), \quad t > 0, \quad \epsilon \in [0, \epsilon_0)$$

As it is done in [3] we separate the estimate for t small and t large.

Choose $\gamma \in (\alpha, 1)$ fixed. Let $\delta > 0$ be a small parameter and let us consider two different cases according to $t \in (0, \delta]$ or $t > \delta$.

i) If $t \in (0, \delta]$ we easily check that

$$\begin{aligned} &\|e^{-A_\epsilon t}(I - P_\epsilon^a) u_\epsilon - e^{-A_0 t}(I - P_0^a) u_\epsilon\|_{H^1(\Omega_0)} \le 2M t^{-\frac{1}{2}} e^{-at} \|u_\epsilon\|_{L^2(\Omega_\epsilon)} \\ &\le M \delta^{\gamma - \frac{1}{2}} t^{-\gamma} e^{-at} \|u\|_{L^2(\Omega_\epsilon)} \end{aligned} \tag{2.4}$$

ii) If $t > \delta$ we proceed as follows. Notice first that we can always choose a positive number $l = l(\delta)$ such that if $z \geq l$ then $ze^{-2zt} \leq \delta t^{-\gamma} e^{-at}$ for all $t \geq \delta$. Since we have $\lambda_k^\epsilon \stackrel{\epsilon\to 0}{\longrightarrow} \lambda_k^0$ and $\lambda_k^0 \stackrel{k\to\infty}{\longrightarrow} +\infty$, there exists $N = N(\delta) > n$ such that $\lambda_k^\epsilon \geq l(\delta)$, $eps \in [0, \epsilon_0)$. Without loss of generality we can assume that we have $\lambda_{N(\delta)}^0 < \lambda_{N(\delta)+1}^0$. Hence, from the spectral decompositions of the linear semigroups, we obtain

$$\begin{aligned}
&\|e^{-A_\epsilon t}(I - P_\epsilon^a)u_\epsilon - e^{-A_0 t}(I - P_0^a)u_\epsilon\|_{H^1(\Omega_0)} \leq \\
&\quad \|\sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^\epsilon t}(u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon - \sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^0 t}(u_\epsilon, \phi_k^0)P_0\phi_k^0\|_{H^1(\Omega_0)} \\
&\quad + \|\sum_{N(\delta)+1}^{\infty} e^{-\lambda_k^\epsilon t}(u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon\|_{H^1(\Omega_0)} + \|\sum_{N(\delta)+1}^{\infty} e^{-\lambda_k^0 t}(u_\epsilon, \phi_k^0)\phi_k^0\|_{H^1(\Omega_0)} = I_1 + I_2 + I_3
\end{aligned} \tag{2.5}$$

Analyzing I_2, I_3 and I_1 respectively, we get

$$\begin{aligned}
I_2 &\leq \sum_{N(\delta)+1}^{\infty} \lambda_k^\epsilon e^{-2\lambda_k^\epsilon t}|(u_\epsilon, \phi_n^\epsilon)|^2 \leq \delta t^{-\gamma} e^{-at}\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \\
I_3 &\leq \sum_{N(\delta)+1}^{\infty} \lambda_k^0 e^{-2\lambda_k^0 t}|(u_0, \phi_n^0)|^2 \leq \delta t^{-\gamma} e^{-at}\|u_\epsilon\|_{L^2(\Omega_\epsilon)}
\end{aligned}$$

$$\begin{aligned}
I_1 &= \|\sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^\epsilon t}(u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon - \sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^0 t}(u_\epsilon, \phi_k^0)\phi_k^0\|_{H^1(\Omega_0)} \\
&\leq \|\sum_{k=n+1}^{N(\delta)} (e^{-\lambda_k^\epsilon t} - e^{-\lambda_k^0 t})(u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon\|_{H^1(\Omega_0)} + \|\sum_{k=n+1}^{N(\delta)} e^{-\lambda_k^0 t}((u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon - (u_\epsilon, \phi_k^0)\phi_k^0)\|_{H^1(\Omega_0)} \\
&\leq \sum_{k=n+1}^{N(\delta)} ((\lambda_k^\epsilon)^{\frac{1}{2}} + 1)|e^{-\lambda_k^\epsilon t} - e^{-\lambda_k^0 t}|\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \\
&+ \sum_{i=r}^{k(\delta)} e^{-\mu_i t}\|\sum_{k=n_i+1}^{n_{i+1}} ((u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon - (u_\epsilon, \phi_k^0)\phi_k^0)\|_{H^1(\Omega_0)}
\end{aligned}$$

Moreover, from the convergence of the eigenvalues and of the spectral projections, we can find $\epsilon_1(\delta) \in (0, \epsilon_0)$ so that

$$\sum_{n=1}^{N(\delta)} ((\lambda_k^\epsilon)^{\frac{1}{2}} + 1)|e^{-\lambda_n^\epsilon t} - e^{-\lambda_n^0 t}| \leq \delta t^{-\gamma} e^{-at}, \quad \epsilon \in (0, \epsilon_1(\delta))$$

$$\begin{aligned}
&\sum_{i=r}^{k(\delta)} e^{-\mu_i t}\|\sum_{k=n_i+1}^{n_{i+1}} ((u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon - (u_\epsilon, \phi_k^0)\phi_k^0)\|_{H^1(\Omega_0)} \leq \\
&\qquad e^{-\lambda_n^0 t}\delta\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C\delta t^{-\gamma} e^{-at}\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \quad \epsilon \in (0, \epsilon_1(\delta)).
\end{aligned}$$

From the estimates for I_1, I_2 and I_3 we obtain

$$\|e^{-A_\epsilon t}(I-P_a^\epsilon)u_\epsilon - e^{-A_0 t}(I-P_a^0)u_\epsilon\|_{H^1(\Omega_0)} \le C\delta t^{-\gamma}e^{-at}\|u_\epsilon\|_{L^2(\Omega_\epsilon)},\ t>\delta,\ \epsilon\in(0,\epsilon_1(\delta)). \tag{2.6}$$

Finally, since δ is an arbitrary small number, the inequalities (2.4) and (2.6) prove the result.

The proof of the first inequality of the proposition is very similar to the one provided for the second inequality. The role of a is played now by b and $P_a^\epsilon = P_a^0 = 0$.

The proof for the third inequality is also very similar. Notice that we are estimating the H^1 norm in $\Omega_\epsilon \setminus \Omega_0$. Step i) is the same and for step ii) the sums I_2 and I_3 are estimated in a similar way. The only difference is now for I_1 where we will have

$$\|\sum_{k=1}^{N(\delta)} e^{-\lambda_k^\epsilon t}(u_\epsilon,\phi_k^\epsilon)\phi_k^\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)} \le e^{-\lambda_1^\epsilon t}\sum_{k=1}^{N(\delta)}|(u_\epsilon,\phi_k^\epsilon)| \sup_{i=1,..,N(\delta)}\{\|\phi_i^\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)}\}$$

$$\le e^{-\lambda_1^\epsilon t}N(\delta)\sup_{i=1,..,N(\delta)}\{\|\phi_i^\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)}\}\|u_\epsilon\|_{L^2(\Omega_\epsilon)}$$

but since $\sup_{i=1,..,N(\delta)}\{\|\phi_i^\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)}\}\to 0$ as $\epsilon\to 0$ we can always choose a $\epsilon_1(\delta)$ small enough such that $N(\delta)\sup_{i=1,..,N(\delta)}\{\|\phi_i^\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)}\}\le\delta$. Hence

$$\|\sum_{k=1}^{N(\delta)} e^{-\lambda_k^\epsilon t}(u_\epsilon,\phi_k^\epsilon)\phi_k^\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)} \le e^{-\lambda_1^\epsilon t}\delta\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \le C\delta t^{-\gamma}e^{-bt}\|u_\epsilon\|_{L^2(\Omega_\epsilon)}$$

and the proof continues as we did above. This concludes the proof of the proposition.

3. UPPER SEMICONTINUITY OF ATTRACTORS AND THE SET OF EQUILIBRIA

In the previous section we have studied in detail the behavior of the linear parts of the operators under the perturbation we are considering and have proved a result on the continuity of the linear semigroups, Proposition 2.1. We will see in this section that the attractors and the stationary states, solutions of the nonlinear elliptic problem, are upper semicontinuous with respect to this perturbations.

To this end we will follow the ideas in [3] that relate the continuity of the linear semigroups with the continuity of the nonlinear semigroups for dissipative parabolic equations by using the variation of constants formula. This in turn will imply the upper semicontinuity of the attractors and the stationary states. See also [11] and [6] for other examples that use a similar technique.

We will show the following result

PROPOSITION 3.1. *There exists a $0\le\gamma<1$ and a function $c(\epsilon)$ with $c(\epsilon)\to 0$ as $\epsilon\to 0$ such that for each $\tau>0$ we have*

$$\|T_\epsilon(t,u_\epsilon) - T_0(t,u_{\epsilon|_{\Omega_0}})\|_{H^1(\Omega_0)} \le M(\tau)c(\epsilon)t^{-\gamma},\quad t\in(0,\tau],\quad u_\epsilon\in\mathcal{A}_\epsilon,\quad \epsilon\in(0,\epsilon_0) \tag{3.1}$$

$$\|T_\epsilon(t,u_\epsilon)\|_{H^1(\Omega_\epsilon\setminus\Omega_0)} \le M(\tau)c(\epsilon)t^{-\gamma}, \quad t\in(0,\tau], \quad u_\epsilon\in\mathcal{A}_\epsilon, \quad \epsilon\in(0,\epsilon_0) \tag{3.2}$$

Moreover the attractors are upper semicontinuous at $\epsilon = 0$ in $H^1(\Omega_0)$, in the sense that

$$\sup_{u_\epsilon\in\mathcal{A}_\epsilon}\left[\inf_{u_0\in\mathcal{A}_0}\{\|u_{\epsilon|_{\Omega_0}} - u_0\|_{H^1(\Omega_0)}\}\right] \to 0, \quad \text{as } \epsilon\to 0 \tag{3.3}$$

$$\sup_{u_\epsilon\in\mathcal{A}_\epsilon}\|u_\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)} \to 0, \quad \text{as } \epsilon\to 0 \tag{3.4}$$

Also, the stationary states are upper semicontinuous at $\epsilon = 0$ in $H^1(\Omega)$, in the sense that if we denote by $\mathcal{E}_\epsilon$, $\epsilon\in[0,\epsilon_0]$ the set of stationary states of (1.1)*, that is, the set of solutions of the nonlinear elliptic problem, then*

$$\sup_{u_\epsilon\in\mathcal{E}_\epsilon}\left[\inf_{u_0\in E_0}\{\|u_{\epsilon|_{\Omega_0}} - u_0\|_{H^1(\Omega_0)}\}\right] \to 0, \quad \text{as } \epsilon\to 0 \tag{3.5}$$

Proof. We will follow the lines given by [3]. Notice that the nonlinear semigroups $T_\epsilon(t)$ are given by the variation of constants formula.

$$T_\epsilon(t,u_\epsilon) = e^{A_\epsilon t}u_\epsilon + \int_0^t e^{A_\epsilon(t-s)}f(T_\epsilon(s,u_\epsilon))ds, \quad \epsilon\in[0,\epsilon_0) \tag{3.6}$$

Hence, calculating $T_\epsilon(t,u_\epsilon) - T_0(t,u_\epsilon\big|_{\Omega_0})$ and with some elementary computations we obtain

$$\begin{aligned}
\|T_\epsilon(t,u_\epsilon) - T_0(t,u_\epsilon\big|_{\Omega_0})\|_{H^1(\Omega_0)} &\le \|e^{A_\epsilon t}u_\epsilon - e^{A_0 t}(u_\epsilon\big|_{\Omega_0})\|_{H^1(\Omega_0)} + \\
&\int_0^t \|e^{A_\epsilon t}f(T_\epsilon(s,u_\epsilon)) - e^{A_0 t}f(T_\epsilon(s,u_\epsilon))\big|_{\Omega_0}\|_{H^1(\Omega_0)}ds \\
&+\int_0^t \|e^{A_0 t}(f(T_\epsilon(s,u_\epsilon))\big|_{\Omega_0} - f(T_0(s,u_\epsilon\big|_{\Omega_0}))\|_{H^1(\Omega_0)}ds \quad \epsilon\in[0,\epsilon_0)
\end{aligned}$$

Applying now Proposition 2.1 we get

$$\begin{aligned}
\|T_\epsilon(t,u_\epsilon) - T_0(t,u_\epsilon\big|_{\Omega_0})\|_{H^1(\Omega_0)} &\le M\theta(\epsilon)t^{-\gamma}e^{-bt}\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \\
&+M\theta(\epsilon)\int_0^t (t-s)^{-\gamma}e^{-b(t-s)}\|f(T_\epsilon(s,u_\epsilon))\|_{L^2(\Omega_\epsilon)} \\
&+M\int_0^t (t-s)^{-1/2}e^{-b(t-s)}C\|T_\epsilon(t,u_\epsilon) - T_0(t,u_\epsilon\big|_{\Omega_0})\|_{H^1(\Omega_0)}
\end{aligned}$$

But since we have uniform bounds in $L^2(\Omega_\epsilon)$ in the attractors and f is a bounded function, the first two terms in the last inequality can be bounded by $M(\tau)\theta(\epsilon)t^{-\gamma}$. Applying now Gronwall's lemma, see [10], we obtain statement (3.1). To obtain, (3.2) we just use the variations of constants formula (3.6) and use the estimates in $H^1(\Omega_\epsilon \setminus \Omega_0)$ of Lemma 2.1

Now, the upper semicontinuity of the attractors in $H^1(\Omega_0)$, statement (3.3) follows directly from (3.1) and the fact that $\mathcal{A}_0$ attracts $\cup_{0<\epsilon\leq\epsilon_0}\mathcal{A}_\epsilon\big|_{\Omega_0}$ in the topology of $H^1(\Omega_0)$, see for instance [3, 8]. Statement (3.4) follows directly from (3.2) and from the invariance properties of the attractors.

To show the upper semicontinuity in $H^1(\Omega_0)$ of the stationary states we will prove that for any sequence of $\epsilon \to 0$ and for any $u_\epsilon \in \mathcal{E}_\epsilon$ we can extract a subsequence, that we still denote by ϵ, and obtain a $u_0 \in E_0$ such that $u_{\epsilon|_{\Omega_0}} \to u_0$ in $H^1(\Omega)$. From the upper semicontinuity of the attractors given by (3.3), we obtain the existence of a $u_0 \in A_0$ such that $u_{\epsilon|_{\Omega_0}} \to u_0$ in $H^1(\Omega)$ as $\epsilon \to 0$. To show that $u_0 \in E_0$ we first observe that for any $t > 0$, $\|u_{\epsilon|_{\Omega_0}} - T_0(t, u_0)\|_{H^1(\Omega)} \to \|u_0 - T_0(t, u_0)\|_{H^1(\Omega)}$. Moreover, for a fixed $\tau > 0$ and for any $t \in (0, \tau)$ we have that,

$$\|u_{\epsilon|_{\Omega_0}} - T_0(t, u_0)\|_{H^1(\Omega)} = \|T_\epsilon(t, u_\epsilon)_{|_{\Omega_0}} - T_0(t, u_0)\|_{H^1(\Omega)}$$

$$\leq \|T_\epsilon(t, u_\epsilon)_{|_{\Omega_0}} - T_0(t, u_{\epsilon|_{\Omega_0}})\|_{H^1(\Omega)} + \|T_0(t, u_0) - T_0(t, u_{\epsilon|_{\Omega_0}})\|_{H^1(\Omega)} \to 0, \quad \text{as } \epsilon \to 0$$

where we have used that u_ϵ is a stationary state, the continuity of the semigroup T_0 in $H^1(\Omega_0)$ and (3.1). In particular we have that for each $t > 0$, $u_0 = T_0(t, u_0)$ which implies that u_0 is a stationary state. This concludes the proof of the Proposition.

Remark 3. 1. Observe that from (3.4) we also have that

$$\sup_{u_\epsilon \in \mathcal{E}_\epsilon} \|u_\epsilon\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \to 0, \quad \text{as } \epsilon \to 0$$

4. CONTINUITY OF EQUILIBRIA AND THEIR UNSTABLE MANIFOLDS

In order to obtain lower semicontinuity of attractors we must ensure that the set of equilibria $\mathcal{E}_\epsilon$ behaves lower-semicontinuously. In this section we prove that, for the sort of domain perturbations considered here, $\mathcal{E}_\epsilon$ is a finite set with constant cardinality; that is, $\mathcal{E}_\epsilon = \{u_1^\epsilon, \cdots, u_n^\epsilon\}$, $0 \leq \epsilon \leq \epsilon_0$. This set behaves continuously with respect to ϵ; that is,

$$\max_{1\leq k\leq n}\{\|u_k^\epsilon - u_k^0\|_{H^1(\Omega_0)} + \|u_k^\epsilon\|_{H^1(\Omega_\epsilon\setminus\Omega_0)}\} \overset{\epsilon\to 0}{\longrightarrow} 0.$$

It follows from the results in Section 3 that if $\epsilon_n \to 0$ and u_{ϵ_n} is a solution for $(P)_\epsilon$, then

$$\|u^{\epsilon_n} - u^0\|_{H^1(\Omega_0)} + \|u^{\epsilon_n}\|_{H^1(\Omega_{\epsilon_n}\setminus\Omega_0)} \to 0$$

where u^0 is a solution to P_0.

We also prove, in this section, that the unstable manifolds of equilibrium solutions are continuous as $\epsilon \to 0$. For that we use the convergence of equilibria to obtain the continuity of the spectrum of the linearization around such equilibria and consequently the continuity of the unstable manifolds.

4.1. Continuity of the Set of Equilibria

Consider the following family of elliptic problems

$$(P)_\epsilon \qquad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega_\epsilon. \end{cases}$$

for each $0 \leq \epsilon \leq \epsilon_0$ ($\epsilon_0 > 0$).

Assume that the problem $(P)_0$ has exactly m distinct solutions, $u_1^0,...,u_m^0$ and that zero is not in the spectrum of any of the operators $\Delta + f'(u_i^0)I : H_n^2(\Omega_0) \subset L^2(\Omega_0) \to L^2(\Omega_0)$, $i = 1, ..., m$. We show that, there is $\epsilon_0 > 0$ such that, for any $0 \leq \epsilon \leq \epsilon_0$, there are yet exactly m distinct solutions $u_1^\epsilon,...,u_m^\epsilon$ for the $(P)_\epsilon$. Furthermore,

$$\|u_k^\epsilon - u_k^0\|_{H^1(\Omega_0)} + \|u_k^\epsilon\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \to 0, \quad \text{as } \epsilon \to 0,\ 1 \leq k \leq m.$$

To show the existence and uniqueness of a solution for $(P)_\epsilon$ in a neighborhood of u_k^0 we proceed as follows. Consider the extension operator $E : H^1(\Omega_0) \to H^1(B)$ and let $u_k^{0,\epsilon} = E(u_k^0)\big|_{\Omega_\epsilon}$ and define the operators

$$\begin{aligned} &\mathcal{A}_{k,\epsilon} : H^1(\Omega_\epsilon) \to H^1(\Omega_\epsilon) \\ &\mathcal{A}_{k,\epsilon}(u_\epsilon) = \left(-\Delta + f'(u_k^{0,\epsilon})I\right)^{-1} \left(f(u_\epsilon) + f'(u_k^{0,\epsilon})u_\epsilon\right). \end{aligned} \tag{4.1}$$

The operator $\mathcal{A}_{k,\epsilon}$ is a continuous compact operator for each $0 \leq \epsilon \leq \epsilon_0$, $k = 1, ..., m$.

We firstly show the following lemma

Lemma 4.1. *There exists $\delta > 0$ such that the operator $\mathcal{A}_{k,\epsilon}$ is a strict contraction from $B_\delta(u_k^{0,\epsilon}) \subset H^1(\Omega_\epsilon)$ into itself.*

Prova: Given v_ϵ, $w_\epsilon \in \overline{B_\delta(u_k^{0,\epsilon})}$, we have that:

$$\begin{aligned} &\|\mathcal{A}_{k,\epsilon}(v_\epsilon) - \mathcal{A}_{k,\epsilon}(w_\epsilon)\|_{H^1(\Omega_\epsilon)} \\ &\quad \leq \| \left(-\Delta + f'(u_k^{0,\epsilon})I\right)^{-1} \|_{L(L^2(\Omega_\epsilon),H^1(\Omega_\epsilon))} \|f(v_\epsilon) - f(v_\epsilon) - f'(u_k^{0,\epsilon})(v_\epsilon - w_\epsilon)\|_{L^2(\Omega_\epsilon)} \\ &\quad \leq C\|f(v_\epsilon) - f(v_\epsilon) - f'(u_k^{0,\epsilon})(v_\epsilon - w_\epsilon)\|_{L^2(\Omega_\epsilon)}. \end{aligned} \tag{4.2}$$

Where we have used Lemma 2.2 to obtain that

$$\{\| \left(-\Delta + f'(u_k^{0,\epsilon})I\right)^{-1} \|_{L(L^2(\Omega_\epsilon),H^1(\Omega_\epsilon))},\ 0 \leq \epsilon \leq \epsilon_0\}$$

is bounded. Next we study $\|f(v_\epsilon) - f(v_\epsilon) - f'(u_k^{0,\epsilon})(v_\epsilon - w_\epsilon)\|_{L^2(\Omega_\epsilon)}$. Note that

$$|f(v_\epsilon(x)) - f(v_\epsilon(x)) - f'(u_k^{0,\epsilon}(x))(v_\epsilon(x) - w_\epsilon(x))| \leq \bar{C}\gamma_\epsilon(x)|v_\epsilon(x) - u_\epsilon|$$

where $\gamma_\epsilon(x) = \min\{1, |v_\epsilon(x) - u_k^{0,\epsilon}(x)| + |v_\epsilon(x) - u_k^{0,\epsilon}(x)|\}$. It follows, from the definition of γ_ϵ, that $\|\gamma_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq 1$, $0 \leq \epsilon \leq \epsilon_0$ and that, for any $1 \leq p < \infty$, $\|\gamma_\epsilon\|_{L^p(\Omega_\epsilon)} \to 0$ as $\epsilon \to 0$. Hence

$$\|f(v_\epsilon) - f(v_\epsilon) - f'(u_k^{0,\epsilon})(v_\epsilon - w_\epsilon)\|_{L^2(\Omega_\epsilon)} \leq \bar{C}\|\gamma_\epsilon(u_\epsilon - v_\epsilon)\|_{L^2(\Omega_\epsilon)}$$

Now if $\varphi_\epsilon = u_\epsilon - v_\epsilon$ we denote by $\tilde{\varphi}_\epsilon = E(\varphi_\epsilon\big|_{\Omega_0})\big|_{\Omega_\epsilon}$. Then

$$\begin{aligned}\|\tilde{\varphi}_\epsilon - \varphi_\epsilon\|_{L^2(\Omega_\epsilon)} &= \|\tilde{\varphi}_\epsilon - \varphi_\epsilon\|_{L^2(\Omega_\epsilon\backslash\Omega_0)} \leq \tfrac{1}{\tau_1^\epsilon}\|\nabla\tilde{\varphi}_\epsilon - \nabla\varphi_\epsilon\|_{L^2(\Omega_\epsilon\backslash\Omega_0)}\\ &\leq C\tfrac{1}{\tau_1^\epsilon}(\|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)} + \|\tilde{\varphi}_\epsilon\|_{H^1(B)}) \leq C\tfrac{1}{\tau_1^\epsilon}(\|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)} + \|\varphi_\epsilon\|_{H^1(\Omega_0)})\\ &\leq C\tfrac{2}{\tau_1^\epsilon}\|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)},\end{aligned}$$

where we have used that $E : H^1(\Omega_0) \to H^1(B)$ is bounded and τ_1^ϵ is the first eigenvalue of $-\Delta$ in $\Omega_\epsilon\backslash\Omega_0$ with Dirichlet boundary condition in $\partial\Omega_0$ and Neumann boundary condition in $\partial\Omega_\epsilon$. Now

$$\begin{aligned}\|\gamma_\epsilon\varphi_\epsilon\|_{L^2(\Omega_\epsilon)} &\leq \|\gamma_\epsilon(\varphi_\epsilon - \tilde{\varphi}_\epsilon\|_{L^2(\Omega_\epsilon\backslash\Omega_0)} + \|\gamma_\epsilon\tilde{\varphi}_\epsilon\|_{L^2(\Omega_\epsilon\backslash\Omega_0)} + \|\gamma_\epsilon\varphi_\epsilon\|_{L^2(\Omega_0)}\\ &\leq \|\gamma_\epsilon\|_{L^\infty(\Omega_\epsilon)}\|\varphi_\epsilon - \tilde{\varphi}_\epsilon\|_{L^2(\Omega_\epsilon\backslash\Omega_0)} + \|\gamma_\epsilon\|_{L^{\frac{2N}{N+2}}(\Omega_\epsilon\backslash\Omega_0)}\|\tilde{\varphi}_\epsilon\|_{H^1(B)} + \|\gamma_\epsilon\|_{L^{\frac{2N}{N+2}}(\Omega_0)}\|\varphi_\epsilon\|_{H^1(\Omega_0)}\\ &\leq \left(C\tfrac{2}{\tau_1^\epsilon} + \tilde{C}|\Omega_\epsilon\backslash\Omega_0|^{\frac{N+2}{2N}} + \|\gamma_\epsilon\|_{L^{\frac{2N}{N+2}}(\Omega_0)}\right)\|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)}\end{aligned}$$

where we have used that $E : H^1(\Omega_0) \to H^1(B)$ is bounded. Now, given $\rho < 1$ choose ϵ_0 such that $\bar{C}C\frac{2}{\tau_1^\epsilon} + \tilde{C}|\Omega_\epsilon\backslash\Omega_0|^{\frac{N+2}{2N}} \leq \frac{\rho}{2}$ and δ small so that $\bar{C}\|\gamma_\epsilon\|_{L^{\frac{2N}{N+2}}(\Omega_0)} < \frac{\rho}{2}$. Then

$$\bar{C}\left(C\frac{2}{\tau_1^\epsilon} + \tilde{C}|\Omega_\epsilon\backslash\Omega_0|^{\frac{N+2}{2N}} + \|\gamma_\epsilon\|_{L^{\frac{2N}{N+2}}(\Omega_0)}\right) \leq \rho < 1$$

and $\mathcal{A}_{k,\epsilon}$ is a contraction from $B_\delta(u_k^{0,\epsilon}) \subset H^1(\Omega_\epsilon)$ into itself.

4.2. Continuity of Unstable Manifolds

In this section we show that the local unstable manifolds of u_k^ϵ are continuous at $\epsilon = 0$. This follows from standard invariant manifold theory and the convergence of the spectrum proved before for the linearization around such equilibria. The invariant manifold result presented here is reproduced from classical invariant manifold results as in [10]. Its proof is adapted to encompass the possibility that the space changes according to a parameter and to track the dependence of the invariant manifold upon the parameter. After this we show that the unstable manifolds are close for small ϵ.

Let u_k^ϵ be a family of solutions for (2.1). Then, we have already proved that

$$\|u_k^\epsilon - u_k^0\|_{H^1(\Omega_0)} + \|u_k^\epsilon\|_{H^1(\Omega_\epsilon\backslash\Omega_0)} \overset{\epsilon\to 0}{\to} 0.$$

Rewriting (1.1) for $w = u - u_k^\epsilon$ to deal with the neighborhood of u_k^ϵ we arrive at

$$\begin{aligned}&w_t = \Delta w + f'(u_k^\epsilon)w + f(w + u_k^\epsilon) - f(u_k^\epsilon) - f'(u_k^\epsilon)w, \quad \text{in } \Omega_\epsilon\\ &\tfrac{\partial u}{\partial n} = 0, \quad \text{in } \partial\Omega_\epsilon\end{aligned} \tag{4.3}$$

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Let $\lambda_1^\epsilon, \lambda_2^\epsilon, \lambda_3^\epsilon, \cdots$ denote the eigenvalues of $(\Delta + f'(u_k^\epsilon)I)$ and $\varphi_1^\epsilon, \varphi_2^\epsilon, \varphi_3^\epsilon, \cdots$ are corresponding orthonormalized eigenfunctions. If $\lambda_1^0, \cdots, \lambda_n^0$ are positive and $\lambda_{n+1}^0, \lambda_{n+2}^0, \cdots$ are negative, let $\beta > 0$ and $\epsilon_0 > 0$ such that $\lambda_1^\epsilon \geq \cdots \geq \lambda_n^\epsilon \geq \beta > 0 > -\beta \geq \lambda_{n+1}^\epsilon \geq \lambda_{n+2}^\epsilon \cdots$, $0 \leq \epsilon \leq \epsilon_0$. Denote by $W_\epsilon = [\varphi_1^\epsilon, \cdots, \varphi_n^\epsilon]$ and $W_\epsilon^\perp = \{\psi \in H^1(\Omega_\epsilon) : \int_{\Omega_\epsilon} \psi\phi = 0,\ \forall \phi \in W\}$. Let $P_\epsilon : H^1(\Omega_\epsilon) \to H^1(\Omega_\epsilon)$ be the orthogonal projections on W_ϵ

$$P_\epsilon \psi = \sum_{i=1}^n \left(\int_{\Omega_\epsilon} \psi \varphi_i^\epsilon \right) \varphi_i^\epsilon$$

and $P_\epsilon^\perp = I - P_\epsilon$.

If $\psi \in W_\epsilon$ then $\psi = \sum_{i=1}^n \left(\int_{\Omega_\epsilon} \psi \varphi_i^\epsilon \right) \varphi_i^\epsilon$ and

$$\|\psi\|_{W_\epsilon} = \left(\sum_{i=1}^n (1 + \lambda_i^\epsilon) \left(\int_{\Omega_\epsilon} \psi \varphi_i^\epsilon \right)^2 \right)^{\frac{1}{2}}$$

and since $\lambda_i^\epsilon \to \lambda_i^0$, $1 \leq i < \infty$, we have that W_ϵ is isomorphic to $\mathbb{R}^n$ through the isomorphism

$$W_\epsilon \ni \psi \stackrel{T_\epsilon}{\to} \left(\int_{\Omega_\epsilon} \psi \varphi_1^\epsilon, \cdots, \int_{\Omega_\epsilon} \psi \varphi_n^\epsilon \right) \in \mathbb{R}^n.$$

T_ϵ is bounded with bounded inverse T_ϵ^{-1} and the norms of T_ϵ and T_ϵ^{-1} are uniformly bounded $0 \leq \epsilon \leq \epsilon_0$.

Now we decompose the equation (4.3) in the following way. If w is a solution to (4.3) we write

$$w = \sum_{i=1}^n v_i \varphi_i^\epsilon + z$$

where $v_i = \displaystyle\int_{\Omega_\epsilon} w \varphi_i^\epsilon$. Hence

$$\dot{v}_i = \lambda_i^\epsilon v_i + \int_{\Omega_\epsilon} \left[f(w + u_k^\epsilon) - f(u_k^\epsilon) - f'(u_k^\epsilon) w \right] \varphi_i^\epsilon$$

and

$$\begin{aligned} z_t &= \Delta z + f'(u_k^\epsilon) z + f(w + u_k^\epsilon) - f(u_k^\epsilon) - f'(u_k^\epsilon) w \\ &\qquad - \sum_{i=1}^n \left(\int_{\Omega_\epsilon} \left[f(w + u_k^\epsilon) - f(u_k^\epsilon) - f'(u_k^\epsilon) w \right] \varphi_\epsilon \right) \varphi_i^\epsilon \\ \frac{\partial z}{\partial n} &= 0. \end{aligned}$$

We write $v = (v_1, \cdots, v_n)^\top$,

$$F_i^\epsilon(v, z) = \int_{\Omega_\epsilon} \left[f\left(\sum_{i=1}^n v_i \varphi_i^\epsilon + z + u_k^\epsilon \right) - f(u_k^\epsilon) - f'(u_k^\epsilon) \left(\sum_{i=1}^n v_i \varphi_i^\epsilon + z \right) \right] \varphi_\epsilon,$$

$F(v,z) = (F_1(v,z), \cdots, F_n(v,z))^\top$ and

$$G_\epsilon(v,z) = f\left(\sum_{i=1}^n v_i\varphi_i^\epsilon + z + u_k^\epsilon\right) - f(u_k^\epsilon) - f'(u_k^\epsilon)w - \sum_{i=1}^n F_i(v,z)\varphi_i^\epsilon.$$

Hence, we have that, $F_\epsilon(0,0) = 0, G_\epsilon(0,0) = 0$ and given $\rho > 0$ there exists $\delta > 0$ such that $\|v\|_{\mathbb{R}^n} + \|z\|_{H^1(\Omega_\epsilon)} < \delta$ implies

$$\begin{aligned}
&\|F_\epsilon(v,z)\|_{\mathbb{R}^n} < \rho,\\
&\|G_\epsilon(v,z)\|_{H^1(\Omega_\epsilon)} < \rho,\\
&\|F_\epsilon(v,z) - F_\epsilon(\tilde{v},\tilde{z})\|_{\mathbb{R}^n} < \rho(\|v-\tilde{v}\|_{\mathbb{R}^n} + \|z-\tilde{z}\|_{H^1(\Omega_\epsilon)}),\\
&\|G_\epsilon(v,z) - G_\epsilon(\tilde{v},\tilde{z})\|_{H^1(\Omega_\epsilon)} < \rho(\|v-\tilde{v}\|_{\mathbb{R}^n} + \|z-\tilde{z}\|_{H^1(\Omega_\epsilon)})
\end{aligned}$$

$0 \leq \epsilon \leq \epsilon_0$. We can extend F_ϵ, G_ϵ outside $B_\delta(u_k^\epsilon)$ without changing the above bounds.

Denote by $A_\epsilon = (\Delta + f'(u_k^\epsilon)I)\big|_{W_\epsilon^\perp}$, $B_\epsilon = \text{diag}(\lambda_1^\epsilon, \cdots, \lambda_n^\epsilon)$. Then, equation (4.3) can be rewritten in the following abstract form

$$\begin{aligned}
\dot{v} &= B_\epsilon v + F_\epsilon(v,z)\\
\dot{z} &= A_\epsilon z + G_\epsilon(v,z),
\end{aligned} \tag{4.4}$$

$v \in \mathbb{R}^n$, $z \in W_\epsilon^\perp$, where

$$\begin{aligned}
&\sup_{v,z}\|F_\epsilon(v,z)\|_{\mathbb{R}^n} < \rho,\\
&\sup_{v,z}\|G_\epsilon(v,z)\|_{H^1(\Omega_\epsilon)} < \rho,\\
&\|F_\epsilon(v,z) - F_\epsilon(\tilde{v},\tilde{z})\|_{\mathbb{R}^n} < \rho(\|v-\tilde{v}\|_{\mathbb{R}^n} + \|z-\tilde{z}\|_{H^1(\Omega_\epsilon)}),\\
&\|G_\epsilon(v,z) - G_\epsilon(\tilde{v},\tilde{z})\|_{H^1(\Omega_\epsilon)} < \rho(\|v-\tilde{v}\|_{\mathbb{R}^n} + \|z-\tilde{z}\|_{H^1(\Omega_\epsilon)}).
\end{aligned}$$

Also, for some positive M, β, independent of ϵ, $0 \leq \epsilon \leq \epsilon_0$

$$\begin{aligned}
&\|e^{A_\epsilon t}z\|_{H^1(\Omega_\epsilon)} \leq Me^{-\beta t}\|z\|_{H^1(\Omega_\epsilon)}, \quad t \geq 0,\\
&\|e^{A_\epsilon t}z\|_{H^1(\Omega_\epsilon)} \leq Mt^{-\frac{1}{2}}e^{-\beta t}\|z\|_{L^2(\Omega_\epsilon)}, \quad t \geq 0,\\
&\|e^{B_\epsilon t}v\|_{\mathbb{R}^n} \leq Me^{\beta t}\|v\|_{\mathbb{R}^n}, \quad t \leq 0.
\end{aligned}$$

Then, for suitably small ρ, there is an unstable manifold for u_k^ϵ

$$S^\epsilon = \{(v,z) : z = \sigma_\epsilon^*(v),\ v \in \mathbb{R}^n\}$$

where $\sigma_\epsilon^* : \mathbb{R}^n \to W_\epsilon^\perp$ is bounded and Lipschitz continuous. Furthermore

$$\sup_{v\in\mathbb{R}^n}\|\sigma_\epsilon^*(v) - \sigma_0^*(v)\|_{H^1(\Omega_0)} \overset{\epsilon\to 0}{\to} 0.$$

Proof of the Results. The first step is to prove the existence of the invariant manifold. For $D > 0$, $\Delta > 0$, $0 < \theta < 1$, given, if $\rho > 0$ is such that

$$\begin{aligned}
&\rho M \beta^{-\frac{1}{2}} \Gamma(\tfrac{1}{2}) \leq D \\
&\rho M^2 (1+\Delta) \beta^{-\frac{1}{2}} \leq \Delta \\
&\beta - \rho M(1+\Delta) \geq \tfrac{\beta}{2} \\
&\rho M \Gamma(\tfrac{1}{2}) \left[\frac{1}{\beta^{\frac{1}{2}}} + \frac{1+\Delta}{\beta - \rho M(1+\Delta)} \right] \leq \theta < 1.
\end{aligned}$$

let $\sigma : \mathbb{R}^n \to W_\epsilon^\perp$ satisfying

$$|||\sigma_\epsilon||| := \sup_{v \in \mathbb{R}^n} \|\sigma_\epsilon(v)\|_{H^1(\Omega_\epsilon)} \leq D, \quad \|\sigma_\epsilon(v) - \sigma_\epsilon(\tilde{v})\|_{H^1(\Omega_\epsilon)} \leq \Delta \|v - \tilde{v}\|_{\mathbb{R}^n}. \tag{4.5}$$

Let $v_\epsilon(t) = \psi(t, \tau, \eta, \sigma_\epsilon)$ be the solution of

$$\frac{dv_\epsilon}{dt} = B_\epsilon v_\epsilon + F_\epsilon(v_\epsilon, \sigma_\epsilon(v_\epsilon)), \quad \text{for} \quad t < \tau,\ v_\epsilon(\tau) = \eta, \tag{4.6}$$

and define

$$\Phi(\sigma_\epsilon)(\eta) = \int_{-\infty}^{\tau} e^{A_\epsilon(\tau - s)} G_\epsilon(v_\epsilon(s), \sigma_\epsilon(v_\epsilon(s))) ds. \tag{4.7}$$

Note that

$$\|\Phi(\sigma_\epsilon)(\cdot)\|_{H^1(\Omega_\epsilon)} \leq \int_{-\infty}^{\tau} \rho M (\tau - s)^{-\frac{1}{2}} e^{-\beta(\tau - s)} ds = \rho M \beta^{-\frac{1}{2}} \Gamma(\frac{1}{2}). \tag{4.8}$$

From the choice of ρ we have that, $\|\Phi(\sigma_\epsilon)(\cdot)\|_{H^1(\Omega_\epsilon)} \leq D$. Next, suppose that σ_ϵ and $\tilde{\sigma}_\epsilon$ are functions satisfying (4.5), η, $\tilde{\eta} \in \mathbb{R}^n$ and denote $v_\epsilon(t) = \psi(t, \tau, \eta, \sigma_\epsilon)$, $\tilde{v}_\epsilon(t) = \psi(t, \tau, \tilde{\eta}, \tilde{\sigma}_\epsilon)$. Then,

$$v_\epsilon(t) - \tilde{v}_\epsilon(t) = e^{B_\epsilon\,(t-\tau)}(\eta - \tilde{\eta}) + \int_{\tau}^{t} e^{B_\epsilon\,(t-s)} [F_\epsilon(v_\epsilon, \sigma_\epsilon(v_\epsilon)) - F_\epsilon(\tilde{v}_\epsilon, \tilde{\sigma}_\epsilon(\tilde{v}_\epsilon))] ds.$$

And

$$
\begin{aligned}
\|v_\epsilon(t)-\tilde v_\epsilon(t)\|_{\mathbb{R}^n} &\leq M e^{\beta(t-\tau)}\|\eta-\tilde\eta\|_{\mathbb{R}^n} + M\int_t^\tau e^{\beta(t-s)}\|F_\epsilon(v_\epsilon,\sigma_\epsilon(v_\epsilon)) - F_\epsilon(\tilde v_\epsilon,\tilde\sigma_\epsilon(\tilde v_\epsilon))\|_{\mathbb{R}^n} ds\\
&\leq M e^{\beta(t-\tau)}\|\eta-\tilde\eta\|_{\mathbb{R}^n}\\
&+\rho M\int_t^\tau e^{-\beta(t-s)}\left(\|\sigma_\epsilon(v_\epsilon)-\tilde\sigma_\epsilon(\tilde v_\epsilon)\|_{H^1(\Omega_\epsilon)}+\|v_\epsilon-\tilde v_\epsilon\|_{\mathbb{R}^n}\right)ds\\
&\leq M e^{\beta(t-\tau)}\|\eta-\tilde\eta\|_{\mathbb{R}^n}\\
&+\rho M\int_t^\tau e^{\beta(t-s)}\left(\|\sigma_\epsilon(\tilde v_\epsilon)-\tilde\sigma_\epsilon(\tilde v_\epsilon)\|_{H^1(\Omega_\epsilon)}+(1+\Delta)\|v_\epsilon-\tilde v_\epsilon\|_{\mathbb{R}^n}\right)ds\\
&\leq M e^{\beta(t-\tau)}\|\eta-\tilde\eta\|_{\mathbb{R}^n}\\
&+\rho M\int_t^\tau e^{\beta(t-s)}\left((1+\Delta)\|v_\epsilon-\tilde v_\epsilon\|_{\mathbb{R}^n}+|||\sigma_\epsilon-\tilde\sigma_\epsilon|||_{H^1(\Omega_\epsilon)}\right)ds\\
&\leq M e^{\beta(t-\tau)}\|\eta-\tilde\eta\|_{\mathbb{R}^n}\\
&+\rho M(1+\Delta)\int_t^\tau e^{\beta(t-s)}\|v_\epsilon-\tilde v_\epsilon\|_{\mathbb{R}^n}ds+\rho M|||\sigma_\epsilon-\tilde\sigma_\epsilon|||_{H^1(\Omega_\epsilon)}\int_t^\tau e^{\beta(t-s)}ds.
\end{aligned}
$$

Let

$$\phi(t)=e^{-\beta\,(t-\tau)}\|v_\epsilon(t)-\tilde v_\epsilon(t)\|_{\mathbb{R}^n}.$$

Then,

$$\phi(t)\leq M\|\eta-\tilde\eta\|_{\mathbb{R}^n}+\rho M\int_t^\tau e^{\beta(\tau-s)}ds|||\sigma_\epsilon-\tilde\sigma_\epsilon|||_{H^1(\Omega_\epsilon)}+M\,\rho\,(1+\Delta)\int_t^\tau\phi(s)ds.$$

By Gronwall's inequality

$$
\begin{aligned}
\|v_\epsilon(t)-\tilde v_\epsilon(t)\|_{\mathbb{R}^n} &\leq [M\|\eta-\tilde\eta\|_{\mathbb{R}^n}e^{\beta\,(t-\tau)}+\rho M\int_t^\tau e^{\beta(t-s)}ds|||\sigma_\epsilon-\tilde\sigma_\epsilon|||_{H^1(\Omega_\epsilon)}]e^{-\rho M(1+\Delta)(t-\tau)}\\
&\leq [M\|\eta-\tilde\eta\|_{\mathbb{R}^n}+\rho M\beta^{-1}|||\sigma_\epsilon-\tilde\sigma_\epsilon|||_{H^1(\Omega_\epsilon)}]e^{-\rho M(1+\Delta)(t-\tau)}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|\Phi(\sigma_\epsilon)(\eta)-\Phi(\tilde\sigma_\epsilon)(\tilde\eta)\|_{H^1(\Omega_\epsilon)} &\leq M\int_{-\infty}^\tau(\tau-s)^{-\frac12}e^{-\beta(\tau-s)}\|G_\epsilon(v_\epsilon,\sigma_\epsilon(v_\epsilon))-G_\epsilon(\tilde v_\epsilon,\tilde\sigma_\epsilon(\tilde v_\epsilon))\|_{L^2(\Omega_\epsilon)}ds\\
&\leq \rho M\int_{-\infty}^\tau(\tau-s)^{-\frac12}e^{-\beta(\tau-s)}\left(\|\sigma_\epsilon(v_\epsilon)-\tilde\sigma_\epsilon(\tilde v_\epsilon)\|_{H^1(\Omega_\epsilon)}+\|v_\epsilon-\tilde v_\epsilon\|_{\mathbb{R}^n}\right)ds\\
&\leq \rho M\int_{-\infty}^\tau(\tau-s)^{-\frac12}e^{-\beta(\tau-s)}\left[(1+\Delta)\|v_\epsilon-\tilde v_\epsilon\|_{\mathbb{R}^n}+|||\sigma_\epsilon-\tilde\sigma_\epsilon|||\right]ds.
\end{aligned}
$$

Using the estimates for $\|v_\epsilon-\tilde v_\epsilon\|_{\mathbb{R}^n}$ we obtain

$$\|\Phi(\sigma_\epsilon)(\eta)-\Phi(\tilde\sigma_\epsilon)(\tilde\eta)\|\leq\rho M\Gamma(\frac12)\left[\beta^{-\frac12}+\frac{1+\Delta}{\beta-\rho M(1+\Delta)}\right]|||\sigma_\epsilon-\tilde\sigma_\epsilon|||+\rho M^2(1+\Delta)\beta^{-\frac12}\|\eta-\tilde\eta\|_{\mathbb{R}^n}.$$

Let

$$I_\sigma(\epsilon)=\rho M\,\Gamma(\frac12)\,\left[\beta^{-\frac12}+\frac{1+\Delta}{\beta-\rho M(1+\Delta)}\right]$$

and

$$I_\eta(\epsilon) = \rho M^2(1+\Delta)\beta^{-\frac{1}{2}}.$$

It is easy to see that, given $\theta < 1$, there exists a ρ_0 such that, for $\rho \le \rho_0$, $I_\sigma(\epsilon) \le \theta$ and $I_\eta(\epsilon) \le \Delta$ and

$$\|\Phi(\sigma_\epsilon)(\eta) - \Phi(\tilde{\sigma}_\epsilon)(\tilde{\eta})\|_{H^1(\Omega_\epsilon)} \le \Delta\|\eta-\eta'\|_{\mathbb{R}^n} + \theta|||\sigma_\epsilon - \tilde{\sigma}_\epsilon|||. \tag{4.9}$$

The inequalities (4.8) and (4.9) imply that G is a contraction map from the class of functions that satisfy (4.5) into itself. Therefore, it has a unique fixed point $\sigma_n^* = \Phi(\sigma_n^*)$ in this class.

It remains to prove that $S = \{(v, \sigma_\epsilon^*(v)) : v \in \mathbb{R}^n\}$ is an invariant manifold for (4.4). Let $(v_0, z_0) \in S$, $z_0 = \sigma_\epsilon^*(v_0)$. Denote by $v_\epsilon^*(t)$ the solution of the following initial value problem

$$\frac{dv}{dt} = B_\epsilon\, v + F_\epsilon(v, \sigma_\epsilon^*(v)), \quad v(0) = v_0.$$

This defines a curve $(v_\epsilon^*(t), \sigma_\epsilon^*(v_\epsilon^*(t))) \in S, \quad t \in \mathbb{R}$. But the only solution of

$$\dot{z} = A_\epsilon z + G_\epsilon(v_\epsilon^*(t), \sigma_\epsilon^*(v_\epsilon^*(t))),$$

which remains bounded as $t \to -\infty$ is

$$z^*(t) = \int_{-\infty}^{t} e^{A_\epsilon(t-s)} f(v_\epsilon^*(s), \sigma_\epsilon^*(v_\epsilon^*(s))ds = \sigma_\epsilon^*(v_\epsilon^*(t)).$$

Therefore, $(v_\epsilon^*(t), \sigma_\epsilon^*(v_\epsilon^*(t)))$ is a solution of (4.4) through (v_0, z_0) and the invariance is proved.

Next we show that the fixed points σ_ϵ^* depend continuously upon ϵ at $\epsilon = 0$. This is accomplished in the following manner. If $0 \le \epsilon \le \epsilon_0$ is such that the unstable manifold is given by the graph of σ_ϵ^*, $0 \le \epsilon \le \epsilon_0$, we want to show that

$$\sup_{\eta\in\mathbb{R}^n} \|\sigma_\epsilon^*(\eta) - \sigma_0^*(\eta)\|_{H^1(\Omega_0)} = |||\sigma_\epsilon^* - \sigma_0^*|||.$$

It follows from Proposition 2.1 that

$$\begin{aligned}
\|\sigma_\epsilon^*(\eta)\big|_{\Omega_0} - \sigma_0^*(\eta)\|_{H^1(\Omega_0)} &\le \int_{-\infty}^{\tau} \|e^{A_\epsilon(\tau-s)} G_\epsilon(v_\epsilon, \sigma_\epsilon^*(v_\epsilon))\big|_{\Omega_0} - e^{A_0(\tau-s)} G_0(v_0, \sigma_0^*(v_0))\|_{H^1(\Omega_0)} ds \\
&\le M\theta(\epsilon) \int_{-\infty}^{\tau} e^{\beta(\tau-s)}(\tau-s)^{-\alpha} \|G_\epsilon(v_\epsilon, \sigma_\epsilon^*(v_\epsilon))\|_{L^2(\Omega_0)} ds \\
&\quad + M\int_{-\infty}^{\tau} e^{\beta(\tau-s)}(\tau-s)^{-\frac{1}{2}} \|G_\epsilon(v_\epsilon, \sigma_\epsilon^*(v_\epsilon))\big|_{\Omega_0} - G_0(v_0, \sigma_0^*(v_0))\|_{L^2(\Omega_0)} ds \\
&\le o(1) + \rho M \beta^{-\frac{1}{2}} \Gamma(\tfrac{1}{2})\, |||\sigma_\epsilon^* - \sigma_0^*||| + \rho M(1+\Delta) \int_{-\infty}^{\tau} e^{-\beta(\tau-s)}(\tau-s)^{-\frac{1}{2}} \|v_\epsilon - v_0\|_{\mathbb{R}^n} ds.
\end{aligned}$$

Thus, it is enough to estimate $\|v_\epsilon - v_0\|_{\mathbb{R}^n}$. Note that

$$
\begin{aligned}
\|v_\epsilon - v_0\|_{\mathbb{R}^n} \leq & \int_t^\tau \|e^{B_\epsilon(t-s)} - e^{B_0(t-s)}\| \, \|F_\epsilon(v_\epsilon, \sigma_\epsilon^*(v_\epsilon))\|_{\mathbb{R}^n} ds \\
& + \int_t^\tau \|e^{B_0(t-s)}\| \, \|F_\epsilon(v_\epsilon, \sigma_\epsilon^*(v_\epsilon)) - F_0(v_0, \sigma_0^*(v_0))\|_{\mathbb{R}^n} ds \\
& \leq \rho M \beta^{-1} [o(1) + |||\sigma_\epsilon^* - \sigma_0^*|||] + \rho M(1+\Delta) \int_t^\tau e^{\beta(t-s)} \|v_\epsilon - v_0\|_{\mathbb{R}^n} ds
\end{aligned}
$$

Therefore

$$
\|v_\epsilon - v_0\|_{\mathbb{R}^n} \leq \rho M \beta^{-1} \left[o(1) + |||\sigma_\epsilon^* - \sigma_0^*||| \right] e^{-\rho M(1+\Delta)(\tau - t)}
$$

which proves that

$$
\sup_{\eta \in \mathbb{R}^n} \|\sigma_\epsilon^*(\eta) - \sigma_0^*(\eta)\|_{H^1(\Omega_0)} \overset{\epsilon \to 0}{\longrightarrow} 0.
$$

We have just proved the continuity of the local unstable manifolds of equilibria and these conclude the proof of lower semicontinuity of attractors.

REFERENCES

1. J. M. Arrieta, PhD Thesis, CDSNS School of Mathematics, Georgia Institute of Technology, Atlanta GA 1991.
2. J. M. Arrieta, Neumann eigenvalue problems on exterior perturbations of the domain. *J. Differential Equations* **118** (1995).
3. J. M. Arrieta, "Spectral Behavior and Uppersemicontinuity of Attractors", International Conference on Differential Equations (EQUADIFF'99) Berlin, World Scientific, 615-621 (2000).
4. J. M. Arrieta and A. N. Carvalho. Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations. *Trans. Amer. Math. Soc.*, 352(1):285–310, 2000.
5. J. M. Arrieta, A. N. Carvalho and A. Rodriguez-Bernal, Attractors for Parabolic Problems with Nonlinear Boundary Condition. Uniform Bounds, *Communications in Partial Differential Equations* **25**, 1-2, pp. 1-37, January 20, 2000.
6. J. M. Arrieta; A. N. Carvalho and A. Rodriguez-Bernal, Upper Semicontinuity of Attractors for Parabolic Problems with Localized Large Diffusion. *Journal of Differential Equations*, Vol 168, no. 1, pp. 33-59, (2000).
7. A. N. Carvalho and E. A. M. Abreu, Attractors for Semilinear Parabolic Problems with Dirichlet Boundary Conditions in Varying Domains, preprint.
8. J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs **25** Amercan Mathematical Society, Providence 1988.
9. J. K. Hale and G. Raugel, Upper semi-continuity of the attractor for a singularly perturbed hyperbolicequation, *Journal of Diff. Equations* **73**, 197–214 (1988).
10. D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin, (1981).
11. M. Prizzi and K. P. Rybakowski, The effect of domain squeezing upon the dynamics of reaction-diffusion equations. *J. Differential Equations* **173** (2001).

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