

Invariants for Bifurcations

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Bifurcation problems with one parameter are studied here. We introduce a topological invariant, the algebraic number of folds. The invariant can be computed by algebraic methods, and we show that it is finite for germs of finite codimension. We conjecture that this value is always attained as the maximum number of folds in a stable unfolding. We also develop a method for computing the number of folds in stable one-parameter unfoldings for any given bifurcation of finite codimension.

We compute these invariants for bifurcations in one dimension and we show that the algebraic number of folds and the Milnor number form a complete set of invariants for simple bifurcations in one dimension. We present an example showing that for modal bifurcations in one dimension these invariants do not classify all problems. October, 2002 ICMC-USP

1. INTRODUCTION

In this paper we study germs of smooth bifurcation problems $f(x, \lambda)$, with n variables $x \in \mathbf{R}^n$ and one parameter $\lambda \in \mathbf{R}$. We consider invariants that arise in the classification of such problems. Our main goal is to bring into the classification of bifurcation problems an approach used in the study of map-germs for the definition of invariants associated to real singularities.

Any invariant associated to a map-germ $f : \mathbf{R}^{n+1}, (0, 0) \rightarrow \mathbf{R}^n, 0$ is also an invariant of the bifurcation problem $f(x, \lambda)$. Expressions that represent completely different bifurcations, as for instance $x^2 + \lambda$ and $x + \lambda^2$, may have the same invariants as maps $f : \mathbf{R}^2 \rightarrow \mathbf{R}$. Thus, in the study of bifurcation problems some invariants have to be introduced, in order to take into account the special role of the parameter λ .

A similar procedure is followed in [10], where a method for counting the number of branches in a bifurcation diagram is obtained using the formula of Eisenbud and Levine [4] for the degree of an isolated singularity. These invariants and their relation to our results are discussed in section 4.

We introduce in section 3 a new topological invariant of a bifurcation problem f , the algebraic number of folds, that we denote by $\beta(f)$. The invariant $\beta(f)$ is the codimension of an ideal associated to the bifurcation f and it counts the number of complex solutions to the equations for fold points. We show that f has finite codimension if and only if $\beta(f) < \infty$.

We use the results in [4, 10] to develop a method for counting the number of folds appearing in a stable one-parameter perturbation of f . The maximum value of this number is the geometric number of folds associated to the bifurcation problem. One open question is whether the geometric and algebraic number of folds coincide for bifurcation problems in one variable, i.e. if there is a stable perturbation with exactly $\beta(f)$ folds. We have computed the invariants for all the simple bifurcations in one variable (section 5) and found the answer to be yes, in the case of simple germs, but it remains open in general.

2. DEFINITIONS AND NOTATION

We study germs of smooth *bifurcation problems* $f(x, \lambda), f : \mathbf{R}^n \times \mathbf{R}, (0, 0) \rightarrow \mathbf{R}^n, 0$. The set of all such germs forms a free module, $\overline{\mathcal{E}}_{x\lambda}$, of rank n over the ring $\mathcal{E}_{x\lambda}$ of germs of smooth functions $g : \mathbf{R}^n \times \mathbf{R}, (0, 0) \rightarrow \mathbf{R}$. The ring $\mathcal{E}_{x\lambda}$, that also has the structure of a real vector space, has a unique maximal ideal $\mathcal{M}_{x\lambda}$ (denoted \mathcal{M} when no ambiguity arises) and a subring \mathcal{E}_λ of germs of functions depending only on λ . The free module over a ring \mathcal{A} generated by f_1, \dots, f_n is denoted $\langle f_1, \dots, f_n \rangle_{\mathcal{A}}$. The most common instance in this article will be the module $\langle f_1, \dots, f_n \rangle_{\mathcal{E}_{x\lambda}} \subset \overline{\mathcal{E}}_{x\lambda}$ and we will drop the subscript $\mathcal{E}_{x\lambda}$ when there is no danger of ambiguity.

To each bifurcation germ $f(x, \lambda)$ we associate its *bifurcation diagram*, the germ of the set $\{(\lambda, x) : f(x, \lambda) = 0\}$, the set of equilibria of the differential equation $\dot{x} = f(x, \lambda)$. Two bifurcation germs have similar bifurcation diagrams if one germ may be transformed into the other by changes of the variables x and of the parameter λ , up to multiplication by an orientation preserving matrix. Thus we say that two germs $f(x, \lambda)$ and $g(x, \lambda)$ are *bifurcation equivalent* (called here *b-equivalent*) if and only if there are germs of maps $S(x, \lambda), X(x, \lambda)$ and $\Lambda(\lambda)$ such that

$$f(x, \lambda) = S(x, \lambda)g(X(x, \lambda), \Lambda(\lambda))$$

where $S(x, \lambda)$ is linear for each (x, λ) and $\det S(0, 0) > 0$ and where $d_x X(0, 0)$ is invertible with $\Lambda'(0) > 0$. If moreover the equality above holds with $X(0, 0) = 0$ and $\Lambda(\lambda) = \lambda$ then we say that the *b-equivalence* of f and g is *strong*. The *b-equivalence* class of f is called the *b-orbit* (resp. the *b-strong orbit*) of f .

An *unfolding* with r parameters of a bifurcation germ $f(x, \lambda)$ is a germ $F : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^r \rightarrow \mathbf{R}^n$ satisfying $F(x, \lambda, 0) = f(x, \lambda)$. The germs $(x, \lambda) \mapsto F(x, \lambda, \alpha)$ can be seen as small deformations of $f(x, \lambda)$, we refer to them as *germs in the unfolding* F of f .

Given two unfoldings $G(x, \lambda, \beta)$ and $F(x, \lambda, \alpha)$ of f , we say that G *factors through* F under b -equivalence if they satisfy:

$$G(x, \lambda, \beta) = S(x, \lambda, \beta)F(X(x, \lambda, \beta), \Lambda(\lambda, \beta), A(\beta))$$

with $S(x, \lambda, 0) = I$, $X(x, \lambda, 0) = x$, $\Lambda(\lambda, 0) = \lambda$ and $A(0) = 0$. An unfolding F of f is *versal* if every other unfolding of f factors through F . If f has a versal unfolding then $\text{cod}(f)$, the *codimension* of f , is the minimum number of parameters in a versal unfolding of f . If no unfolding of f is versal we say that $\text{cod}(f) = \infty$.

For $n = 1$, all bifurcation problems of codimension seven or less have been classified by Keyfitz [8]. The only non trivial bifurcations of codimension zero, called *stable* bifurcations, are the folds $f(x, \lambda) = \pm x^2 \pm \lambda$ and their suspensions in \mathbf{R}^n .

Given a finite codimension bifurcation problem f , and a one-parameter unfolding $F(x, \lambda, t)$ of f we say that F is a *stabilization* of f when each germ $F_t(x, y)$ for $t \neq 0$ is stable.

A bifurcation germ $f \in \mathcal{E}_{x\lambda}$ is *adjacent* to $g \in \mathcal{E}_{x\lambda}$ if f is b -equivalent to some germ in an unfolding of g . A bifurcation germ $f(x, \lambda)$ is *simple* if there is only a finite number of b -equivalence classes of germs adjacent to f .

The tangent space at f to the b -strong-equivalence class of f , called the *restricted tangent space* of f , is

$$\text{RT}(f) = \left\langle f_i e_j, x_i \frac{\partial f}{\partial x_j}, \lambda \frac{\partial f}{\partial x_j} \quad i, j = 1, \dots, n \right\rangle_{\mathcal{E}_{x\lambda}} \subset \overrightarrow{\mathcal{E}_{x\lambda}}$$

where e_j denotes the elements of the standard basis of \mathbf{R}^n . It can be shown [3] that f has finite codimension if and only if $\text{RT}(f)$ has finite codimension in $\overrightarrow{\mathcal{E}_{x\lambda}}$.

3. INVARIANTS — CRITICAL POINTS AND FOLDS

If in the definition of b -equivalence we drop the requirement that the change of parameter Λ does not depend on x we obtain the usual definition of contact equivalence of maps: two germs $f, g : \mathbf{R}^n, 0 \rightarrow \mathbf{R}^p, 0$ are \mathcal{K} -equivalent if there is a germ of a diffeomorphism $Y : \mathbf{R}^n, 0 \rightarrow \mathbf{R}^n, 0$ and a germ of a map $S : \mathbf{R}^n, 0 \rightarrow \mathcal{GL}(p)$ (i.e. $S(y)$ is an invertible linear map on \mathbf{R}^p) such that:

$$g(y) = S(y)f(Y(y))$$

Every invariant of \mathcal{K} -equivalence of germs in $\overrightarrow{\mathcal{E}_{x\lambda}}$ is also invariant under b -equivalence, the second being a specialization of the first. One example is the Milnor number $\mu(f)$ of a bifurcation f . If f has finite codimension with respect to \mathcal{K} -equivalence, there is no loss of generality in assuming that f is real analytic. Let $f_{\mathbf{C}}$ be its complexification. Then, the complex hypersurface $f_{\mathbf{C}}^{-1}(0)$ defines a complete intersection with isolated singularity, and $\mu(f) = \mu(f_{\mathbf{C}})$ is defined as the rank of the middle dimensional homology of the Milnor fiber of $f_{\mathbf{C}}$ (see [9]).

For $n = 1$, $\mu(f)$ is the codimension of the ideal $\mathcal{I}_{\mu}(f) \subset \mathcal{E}_{x\lambda}$ given by

$$\mathcal{I}_{\mu}(f) = \langle f_x(x, \lambda), f_{\lambda}(x, \lambda) \rangle.$$

It gives an upper bound for the number of Morse critical points appearing in any germ in the unfolding of f . Note that these critical points do not have to be zeros of f . It also follows that a b -versal unfolding of a bifurcation $f(x, \lambda)$ is also a \mathcal{K} -versal unfolding of the germ f .

In addition to the \mathcal{K} invariants we have some special invariants for b -equivalence that account for the relative positions of the bifurcation diagram and the λ fibration. The algebraic number of folds $\beta(f)$ of a bifurcation f is the codimension of the ideal $\mathcal{B}(f) \subset \mathcal{E}_{x\lambda}$ given by $\mathcal{B}(f) = \langle f_1(x, \lambda), \dots, f_n(x, \lambda), J_x f(x, \lambda) \rangle$, where $J_x f$ denotes the determinant of $d_x f$. It counts the number of simple solutions $(x, \lambda) \in \mathbf{C}^{n+1}$ of $F(x, \lambda, \alpha) = 0, J_x F(x, \lambda, \alpha) = 0$ for generic fixed α on an unfolding F of f , as we show below.

THEOREM 3.1. *A bifurcation problem $g : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$ has finite codimension if and only its algebraic number of folds is finite. In this case, $\beta(f)$ is an invariant for b -equivalence giving the number of folds appearing in a stable deformation of the complexification of f .*

Proof: Assume f has finite codimension and therefore $\text{RT}(f) \supset \mathcal{M}_{x\lambda}^k \overrightarrow{\mathcal{E}_{x\lambda}}$ for some k . Following the argument of Gaffney in [5], Lemma 2.12, we will show that $\mathcal{B}(f)$ contains $\mathcal{M}_{x\lambda}^{kn}$. In fact, let $u \in \mathcal{M}_{x\lambda}^{kn}$, say $u = \prod_{i=1}^n u_i$ where $u_i \in \mathcal{M}_{x\lambda}^k$. Since $\text{RT}(f)$ contains the germs $u_i e_i$, then the matrix equation

$$d_x f \cdot A = \begin{pmatrix} u_1 & 0 & \dots & 0 \\ 0 & u_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & u_n \end{pmatrix}$$

has a solution A with entries in $\mathcal{E}_{x\lambda}$, modulo $\langle f_i e_j \rangle_{i,j=1,\dots,n}$. Taking the determinant of each side it follows that $u \in \langle J_x f \rangle \pmod{\langle f_1, \dots, f_n \rangle}$, and thus the condition is necessary.

To see the sufficiency, assume $\mathcal{B}(f) \supset \mathcal{M}_{x\lambda}^k$ for some k . It is enough to prove that $d_x f(\overrightarrow{\mathcal{M}_{x\lambda}}) \supset \langle J_x f e_i \mid i = 1, \dots, n \rangle_{\mathcal{E}_{x\lambda}}$. The proof is again just a parametrized version of Gaffney's argument.

Let $M(i, j)$ be the cofactor of the element $\frac{\partial f_i}{\partial x_j}$ in the matrix $d_x f$. Then we can write the vector whose only nonzero entry is $J_x f$ in the l -th position as:

$$\begin{pmatrix} 0 \\ \vdots \\ J_x f \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_l}{\partial x_1} & \dots & \frac{\partial f_l}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} M(l, 1) \\ \vdots \\ M(l, l) \\ \vdots \\ M(l, n) \end{pmatrix}$$

proving the first statement.

For the second statement, let $\tilde{F} : \mathbf{C}^n \times \mathbf{C} \times \mathbf{C}^u \rightarrow \mathbf{C}^n$ be the versal unfolding of the complexification of f , and let Σ be the set of $u \in \mathbf{C}^u$ such that $\tilde{F}_u = \tilde{F}(\cdot, \cdot, u)$ is more degenerate than a fold. Then, Σ is connected and for any u and \tilde{u} in the complement of Σ we have that \tilde{F}_u and $\tilde{F}_{\tilde{u}}$ are b -equivalent. Thus, for f complex the number of folds does not depend on the choice of stabilization.

Let F_α be a representative of a stabilization of the complexification of f defined in a neighbourhood of the origin. Then for each $\alpha \neq 0$

$$\beta(F_\alpha) = \sum_{(x_i, \lambda_i)} \dim_{\mathbf{C}} \frac{\mathcal{E}_{x\lambda}|_{(x_i, \lambda_i)}}{\left\langle J_x F_\alpha|_{(x_i, \lambda_i)}, F_\alpha(x_i, \lambda_i) \right\rangle}$$

where the summation is taken over all the (x_i, λ_i) that are fold points and $\cdot|_{(x_i, \lambda_i)}$ stands for germs at (x_i, λ_i) .

Each fold point contributes 1 to the summation, since $\dim(\mathcal{E}_{x\lambda}/\mathcal{B}(x^2 \pm \lambda)) = 1$ and thus $\beta(F_\alpha)$ is the number of folds for F_α .

Since $\mathcal{B}(F_\alpha)$ defines a family of complete intersection with isolated singularity, it follows [9] that the multiplicity of $\beta(F_\alpha)$ does not depend on α , that is $\beta(F_\alpha) = \beta(f)$. \square

The invariant $\beta(f)$ counts the number of complex solutions of $F_\alpha = 0, J_x F_\alpha = 0$, with multiplicity, for an unfolding F of f . In the real case, the complement in \mathbf{R}^u of the singular set Σ is not connected in general. For a stabilization F_α , the number $\mathbf{b}(F_\alpha)$ of real solutions of $F_\alpha = 0, J_x F_\alpha = 0$ depends on the choice of stabilization. Its maximum $\mathbf{b}_{max}(f)$ over all stabilizations of a given germ f is clearly an invariant of b -equivalence.

A natural question is whether $\mathbf{b}_{max}(f) = \beta(f)$. In other words, we want to know if the geometric and algebraic number of folds coincide. This is the case for all simple bifurcations in one spatial dimension studied in [8] (see section 5).

An analogous result holds for \mathcal{K} -versal unfoldings of germs of maps in \mathbf{R}^2 [1]: if $g : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ has finite \mathcal{K} -codimension and if $G(y, \alpha)$ is a \mathcal{K} -versal unfolding of $g = (g_1, g_2)$ then there is a germ of an open, path connected set A of parameters α with the origin in the closure of A , such that for each $\alpha \in A$ the map $y \mapsto G(y, \alpha)$ has exactly m zeros where m is the codimension of $\langle g_1, g_2 \rangle_{\mathcal{E}_y}$.

CONJECTURE 3.1. *Let $f : \mathbf{R} \times \mathbf{R}, (0, 0) \rightarrow \mathbf{R}, 0$ be a finite codimension bifurcation problem and $F(x, \lambda, \alpha)$ a b -versal unfolding of f . Then there is a germ of an open, path connected set A of parameters α with the origin in the closure of A such that for each $\alpha \in A$ the bifurcation diagram of $x \mapsto F(x, \lambda, \alpha)$ contains exactly $\beta(f)$ fold points.*

A natural way to prove Conjecture 3.1 would be to reduce it to the \mathcal{K} -equivalence case by considering the germ $g(x, \lambda) = (f(x, \lambda), f_x(x, \lambda))$ and a \mathcal{K} -versal unfolding of the form $G(x, \lambda, \alpha) = (F(x, \lambda, \alpha), F_x(x, \lambda, \alpha))$. Then for suitable α , the germ G would have precisely $\beta(f)$ zeros and F would factor through any b -versal unfolding of f . Unfortunately it is often not possible to find a \mathcal{K} -versal unfolding of the form (F, F_x) . For instance, consider the

simple bifurcation problem $f(x, \lambda) = x^3 - x\lambda$. An unfolding $G(x, \lambda, \alpha)$ of $g(x, \lambda) = (x^3 - x\lambda, 3x^2 - \lambda)$ is \mathcal{K} -versal if and only if $\langle \frac{\partial G}{\partial \alpha_i} \rangle_{\mathbf{R}}$ forms a complement for $\mathbb{T}_{\mathcal{K}}g$ in the $\mathcal{E}_{x\lambda}$ -module of germs of maps from the plane to the plane, where $\mathbb{T}_{\mathcal{K}}g = \{(u, v)u, v \in \mathcal{M}^2 + \langle \lambda \rangle\} + \langle (x, 1) \rangle_{\mathcal{E}_{x\lambda}}$. Clearly, the constant germ $(0, 1)$ cannot be written as $(h, h_x) + \eta$ with $h \in \mathcal{E}_{x\lambda}$, $\eta \in \mathbb{T}_{\mathcal{K}}g$.

4. INVARIANTS — BRANCHES AND FOLDS

Another \mathcal{K} -invariant of map germs $g : \mathbf{R}^{n+1}, 0 \rightarrow \mathbf{R}^n, 0$ is the number of half-branches in the bifurcation diagram of g , i.e. the number of connected components of $g^{-1}(0) - \{0\}$, denoted here as $\mathbf{r}(g)$. The corresponding invariants for b-equivalence are $\mathbf{r}_+(g)$ and $\mathbf{r}_-(g)$, the number of half branches with $\lambda > 0$ (resp. $\lambda < 0$) in the bifurcation diagram of g and $\mathbf{r}_{\pm}(g) = \mathbf{r}_+(g) - \mathbf{r}_-(g)$.

Nishimura, Fukuda and Aoki [10], have shown that $\mathbf{r}(g)$ is twice the topological degree of the germ $\Phi_1 : \mathbf{R}^{n+1}, 0 \rightarrow \mathbf{R}^{n+1}, 0$, given by $\Phi_1(y, \lambda) = (g(y, \lambda), \lambda J_y g(y, \lambda))$, where $J_y g(y, \lambda)$ is the Jacobian of g . Similarly, $\mathbf{r}_{\pm}(g)$ is twice the topological degree of the germ $\Phi_2 : \mathbf{R}^{n+1}, 0 \rightarrow \mathbf{R}^{n+1}, 0$, given by $\Phi_2(y, \lambda) = (g(y, \lambda), J_y g(y, \lambda))$. Therefore, $\mathbf{r}(g)$, $\mathbf{r}_+(g)$ and $\mathbf{r}_-(g)$ are topological invariants and the degrees of Φ_1 and Φ_2 can be computed using a result of Eisenbud and Levine [4] that we shall describe briefly.

Let $\langle \Phi \rangle$ be the ideal in \mathcal{E}_y generated by the components of a map $\Phi(y), \Phi : \mathbf{R}^m, 0 \rightarrow \mathbf{R}^m, 0$. If $\langle \Phi \rangle$ has finite codimension, consider the algebra $Q(\Phi) = \mathcal{E}_y / \langle \Phi \rangle$ and let I be an ideal of $Q(\Phi)$ that is maximal with respect to the property $I^2 = 0$. Then Eisenbud and Levine [4] show that if $Q(\Phi)$ is finite-dimensional then $|\text{degree}(\Phi)| = \dim_{\mathbf{R}} Q(\Phi) - 2 \dim_{\mathbf{R}} I$.

The algebra $Q(\Phi)$ has a minimal ideal, called the socle, generated by the class J_0 of $J_y \Phi$. Another way to obtain the degree, from [4], is to consider any linear functional l in $Q(\Phi)$ such that $l(J_0) > 0$. Then the degree of Φ is the signature of the bilinear form $L : Q(\Phi) \times Q(\Phi) \rightarrow \mathbf{R}$ given by $L(p, q) = l(pq)$.

Given a stabilization $F(x, \lambda, t)$ of a finite codimension bifurcation problem $f(x, \lambda)$, consider the algebras

$$Q_1(F) = \mathcal{E}_{x\lambda t} / \langle F, J_x F, tJ_{x\lambda}(F, J_x F) \rangle$$

and

$$Q_2(F) = \mathcal{E}_{x\lambda t} / \langle F, J_x F, J_{x\lambda}(F, J_x F) \rangle .$$

LEMMA 4.1. *If f has finite codimension and F is a stabilization of f then the algebras $Q_1(F)$ and $Q_2(F)$ have finite dimension as real vector spaces.*

Proof: Without loss of generality we may assume F is analytic, since f has finite codimension. Let $F_{\mathbf{C}} : \mathbf{C}^{n+1} \times \mathbf{C} \rightarrow \mathbf{C}^n$ be the complexification of F , that we will denote by F in the remainder of this proof. If we show that the varieties of the ideals $\langle F, J_x F, tJ_{x\lambda}(F, J_x F) \rangle$ and $\langle F, J_x F, J_{x\lambda}(F, J_x F) \rangle$ reduce to 0 then the result follows from Hilbert Nullstellensatz.

We start by the second ideal. The equations $F = 0$, $J_x F = 0$ are defining conditions for folds in the unfolding F . We claim that folds satisfying the last equation $J_{x\lambda}(F, J_x F) = 0$

are degenerate: at these points the matrix $D_{x\lambda}(F, J_x F)$ is singular, and thus either the gradient $\nabla_x(J_x F)$ is equal to zero or $D_{x\lambda}F$ is not surjective. Both possibilities characterize points that are more degenerate than folds, as claimed. This implies $t = 0$, since F is stable for $t \neq 0$. For $t = 0$, the first two equations reduce to $f(x, \lambda) = 0$, $J_x f(x, \lambda) = 0$ and have a unique solution $(x, \lambda) = (0, 0)$ since we have already shown that the fold ideal has finite codimension when $\text{cod}(f) < \infty$ and thus the claim holds.

For the first ideal, there are two possibilities, $t = 0$ and $t \neq 0$, both covered by the arguments above. □

Each one of the algebras $Q_1(F)$ and $Q_2(F)$ has a socle generated, respectively, by the residue classes of

$$s_1 = J_{x\lambda t}(F, J_x F, tJ_{x\lambda}(F, J_x F)) \quad \text{and} \quad s_2 = J_{x\lambda t}(F, J_x F, J_{x\lambda}(F, J_x F)) .$$

In each algebra $Q_i(F)$, let l_i be a linear functional satisfying $l_i(s_i) > 0$ and let L_i be the bilinear form $L_i(p, q) = l_i(pq)$, as in the Eisenbud and Levine [4] result.

PROPOSITION 4.1. *If F is a stabilization of a bifurcation problem f of finite codimension, then the number $\mathbf{b}_+(F)$ of folds in F for $t > 0$ is $|\mathbf{b}_+(F)| = \text{signature}(L_1) + \text{signature}(L_2)$.*

Proof: Consider the map germs Φ_1 and $\Phi_2 : \mathbf{R}^{n+2} \rightarrow \mathbf{R}^{n+2}$ given by

$$\Phi_1 = (F, J_x F, tJ_{x\lambda}(F, J_x F)) \quad \text{and} \quad \Phi_2 = (F, J_x F, J_{x\lambda}(F, J_x F)) .$$

Their local algebras, $Q_1(F)$ and $Q_2(F)$, are finite dimensional by Lemma 4.1. Therefore, by the results of [4] the degree of each Φ_i is the signature of L_i .

On the other hand, the fold points of F are precisely the zeros of

$$G(x, \lambda, t) = (F(x, \lambda, t), J_x F(x, \lambda, t))$$

i.e., the branches of the bifurcation problem $G(y, t)$ with $y = (x, \lambda)$, and bifurcation parameter t . Therefore, the total number of fold points of F for $t > 0$ and $t < 0$ equals the total number, $\mathbf{r}(G)$, of half-branches of G and the difference between the number of folds with $t > 0$ and $t < 0$ is $\mathbf{r}_\pm(G)$. From [10] it follows that $\mathbf{r}(G) = 2 \text{degree}(\Phi_1)$ and $\mathbf{r}_\pm(G) = 2 \text{degree}(\Phi_2)$. □

Another formulation of Proposition 4.1, less easy to compute, can be obtained using the results of [4]:

PROPOSITION 4.2. *If F is a stabilization of a bifurcation problem f of finite codimension, then the number $\mathbf{b}(F)$ of folds in F for $t > 0$ is $|\mathbf{b}(F)| = \dim(Q_1(F)) - \dim(Q_2(F)) - 2 \dim_{\mathbf{R}}(I_1) + 2 \dim_{\mathbf{R}}(I_2)$, where each $I_i \subset Q_i(F)$ is an ideal that is maximal with respect to the property $I_i^2 = 0$.*

COROLLARY 4.1. *If F is a stabilization of a bifurcation problem f of finite codimension, then the number of folds for $t > 0$ is congruent modulo 2 to the number of folds for $t < 0$.*

It is shown in [6] that generic points in the singular set Σ in the unfolding of a bifurcation problem of finite codimension (defined in section 3 in the proof of Theorem 3.1) are hysteresis, bifurcation or double limit points. Crossing a double limit point only changes the relative position of folds in a bifurcation diagram. Crossing a hysteresis or a bifurcation point creates (or destroys) a pair of folds. Thus it follows:

COROLLARY 4.2. *If F is a stable germ in a versal unfolding of a bifurcation problem f of finite codimension, then the number of folds is constant modulo 2.*

5. EXAMPLES

In this section we compute the invariants for many examples of bifurcation problems. Recall that the Milnor number $\mu(f)$ (section 3) and the number, $\mathbf{r}(f)$, of half-branches in the bifurcation diagram of f , i.e. the number of connected components of $f^{-1}(0) - \{0\}$ (section 4) are invariants for \mathcal{K} -equivalence. Invariants for b-equivalence are the algebraic number of folds, $\beta(f)$ (section 3) and the number, $\mathbf{r}_+(f)$ (resp. $\mathbf{r}_-(f)$), of half branches with $\lambda > 0$ (resp. $\lambda < 0$) in the bifurcation diagram of f as well as $\mathbf{r}_\pm(f) = \mathbf{r}_+(f) - \mathbf{r}_-(f)$ (section 4).

We start with simple bifurcations in one spatial dimension, studied in [8]:

Table 1 – invariants for simple bifurcations, $n = 1$

normal form $f(x, \lambda)$	k	$\text{cod}(f)$	$\beta(f)$	$\mu(f)$	$\mathbf{r}(f)$	$\mathbf{r}_\pm(f)$
$\pm x^3 \pm \lambda^2$		3	4	2	2	0
$\pm x^k \pm \lambda$	$k \geq 2$	k even k odd	$k - 2$	$k - 1$	0	2 2 ± 2 0
$\pm x^2 \pm \lambda^k$	$k \geq 2$	k even k odd	$k - 1$	k	$k - 1$	4 2 0 ± 2
$\pm x^k \pm x\lambda$	$k \geq 3$	k even k odd	$k - 1$	k	1	4 4 0 ± 2

For these simple bifurcations it is easy to find a suitable unfolding explicitly and to check that Conjecture 3.1 holds, i.e., $\mathbf{b}_{\max}(f) = \beta(f)$. No two such problems have the same invariants $\beta(f)$ and $\mu(f)$. Modal bifurcations in one dimension, however, are not

classified by the Milnor number and the algebraic number of folds, as can be seen in Table 2:

Table 2 – invariants for some modal bifurcations

normal form	restrictions	cod(f)	$\beta(f)$	$\mu(f)$
$\pm x^k \pm x\lambda^2$	$k \geq 4$	$2k - 1$	$2k$	$k + 1$
$\pm x^k \pm x\lambda^2 \pm \lambda^3$	$k \geq 4$	$2k - 2$	$2k$	$k + 1$
$\pm x^{k+1} \pm x^{k-1}\lambda^2 \pm \lambda^3$	$k \geq 3$	$2k - 2$	$2k$	$k + 1$
$c_m(x, \lambda) = \pm(x^3 - 3mx\lambda^2 \pm 2\lambda^3)$	$m \neq 0, 1$	5	6	4
$q_m(x, \lambda) = \pm x^4 + 2mx^2\lambda \pm \lambda^2$	$m \neq 0, \pm 1$	5	6	3
$\pm x^k \pm \lambda^2$	$k \geq 4$	$2k - 3$	$2k - 2$	$k - 1$
$\pm x^k \pm x^{k-2}\lambda \pm \lambda^2$	$k \geq 5$	$2k - 4$	$2k - 2$	$k - 1$

Even for bifurcation problems of the same codimension, the invariants β and μ are not complete. For instance, the bifurcations $\pm x^4 \pm x\lambda^2 \pm \lambda^3$ and $\pm x^5 \pm x^3\lambda^2 \pm \lambda^3$ both have codimension 6 and $\beta = 8, \mu = 5$. Both examples, with all choices of sign \pm , have $\mathbf{r}_{\pm}(f) = 0$. For the first, $f_1 = x^4 + x\lambda^2 + \lambda^3$ we have $\mathbf{r}(f_1) = 4$ while $f_2 = x^5 + x^3\lambda^2 + \lambda^3$ satisfies $\mathbf{r}(f_2) = 2$.

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