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ON LOCAL DIFFEOMORPHISMS OF $\mathbb{R}^n$ THAT ARE INJECTIVE

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Let $X: \mathbb{R}^2 \to \mathbb{R}^2$ be a map of class C^1 and let $\text{Spec}(X)$ be the set of (complex) eigenvalues of the derivative DX_p when p varies in $\mathbb{R}^2$. If, for some $\epsilon > 0$, either $spec(X) \cap (-\epsilon, 0] = \emptyset$ or $spec(X) \cap [0, \epsilon) = \emptyset$, then X is injective. Some partial extensions to $\mathbb{R}^n$, of this result, will be presented. October, 2002
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1. INTRODUCTION

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. We denote by $\text{Spec}(F)$ the set of (complex) eigenvalues of the derivative DF_p, as p varies in $\mathbb{R}^n$. One of the several equivalent formulations of the famous KELLER JACOBIAN CONJECTURE states that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map having constant non-zero Jacobian, then F is injective. The WEAK MARCUS-YAMABE CONJECTURE states that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map such that $\text{Spec}(F) \subset \{z \in \mathbb{C} : \Re(z) < 0\}$, then F is injective. The CHAMBERLAND CONJECTURE [4] states that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a map of class C^1 such that, for some $\epsilon > 0$, $\text{Spec}(F) \cap \{z \in \mathbb{C} : |z| < \epsilon\} = \emptyset$, then F is injective. We shall see that the Chamberland Conjecture implies the Weak Marcus-Yamabe.

As a consequence of the work by H. Bass, E. Conell, D. Wright and A. Yagzhev ([1], [19]) we know that if the Keller Jacobian Conjecture is true for polynomial maps $P : \mathbb{R}^n \to \mathbb{R}^n$ of degree ≤ 3, for all $n \geq 2$, and such that $\text{Spec}(P) = \{-1\}$, then it is also true for general polynomial maps (see [17], Proof of Proposition 8.1.8). As a consequence, if either the Chamberland or the Weak Marcus-Yamabe conjectures is true for the set of polynomial

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maps $P : \mathbb{R}^n \to \mathbb{R}^n$ of degree ≤ 3, for all $n \geq 2$, then the Keller Jacobian Conjecture is also true. For more details about these conjectures, we refer the reader to the book by A. van den Essen [17] and the article by S. Nollet and F. Xavier [12].

Pinchuck [14] (See also [17], page 241), constructed non-injective polynomial maps $P : \mathbb{R}^2 \to \mathbb{R}^2$ such that $0 \notin \mathrm{Spec}(P)$. Also B. Smith and F. Xavier ([16], Theorem 4) proved that there exist integers $n > 2$ and non-injective polynomial maps $P : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathrm{Spec}(P) \cap [0, \infty) = \emptyset$. These examples lead us to believe that the Weak Marcus Yamabe Conjecture and the Chamberland Conjectures may not be true in general. In this paper we study cases in which both conjectures are true. The nice survey article of S. Nollet and F. Xavier [12] provides other sufficient conditions for the existence of Global diffeomorphism of $\mathbb{R}^n$.

Our main result is the following theorem:

Theorem 2.1. *Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a map of class C^1. If, for some $\epsilon > 0$, either $spec(X) \cap (-\epsilon, 0] = \emptyset$ or $spec(X) \cap [0, \epsilon) = \emptyset$, then X is injective.*

Comments:

(1) The theorem is optimal in the following sense. If the assumptions are relaxed to $0 \notin \mathrm{Spec}(X)$, then the conclusion, even for polynomials map X, need no longer be true, as shown by Pinchuck's counterexample.

(2) This theorem implies Fessler [6] and Gutierrez [7] injectivity result, which requires that $spec(X) \cap [0, \infty) = \emptyset$. As a consequence, Theorem 2.1 together with Olech's work [13] provide an alternative and short proof of the bidimensional Marcus-Yamabe Conjecture ([11], [6], [7]).

(3) This theorem confirms, in a stronger way, the Chamberland Conjecture in dimension 2.

(4) This theorem does not imply the bidimensional real Keller Jacobian Conjecture, since given an even natural n, the polynomial map $X(x, y) = (-y, x + y^n)$ has constant Jacobian equal to one and satisfies $\mathrm{Spec}(X) = \mathbb{S}^1 \cup (\mathbb{R} \setminus \{0\})$

(5) Campbell [3] classified the two–dimensional C^1 maps whose eigenvalues are both 1. All such maps have an explicit inverse. The class of functions considered in Theorem 2.1 is much broader, but no explicit inverse is given.

We acknowledge that M. Cobo and A. Sarmiento obtained independently Theorem 2.1.

A well known result of Hadamard says that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a locally diffeomorphic proper map of class C^1, then F is a local diffeomorphism. In other direction to Hadamard's Theorem and the Smyth and Xavier examples [16], we prove an injectivity result which is valid for maps which need not be proper:

Theorem 3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map of class C^1 such that $\mathrm{Spec}(F)$ is disjoint of a sequence $\{t_m\}$ of real numbers which converges to 0 as $m \to \infty$. If there exist $R > 0$ and $0 < \alpha < 1$ such that, for all x in $\mathbb{R}^n$ with $\|x\| > R$, $\|F(x)\| \leq \|x\|^\alpha$, then F is injective.*

We will then prove the following result

Theorem 3.4. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 Lipschitz local diffeomorphism. Suppose that there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbb{C}$ (with non-empty interior), centered* *at points* t_m *of the real axis, such that* $\lim_{m\to\infty} t_m = 0$ *and*

$$\mathrm{Spec}(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

Then, F *is injective.*

Corollary The Chamberland Conjecture and the Weak Marcus Yamabe Conjecture are true for C^1 Lipschitz maps.

This paper is organized as follows. The first section is devoted to prove Theorem 2.1 under a stronger assumption. The proof of Theorem 2.1 is completed in section 2. Section 3 is devoted to the proof of Theorems B and C.

1. HALF-REEB COMPONENTS

In this section we prove the following result of Cobo, Gutierrez and Llibre [5]. The proof is as appears in [5], and we included it for completeness.

Theorem 1.1. *Let* $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^1 *map. If, for some* $\epsilon > 0$, *spec*$(DX) \cap (-\epsilon, \epsilon) = \emptyset$, *then* X *is injective.*

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 submersion. For $q \in \mathbb{R}^2$ we denote by $X_f(q) = (-f_y(q), f_x(q))$ the planar Hamiltonian vector field for Hamiltonian f. As usual $\nabla f(p) = (f_x(p), f_y(p))$ denotes the gradient of f. Let $g(x, y) = xy$ and consider the set

$$B = \{(x, y) \in [0, 2] \times [0, 2] : x + y \leq 2\} \setminus \{(0, 0)\} \ .$$

Definition 1. We will say that $\mathcal{A} \subset \mathbb{R}^2$ is a *half-Reeb component* for X_f (or simply a *hRc* for X_f) if there is a homeomorphism $h : B \to \mathcal{A}$ which is a topological equivalence between $X_f|_{\mathcal{A}}$ and $X_g|_B$ and such that

(1) The segment $\{(x, y) \in B : x + y = 2\}$ is sent by h onto a transversal section for the flow of X_f in the complement of $h(1, 1)$; this section is called the *compact edge* of $\mathcal{A}$.

(2) Both segments $\{(x, y) \in B : x = 0\}$ and $\{(x, y) \in B : y = 0\}$ are sent by h onto full half-trajectories of X_f. These two semi-trajectories of X_f are called the *non–compact edges* of $\mathcal{A}$.

The connection between half-Reeb components and injectivity is given by the following result.

Proposition 1.2. *Suppose that* $X = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ *is a* C^1 *map such that* $0 \notin \mathrm{Spec}(X)$. *If* X *is not injective, then both* X_f *and* X_g *have hRc's.*

Proof:

Suppose by contradiction that X_f has no half-Reeb components. By assumption, the Hamiltonian vector field X_f has no singularities. Hence, by Kaplan's classification of planar foliations [9], [10], we obtain that X_f is topologically equivalent to the horizontal foliation

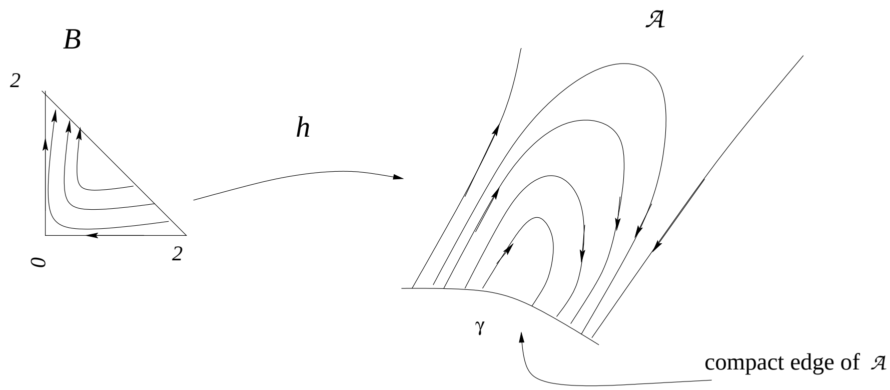

FIG. 1. A half-Reeb component.

of $\mathbb{R}^2$. Since f is also a submersion we have that each non-empty level curve of f must have exactly one connected component. As g restricted to each level curve of f is strictly monotone, we arrive at the contradiction that X is injective. ■

For each $\theta \in \mathbb{R}$ let R_θ denote the linear rotation

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The following proposition will be useful. Its proof is contained in [7, Lemma 2.5].

Proposition 1.3. *Let $X = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a non-injective C^1-map such that $0 \notin \mathrm{Spec}(X)$. Let $\mathcal{A}$ be a hRc of X_f and let $(f_\theta, g_\theta) = R_\theta \circ X \circ R_{-\theta}$, $\theta \in \mathbb{R}$. Then there is an $\epsilon > 0$ such that, for all $\theta \in (-\epsilon, 0) \cup (0, \epsilon)$, the vector field X_{f_θ} has a hRc whose projection on the x–axis is an interval of infinite length.*

Proof of Theorem 1.1

Suppose by contradiction that $X = (f, g)$ is not injective. Since non–injective maps with the assumptions of Theorem 1.1 are open in the Whitney C^1 topology; we will assume that X is smooth.

By Proposition 1.2, X_f has a half-Reeb component $\mathcal{A}$. Let $\Pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate. By composing with a rotation if necessary (see Proposition 1.3) we may assume that $\Pi(\mathcal{A})$ is an unbounded interval. To simplify matters, let us suppose that $[b, \infty) \subset \Pi(\mathcal{A})$.

By Thom's Transversality Theorem [8], we can assume the following:

(a1) The set

$$T = \{(x, y) \in \mathbb{R}^2 : f_y(x, y) = 0\}$$

is composed of regular curves;

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(a2) There is a discrete subset Δ of T such that if $p \in T \setminus \Delta$ (resp. $p \in \Delta$), X_f has quadratic contact (resp. cubic contact) with the vertical foliation of $\mathbb{R}^2$.

Then, if $a > b$ is sufficiently large,

(b) For any $x \geq a$, the vertical line $\Pi^{-1}(x)$ intersects exactly one trajectory $\alpha_x \subset \mathcal{A}$ of $X_f|_{\mathcal{A}}$ such that $\Pi(\alpha_x) \cap (x, \infty) = \emptyset$. In other words, x is the maximum for the restriction $\Pi|_{\alpha_x}$.

It follows that

(c) If $x \geq a$ and $p \in \alpha_x \cap \Pi^{-1}(x)$ then $p \in T \cap \mathcal{A} \setminus \Delta$.

Let T_x be the set of $p \in \mathcal{A}$ such that $p \in \alpha_x \cap \Pi^{-1}(x)$, $x \geq a$. Notice that, for every $x \geq a$, $\alpha_x \cap \Pi^{-1}(x)$ is a finite set. Nevertheless, by (b), (c) and the Thom's Transversality Theorem, we may obtain the following stronger statement:

(d) There is a sequence $F = \{a_1, a_2, \cdots, a_i, \cdots\}$ in $[a, \infty)$, which may be at most countable, such that if $x \in F$ (resp. $x \in [a, \infty) \setminus F$), then $\Pi^{-1}(x) \cap T_x$ is a two-point-set (resp. a one-point-set).

If $x \in [a, \infty) \setminus F$, define $\eta(x) = (x, \eta_2(x)) = \Pi^{-1}(x) \cap T_m$. Observe that $\eta : [a, \infty) \setminus F \to T_m$ is a smooth embedding. Since $f|_{\mathcal{A}}$ is bounded,

(e) $F \circ \eta$ extends continuously to a strictly increasing bounded map defined in $[a, \infty)$ such that,

for all $\quad x \in [a, \infty) \setminus F, \quad f_x(\eta(x)) \quad$ has constant sign.

Therefore, there exists a real constant K such that

$$\begin{aligned} K = \int_{a_1}^{\infty} \frac{d}{dx} f(\eta(x))\, dx &= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} \frac{d}{dx} f(\eta(x))\, dx \\ &= \sum_{i=1}^{\infty} \int_{a_i}^{a_{i+1}} f_x(\eta(x)) \end{aligned}$$

This observation and (e) imply that, for some sequence $x_n \to \infty$, $\lim_{n\to\infty} f_x(\eta(x_n)) = 0$. This contradicts the assumption $\mathrm{Spec}(X) \cap (-\epsilon, \epsilon) = \emptyset$. We have proved Theorem 1.1. ∎

2. MAIN LEMMA AND MAIN RESULT

This section is devoted to the proof of the following

Theorem 2.1. *Let $X\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map of class C^1. If, for some $\epsilon > 0$, either spec$(DX) \cap (-\epsilon, 0] = \emptyset$ or spec$(DX) \cap [0, \epsilon) = \emptyset$, then X is injective.*

We shall need the following important

Lemma 2.2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map such that* $\det(F'(x)) \neq 0$ *for all x in $\mathbb{R}^n$. Given $t \in \mathbb{R}$, let $F_t \colon \mathbb{R}^n \to \mathbb{R}^n$ denote the map $F_t(x) = F(x) - tx$. If there exists a sequence $\{t_m\}$ of real numbers converging to 0 such that every map $F_{t_m} \colon \mathbb{R}^n \to \mathbb{R}^n$ is injective, then F is injective.*

Proof:
Choose $x_1, x_2 \in \mathbb{R}^n$ such that $F(x_1) = y = F(x_2)$. We will prove $x_1 = x_2$. Define $G\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ by $G(x,t) = F(x) - tx$. Since $D_x G(x,0) = F'(x)$ is nonsingular for all $x \in \mathbb{R}^n$, it follows from the Implicit Function Theorem that there exist C^1 maps $\phi_1, \phi_2 \colon (-\delta, \delta) \to \mathbb{R}^n; \delta > 0$, such that $\phi_1(0) = x_1$, $\phi_2(0) = x_2$ and $G(\phi_1(t), t) = y = G(\phi_2(t), t)$ for every $-\delta < t < \delta$. Therefore, if m is large enough, F_{t_m} is injective and $t_m \in (-\delta, \delta)$. Thus, $\phi_1(t_m) = \phi_2(t_m)$. and so, $x_1 = \lim_{m\to\infty} \phi_1(t_m) = \lim_{m\to\infty} \phi_2(t_m) = x_2$. ■

Remark 2.3. *Even if $n = 1$ and the maps F_{tm} in Lemma 2.2 are diffeomorphisms, we cannot conclude that F is a diffeomorphism. For instance, if $F : \mathbb{R} \to (0,1)$ is an orientation reversing diffeomorphism, then for all $t > 0$, the map $F_t : \mathbb{R} \to \mathbb{R}$ will be an orientation reversing global diffeomorphism.*

Proof of Theorem 2.1

We need only consider the case where $spec(DX) \cap [0, \epsilon) = \emptyset$. We claim that for each $0 < t < \epsilon$, the map $F_t \colon \mathbb{R}^2 \to \mathbb{R}^2$, given by $F_t(x) = F(x) - tx$, is injective. In fact, if $0 < a < \min\{t, \epsilon - t\}$, then $spec(DF_t) \cap (-a, a) = \emptyset$. This theorem follows immediately from Lemma 2.2 and Theorem 1.1. ■

3. THEOREM 2.1 VERSUS HIGHER DIMENSIONS

the following theorem is in the opposite direction from the Smyth and Xavier example [[16], Theorem 4]

Theorem 3.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a map of class C^1 such that* $\mathrm{Spec}(DF)$ *is disjoint of a sequence $\{t_m\}$ of real numbers which converges to 0 as $m \to \infty$. If there exist $R > 0$ and $0 < \alpha < 1$ such that, for all x in $\mathbb{R}^n$ with $\|x\| > R$, $\|F(x)\| \leq \|x\|^\alpha$, then F is injective.*

Proof:
Define $F_{t_m} \colon \mathbb{R}^n \to \mathbb{R}^n$ by $F_{t_m}(x) = F(x) - t_m x$. Since every t_m is not in $\mathrm{Spec}(DF)$, we have that every F_{t_m} is a local diffeomorphism. By the assumptions, $F_{t_m}(x) \to \infty$ as $x \to \infty$, which implies that F_{t_m} is proper. It follows from Hadamard Theorem that F_{t_m} is injective, for every t_m. Therefore, we conclude from Lemma 2.2 that F is injective. ■

We shall need the following Hadamard - Plastock Theorem ([2], 5.1.5 or [15], Thm. 4.2).

Theorem 3.2. *A local C^1 diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ is bijective if*

$$\int_0^\infty \inf_{|x|=r} \|DF(x)^{-1}\|^{-1} dr = \infty.$$

We now prove the Chamberland Conjecture for Lipschitz maps.

Theorem 3.3. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 Lipschitz map. Suppose that for some $\epsilon > 0$,* $\mathrm{Spec}(F) \cap \{z \in \mathbb{C} : |z| \leq \epsilon\} = \emptyset$*. Then F is bijective.*

Proof:
Let $K > 0$ be the Lipschitz constant for F. Since F is a C^1 map, for all $x \in \mathbb{R}^n$,

$$\|DF(x)\| := \sup\{DF(x)v \ : \ |v| = 1\} \leq K,$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^n$. Let $\|\cdot\|_M$ be the norm, on the space of real matrices $n \times n$, given by

$$\|A\|_M := \sup\{|a_{ij}| \ : \ 1 \leq i, j \leq n\},$$

where $A = \{a_{ij}\}$. As the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent, there exists $K_1 > 0$ such that, for all $x \in \mathbb{R}^n$, $\|DF(x)\|_M \leq K_1$. Therefore, there exists a positive constant $K_2 > 0$ such that the classic adjoint matrix $A(x)$ of $DF(x)$ satisfies, for all $x \in \mathbb{R}^n$,

$$\|A(x)\|_M \leq K_2.$$

By the assumptions on $\mathrm{Spec}(F)$, we have that for all $x \in \mathbb{R}^n$, $|\det(DF(x))| \geq \epsilon^n$. Therefore, for all $x \in \mathbb{R}^n$,

$$\|DF(x)^{-1}\|_M \leq K_3,$$

where $K_3 = K_2/\epsilon^n > 0$ is constant. Again since the norms $\|\cdot\|_M$ and $\|\cdot\|$ are equivalent, there exists $K_4 > 0$ such that for all $x \in \mathbb{R}^n$, $\|DF(x)^{-1}\| \leq K_4$. This theorem follows after applying Theorem 3.2. ■

We now prove that the Weak Marcus Yamabe Conjecture is true for Lipschitz maps.

Theorem 3.4. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 Lipschitz local diffeomorphism. Suppose that there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbb{C}$ (with non-empty interior), centered at points t_m of the real axis, such that $\lim_{m\to\infty} t_m = 0$ and*

$$\mathrm{Spec}(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

Then, F is injective.

Proof:
Let $Id : R^n \to \mathbb{R}^n$ denote the Identity Map. Then, for every t_m,

$$\mathrm{Spec}(F - t_m Id) \cap (D_m - t_m) = \emptyset,$$

where $D_m - t_m = \{z \in \mathbb{C} : z + t_m \in D_m\}$ is a compact disc centered at 0. Since $F - t_m Id : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 Lipschitz map, applying Theorem 3.3, we obtain that, for all t_m, the map $F - t_m Id$ is a (global) diffeomorphism. Lemma 2.2 allow us to conclude that F is injective ■

Corollary 3.5. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 Lipschitz map. Suppose that* $\mathrm{Spec}(F) \cap \{z \in \mathbb{C} : \Re(z) \geq 0\} = \emptyset$. *Then F is injective.*

Remark 3.6. *Corollary 3.5 is stronger than Vidossich's Theorem 2,a [18] which has the additional assumption that, for some constant $A > 0$ and for all $x \in \mathbb{R}^n$,* $\mathrm{Trace}(DF(x) \leq -A$. *The proof of Vidossich fails when he claims that the linear map h is invertible ([18], page 972). Nevertheless, his proof still works if the assumptions are strengthened, for instance, to the following one: "for some real constant $A > 0$,* $\mathrm{Spec}(F) \cap \{z \in \mathbb{C} : \Re(z) \geq -A\} = \emptyset$.*"*

4. EQUIVALENT STATEMENTS

It is interesting to know if these conjectures are true for other family of maps. We prove the equivalence of these conjectures in another class of maps.

Let $\mathcal{L}$ be a linear subspace of the C^1 self-maps of $\mathbb{R}^n$ which contains the identity map Id of $\mathbb{R}^n$. Let $A > 0$ be a real constant. We state

(1) A-Weak Marcus Yamabe Conjecture for $\mathcal{L}$: If $F \in \mathcal{L}$ satisfies $\mathrm{Spec}(F) \subset \{z \in \mathbb{C} : \Re(z) < -A\}$, then F is injective.

(2) Strong Chamberland Conjecture for $\mathcal{L}$: If $F \in \mathcal{L}$ is a local diffeomorphism which has the property that there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of compact discs of $\mathbb{C}$ (not reduced to points), centered at points t_m of the real axis, such that $\lim_{m\to\infty} t_m = 0$ and

$$\mathrm{Spec}(F) \cap (\cup_{m=1}^{\infty} D_m) = \emptyset.$$

then, F is injective.

Proposition 4.1. *The Weak Marcus Yamabe Conjecture for $\mathcal{L}$ is true if, and only if, A-Weak Marcus Yamabe Conjecture for $\mathcal{L}$ is true*

Proof:

Suppose that the A-Weak Marcus Yamabe Conjecture for $\mathcal{L}$ is true. Let t be any positive real constant. Choose $F \in \mathcal{L}$ such that $\mathrm{Spec}(F) \subset \{z \in \mathbb{C} : \Re(z) < -t\}$. Then $(A/t)F = G \in \mathcal{L}$ and $\mathrm{Spec}(G) \subset \{z \in \mathbb{C} : \Re(z) < -A\}$. G is injective and so F is injective. This implies the t-Weak Marcus Yamabe Conjecture for $\mathcal{L}$. Since $\mathcal{L}$ is a linear space containing Id, we apply Lemma 2.2 to show that the Weak Marcus Yamabe Conjecture for $\mathcal{L}$ is also true. The converse is obvious. ∎

When $\mathcal{L}$ consists of the set of C^1 Lipschitz Local diffeomorphism of $\mathbb{R}^n$ into itself, Proposition 4.1 together with the version of Vidossich Theorem, suggested at the end of Remark 3.6, imply Proposition 3.5.

Proposition 4.2. *The Strong Chamberland Conjecture for $\mathcal{L}$ and the Chamberland Conjecture for $\mathcal{L}$ are equivalent*

Proof:

It is obvious that the Strong Chamberland Conjecture for $\mathcal{L}$ implies the Chamberland Conjecture for $\mathcal{L}$. The proof of the converse follows the lines of the proof of Theorem 3.4. ■

It is obvious that the Chamberland Conjecture for $\mathcal{L}$ implies the A-Weak Marcus Yamabe Conjecture for $\mathcal{L}$. Therefore,

Corollary 4.3. *The Chamberland Conjecture for $\mathcal{L}$ implies the Weak Marcus Yamabe Conjecture for $\mathcal{L}$.*

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REFERENCES

1. H. Bass, E. Conell and D. Wright, *The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse,* Bull. of the A.M.S., **7** (1982), 287–330.
2. M. Berger, *Nonlinearity and Functional Analysis,* Pure and Applied Mathematics **74**, Academic Press, New York 1977.
3. L.A. Campbell, *Unipotent Jacobian matrices and univalent maps,* Contemp. Math. **264** (2000), 157–177.
4. M. Chamberland and G. Meisters, *A mountain pass to the Jacobian Conjecture*, Canadian Math. Bull. **41** (1998), 442–451.
5. M. Cobo, C. Gutierrez, J. Llibre, *On the injectivity of C^1 maps of the real plane* To appear in Canadian Journal of Mathematics.
6. R. Fessler, *A proof of the two dimensional Markus–Yamabe Stability Conjecture and a generalization*, Annales Polonici Math. **LXII** (1995), 45–74.
7. C. Gutierrez, *A solution to the bidimensional global asymptotic stability conjecture.* Ann. Inst. H. Poincaré. Analyse non linearaire 12 (1995) 627-671.
8. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Graduate Text in Math. **14**, Springer–Verlag, 1973.
9. W. Kaplan, *Regular curve–families filling the plane I*, Duke Math. J. **7** (1940), 154-185.
10. W. Kaplan, *Regular curve–families filling the plane II*, Duke Math. J. **8** (1941), 11–46.
11. L. Markus and H. Yamabe, *Global stability criteria for differential systems*, Osaka Math. J. **12** (1960), 305–317.
12. S. Nollet and F. Xavier, *Global inversion via the Palais-Smale Condition,* Discrete and Cont. Dyn. Sys. **8**, No. 1, (01/2002), 17–28.
13. C. Olech, *On the global stability of an autonomous system on the plane,* Cont. to Diff. Eq., **1**, (1963), 389–400.
14. S. Pinchuck, *A counterexample to the strong Jacobian conjecture*, Math. Z. **217** (1994), 1–4.
15. R. Plastock, *Homeomorphisms between Banach spaces,* Trans. Amer. Math. Soc. **200** (1974), 169–183

16. B. Smyth, F. Xavier, *Injectivity of local diffeomorphisms from nearly spectral conditions,* J. Diff. Equations 130, (1996), 406–414.

17. A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture,* Progress in Mathematics **190**, Birkhauser Verlag, Basel, 2000.

18. G. Vidossich, *Two remarks on the stability of ordinary differential equations,* Nonlinear Analysis, Theory Methods & Applications **4**, No. 5, 967–974.

19. A. V. Yagzhev, *On Keller's problem,* Siberian Math. J., bf 21 (1980), 747–754.

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