

Dupin indicatrices, families of curve congruences and the zero curves

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We study a number of natural families of binary differential equations (BDE's) on a smooth surface M in \mathbb{R}^3 . One, introduced in [19], interpolates between the asymptotic and principal BDE's, another between the characteristic and principal BDE's. The locus of singular points of the members of these families determine curves on the surface labelled the zero curves. These are new features of the surface and we initiate their study here. In these two cases they are the tangency points of the discriminant sets with the characteristic (resp. asymptotic) BDE.

More generally we consider a natural class of BDE's on such a surface M , and show how the pencil of BDE's joining certain pairs are related to a third BDE of the given class, the so-called polar BDE. This explains, in particular, why the principal, asymptotic and characteristic BDE's are intimately related.

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1. INTRODUCTION

Asymptotic curves and lines of curvature have been considered as separate entities. However, in [19] a natural 1-parameter family of Binary Differential Equations (BDE's) is constructed (see Section 2 for definition) interpolating between the asymptotic BDE and that of the lines of curvature. This family is referred to as the *conjugate congruence* and is denoted by \mathcal{C}_α .

There is another BDE on the surface, namely that determining the characteristic directions (those conjugate directions at an elliptic point which are inclined at a minimum angle). This is a classical BDE which does not appear to have been much studied, and plays a similar role to the asymptotic BDE but in the elliptic region. We exhibit in this paper a second natural congruence, labelled *reflected conjugate congruence* and denoted by \mathcal{R}_α , that interpolates between the principal and characteristic BDE's.

The discriminants of the \mathcal{C}_α (resp. \mathcal{R}_α) foliate the elliptic (resp. hyperbolic) region of the surface. There are points on these discriminants where \mathcal{C}_α (resp. \mathcal{R}_α) has a well-folded singularity or worse (see Section 2 for definitions). We shall call the set of such points *the conjugate* (resp. *reflected*) *zero curve* of \mathcal{C}_α (resp. \mathcal{R}_α) and label it \mathcal{Z}_C (resp. \mathcal{Z}_R). These are new robust features (in the sense that they can be marked on an evolving surface), and we shall initiate their study here. Interestingly these zero curves have another characterisation, For the conjugate curve congruence the zeros are the points of tangency of the characteristic BDE and the natural foliation given by the constant eccentricity of the Dupin ellipses or hyperbola. For the reflected curve congruence it is the set of tangency points of this foliation and the asymptotic BDE. So in these examples the roles of the asymptotic and characteristic BDE's are interchanged. The families \mathcal{C}_α and \mathcal{R}_α originate with the pair of involutions on the tangent directions at a non-umbilic point on a surface, given by conjugation or reflection in either principal directions. There is a third family (indeed a variety of them) which interpolates between the asymptotic and characteristic BDE's with the auxilliary role played by the principal BDE's, showing that these three BDE's are intimately and symmetrically related.

These families are examples of a more general phenomena which we study. Indeed we consider *curvature* BDE's which, in a principal co-ordinate system, can be written (in an invariant way) in terms of the principal curvatures. We show that they extend across isolated umbilic points and so determine globally defined BDE's. However we can also view them as points in the projective plane over the field of rational functions in the curvatures. We use some elementary plane geometry to explain how the curvature, asymptotic and principal BDE's (the three classically best known natural BDE's) are in fact intimately related. We produce further examples of such triples.

The paper is arranged as follows. In the next section we recall the classification of singularities of BDEs of codimension ≤ 1 . Section 3 reviews some basic properties of conjugacy, introduces the characteristic BDE and establishes its properties. Section 4 introduces the conjugate and reflection curve congruences \mathcal{C}_α and \mathcal{R}_α . In Section 5 we define the zero curves and relate them with the bifurcations of the family \mathcal{C}_α and \mathcal{R}_α . In Section 6 we establish that the folding curve of the spherical representation of a BDE on a surface corresponds to the geodesic inflexions of the conjugate BDE. In Section 7 we discuss binary quadratic forms and relate their properties to binary differential equations. In Section 8 we define curvature BDE's, show that they determine global BDE's on the surface and determine some of their elementary properties. Finally in Section 9 we give some further examples.

2. SINGULARITIES OF BDE'S CODIMENSION ≤ 1

An implicit differential equation (IDE) is an equation of the form

$$F(x, y, \frac{dy}{dx}) = 0 \tag{1}$$

where F is a smooth function in $(x, y, p) \in \mathbb{R}^3$ (in this paper smooth means infinitely differentiable). We are interested in the qualitative local behaviour of such an equation.

We assume, without loss of generality, that the point of interest is the origin and suppose $\frac{dy}{dx} = p = 0$ by a rotation of the (x, y) -plane. When the partial derivative $F_p \neq 0$ at the origin, the IDE can be written locally in the form $\frac{dy}{dx} = g(x, y)$ and studied using methods from the theory of ordinary differential equations.

A new approach for investigating IDE's that define at most two directions (i.e. $F = F_p = 0$ but $F_{pp} \neq 0$) was initiated in [24], [13], and [25]. It consists of lifting the bi-valued direction field defined in the plane to a single field ξ on the surface $\tilde{M} = F^{-1}(0)$ in \mathbb{R}^3 . This field is given by

$$\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p},$$

(see for example [2]), and is determined by the restriction of the standard contact form $dy - p dx$ in \mathbb{R}^3 to the surface.

If 0 is a regular value of F then \tilde{M} is smooth and the projection $\pi : \tilde{M} \rightarrow \mathbb{R}^2$ given by $\pi(x, y, p) = (x, y)$ is a fold at points where $F_p = 0$ and $F_{pp} \neq 0$ (that is, π can be written in an appropriate system of coordinates in the source and target in the form $(u, v) \mapsto (u, v^2)$). The critical set of this projection is called the *criminant* and its image is the *discriminant* of the equation. The configuration of the solution curves of F at a point on the discriminant is determined by the pair (ξ, σ) , where σ is the involution on M that interchanges points with the same image under the projection to \mathbb{R}^2 .

It is shown in [12], [13] that if ξ does not vanish at the point in question then locally the IDE can be reduced by smooth changes of coordinates in the plane to $dy^2 - x dx^2 = 0$. The integral curves in this case is a family of cusps transverse to the discriminant, which is a smooth curve.

If ξ has an elementary singularity (saddle/node/focus) with non resonant eigenvalues, separatrices transverse to the criminant and not killed under projection, then it is called a *well-folded singularity* ([14]). Note that a zero (i.e a singularity) of ξ occurs at points where $F = F_p = F_y + pF_x = 0$. These are also the points where the unique direction on the discriminant determined by the IDE is tangent to the discriminant. At well-folded singularities, the equation is locally smoothly equivalent to $dy^2 + (-y + \lambda x^2) dx^2 = 0$, with $\lambda \neq 0, \frac{1}{16}$ ([14]). There are three topological models, a well-folded saddle if $\lambda < 0$, a well-folded node if $0 < \lambda < \frac{1}{16}$ and a well-folded focus if $\frac{1}{16} < \lambda$; Figure 1. (See also [17] for applications to control theory).

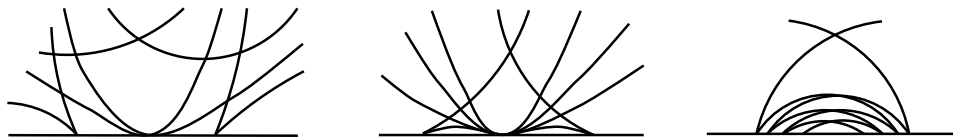


FIG. 1. Well-folded saddle (left), node (centre) and focus (right).

The family of cusps and the well-folded singularities are the only locally structurally stable configurations of singular IDE's with $F_{pp} \neq 0$. The bifurcations in generic 1-parameter families of these IDE's have also been established. One of these is the *well-folded saddle*

node bifurcations ($\lambda = 0$ above) and occurs when the discriminant is smooth and the lifted field ξ has a saddle-node singularity. Then the equation is locally smoothly equivalent to $dy^2 + (-y + x^3 + \mu x^4)dx^2 = 0$ ([15]). See Figure 2.

When $\lambda = \frac{1}{16}$ we have a transition from well-folded node to well-folded focus and the IDE is locally smoothly equivalent to $dy^2 + (-y + \frac{1}{16}x^2)dx^2 = 0$ ([16]).

It is not difficult to show that a generic 1-parameter family of IDE's with a well-folded saddle node (resp. well-folded node-focus change) singularity at $t = 0$ is (fibre) topologically equivalent to $dy^2 + (-y + x^3 + \mu x^4 + tx)dx^2 = 0$ (resp. $dy^2 + (-y + (\frac{1}{16} + t)x^2)dx^2 = 0$) (see for example [8] for definition of fibre-topological equivalent of IDE's).

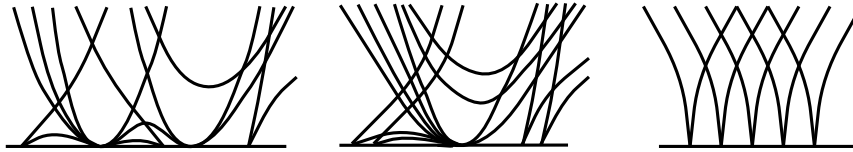


FIG. 2. Well-folded saddle node bifurcations.

Bifurcations can also occur when the discriminant has a Morse singularity. These equations are labelled *Morse Type 1* in [4]. Generic Morse Type 1 singularities are locally topologically equivalent to $dy^2 + (\pm x^2 \pm y^2) = 0$ ([4]; see also [22]). A generic 1-parameter family of IDE's with a Morse Type 1 singularity at $t = 0$ is (fibre) topologically equivalent to $dy^2 + (\pm x^2 \pm y^2 + t) = 0$ ([4]). As t varies near the origin, two well-folded saddles or foci singularities appear on one side of the transition and none on the other; Figure 3. The saddle or focus type are distinguished by the sign of x^2 in the normal form (+ for focus and - for saddle).

When $F = F_p = 0$, and $F_{pp} \neq 0$ the equation defines locally at most two directions in the plane. Another way for this to happen is if the IDE is simply given by a quadratic equation of the form

$$a(x, y)dy^2 + 2b(x, y)dx dy + c(x, y)dx^2 = 0$$

where a, b, c are smooth real functions in (x, y) . (Note that in case $F = F_p = 0$, and $F_{pp} \neq 0$ one can locally reduce F to a quadratic equation in p ([9]).) In this case the IDE is sometimes referred to as a Binary Differential Equation (BDE). Naturally this includes the situation where all the coefficients a, b, c vanish at the origin. (This is an infinitely degenerate equation in the set of all IDE's but of finite codimension within the set of BDE's.) A BDE defines two directions in the region where $\delta = (b^2 - ac)(x, y) > 0$, a double direction on the set $\delta = 0$ and no direction where $\delta < 0$. The set $\delta = 0$ is the *discriminant* of the BDE.

Consider the case when a, b, c all vanish at the origin. One way to proceed here is given in [3]. Consider the set $\tilde{M} = \{(x, y, [\alpha : \beta]) : a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\}$. The discriminant function $\delta = b^2 - ac$ of the BDE plays a key role. When δ has a Morse singularity the surface \tilde{M} is smooth and the projection $\pi : \tilde{M} \rightarrow \mathbb{R}^2$ is a double cover of the set $\{(x, y) : \delta(x, y) \geq 0\}$. (We label these BDE's *Morse Type 2*.) The bi-valued field lifts to a single field ξ on \tilde{M} and extends smoothly to $\pi^{-1}(\delta = 0)$. Note here that $0 \times \mathbb{R}P^1 \subset \tilde{M}$ and

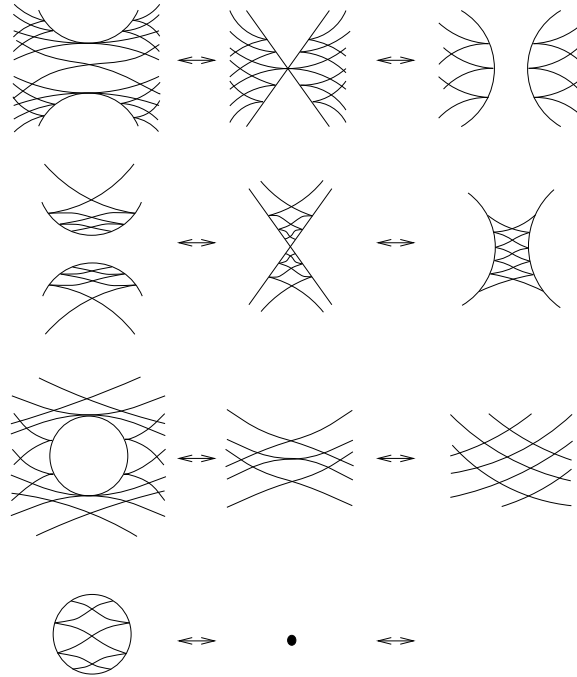


FIG. 3. Morse Type 1 bifurcations.

is an integral curve of the lifted field. The involution σ on $\tilde{M} - 0 \times \mathbb{R}P^1$ that interchanges points with the same image under the projection to \mathbb{R}^2 also extends to \tilde{M} . Taking an affine chart $p = \beta/\alpha$, then $\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p}$ as before, with $F = ap^2 + 2bp + c$. The zeros of ξ on the exceptional fibre are given by the roots of the cubic

$$\phi(p) = (F_x + pF_y)(0, 0, p).$$

A case of interest in this paper is when the discriminant is an isolated point Morse singularity. Then the BDE is generically topologically equivalent to one of the following normal forms: ([3], [21], [22])

- (a) Lemon $ydy^2 + 2xdxdy - ydx^2 = 0$
- (b) Star $ydy^2 - 2xdxdy - ydx^2 = 0$
- (c) Monstar $ydy^2 + \frac{1}{2}xdxdy - ydx^2 = 0.$

Morse Type 2 singularities occur generically in 1-parameter families of BDE's. Their bifurcations are studied in [8], and models of generic families are obtained by simply adding a parameter t to the coefficient of dy^2 in the above normal forms. For pictures of these bifurcations see Figure 4. (See [6] and [8] for the case when the discriminant is a pair of crossing curves.)

The principal direction BDE is singular at umbilic points. At such points all the coefficients of the BDE vanish and we have generically a Morse Type 2 singularity. We can consider the surface in Monge form $h = \frac{\kappa}{2}(x^2 + y^2) + C(x, y) + O(4)$, in such a way that the cubic part $C(x, y)$ is given in complex form $\phi = \operatorname{Re}(z^3 + \beta z^2 \bar{z})$, where $\beta = u + iv$ is a complex number. We can then read information about the umbilic point from the position of β in the complex-plane. The cubic ϕ above depends only on the cubic C and has a double root when β belongs to the hypocycloid $2e^{i\theta} + e^{2i\theta}$. Off this curve the lifted field has one zero of type saddle so the umbilic is a lemon. On the circle $|\beta| = 3$ the discriminant of the BDE fails to be Morse so the surface of the equation M is no longer smooth. (This circle also corresponds to points where a natural folding map is not versal [26].) Inside the circle ξ has 3 zeros of type saddle so the umbilic is a star, and outside the circle we have two saddles and one node and the umbilic is a monstar. On the circle we have the birth of two umbilic points (a star and a monstar).

3. CONJUGATE AND CHARACTERISTIC DIRECTIONS

We start with a discussion of conjugate directions. Suppose given an (affine) conic section, and a direction. The lines parallel to this direction will meet the conic in 0 or 2 points, or be tangent (one repeated point), or pass through a singular point in the case of a line pair. In the case of a pair of intersection points consider their midpoint. These midpoints are collinear and determine the *conjugate direction*. If l is conjugate to l' it is an elementary fact that l' is the conjugate of l . Given a surface M in \mathbb{R}^3 , at each point p we have a Dupin indicatrix. This is the conic approximation of the intersection of the surface at the point with a small translate of the tangent plane in the normal direction (at an elliptic point this has to be the right direction). Two directions are conjugate if they are conjugate with respect to the Dupin indicatrix. For the ellipse $x^2/a^2 + y^2/b^2 = 1$ (respectively the hyperbola $x^2/a^2 - y^2/b^2 = 1$) the directions $y = m_1x$, $y = m_2x$ are conjugate if and only if $m_1m_2 = -b^2/a^2$ (respectively $m_1m_2 = b^2/a^2$). At hyperbolic points it is sometimes traditional to consider the indicatrix to be the union of hyperbolas obtained by translating in both directions, i.e. $x^2/a^2 - y^2/b^2 = \pm 1$. At parabolic points where the second fundamental form is not identically zero the indicatrix is a pair of parallel lines and each direction is conjugate to itself.

We need some notation. Given an oriented surface M in \mathbb{R}^3 with a family of normals N we have a Gauss map $N : M \rightarrow S^2$. At a point p the map $-dN(p) : T_pM \rightarrow T_{N(p)}S^2$ can be thought of as an automorphism of T_pM , this is the classical shape operator S_p , or simply S . If M is parametrised by $\mathbf{r}(x, y)$ with shape operator S , the coefficients of the first (resp. second) fundamental form E, F, G (resp. l, m, n) are given by

$$\begin{aligned} E &= \mathbf{r}_x \cdot \mathbf{r}_x, & F &= \mathbf{r}_x \cdot \mathbf{r}_y, & G &= \mathbf{r}_y \cdot \mathbf{r}_y \\ l &= S(\mathbf{r}_x) \cdot \mathbf{r}_x = N \cdot \mathbf{r}_{xx}, \\ m &= S(\mathbf{r}_x) \cdot \mathbf{r}_y = S(\mathbf{r}_y) \cdot \mathbf{r}_x = N \cdot \mathbf{r}_{xy}, \\ n &= S(\mathbf{r}_y) \cdot \mathbf{r}_y = N \cdot \mathbf{r}_{yy}. \end{aligned}$$

REMARKS 3.1. Here are some properties and interpretations of conjugacy:

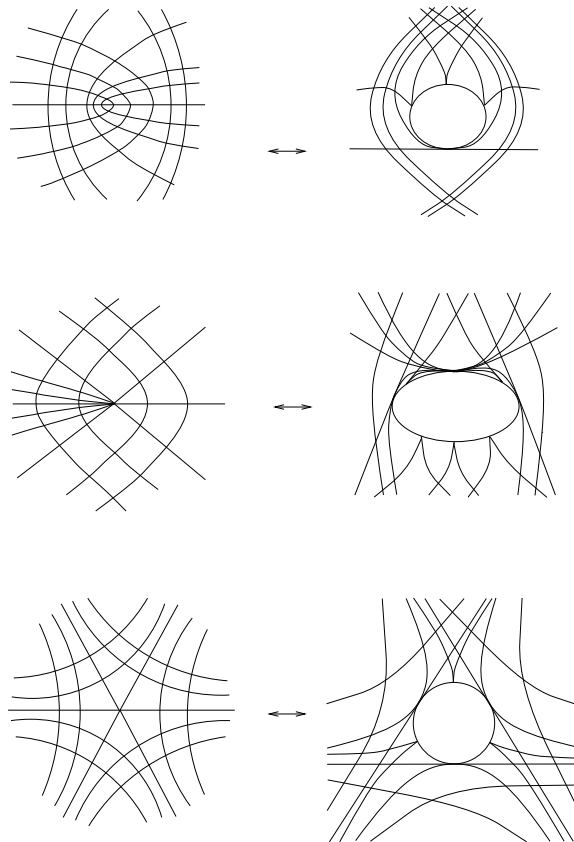


FIG. 4. Morse Type 2 bifurcations: Lemon, Monstar and Star.

1. Two vectors v and $w \in T_pM$ are conjugate if and only if $II(v, w) = 0$, where II is the second fundamental form at p .
2. The notion of conjugacy is invariant under affine and inversive transformations.
3. Let v be a direction in the tangent plane T_pM at p and consider a curve on M through p in the direction of v . Then the characteristic at p of the envelope of planes determined by this curve is a line in the direction conjugate to v .
4. If we parallel project a surface M in the direction v , then the conjugate direction at a singular point p of the projection is the tangent to the singular set of the projection. (This is a consequence of the characterisation given in (3).)
5. If v, \bar{v} are conjugate directions at $p \in M$ then $S(v)$ and \bar{v} are orthogonal. (This is just the definition of conjugacy in another guise.)
6. The angle between conjugate directions v, \bar{v} and the vectors $S(v), S(\bar{v})$ are equal or supplementary according as the point is hyperbolic or elliptic.

7. For any $v \in T_p M$ (that is not an asymptotic direction at a parabolic point) let α be the signed angle between v and \bar{v} . Then from (5)

$$\sin \alpha = \frac{S(v).v}{\|S(v)\|\|v\|}.$$

8. The sum of the radii of curvature in conjugate (non-asymptotic) directions is constant, and consequently equal to the sum of the principal radii of curvature.

9. The only orthogonal conjugate directions are those given by the principal directions.

10. At non-parabolic points the only self conjugate directions are the asymptotic directions.

11. Two direction fields on a surface M are said to be *conjugate* if the corresponding pairs of directions are conjugate everywhere. So for example (away from umbilics) the direction fields determined by the principal directions are conjugate. In a rather degenerate way the direction field in the hyperbolic region given locally by (one of the) families of asymptotic directions are self conjugate. One way of finding conjugate fields (indeed families of integral curves of conjugate fields) on a surface is by fixing a line L and considering (a) the planar sections of the surface by the pencil determined by L , and (b) the set of singular points for central projections of M from points of L . Of course any direction field on a surface determines a conjugate field.

Given a direction v in $T_p M$ consider the conjugate direction \bar{v} . At hyperbolic points these directions can coincide (the asymptotic directions). At elliptic points however there is a unique pair of conjugate directions for which the included angle (i.e the angle between these directions) is minimal. This pair is determined by a binary differential equation on the surface.

PROPOSITION 3.1. *At an elliptic point p of M there is a unique conjugate pair of directions for which the angle between them is a minimum. These conjugate directions are reflections of each other in either of the lines of symmetry of the Dupin ellipse (corresponding to the principal directions at p).*

The BDE determining the above pair is given, in terms of the coefficients of the first and second fundamental forms, by

$$\begin{aligned} & (2m(Gm - Fn) - n(Gl - En))dy^2 + \\ & \quad 2(m(Gl + En) - 2Fln)dydx + \\ & (l(Gl - En) - 2m(Fl - Em))dx^2 = 0. \end{aligned}$$

Away from umbilics this can be written, with respect to a co-ordinate system given by lines of curvature, as

$$\kappa_2 dy^2 - \kappa_1 dx^2 = 0$$

where κ_1 (resp. κ_2) is the principal curvature in the x - (resp. y -) direction.

Proof: See for example [18], p129, or Proposition 4.2. \square

DEFINITION 3.1. The directions determined by this BDE are called characteristic directions. The integral curves of the characteristic directions are called the characteristic curves.

REMARKS 3.2.

1. Each elliptic point has two characteristic directions; each parabolic point one (coinciding with the asymptotic direction there); each hyperbolic point none.
2. The characteristic directions are, in many ways, the analogue of the asymptotic directions for the elliptic region of a surface. (Recall that the asymptotic directions are given by $\kappa_2 dy^2 + \kappa_1 dx^2 = 0$, a switch of signs from that for the equation above.)
3. The characteristic directions are also characterised as the conjugate directions where the radii of normal curvature are equal, or as stated above the conjugate directions bisected by the principal directions.
4. The solution curves of the characteristic BDE, as the asymptotic BDE, yield some interesting information. Indeed the angle between them determines the eccentricity of the Dupin indicatrix (the ratio of principal curvatures), and the lines themselves determine its orientation. The same holds for the asymptotic lines in the hyperbolic region. (Note that the principal directions, which are the bisectors of the characteristic and asymptotic directions, only determine the orientation of the Dupin indicatrix.)

COROLLARY 3.1. *The discriminant of the characteristic BDE consists of the parabolic set and umbilic points, and the singular points of the characteristic equation are given by the cusps of Gauss the umbilics.*

Proof: The discriminant is given by

$$K((En - Gl)^2 - 4(Em - Fl)(Fn - Gm)) = 0.$$

where K is the Gauss curvature. The second component is the discriminant of the lines of curvature BDE and vanishes precisely at umbilic points.

At umbilics any two conjugate directions are orthogonal, and one can check that all the coefficients of the characteristic BDE vanish there. These points are therefore singular points of the equation.

Away from umbilics, the singular points of the BDE are given by the equations $F = F_p = F_x + pF_y = 0$, where $F = \kappa_2 p^2 - \kappa_1$ (see Section 2). A quick calculation shows that this occurs when $\kappa_1 = \partial\kappa_1/\partial x = 0$, (or $\kappa_2 = \partial\kappa_2/\partial y = 0$). This means that we are at the intersection of the parabolic set (e.g. $\kappa_1 = 0$) and the ridge set ($\partial\kappa_1/\partial x = 0$), and these are just the cusps of Gauss. \square

We next need to analyse the zeros of the characteristic BDE. The umbilic points are dealt with in a more general context in Proposition 5.2(5). We relate below the nature

of the zeros of the characteristic and asymptotic BDEs at cusps of Gauss. Recall that a well-folded saddle singularity has index $-\frac{1}{2}$ and a well-folded node and focus have index $+\frac{1}{2}$.

PROPOSITION 3.2. *At a cusp of Gauss the asymptotic and characteristic BDE's have well-folded singularities of opposite indices, that is, on one side of the parabolic curve we have a well-folded saddle and on the other a well-folded node or focus.*

Proof: We can write the surface at a cusp of Gauss on the form $z = h(x, y) = y^2 + ax^2y + bxy^2 + cy^3 + dx^4 + \dots$ and assume that the parabolic set is smooth, so that $b \neq 0$. Then the equations of the asymptotic and characteristic directions are both locally smoothly equivalent to a BDE of the form $dy^2 - (y - \lambda x^2) = 0$, with $\lambda = \lambda_1 = 3(-b^2 + 4d)/(2b^2)$ for the asymptotic BDE and $\lambda = \lambda_2 = -3(-b^2 + 4d)/(2b^2)$ for the characteristic BDE. We have a well-folded singularity provided $\lambda \neq 0, \frac{1}{4}$, and the singularity is of type well-folded saddle (resp. node or focus) if $\lambda < 0$ (resp. $0 < \lambda < \frac{1}{4}$ or $\frac{1}{4} < \lambda$) ([14]). Considering λ_1 and λ_2 in the (b, d) -plane we find that the only combinations for the types of singularities of asymptotic and characteristic BDE's at a cusp of Gauss are (well-folded saddle, well-folded node), (well-folded saddle, well-folded focus) or vice-versa. \square

In the next section we construct families of BDEs on a surface M . The key idea is the following. Consider the space of all tangent directions through a point p on M which is neither an umbilic nor parabolic point. Conjugation gives an involution in T_pM , $v \mapsto \bar{v} = C(v)$. There is another involution in T_pM which is simply reflection in (either of) the principal directions, $v \mapsto R(v)$. These determine a mapping on the set of directions through p . We use C and $C \circ R$ to determine families of BDE's by asking that the angle between a direction v and the image of v under one of these mappings is constant. The following is straightforward.

- PROPOSITION 3.3.**
1. $R \circ C = C \circ R$, so $R(\bar{v}) = R(\bar{v})$.
 2. Let v be a direction; then the set $\{v, C(v), R(v), C(R(v))\}$ has less than four elements if and only if v is either a principal direction ($v = R(v)$), an asymptotic direction ($v = C(v)$), or a characteristic direction ($v = R(C(v))$).
 3. Given a line field ξ , we can create 4 other fields from it using this pair of involutions. Naturally these can be arranged in three ways as pairs of fields, or BDE's. These pairs will be closed under C , R or $R \circ C = C \circ R$.
 4. The angle between v and resp. $C(v)$, $C \circ R(v)$, $R(v)$ is zero (resp. $\frac{\pi}{2}$) if and only if v is resp. asymptotic, characteristic, principal (resp. principal, principal, bisectors of the principal directions).

4. THE CONJUGATE AND REFLECTED CONGRUENCES

In what follows we obtain one parameter families of BDE's by orienting our surface M and considering those directions that make an oriented angle α between v and $C(v)$ or between v and $R \circ C(v)$. The first family, labelled *conjugate congruence*, interpolates

between the asymptotic and principal BDE's and was first introduced in [19]; the second, labelled *reflected congruence*, interpolates between the characteristic and principal BDE's. The first set of BDE's is closed under R ; the second under C .

DEFINITION 4.1. ([19]) Let $\Theta : TM \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ be given by $\Theta(p, v) = \alpha$ where α denotes the oriented angle between v and $\bar{v} = C(v)$. (Note that Θ is not well defined at points of the tangent bundle TM corresponding to asymptotic directions at parabolic points.) The conjugate curve congruence, for a fixed α , is defined to be $\Theta^{-1}(\alpha)$ which we denote \mathcal{C}_α .

It is clear that the set of all asymptotic directions is \mathcal{C}_0 and $\mathcal{C}_{\pm\frac{\pi}{2}}$ is the set of all principal directions. Note that, for any α , \mathcal{C}_α contains the asymptotic directions at parabolic points. It is shown in [19] that \mathcal{C}_α is given by a BDE. (We present here a more compact proof.)

PROPOSITION 4.1. ([19]) 1. *The conjugate curve congruence \mathcal{C}_α of a parametrised surface is given by the BDE*

$$\begin{aligned} & (\sin \alpha(Gm - Fn) - n \cos \alpha \sqrt{EG - F^2})dy^2 + \\ & (\sin \alpha(Gl - En) - 2m \cos \alpha \sqrt{EG - F^2})dydx + \\ & (\sin \alpha(Fl - Em) - l \cos \alpha \sqrt{EG - F^2})dx^2 = 0. \end{aligned}$$

2. *The discriminant of \mathcal{C}_α , which we denote $\Delta_{\mathcal{C}_\alpha}^\alpha$, is given by*

$$H^2(x, y) \sin^2 \alpha - K(x, y) = 0,$$

where H is the mean curvature and K is the Gauss curvature.

3. *Away from umbilics the BDE \mathcal{C}_α can be written, with respect to a co-ordinate system given by lines of curvature, by*

$$\kappa_2 \cos \alpha dy^2 + (\kappa_2 - \kappa_1) \sin \alpha dydx + \kappa_1 \cos \alpha dx^2 = 0.$$

Proof: Write the first and second fundamental form as matrices A and B respectively. Then the shape operator S is given by $S(v) = A^{-1}Bv$. If $v = (\alpha, \beta)^T$ then write $v^* = (-\beta, \alpha)^T$. Then $A^{-1}v^* = v^\perp$ is orthogonal to v . Now

$$\begin{aligned} S(v).v &= v^T Bv, \\ S(v).v^\perp &= v^T B A^{-1}v^*. \end{aligned}$$

Moreover $\|v\| = \sqrt{v^T A v}$, $\|v^\perp\| = \sqrt{v^{*T} A^{-1} v^*}$, $\|v^\perp\| = \|v\|/\sqrt{EG - F^2}$. If α is the angle between v and v^\perp we have already seen that $\sin \alpha = S(v).v/\|S(v)\|.\|v\|$. However the angle between $S(v)$ and v^\perp is $\pi - \alpha$ so $\cos \alpha = -S(v).v^\perp/\|S(v)\|.\|v^\perp\|$. It follows that $(S(v).v^\perp) \sin \alpha \sqrt{EG - F^2} = -(S(v).v) \cos \alpha$, which is a quadratic equation in α , β .

Replacing α by dx and β by dy we obtain the quadratic equation determining \mathcal{C}_α . The rest now is straightforward. \square

REMARKS 4.3. 1. It is clear that the discriminant Δ_C^0 is the parabolic curve and $\Delta_C^{\pm\frac{\pi}{2}}$ consists of the umbilic points. For any α the discriminant curves occur in the non hyperbolic region of the surface. The discriminant curves clearly foliate this region.

2. Note that whilst $\mathcal{C}_\alpha \neq \mathcal{C}_{-\alpha}$ generally, we have $\Delta_C^\alpha = \Delta_C^{-\alpha}$. Indeed it is not hard to see that the pairs of directions determined by \mathcal{C}_α have as their conjugates the pairs of directions determined by $\mathcal{C}_{-\alpha}$. In other words these are conjugate BDE's. On the discriminant $\Delta_C^\alpha = \Delta_C^{-\alpha}$ the BDE \mathcal{C}_α determines a repeated direction which is one of the characteristic directions. The other corresponds to the repeated direction determined by $\mathcal{C}_{-\alpha}$.

3. The curve Δ_C^α can also be characterised the set of points where the characteristic directions make an angle $\pm\frac{\alpha}{2}$ with one of the principal directions.

4. It is not difficult to see that $H^2/K = \text{constant}$ is (away from umbilics) equivalent to the ratio of the principal curvatures $\kappa_1/\kappa_2 = e$ being constant. Indeed one easily checks that $4H^2/K = e + 2 + e^{-1}$, and such an equation has two solutions e which are mutually inverse. Of course the umbilic points correspond to $H^2/K = 1$, and in the elliptic region this is a minimum value for H^2/K .

5. At heart much of this paper is concerned with indicatrices of Dupin. Another interpretation of the invariant H^2/K is that it determines the eccentricity of these ellipses and hyperbola, that is their type up to a Euclidean motion and dilation. Of course any relation concerning angles between conjugate directions will be invariant under these changes.

6. We shall separate the cases where $\alpha \geq 0$ and $\alpha \leq 0$ and denote by \mathcal{C}_α the conjugate curve congruence with $\alpha \in [0, \frac{\pi}{2}]$ and by $\mathcal{C}_{-\alpha}$ for the case when $-\alpha \in [-\frac{\pi}{2}, 0]$.

We now consider the congruence with $R \circ C$ replacing C . Again we suppose that M is an oriented surface in \mathbb{R}^3 .

DEFINITION 4.2. Let $\Phi : TM \rightarrow \mathbb{R}$ by $\Phi(p, v) = \alpha$ where α is the angle between v and $R(\bar{v})(= R \circ C(v))$. An alternative way of defining Φ at (p, v) is as the sum of the signed angles between v and a principal direction e and \bar{v} and e . (Note that Φ is not well defined at umbilics.) Then the reflected conjugate curve congruence, for a fixed α , is defined to be $\Phi^{-1}(\alpha)$, which we denote \mathcal{R}_α .

There are clearly ambiguities in this definition. Replacing e by $-e$ shifts the total angle by 2π . We can also use the other principal direction to measure from; if we do that the total angle is now $\pi - \alpha$. This means that we need only consider \mathcal{R}_α for $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$. Finally if we reverse orientation we obtain the summed angle $-\alpha$; but note that $\mathcal{R}_\alpha \neq \mathcal{R}_{-\alpha}$ once an orientation is chosen.

It is clear that the set $\mathcal{R}_{\frac{\pi}{2}}$ is the set of principal directions and the set \mathcal{R}_0 is the set of characteristic directions. So \mathcal{R}_α interpolates between the characteristic BDE and the prin-

cial BDE and indeed interchanges the roles played by the characteristic and asymptotic BDE's.

PROPOSITION 4.2. 1. *The reflected conjugate congruence \mathcal{R}_α is given by the BDE*

$$\begin{aligned} & \{(2m(Gm - Fn) - n(Gl - En)) \cos \alpha + (Fn - Gm) \frac{2Fm - Gl - En}{\sqrt{(EG - F^2)}} \sin \alpha\} dy^2 + \\ & \quad \{2(m(Gl + En) - 2Fln) \cos \alpha + (En - Gl) \frac{2Fm - Gl - En}{\sqrt{(EG - F^2)}} \sin \alpha\} dydx + \\ & \quad \{(l(Gl - En) - 2m(Fl - Em)) \cos \alpha + (Em - Fl) \frac{2Fm - Gl - En}{\sqrt{(EG - F^2)}} \sin \alpha\} dx^2 = 0. \end{aligned}$$

2. *The discriminant consists of umbilic points together with the set*

$$K \cos^2 \alpha + H^2 \sin^2 \alpha = 0,$$

which we denote by Δ_R^α .

3. *Away from umbilics the equation for \mathcal{R}_α is given, in the principal co-ordinate system, by*

$$\kappa_2 \cos \alpha dy^2 - (\kappa_1 + \kappa_2) \sin \alpha dydx - \kappa_1 \cos \alpha dx^2 = 0.$$

Proof: In the principal co-ordinate system e_1, e_2 two directions $u = e_1 + pe_2, \bar{u} = e_1 + re_2$ are conjugate if and only if $\kappa_1 + \kappa_2 pr = 0$, so $r = -\kappa_1/\kappa_2 p$. Now if we measure angles from the first principal direction then these two directions are at angles θ_1, θ_2 where $\tan \theta_1 = p, \tan \theta_2 = r$. Setting the formula for $\tan(\theta_1 + \theta_2)$ equal to $\tan \alpha$ we get

$$\frac{p + r}{1 - pr} = \frac{\sin \alpha}{\cos \alpha}. \tag{2}$$

The equation of the conjugate curve congruence in the principal co-ordinate system now follows (where $p = dy/dx$ as usual).

We need now to work with a general parametrisation $\mathbf{r}(x, y)$ of the surface. Two directions $u = \mathbf{r}_x + \xi \mathbf{r}_y, \bar{u} = \mathbf{r}_x + \bar{\xi} \mathbf{r}_y$ are conjugate if and only if $n\xi\bar{\xi} + m(\xi + \bar{\xi}) + l = 0$ where l, m, n are the coefficients of the second fundamental form. Then

$$\bar{\xi} = -\frac{l + m\xi}{m + n\xi}. \tag{3}$$

Let $e_1 = M_1 \mathbf{r}_x + M_2 \mathbf{r}_y$ and $e_2 = N_1 \mathbf{r}_x + N_2 \mathbf{r}_y$ be the expressions for the principal directions obtained by solving the principal directions BDE

$$(Fn - Gm)dy^2 + (En - Gl)dydx + (Em - Fl)dx^2 = 0.$$

Comparing the co-ordinates of u and \bar{u} in the systems e_1, e_2 and $\mathbf{r}_x, \mathbf{r}_y$, yields

$$p = -\frac{M_2 - M_1\xi}{N_2 - N_1\xi}, \text{ and } r = -\frac{M_2 - M_1\bar{\xi}}{N_2 - N_1\bar{\xi}}. \quad (4)$$

The general equation of the reflected conjugate curve congruence follows now from (2) by substituting $p, r, \bar{\xi}$ with their expressions in term of ξ form (3) and (4). The result on the discriminants is now straightforward. \square

REMARKS 4.4. 1. Clearly \mathcal{R}_α is invariant under conjugation (it simply interchanges the two directions). In the previous case we saw that the BDE was not closed under conjugation; indeed the BDE conjugate to \mathcal{C}_α is $\mathcal{C}_{-\alpha}$.

2. There are consequently two ways of moving from the principal BDE to the characteristic BDE: as α moves from $\frac{\pi}{2}$ to 0 or from $-\frac{\pi}{2}$ to 0. We shall write \mathcal{R}_α for the BDE's with $\alpha \in [0, \frac{\pi}{2}]$ and $\mathcal{R}_{-\alpha}$ the cases when $-\alpha \in [-\frac{\pi}{2}, 0]$.

3. Following the same argument as in Remark 4.3(4), the curves Δ_R^α coincide with the curves $\kappa_1/\kappa_2 = \text{constant}$. The set Δ_R^α foliates the hyperbolic region of the surface, and $\Delta_R^\alpha = \Delta_R^{-\alpha}$. Note that Δ_R^0 is the parabolic curve, and $\Delta_R^{\frac{\pi}{2}}$ is the curve $H = 0$.

4. On the discriminant Δ_R^α the pair defined by \mathcal{R}_α reduces to an asymptotic direction. Consequently, the discriminant Δ_R^α can also be characterised as the set of points where the angle between an asymptotic direction and the chosen principal direction is $\pm\frac{\alpha}{2}$.

5. Clearly at points where $H = 0$ we cannot canonically order the principal directions. So in a neighbourhood of the curve $H = 0$ there are generally two curves of points whose indicatrices have the same eccentricity.

6. The angle between any two conjugate directions at an umbilic is always $\frac{\pi}{2}$, and these were our special points for our first family (they only appeared in the single member of the family corresponding to principal curves). Umbilics are also special points for \mathcal{R}_α . At such points all directions are principal, so for any given α an any direction v , we can find a principal direction e so that the sum of the signed angles between v and this principal direction e and \bar{v} and e equals α . At a point where the mean curvature is zero the Dupin indicatrix is a rectangular hyperbola, and here the angle constructed above is always $\frac{\pi}{2}$. So these points are exceptional for the new family. (Again they only appear in the single member of the family corresponding to principal curves). Note that $\mathcal{R}_{\frac{\pi}{2}}$ is the equation of the principal directions multiplied by H .

5. THE CONJUGATE AND REFLECTED ZERO CURVES

We are interested in the locus of points where the BDE \mathcal{C}_α (resp. \mathcal{R}_α) is singular, i.e. has a well-folded singularity or worse. If a well-folded singularity of \mathcal{C}_{α_0} occurs on $\Delta_C^{\alpha_0}$ we expect it to persist in \mathcal{C}_α on Δ_C^α , with α close to α_0 , as the well-folded singularities are stable. Furthermore, the stability of these singularities shows that the locus of the zeros

of \mathcal{C}_α , for α close to α_0 , form a smooth curve near a zero of \mathcal{C}_α . This leads us to make the following definition.

DEFINITION 5.1. The conjugate zero curve \mathcal{Z}_C is the locus of zeros of the members of the family \mathcal{C}_α . We shall denote by \mathcal{Z}_C^+ (resp. \mathcal{Z}_C^-) the locus of zeros of \mathcal{C}_α for $\alpha \in [0, \frac{\pi}{2}]$. (resp. $\alpha \in [-\frac{\pi}{2}, 0]$), so that \mathcal{Z}_C is the union of \mathcal{Z}_C^+ and \mathcal{Z}_C^- .

REMARKS 5.5. 1. By Remark 4.3(1) we see that the curve \mathcal{Z}_C lies in the non-hyperbolic region of the surface. It contains the zeros of \mathcal{C}_0 which are the cusps of Gauss and the zeros of $\mathcal{C}_{\pm \frac{\pi}{2}}$, the umbilic points.

2. On the discriminant Δ_C^α the unique solution of \mathcal{C}_α is a characteristic direction. So an alternative definition of \mathcal{Z}_C is as the locus of tangency points of the pair of foliations determined by the characteristic BDE and that given by equal eccentricity curves of the Dupin ellipses. Of course at most elliptic points the characteristic foliation determines two fields and so two standard foliations. The tangency of one with the eccentricity curves yields \mathcal{Z}_C^+ and with the other \mathcal{Z}_C^- .

3. We do not expect $\mathcal{Z}_C^+/\mathcal{Z}_C^-$ to be singular at a smooth point of a discriminant.

4. Note that given any BDE in the plane and foliation one can study the set of tangency points between the foliation and the integral curve. In our case the two are related of course: the foliation is by the eccentricity of the indicatrices; the BDE determines the orientation of the indicatrices, and the angle between the two directions the eccentricity. That angle is constant on the discriminant. Nevertheless one might expect, as models, the generic curves arising from the tangency of a BDE and foliation in general position. For this reason singular points of the BDE and foliation will be important.

5. The equation of the zero curve is now easy to write down with respect to the usual principal co-ordinate system: $\kappa_1(\partial(\kappa_1/\kappa_2)/\partial y)^2 = \kappa_2(\partial(\kappa_1/\kappa_2)/\partial x)^2$.

6. Away from the parabolic points the Gauss map is a local diffeomorphism and as we have seen preserves the angle between conjugate directions. So if we wish to consider the configuration of the characteristic curves and their tangency with the discriminants, then by composing the Gauss map with stereographic projection we can reduce to a BDE in the plane. We are then considering the tangency of its integral curves with the curves given by fixing the angle between the pairs of directions. Of course for the BDE $ap^2 + 2bp + c = 0$ the angles the two directions (when $b^2 - 4ac > 0$) make with the x -axis, θ_1, θ_2 ensure that $p = t_j = \tan \theta_j$ are roots of this quadratic equation. So $\tan(\theta_1 - \theta_2) = (t_1 - t_2)/(1 + t_1 t_2)$. If this is constant so is its square, that is $\{(t_1 + t_2)^2 - 4t_1 t_2\}/\{1 + t_1 t_2\}^2 = 4(b^2 - ac)/(a + c)^2$, and we are considering the tangency of our BDE with these curves.

The zeros of the family \mathcal{R}_α define, in an analogous way to \mathcal{C}_α , a curve in the non-elliptic region of the surface. We have the following definition.

DEFINITION 5.2. The reflected conjugate zero curve \mathcal{Z}_R is the locus of zeros of the members of the family \mathcal{R}_α . We shall denote by \mathcal{Z}_R^+ (resp. \mathcal{Z}_R^-) the locus of zeros of \mathcal{R}_α for $\alpha \in [0, \frac{\pi}{2}]$. (resp. $\alpha \in [-\frac{\pi}{2}, 0]$), so that \mathcal{Z}_R is the union of \mathcal{Z}_R^+ and \mathcal{Z}_R^- .

- REMARKS 5.6. 1. By Remark 4.4(3) we see that \mathcal{Z}_R lies in the non-elliptic region of the surface. It contains the cusps of Gauss (the zeros of \mathcal{R}_0), umbilics (the common, generically isolated, zero to all \mathcal{R}_α) and the curve $H = 0$ (degenerate zeros of $\mathcal{R}_{\pm \frac{\pi}{2}}$).
2. On the discriminant Δ_R^α the unique solution of \mathcal{R}_α is an asymptotic direction. So an alternative definition of \mathcal{Z}_R is as the locus of tangency points of the pair of foliations determined by the asymptotic BDE and that given by equal eccentricity curves of the Dupin hyperbola. One of the asymptotic foliation yields \mathcal{Z}_R^+ and the other \mathcal{Z}_R^- .
3. The equation of the reflected zero curve is easy to write down with respect to principal directions co-ordinate system: $\kappa_1(\partial(\kappa_1/\kappa_2)/\partial y)^2 = -\kappa_2(\partial(\kappa_1/\kappa_2)/\partial x)^2$.
4. We do not expect $\mathcal{Z}_R^+/\mathcal{Z}_R^-$ to be singular at a smooth point of a discriminant.

The congruences \mathcal{C}_α and \mathcal{R}_α consist of a 1-parameter family of BDE's, therefore we expect the singularities of BDE's of codimension 1 in Section 2 to occur in \mathcal{C}_α and \mathcal{R}_α . We relate the zero curves to the singularities.

We need the following definitions. As observed before the well-folded singularities occur when the unique direction defined by the BDE on the discriminant is tangent to the discriminant. If the family of BDE's is given by $ady^2 + 2bdydx + cdx^2 = 0$ with $a \neq 0$, then the zero curve is obtained by eliminating the parameter α from the equations $\delta = b^2 - ac = 0$ and $a\delta_x - b\delta_y = 0$. In our particular case we have $\delta = H^2 \sin^2 \alpha - K$, and we shall call

$$\mathcal{DS}_C = \{(x, y, \alpha) : H^2 \sin^2 \alpha - K = 0\}$$

the *discriminant surface* and the surface

$$\mathcal{ZS}_C = \{(x, y, \alpha) : a\delta_x - b\delta_y = 0\}$$

the *zeros surface*, where a, b, c are the coefficients of the BDE \mathcal{C}_α . (We make analogous definitions for \mathcal{R}_α .) The conjugate zero curve is then the projection to the (x, y) -plane of the intersection set of the surfaces \mathcal{DS}_C and \mathcal{ZS}_C . Note that the discriminant curve has an intrinsic meaning; the zeros surface is simply defined so that its intersection with the discriminant curve is the zero set.

- PROPOSITION 5.1. 1. *The conjugate zero curve \mathcal{Z}_C is tangent to the parabolic curve at the cusp of Gauss, changing from \mathcal{Z}_C^+ to \mathcal{Z}_C^- there.*
2. *Let p be a smooth point of Δ_C^α . Then p is a well-folded saddle-node singularity of \mathcal{C}_α (resp. $\mathcal{C}_{-\alpha}$) if and only if \mathcal{Z}_C^+ (resp. \mathcal{Z}_C^-) is tangent to Δ_C^α at p .*
3. *Let p be a hyperbolic point. Then p is a Morse Type 1 singularity of \mathcal{C}_α (and $\mathcal{C}_{-\alpha}$) if and only if \mathcal{Z}_C^+ and \mathcal{Z}_C^- cross at p . Furthermore, at a maximum/minimum singularity of Δ_C^α , \mathcal{Z}_C^+ and \mathcal{Z}_C^- are of the same type. More precisely, if p is a maximum/minimum of the height function α on the discriminant surface and the shrinking discriminant encloses the region where 0 (resp. 2) solutions exist then \mathcal{C}_α is of saddle type (resp. focus type). At a saddle singularity of Δ_C^α , \mathcal{Z}_C^+ can be either of saddle or focus type, and \mathcal{Z}_C^- does not necessarily have the same type as \mathcal{Z}_C^+ .*

- 4. There are three smooth \mathcal{Z}_C curves at a star and monstar and one such curve at a lemon. The curves change from \mathcal{Z}_C^+ to \mathcal{Z}_C^- as they pass through the umbilic point. (See Figure 5.)
- 5. At a focus-node change, the zero curves are smooth and transverse to the discriminant curves. So this change is not detected by the zero curve.

See Figure 7 (right) for illustrations.

Proof: We take the surface in Monge form $z = h(x, y)$ where

$$h(x, y) = \frac{1}{2} \sum_{i=0}^2 \binom{i}{2} a_i x^{2-i} y^i + \frac{1}{6} \sum_{i=0}^3 \binom{i}{3} b_i x^{3-i} y^i + \frac{1}{24} \sum_{i=0}^4 \binom{i}{4} c_i x^{4-i} y^i + \dots$$

and (x, y) in a neighbourhood of the origin. We rotate the coordinates axes so that $(1, 0)$ is an element of \mathcal{C}_{α_0} at the origin. Then $\sin \alpha_0 = a_0 / (a_0^2 + a_1^2)^{\frac{1}{2}}$ and $\cos \alpha_0 = -a_1 / (a_0^2 + a_1^2)^{\frac{1}{2}}$. We shall assume that $\alpha \neq \frac{\pi}{2}$, that is $(1, 0)$ is not a principal direction, equivalently, $a_1 \neq 0$.

1. At a cusp of Gauss, the discriminant surface is smooth and symmetrical with respect to the plane $\alpha = 0$. The zeros surface is also smooth and intersects the discriminant surface transversally. In general the tangent direction to this curve is not along the α -axis, so projecting to the (x, y) -plane yields a smooth curve tangent to the parabolic set.

2. We first observe that all the singularities of BDE's of codimension ≤ 1 depend only on the 2-jet of the coefficients a, b, c at the point in question ([10]). We reduce the 2-jet of the coefficients of \mathcal{C}_{α_0} following the algorithm given in [7], to the form

$$dy^2 + (A_0 + A_1x + A_2y + A_3x^2)dx^2. \tag{5}$$

The discriminant is then given by $A_0 + A_1x + A_2y + A_3x^2 + \dots = 0$. The origin is a point on $\Delta_C^{\alpha_0}$ if, and only if

$$A_0 = a_0^2 - a_0a_2 + 2a_1^2 = 0.$$

It is a singular point of \mathcal{C}_{α_0} on a smooth discriminant if furthermore $A_1 = 0$ and $A_2 \neq 0$. This occurs when

$$b_0a_1 - b_1a_0 = 0 \text{ and } b_1a_1 - b_2a_0 \neq 0.$$

The saddle-node bifurcations occurs when, in addition, $A_3 = 0$, that is, when

$$4a_1^3(a_0^2 + a_1^2) (a_0a_1(a_0^2 + a_1^2) + (a_1c_0 - a_0c_1)) - a_0 ((2a_1^2 + a_0^2)b_1 - a_0a_1b_2) ((3a_1^2 + a_0^2)b_1 - 2a_0a_1b_2) = 0.$$

We compute the linear part of the zero curve and show that the coefficient of x vanishes precisely at a saddle-node bifurcation.

- 3. From Remark 5.5(5), the curve \mathcal{Z}_C^+ is given by $\nabla(\frac{\kappa_1}{\kappa_2}).(1, \sqrt{\frac{\kappa_1}{\kappa_2}}) = 0$ and \mathcal{Z}_C^- by $\nabla(\frac{\kappa_1}{\kappa_2}).(1, -\sqrt{\frac{\kappa_1}{\kappa_2}}) = 0$. So the two curves intersect if and only if $\nabla(\frac{\kappa_1}{\kappa_2}) = 0$, that is generically, if and only if \mathcal{C}_α has a Morse Type 1 singularity.

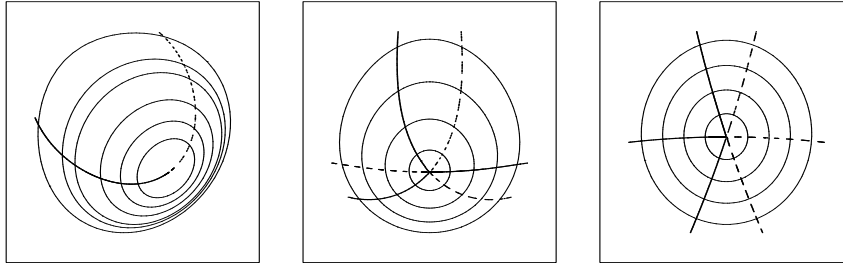


FIG. 5. A Maple plot of the conjugate zero curve at a lemon (left), star (centre), monstar (right) where \mathcal{Z}_C^+ (resp \mathcal{Z}_C^-) are the continuous (resp. dashed) thick curves and $\Delta_C^{\mathcal{C}}$ are the closed curves.

Suppose that p is a minimum/maximum of the discriminant surface $\mathcal{D}\mathcal{S}_C$. At some nearby level the discriminant is a circle which either bounds the points where the BDE has 2 solutions or 0. We can lift a neighbourhood of this disc to the projective tangent bundle; in the first case we obtain, a sphere, and in the second a cylinder. But the lift of the vector field has two foci or two saddles. Clearly we must have the foci when the lift is a sphere, and saddles when it is a cylinder.

An alternative proof is by direct calculations. We take p as the origin and write the discriminant surface as the graph of a function $\alpha = \alpha(x, y)$. At a local extremum we have $\alpha = l_0x^2 + 2l_1xy + l_2y^2 + \dots$ for some scalars l_1, l_2, l_3 . Now the 2-jet of the BDE can be reduced to $dy^2 + (s_0x^2 + 2s_1xy + s_2y^2)dx^2$, and a calculation shows that s_0 and l_0 have the same sign in a neighbourhood of $(0, 0)$. So if $l_0 < 0$ the BDE is topologically equivalent to $dy^2 + (-x^2 - y^2)dx^2 = 0$, and $l_0 > 0$ it is equivalent to $dy^2 + (x^2 + y^2)dx^2 = 0$.

An analysis of the coefficients s_i and l_i above also shows that both types (saddle and focus) occur at a saddle singularity of the discriminant, and the type of \mathcal{Z}^- is not necessarily the same as that of \mathcal{Z}^+ .

4. At a generic umbilic the discriminant surface is (half) a cone ([8]). It is not difficult to see that $\mathcal{C}_{-\alpha}$ is the same as $\mathcal{C}_{\pi-\alpha}$, so we can consider the zero curve around the whole cone at $\frac{\pi}{2}$, with \mathcal{Z}_C^+ on one half and \mathcal{Z}_C^- on the other half of the cone. A result in [8] states that there are generically 1 or 3 smooth curves of zeros on the discriminant surface passing through the cone point. These are of the same type as the zeros at the umbilic (3 lines of saddles at a star, 2 lines of saddles and 1 line of nodes at a monstar, and 1 line of saddles at a lemon).

There are two possible configurations when there are three zero curves at umbilics: going around the umbilic one meets the \mathcal{Z}_C^+ and \mathcal{Z}_C^- alternately or not. The only way a change can occur from one configuration to another is through tangency of the zero curves which corresponds to the cubic ϕ being on the circle $|\beta| = 3$ (see Section 2). Hence the configuration of the zero curves is the same for all star and all monstar umbilics. It turns out these are as in Figure 5. \square

We turn now to the reflected zero curve.

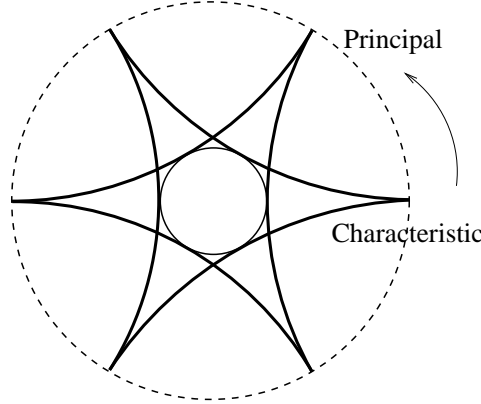


FIG. 6. Partition of the (u, v) -plane: the hypercycloid Γ_α rotates around the circle $\beta = 3$.

- PROPOSITION 5.2.
1. The reflected zero curve Z_R is tangent to the parabolic curve at the cusp of Gauss, changing from Z_R^+ to Z_R^- there.
 2. Let p be a smooth point of Δ_R^α . Then p is a well-folded saddle-node singularity of \mathcal{R}_α (resp. $\mathcal{R}_{-\alpha}$) if and only if Z_R^+ (resp. Z_R^-) is tangent to Δ_R^α at p .
 3. Let p be a hyperbolic point. Then p is a Morse Type 1 singularity of \mathcal{R}_α (and $\mathcal{R}_{-\alpha}$) if and only if Z_R^+ and Z_R^- cross at p . Furthermore, at a maximum/minimum singularity of Δ_R^α , Z_R^+ and Z_R^- are of the same type. More precisely, if p is a maximum/minimum of the height function α on the discriminant surface and the shrinking discriminant encloses the region where 0 (resp. 2) solutions exist then C_α is of saddle type (resp. focus type). At a saddle singularity of Δ_R^α , Z_R^+ can be either of saddle or focus type, and Z_R^- does not necessarily have the same type as Z_R^+ .
 4. The curve $H = 0$ is a degenerate component of the Z_R . The non-degenerate component of Z_R is in general smooth and transverse to $H = 0$ at points where an asymptotic direction is tangent to $H=0$.
 5. Umbilic points are common isolated zeros for all of the \mathcal{R}_α . In the notation of Section 2 (last paragraph), for a fixed α the umbilic is of type star if $\beta = u + iv$ is inside the circle $u^2 + v^2 - 9 = 0$, of type lemon if β is outside the hypercycloid $3e^{i\frac{2\alpha}{3}}(2e^{i\theta} + e^{-2i\theta})$ and of type monstar in the remaining regions of the complex plane. As α varies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, a star umbilic remains a star, a lemon with β outside the circle $u^2 + v^2 = 81$ remains a lemon, and umbilics in between the two circles change from lemon to monstar and back again, and vice-versa.

See Figure 7 (right) for illustrations.

Proof: We take the surface, in Monge form $z = h(x, y)$ as in the proof Proposition 5.1 and fix $(1, 0)$ to be an asymptotic direction at the origin, so that $a_0 = 0$. This direction is a member of \mathcal{R}_α if $\tan(\alpha) = \frac{2a_1}{a_2}$. The cases 1-4 now follow by direct calculations in the same way in the proof of Proposition 5.1.

We consider now the situation at umbilics. We observed in Proposition 4.2 that umbilic points are common singularities for all the \mathcal{R}_α . Writing the cubic $C(x, y) = \operatorname{Re}(z^3 + (u + iv)z^2\bar{z}) = (1 + u)x^3 - vx^2y + (u - 3)xy^2 - vy^3$ for some $u, v \in \mathbb{R}$, yields the 1-jet of \mathcal{R}_α of the form $ady^2 + 2bdxdy - adx^2$ with

$$\begin{aligned} a &= -((u + 3)\cos\alpha + v\sin\alpha)x + (-v\cos\alpha + (u - 3)\sin\alpha)y, \\ b &= (-v\cos\alpha + (u + 3)\sin\alpha)x + ((u - 3)\cos\alpha + v\sin\alpha)y. \end{aligned}$$

As observed in Section 2, there are two curves whose complement cuts the (u, v) -plane into the three umbilic types lemon, monstar, star regions. One is given by the set of points where the discriminant of the BDE fails to be Morse. This curve is the circle $u^2 + v^2 = 9$ (does not depend on α).

The other curve is the set of points where the cubic $\phi_\alpha(p)$ has repeated roots. We have

$$\begin{aligned} \phi_\alpha(p) &= (-v\cos\alpha + (u - 3)\sin\alpha)p^3 + ((u - 9)\cos\alpha + v\sin\alpha)p^2 + \\ &\quad (-v\cos\alpha + (u + 9)\sin\alpha)p + (u + 3)\cos\alpha + v\sin\alpha. \end{aligned}$$

A calculation shows that this cubic can be written in the form $\operatorname{Re}(3iz^3 + e^{i\frac{2\alpha}{3}}(u - iv)z^2\bar{z})$, and therefore has repeated roots if and only if

$$\Gamma_\alpha : u + iv = 3e^{i\frac{2\alpha}{3}}(2e^{i\theta} + e^{-2i\theta}), \quad \theta \in [0, 2\pi].$$

This is a hypocycloid rotating around the circle $u^2 + v^2 = 9$ as α varies in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Its three cusps trace the circle $u^2 + v^2 = 81$ once, each turning through an angle of $\frac{2\pi}{3}$.

For a fixed α the umbilic is of type lemon if the cubic C is outside Γ_α , of type star if it is inside the circle, and of type monstar in the remaining region. The result now follows. (See Figure 6 \square)

We plot some \mathcal{Z}_C and \mathcal{Z}_R curves using the computer algebra package Maple. As observed before, the conjugate zero curve lies in the elliptic region and the reflected zero curve in the hyperbolic region of the surface. We start by taking a surface patch $z = h_0(x, y)$ which is hyperbolic at the origin and deform it until we create an elliptic region. The simplest way to do this is to go through an A_3 -transition of the height function h_0 ([5]). We take h_t of the form

$$h_t(x, y) = x^2 + ty^2 - x^2y^2 - \frac{1}{5}y^4.$$

with $t = 0.4$ and plot the various curves in the neighborhood of the origin $[-1, 1] \times [-1.2, 1.2]$. In Figure 7 (left and right) the parabolic set is the dashed curves. It consists of a closed loop (created by the A_3 -transition) and two other curves. Points inside the closed parabolic loop are elliptic points. The region in between the parabolic curves is hyperbolic which contains the curve $H = 0$ (dotted curve in Figure 7 right). The thick curves are the zero curves (continuous for \mathcal{Z}_C^+ and \mathcal{Z}_R^+ and dashed for \mathcal{Z}_C^- and \mathcal{Z}_R^-). The discriminants Δ_C^α (Figure 7 left) and Δ_R^α (Figure 7 right) are the continuous thin curves.

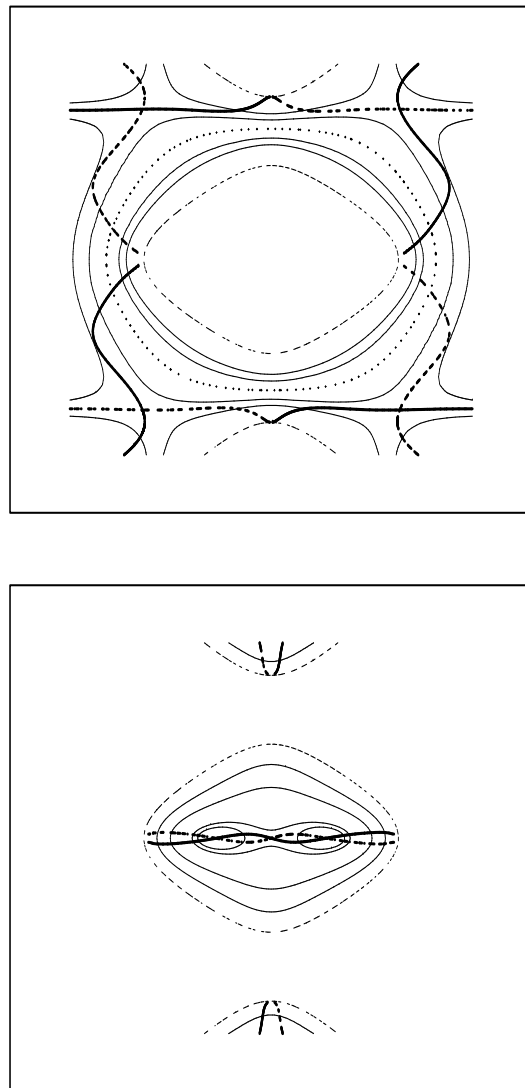


FIG. 7. A Maple plot of the zero curves: Z_C left and Z_R right

The singularities of the discriminants Δ_C^α (resp. Δ_R^α) are clearly captured by the intersections of \mathcal{Z}_C^+ and \mathcal{Z}_C^- (resp. \mathcal{Z}_R^+ and \mathcal{Z}_R^-) in Figure 7 left (resp. Figure 7 right). Figure 7 right also shows the well-folded singularities of \mathcal{R}_α at points of tangency of \mathcal{Z}_R with Δ_R^α .

We have two cusps of Gauss on the closed parabolic curve and one on each of the other parabolic curves. These are also shown as points where \mathcal{Z}_C (resp. \mathcal{Z}_R) is tangent to the parabolic curve changing there from \mathcal{Z}_C^+ to \mathcal{Z}_C^- (resp. \mathcal{Z}_R^+ to \mathcal{Z}_R^-).

6. THE FOLD CURVES OF THE SPHERICAL MAP

Here we briefly take an alternative look at our BDE's. At each non-singular point of a direction field on a surface in \mathbb{R}^3 we have a direction, yielding a point in the projective plane (or coherently orienting directions in the unit sphere). Of course locally a BDE determines 0, 1 or 2 direction fields. The next result relates this spherical representation of conjugate vector fields to the geometric properties of their integral curves. (The final part first appeared in [19].)

PROPOSITION 6.1.

1. Let ξ be a smooth unit vector field on a surface $M \in \mathbb{R}^3$, and let $\bar{\xi}$ be another unit vector field with ξ and $\bar{\xi}$ conjugate at each point of M ; we suppose further that these vector fields are distinct; that is ξ is not a field of asymptotic directions. The fields ξ and $\bar{\xi}$ determine smooth maps $X \rightarrow S^2$. Then ξ (respectively $\bar{\xi}$) is singular at p if and only if its derivative in the direction of $\bar{\xi}$ (respectively ξ) is zero at p . Note that ξ has a cusp or worse if the fold curve of ξ is tangent to $\bar{\xi}$.
2. Suppose now (only) that the angle between ξ and $\bar{\xi}$ is constant. Then an integral curve for ξ has a geodesic inflexion if and only if $\partial\xi/\partial t$ is normal to the surface if and only if $\partial\bar{\xi}/\partial t$ is normal to the surface. (Similarly for $\bar{\xi}$.)
3. Finally suppose that both hypotheses hold that is ξ and $\bar{\xi}$ are conjugate and inclined at a constant angle. Then singular points of the map $\xi : X \rightarrow S^2$ correspond to the points of geodesic inflexion of the integral curves of $\bar{\xi}$ (and both to $\partial\xi/\partial\tau = 0$) and vice-versa.

Proof: 1. Let N denote a unit normal vector field on X . Choose a curve $\delta(s)$ transverse to both fields, and consider integrals γ and $\bar{\gamma}$, so that $\partial\gamma(s, t)/\partial t = \xi(\gamma(s, t))$, $\partial\bar{\gamma}(s, \tau)/\partial\tau = \xi(\bar{\gamma}(s, \tau))$, and $\gamma(s, 0) = \bar{\gamma}(s, 0) = \delta(s)$. Now consider the identities $N \cdot \xi = N \cdot \bar{\xi} = 0$. Composing these with γ and $\bar{\gamma}$ and differentiating with respect to t and τ we find that (setting $t = \tau = 0$ and using $S(\xi) \cdot \bar{\xi} = S(\bar{\xi}) \cdot \xi = 0$)

$$\begin{aligned} S(\xi(\delta(s))) \cdot \xi(\delta(s)) + N(\delta(s)) \cdot \gamma''(\delta(s)) &\equiv 0 \\ S(\bar{\xi}(\delta(s))) \cdot \bar{\xi}(\delta(s)) + N(\delta(s)) \cdot \bar{\gamma}''(\delta(s)) &\equiv 0 \\ N(\delta(s)) \cdot \partial(\xi(\bar{\gamma}(s, \tau)))/\partial\tau|_{\tau=0} &\equiv 0 \\ N(\delta(s)) \cdot \partial(\bar{\xi}(\gamma(s, t)))/\partial t|_{t=0} &\equiv 0 \end{aligned}$$

From the fact that ξ and $\bar{\xi}$ are unit vectors we also find that $\partial(\xi(\bar{\gamma}(s, \tau)))/\partial\tau|_{\tau=0}$ (respectively $\partial(\bar{\xi}(\gamma(s, t)))/\partial t|_{t=0}$) are orthogonal to $\xi(\delta(s))$ (respectively $\bar{\xi}(\delta(s))$). So at any point where ξ and $\bar{\xi}$ are independent (that is provided neither is an asymptotic direction) $\partial(\xi(\bar{\gamma}(s, \tau)))/\partial\tau|_{\tau=0}$ (respectively $\partial(\bar{\xi}(\gamma(s, t)))/\partial t|_{t=0}$) is zero if and only if they are orthogonal to $\bar{\xi}$ (respectively ξ). Note also that if ξ (respectively $\bar{\xi}$) is singular at a point then $\partial\xi/\partial\tau$ (respectively $\partial\bar{\xi}/\partial t$) is a multiple of γ'' (respectively $\bar{\gamma}''$). If it is a non-zero multiple then we deduce that $S(\xi).\xi$ ($S(\bar{\xi}).\bar{\xi}$) is zero. So if ξ is not asymptotic then ξ (respectively $\bar{\xi}$) is singular if and only if $\partial\xi/\partial\tau$ (respectively $\partial\bar{\xi}/\partial t$) is zero.

2. Now suppose that our families have the property that the angles between the ξ and $\bar{\xi}$ are constant, so that $\xi.\bar{\xi}$ is constant. Composing with γ and differentiating with respect to t we find that

$$(\gamma''.\bar{\xi} + \xi.\partial\bar{\xi}/\partial t)(\delta(s)) \equiv 0$$

Now γ has a geodesic inflexion if and only if γ'' is a multiple of n . From the equation (provided the constant angle is not 0) this happens if and only if $\xi.\partial\bar{\xi}/\partial t = 0$. Since $\bar{\xi}$ is a unit vector it is also orthogonal to $\partial\bar{\xi}/\partial t$, and the result follows.

3. If, for example, γ has an inflexion then $\partial\bar{\xi}/\partial t$ is normal from part (2). But from the equations in (1) it is orthogonal to n , so it must be zero, and in particular $\bar{\xi}$ has rank 1. Conversely if $\bar{\xi}$ has rank 1, then $\partial\bar{\xi}/\partial t$ is zero from (1) and then we have γ'' orthogonal to $\bar{\xi}$ and ξ , that is we have a geodesic inflexion. \square

Now if we have a BDE F on a surface M then we can consider away from any zeros the subset of the projective tangent sphere bundle $\Gamma(F)$ which this determines; generically this is smooth. We can consider the natural map $\Gamma(F) \rightarrow \mathbb{R}P^2$ which assigns to a point (p, v) the corresponding tangent direction. We denote the fold curve of F by $\mathcal{F}(F)$.

COROLLARY 6.1. *Suppose that $\alpha \neq 0$. Then away from its discriminant $\mathcal{F}(F_\alpha)$ is the set of geodesic inflexions of the integral curves of $F_{-\alpha}$.*

We next obtain some information on the asymptotic and characteristic directions, some of which reproves earlier results.

PROPOSITION 6.2.

1. *Let ξ be a smooth unit field of asymptotic directions on a surface M and let $\tilde{\xi}$ be another unit field of asymptotic directions. Then the angle between ξ and $\tilde{\xi}$ is constant on the discriminant sets Δ_C^α . (Remark 4.3 (3).) Away from the discriminant (the parabolic curve) ξ has rank zero at a point p if and only if the corresponding asymptotic curve has a geodesic inflexion at p .*
2. *Let ξ be a smooth unit field of characteristic directions and let $\bar{\xi}$ be a unit field representing the conjugate directions. Then the angle between the characteristic directions is constant on the discriminants Δ_R^α . (Remark 4.6 (4).)*

Proof: 1. One easily checks that $\xi + \tilde{\xi}$ and $\xi - \tilde{\xi}$ are principal directions. Apply S to both vectors and taking the inner products with these principal vectors we deduce that $S(\xi).\tilde{\xi} = S(\tilde{\xi}).\xi$ and $\xi.\tilde{\xi} = (\alpha+1)/(\alpha-1)$ where α is one of the ratios of principal curvatures.

From the same arguments as in Proposition 6.1 and the fact that $S(\xi).\xi \equiv 0$ we deduce that $n.\gamma'' \equiv 0$, where γ are the integral curves of ξ (the asymptotic curves). Now we know that $n.\xi$ is identically zero along any curve $\delta(s)$. If we differentiate the identity we find that $S(\delta').\xi + n.\partial\xi/\partial s = 0$. Now if ξ has rank 1 $\partial\xi/\partial s$ is a multiple of γ'' , and we can deduce that either $\gamma'' = 0$ or that $S(\delta').\xi = 0$ for any vector δ' . In the latter case S has rank ≤ 1 , and we are on the parabolic curve, the discriminant which we exclude. So $\gamma'' = 0$ and we have a geodesic inflexion. Conversely if $\gamma'' = 0$ then $D\xi(\gamma') = 0$ and ξ has rank ≤ 1 .

2. Similarly $\xi + \tilde{\xi}$ and $\xi - \tilde{\xi}$ are principal directions. Applying the same procedure as in (i) we deduce the same result. For the second part note first that since the directions are conjugate we have, as before, that $\tilde{\xi}$ is singular if and only if $\partial\tilde{\xi}/\partial t = 0$. \square

7. PENCILS OF FORMS AND BDE'S

Our intention in the next two sections is to understand why the classical triple of BDE's (the asymptotic, characteristic and principal BDE's) are intimately related and to seek to generalise this example.

Clearly we can write down the BDE's on a surface in any co-ordinate system. However as we have seen the interpretation of the coefficients is then less than clear. The approach we have taken here is to consider (oriented) surfaces which have isolated umbilic points. Those surfaces in \mathbb{R}^3 without this property form a set of infinite codimension in a very natural sense. Away from the umbilics we may select a principal co-ordinate system. This is simply a co-ordinate system with the lines $x = \text{constant}$ and $y = \text{constant}$ the lines of curvature. We are particularly interested in those BDE's which, in such a system, can be expressed in the form

$$a(\kappa)p^2 + 2b(\kappa)pq + c(\kappa)q^2 = 0$$

where a, b, c are functions of the principal curvatures κ_1, κ_2 in the x, y -directions respectively, and the ratio p/q determines the slope of the given direction with respect to the given axes.

It will be helpful to recall some elementary facts concerning binary quadratic forms and plane conics. So let k denote the field of real or complex numbers. Write L for a field of characteristic zero. The examples we are interested in are the field of rational functions in x_1, \dots, x_n with coefficients in $k = \mathbb{R}, \mathbb{C}$, in particular where $n = 1, 2$, which we denote by $k(x)$. We shall be considering binary quadratic forms with coefficients in L .

REMARKS 7.7. 1. We do not distinguish non-zero L multiples of such forms. Consequently we are considering elements of the projective plane PL^2 ; where $f = ap^2 + bpq + cq^2$ corresponds to the point $(a : b : c)$. Clearly the plane contains the conic Δ of singular forms given by $b^2 = 4ac$. A singular form is one of the type $(rp + sq)^2$. If $(r : s) \in PL^1$ then clearly $(r : s) \mapsto (r^2 : 2rs : s^2)$ is the usual parametrisation of the conic Δ .

2. Motivated by our previous work we will be considering pairs of forms and the pencils determined by them. These can be viewed as lines in PL^2 . Any such line will have 0, 1 or 2 points on the pencil determining singular forms. (Clearly even when $k = \mathbb{C}$ the field of rational functions $L = k(x)$ is not algebraically closed.)

Given a conic in the projective plane PL^2 , then any point in PL^2 determines a polar line, and given a line there is a corresponding polar point. Geometrically if the line meets the conic then the tangents at the points of intersection meet in the polar point. Three points in the plane are said to be *self-polar* (as is the triangle determined by them) if the polar of any vertex is the line through the remaining two points. The next series of results are well known and elementary, but very useful. They relate some of the invariants of pairs of binary forms to the geometry of the conic Δ of singular forms.

PROPOSITION 7.1. 1. *Let ω be a binary quadratic form with distinct roots in L , determining a point in the plane PL^2 . Then the polar line of ω with respect to the conic of singular quadratics Δ consists of the line through the two forms which are the squares of the factors of ω . In other words the tangents to the conic at these two points pass through ω . We refer to this intersection as the polar form of the pencil. Conversely given any pencil meeting the conic Δ , the corresponding polar form is the binary form whose factors are the repeated factors at the two singular members of the pencil.*

2. *This polar form of the pencil is given by the classical Jacobian of any two of the forms in the pencil, that is the 2×2 determinant of the matrix of partial derivatives of the forms with respect to p and q . The Jacobian is non-zero provided we have a genuine pencil, and is a square if and only if the forms have a factor in common.*

3. *Fixing two forms $\omega = ap^2 + bpq + cq^2$, $\Omega = Ap^2 + Bpq + Cq^2$ we write $D(\alpha : \beta)$ for the discriminant of $\alpha\omega + \beta\Omega$; this is another binary quadratic form. We can write it as $D(\omega)\alpha^2 + E(\omega, \Omega)\alpha\beta + D(\Omega)\beta^2$, where $D(\omega) = (b^2 - 4ac)$, $D(\Omega) = (B^2 - 4AC)$, $E(\omega, \Omega) = 2(bB - 2aC - 2Ac)$. The associated polar point of the pencil, the Jacobian, determined by ω, Ω is*

$$Jac(\omega, \Omega) = (aB - Ab)p^2 + 2(aC - Ac)pq + (bC - Bc)q^2.$$

Note in particular that if the two forms ω and Ω have their coefficients in some ring R then so does their Jacobian. (So if f and F have entries in $k[\kappa_1, \kappa_2]$ or $k(\kappa_1, \kappa_2)$ the same is true of their Jacobian.)

4. *Pairs of forms ω, Ω with the term $E(\omega, \Omega)$ above zero are said to be apolar. This is equivalent to the condition that the corresponding four roots harmonically separate each other, or that the forms lie on each others polars with respect to the conic Δ , that is are conjugate. The Jacobian of any two forms is apolar with respect to all the elements of the pencil determined by them.*

5. *Three forms determine a self-polar triangle with respect to the conic Δ if and only if each is the Jacobian of the other two. There are a variety of ways of obtaining self-polar triples. Any form ω determines a polar line. Choose an arbitrary form say Ω on the line; this has a polar line which passes through ω . Consider the intersection point of these two polar lines; this gives a third form μ , with ω, Ω, μ self-polar. Any self-polar triple arises in*

this way. Also if ω, Ω are conjugate, then the triple $\omega, \Omega, \text{Jac}(\omega, \Omega)$ is self polar. Finally if the vertices of a quadrangle lie on Δ then the diagonal triangle (the triangle whose vertices are intersections of the lines joining distinct pairs of distinct points) is self-polar.

6. The discriminants, the invariant E , and the Jacobian of a pair of forms ω, Ω are related as follows:

$$\text{Jac}^2(\omega, \Omega) - 4D(\Omega)\omega^2 - 4D(\omega)\Omega^2 + 4E(\omega, \Omega)\omega\Omega = 0.$$

We now suppose that L is the field of rational functions $k(x)$ in n -variables, U is an open subset of k^2 and that we have a smooth map $X : U \rightarrow k^n$. Given a form $\omega = ap^2 + bpq + cq^2$, with $a, b, c \in k[x]$ we can define a BDE on U by considering

$$a(X)p^2 + b(X)pq + c(X)q^2 = 0$$

where at $(x, y) \in k^2$ the solution $(p : q) = (p_0, q_0)$ is the direction with slope p_0/q_0 ; this BDE is denoted $X^*\omega$. As usual we suppose given a pair of forms ω, Ω and consider the pencil of BDE's, $X^*(\alpha\omega + \beta\Omega) = \alpha X^*\omega + \beta X^*\Omega$, but with $\alpha, \beta \in k$. Each $(\alpha : \beta) \in Pk^1$ determines a BDE, and this will have a discriminant $D_{\alpha, \beta}$ in U .

PROPOSITION 7.2. 1. If we fix a point in U there are 2 or 1 (respectively 2, 1 or 0) discriminant curves passing through it when $k = \mathbb{C}$ (respectively \mathbb{R}). The condition that there is just one is $E^2(\omega, \Omega) = 4D(\Omega)D(\omega)$.

2. Suppose we now consider a point p of U through which two discriminant curves pass, so there are two values of $(\alpha : \beta)$ for which D vanishes at the point. Each corresponds to a BDE with a repeated direction through p ; putting these two directions together at each point yields a new BDE, which is just the pull-back by X of the Jacobian or polar form

$$X^*((aB - Ab)p^2 + 2(aC - Ac)pq + (bC - Bc)q^2) = 0.$$

(It can be written in a number of other ways e.g. if $(aB - Ab) \neq 0$ as $X^*(D(\omega)(Ap + Bq)^2 - 2E(\omega, \Omega)(ap + bq)(Ap + Bq) + D(\Omega)(ap + bq)^2) = 0$.)

Proof: Part (1) is clear. For (2) we write $f = X^*(\omega)$ and $F = X^*(\Omega)$, and simply write a for $X^*a = a \circ X$ etc. If we have a point through which only one discriminant curve passes then it satisfies $\alpha^2 D_1 + E\alpha\beta + \beta^2 D_2 = 0$ with one of α and β non-zero. Now if the roots are (α_j, β_j) , $j = 1, 2$ then $\alpha_j(ap^2 + 2bpq + cq^2) + \beta_j(Ap^2 + 2Bpq + Cq^2)$ has a repeated root, so must be of the form $p/q = (-b\alpha_j - B\beta_j)/(a\alpha_j + A\beta_j)$. A little manipulation gives the resulting BDE, in the second form above.

Alternatively and more directly the pencil meets the conic of singular quadratics in two points. We are simply considering the quadratic whose roots are each of the repeated roots, and the result follows from the elementary properties above. \square

COROLLARY 7.1. 1. The set of zeros of the pencil determined by f and F coincides with the set of tangency points of the polar BDE of the pencil with the discriminants. (In the case

when $n = 1$, the field is $K = k(\lambda)$, and the map X is given by $X(x, y) = (\kappa_2(x, y)/\kappa_1(x, y))$, where the κ_j are the principal curvatures in some order at a non-umbilic point, then the discriminants are given by $\kappa_2/\kappa_1 = \text{constant}$.)

2. The set of points through which only one discriminant curve passes (roughly the envelope of the discriminant curves) is given by $E^2 = 4D_1D_2$, that is $(aC - Ac)^2 = (aB - Ab)(bC - cB)$. This is the discriminant of the polar BDE to the pencil (the Jacobian form). (The statement can then be read as: the envelope of the discriminants of the pencil is the discriminant of the polar or Jacobian form.)

3. In our previous work we considered families of BDE's F_α whose discriminants D_α actually satisfy $D_\alpha = D_{-\alpha}$. So it is natural to consider those forms f and F which have the property that $D_{\alpha,\beta} = D_{-\alpha,\beta}$. The BDE's f and F arising from the forms ω, Ω have the property that the discriminant of $\alpha f + \beta F$ coincides with that of $\alpha f - \beta F$ if ω, Ω are apolar (or equivalently conjugate with respect to the conic Δ of singular forms). Conversely if any pencil of BDE's induced from ω, Ω has this property then these forms are apolar.

4. We can obtain a triple of BDE's with the property that each is the polar BDE of the pencil determined by the other two using the constructions above for self-polar triples.

REMARKS 7.8. 1. The first result tells us that the zero points of the families of the pencil will pass through the zeros of the polar form, since in any neighbourhood of a singular point of a BDE there will be tangency points with any foliation.

2. We extend the terminology for forms (polar, Jacobian, apolar) to BDE's in the natural way.

3. Note that if we were considering the full pencil joining f and F then each point of the pencil would determine a second conjugate point of the pencil. However if we are only considering points $\alpha f + \beta F$ with $\alpha, \beta \in k$ (rather than $L = k(x)$) then this is not necessarily true.

EXAMPLE 7.1. The set of asymptotic, characteristic and principal BDE's ($F = \kappa_2 p^2 + \kappa_1 q^2$, $G = pq$, $H = \kappa_2 p^2 - \kappa_1 q^2$) is a self-polar triple.

The polars of certain forms share some interesting properties; in what follows the angle θ associated to $(p : q)$ satisfies, as usual, $\tan \theta = p/q$.

EXAMPLE 7.2. 1. The polar of $p^2 + q^2$ is the set of forms whose roots have product -1 (directions are orthogonal).

2. The polar of $p^2 - q^2$ is the set of forms whose roots have product 1 (angles sum to $\pi/2$).

3. The polar of pq is the set of forms whose roots have sum 0 (angles sum to 0).

4. The polar of $\kappa_1 p^2 + \kappa_2 q^2$ is the set of forms whose roots determine conjugate directions.

Indeed it is easy to check that the polar of a point $(a, b, c) \in kP^2$ is the set of forms for which the sum of the angles is constant if and only if $a + c = 0$.

8. CURVATURE-BDE'S

In our examples the coefficients of the BDE's we are considering are given, in a principal co-ordinate system, by polynomials in the principal curvatures. In other words we have taken $n = 2$ above and the map X given by $x \mapsto (\kappa_1(x), \kappa_2(x))$. A key issue is that at each point we cannot canonically select a pair of ordered principal directions. For this reason we need the following discussion. At a point z of our surface we can consider an ordered pair (consistent with the orientation) of principal directions e_1, e_2 , yielding an ordered pair of principal vector fields nearby. Suppose given a BDE near $z \in M$ which in a principal co-ordinate system can be written in the form $a(\kappa)p^2 + 2b(\kappa)pq + c(\kappa)q^2 = 0$, with $a, b, c \in k[\kappa_1, \kappa_2]$. Here the solution $(p : q)$ at (x, y) correspond to the tangent direction making an angle of $\arctan p/q$ to the principle direction e_1 . If this is to be well defined in a global sense then we need the same solutions if we replace e_1, e_2 by $e_2, -e_1$ (and κ_1, κ_2 by κ_2, κ_1). Label the alternative pair of principal directions d_1, d_2 , with $Q, -P$ replacing p, q . Then the new BDE is given by

$$\bar{a}(\kappa)Q^2 - 2\bar{b}(\kappa)PQ + \bar{c}(\kappa)P^2 = 0$$

where $\bar{h}(\kappa_1, \kappa_2) = h(\kappa_2, \kappa_1)$. So we need (a, b, c) to be a non-zero multiple of $(\bar{c}, -\bar{b}, \bar{a})$. In what follows we write τ for $\kappa_1 - \kappa_2$.

PROPOSITION 8.1. *1. We have (a, b, c) a non-zero multiple of $(\bar{c}, -\bar{b}, \bar{a})$ if and only if the BDE is in one of the following forms:*

- (a) $a(\kappa)p^2 + b(\kappa)pq + \bar{a}(\kappa)q^2$, with $b + \bar{b} = 0$ (type I), or
 (b) $a(\kappa)p^2 + b(\kappa)pq - \bar{a}(\kappa)q^2$, with $b = \bar{b}$ (type II).

2. *Multiplication by τ interchanges type. In the first case b is of the form $B - \bar{B}$, in the second $B + \bar{B}$ for some B . Alternatively in the second case we can write $b = h(H, K)$ and in the first $b = \tau h(H, K)$ for some h , where $H = (\kappa_1 + \kappa_2)/2$ is the mean curvature and $K = \kappa_1\kappa_2$ is the Gauss curvature.*

3. *The BDE corresponding to the directions conjugate to those determined by (1) and (2) above is obtained by replacing (a, b) by $(\bar{a}\kappa_2^2, \bar{b}\kappa_1\kappa_2)$.*

4. *Given two BDE's of type I (or II) then an \mathbb{R} or \mathbb{C} linear combination of them is of the same type. Indeed if $k_s[\kappa_1, \kappa_2]$ (respectively $K_s = k_s(\kappa_1, \kappa_2)$) is the subring (subfield) of $k[\kappa_1, \kappa_2]$ ($k(\kappa_1, \kappa_2)$) consisting of symmetric functions of κ_1, κ_2 , in other words polynomial (rational) functions in H and K , then $k_s[\kappa_1, \kappa_2]$ -(K_s -)linear combinations of BDE's of a given type are of the same type.*

Proof: This is easy to prove. For the second part of (2) note that if $b = \bar{b}$ then b is symmetric in κ_1 and κ_2 and any such function can be written as a function of their sum and product. If $b + \bar{b} = 0$ then $b(t, t)$ is identically zero and so we can write $b = (\kappa_2 - \kappa_1)b'$ and one easily checks that b' is symmetric. \square

DEFINITION 8.1. We say that a BDE of one of the above types is a curvature BDE, or CBDE for short.

- REMARKS 8.9. 1. We shall invariably select a, b, c to be polynomials. Note that if the CBDE only depends on the ratio of the curvatures (that is the indicatrix up to similarity) then a, b, c are homogeneous polynomials in κ_1, κ_2 of a given degree.
2. Note also that the map $a \mapsto \bar{a}$ is an automorphism of order 2 of the field $K = k(\kappa_1, \kappa_2)$, with fixed set K_s .
3. The set of CBDE's do not sit inside KP^2 ; a $K = k(\kappa_1, \kappa_2)$ -multiple of a CBDE is not necessarily a CBDE, but a K_s -multiple is.
4. Multiplication by τ interchanges BDE's of types I and II, and since $\kappa_1 = \kappa_2$ only at umbilics, it does not change the integral curves of the BDE's. For this reason it is sufficient to study BDE's of type I say.

We now show that a CBDE extends across the umbilic points, justifying their name.

PROPOSITION 8.2. *Let M be an oriented surface in \mathbb{R}^3 with isolated umbilic points. Then any CBDE with polynomial entries determines a smooth BDE on the whole of M ; that is extends smoothly across umbilic points.*

Proof: Obviously we cannot take a special principal co-ordinate system at an umbilic; this is the key problem! So we take a general parametrisation $\mathbf{r}(x, y)$ at an umbilic and change coordinates.

Let $ap^2 + 2bpq + cq^2 = 0$ be a BDE with 2 solutions u_1, u_2 at a point in the plane. Suppose that we have a (parametrised) linear action on the vectors u_1, u_2 by a matrix

$$M = \begin{pmatrix} M_1 & N_1 \\ M_2 & N_2 \end{pmatrix}$$

and let $v_1 = Mu_1, v_2 = Mu_2$. Then v_1, v_2 are solutions at this point of the BDE $Ap^2 + 2Bpq + Cq^2 = 0$ with

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} M_1^2 & -2M_1N_1 & N_1^2 \\ -M_1M_2 & M_1N_2 + M_2N_1 & -N_1N_2 \\ M_2^2 & -2M_2N_2 & N_2^2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Now we work in a neighbourhood of an umbilic, suppose that $ap^2 + 2bpq + cq^2 = 0$ is a CBDE, with the map M taking the co-ordinate directions with respect to the parametrisation \mathbf{r} to the principal directions. So M has entries as above where $M_1\mathbf{r}_x + M_2\mathbf{r}_y$ and $N_1\mathbf{r}_x + N_2\mathbf{r}_y$ are the principal directions. These in turn are obtained by solving the principal directions BDE

$$(Fn - Gm)dy^2 + (En - Gl)dydx + (Em - Fl)dx^2 = 0.$$

So we replace dy by M_2 (respectively M_1) and dx by N_2 (respectively N_1) and solve the corresponding equations. If say $Fn - Gm \neq 0$ then the expressions for $M_2/M_1, N_2/N_1$ are

$\{-(En - Gl) \pm 2(EG - F^2)\sqrt{H^2 - K}\}/2(Fn - Gm)$. Now the BDE $ap^2 + 2bpq + cq^2$ starts out with entries which are polynomials in the curvatures, satisfying those conditions which make them CBDE's (of type I or II). We can extend smoothly to umbilics if after changing coordinates, we can get rid of the 'discriminant' $\sqrt{H^2 - K}$ in the new coefficients (A, B, C) . Of course if κ_1, κ_2 are the principal curvatures then $\tau = \kappa_1 - \kappa_2 = \pm 2\sqrt{H^2 - K}$. Since our BDE is of the form I or II, in one case the expressions for A, B, C do not involve τ and in the other τ is a factor of each of them, so can be discounted. \square

We now wish to study CBDE's; we note that when, for example, we refer to the CBDE $\kappa_2 p^2 + \kappa_1 q^2$ we will be thinking of the BDE determined by this on the whole surface M (in this case the asymptotic BDE). As remarked above if $h \in k[\kappa_1, \kappa_2]$ and f is a CBDE then the product $h.f$ need not be unless $h \in K_s$. Working over K_s however we can think of our two types of CBDE's as each determining a projective plane over K_s .

PROPOSITION 8.3. 1. Let $\tau = \kappa_1 - \kappa_2$. The map $K_s^3 \rightarrow K^3, (u, v, w) \mapsto (u + \tau w, \tau v, u - \tau v)$ is injective and has image the set of CBDE's of type I. The map $(u, v, w) \mapsto (u + \tau w, v, -u + \tau w)$ is injective and has image the set of CBDE's of type II. We abuse notation and write P_I (respectively P_{II}) for the projective planes PK_s^3 with the above maps into PK^3 .

2. The pull back of the discriminant conic in P_I (respectively P_{II}) is given by $u^2 - \tau^2(v^2 + w^2) = 0$ (respectively $u^2 + v^2 - \tau^2 w^2$). (Here u, v, w are rational functions of H and K and $\tau^2 = 4(H^2 - 4K)$.)

3. The polar BDE of two CBDE's of type I (respectively of type II, of types I and II) are CBDE's of type II (respectively II, I).

The discussion above now works in the projective planes P_1 and P_2 . Note however that if we wish to discuss self-polar triples then the natural types of CBDE's to consider are type I, since they are closed under this operation. Since the families are essentially equivalent this is no real restriction.

One final note. The projective line of directions through a point of our surface clearly has further structure inherited from the Euclidean metric on the tangent space. Indeed the metric $p^2 + q^2$ means we can identify the projective line P^1 with its dual P^{1*} . Elements of the dual of course also determine directions at the origin and we have the following result.

PROPOSITION 8.4. If L is any field then the form $ap^2 + bpq + cq^2$ determines 0, 1 or 2 directions in the plane. The dual directions are 'orthogonal' to these and determined by $cp^2 - 2bpq + aq^2$. We refer to this as the dual equation. Replacing a, b by \bar{a}, \bar{b} in cases I or II above we obtain the corresponding dual BDE.

9. ANGLES, MAPS, INVOLUTIONS AND FURTHER FAMILIES

There are a variety of other possible families (pencils) one can consider.

EXAMPLE 9.1. As pointed out above it is natural to consider the pencil containing the characteristic and asymptotic BDE's; we now know that the principal directions will be the associated polar BDE. It is less clear here how to choose the right family which has a nice geometric interpretation. Again this highlights the fact that the pencils we naturally consider are over k ; since we can multiply our base CBDE's by any element of K_s there are a number of possible k -pencils corresponding to a given K_s -pencil. One way that we can obtain a family is as follows. Consider a direction v and its conjugate \bar{v} . If these make angles θ and ϕ with one of the principal directions then we can consider those v with $\tan(\theta + \phi)/\tan(\theta - \phi)$ constant, say $\tan(\alpha)$. It is not hard to see that this is given (in a principal co-ordinate system) by

$$F_\alpha = (\kappa_1 - \kappa_2)(\kappa_2 p^2 - \kappa_1 q^2) \sin \alpha + (\kappa_1 + \kappa_2)(\kappa_2 p^2 + \kappa_1 q^2) \cos \alpha = 0$$

This pencil has the following properties:

1. It interpolates between the asymptotic and principal BDE's, and all are CBDE's.
2. The discriminants D_α satisfy $D_\alpha = D_{-\alpha}$ (this follows because the pair of forms are apolar).
3. If a tangent direction v at a point satisfies $F_\alpha = 0$, so does \bar{v} . Moreover $Rv, R\bar{v}$ are the two directions at the same point determined by $F_{-\alpha}$.
4. The discriminants are given by $\kappa_1/\kappa_2 = \text{constant}$, and on the D_α the repeated direction is principal.

Note, to illustrate the point about the choices of k -pencil available, that we could determine another suitable family by asking that the quotient $\sin 2\phi/\sin 2\theta$ is constant. This again joins the principal and characteristic BDE's (or alternatively the BDE's $p^2 + q^2$ and $\kappa_2^2 p^2 + \kappa_1^2 q^2$).

Of course the most obvious family to choose is $(\kappa_2 p^2 + \kappa_1 q^2) \cos \alpha + (\kappa_2 p^2 - \kappa_1 q^2) \sin \alpha$. This k -pencil has the obvious disadvantage that it actually meets the singular conic (when $\tan \alpha = \pm 1$). So for these two values of α we obtain a line field (naturally the principal directions), and for these values of α the discriminant D_α is the whole plane.

We now describe a general class of examples. We first recall that an invertible linear map $(p : q) \mapsto (ap + bq : cp + dq)$ yields an automorphism of the projective line, and the fixed points (or double points) of the map are given by $cp^2 + (d - a)pq - bq^2 = 0$. The map is an involution if and only if $a + d = 0$, and its double points are distinct. The involution is said to be hyperbolic if the fixed points are in L and elliptic otherwise.

EXAMPLE 9.2. Consider an element Γ of $PGL(2, K)$, that is an invertible 2×2 matrix with entries in K up to projective equivalence which we can view as acting on the set of tangent directions at points of the surface. (It actually acts on the tangent directions in the κ -space.) With the usual conventions on entries we need then to consider the map of directions $p/q \mapsto (ap + bq)/(cp + dq) = P/Q$, which we write as $\Gamma(p, q)$. We can take the angle between the lines with slopes $(p : q)$ and $(P : Q)$ and ask that this is a given constant. As this varies we obtain the pencil joining $F = ap^2 + (b + c)pq + dq^2$ and $G = cp^2 + (d - a)pq - bq^2$. The first of these gives the directions for which $(p, q), \Gamma(p, q)$ are orthogonal, the second, as noted above, the fixed directions of the map. Note that F

and G are apolar (so that the discriminants of $F_{\alpha,\beta} = \alpha F + \beta G$ and $F_{\alpha,-\beta}$ coincide) if and only if $b = c$ or $a + d = 0$. In the latter case the element of $PGL(2, K)$ is an involution. In either case we get a self-polar triple in the usual way. Interestingly if F and G are distinct then $b = c$ if and only if $p^2 + q^2$ lies on the line joining F and the polar BDE, and $a + d = 0$ if and only if it lies on the line joining G and the polar BDE. Finally note that F and G are not arbitrary; indeed with the usual conventions they are precisely those pairs of forms conjugate with respect to the conic $(a - c)^2 + b^2 = 0$.

As noted above we are largely interested in the case of CBDE's.

PROPOSITION 9.1. 1. We get CBDE's above if and only if the matrix Γ is of the form $(a, b, -\bar{b}, \bar{a})$, with both CBDE's of type I, and $(a, b, \bar{b}, -\bar{a})$ which makes them both of type II. We refer to these matrices as type I and II respectively. Again multiplication by τ interchanges type, and we may restrict to matrices of type I from now on.

2. Any CBDE of type I can be obtained from a matrix of type I, and this then determines a second CBDE of type I. If f and F_1 and f and F_2 both come from a matrix of type I then F_1 differs from F_2 by a K_s -multiple of $p^2 + q^2$.

3. A matrix of either form gives the same pencil as that obtained by replacing (a, b) by $(\bar{b}, -\bar{a})$.

4. Both sets of matrices (dropping the invertibility hypothesis) of the given form are k -convex sets. Indeed if $\alpha, \beta \in K_s$, and Γ_1, Γ_2 are both of type I (or II) then so is $\alpha\Gamma_1 + \beta\Gamma_2$.

5. The matrices of type I form a subgroup of $PGL(2, K)$. When $k = \mathbb{R}$ the set of matrices in the form I but of determinant zero just consists of the zero matrix.

6. The discriminants of the two BDE's of type I constructed above over $k = \mathbb{R}$ are given by $a\bar{a} = 0$, $b = \bar{b}$ and $b\bar{b} = 0$, $a = \bar{a}$.

7. The BDE's F and G are apolar (so that the discriminants of $F_{\alpha,\beta} = \alpha F + \beta G$ and $F_{-\alpha,\beta}$ are coincident) if for matrices of type I $a + \bar{a} = 0$ or $b + \bar{b} = 0$.

Proof: Generally these are straightforward calculations. We do, however, prove part (5), that when $k = \mathbb{R}$ if $a\bar{a} + b\bar{b} = 0$ then $a = b = 0$. To see this first clear denominators so that we may suppose that a, b lie in $\mathbb{R}[\kappa_1, \kappa_2]$. Write $a = \tau^m a_1$ and $b = \tau^n b_1$, with $\tau = (\kappa_1 - \kappa_2)$ not dividing a_1 or b_1 and suppose without loss of generality that $m \geq n$. Then $a\bar{a} + b\bar{b} = 0$ implies that $(-1)^{(m-n)} \tau^{2(m-n)} a_1 \bar{a}_1 + b_1 \bar{b}_1 = 0$. Since $\mathbb{R}[\kappa_1, \kappa_2]$ is a unique factorisation domain $m = n$. Now evaluate at (t, t) to obtain $(a_1^2 + b_1^2)(t, t) = 0$. Clearly we deduce that $a_1(t, t) = b_1(t, t) = 0$, and given our hypotheses $a_1 = b_1 = 0$. This is used in part (6) where each of the discriminants is an expression of the form $A\bar{A} + B\bar{B}$. \square

EXAMPLE 9.3. We have already considered several cases of this type.

1. The map $p \mapsto -\kappa_1/\kappa_2 p$ yields the first of our families.
2. The map $p \mapsto \kappa_1/\kappa_2 p$ yields the second.
3. There is no choice of Γ that will yield a family joining (any multiple of) the asymptotic BDE to (any multiple of) the characteristic BDE. (Indeed two BDE's of the form $ap^2 + cq^2$ arise as F and G above if and only if one of them is $p^2 + q^2$.)

4. Consider the map $p \mapsto -p$. This yields the pair of forms $p^2 - q^2$ and pq . The associated family is simply that obtained from the principal BDE's by rotating through a fixed angle. The polar BDE to this pencil is $p^2 + q^2$ (which emphasises that the only zeros of this family are the umbilics). This suggests we consider the pencils joining pq to $p^2 + q^2$ and $p^2 - q^2$ to $p^2 + q^2$. It is easy to see that neither can arise from this construction. In fact it is clear that in the first case we are considering those BDE's with directions inclined at angles θ and $\pi/2 - \theta$ to either of the principle directions, θ parameterising the family, and in the second case the angles are θ and $-\theta$, again with θ parameterising the family.
5. Consider the map $p \mapsto 1/p$. This actually gives the same family as $p \mapsto -p$, illustrating part (3) of the above Proposition.

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