

A Massera Type Criterion for a Partial Neutral Functional Differential Equation.

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We prove the existence of a periodic solution for a partial neutral functional differential equation with delay, using a Massera type criterion. October, 2002
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1. INTRODUCTION

By using a Massera type criterion, we prove the existence of a periodic solution for a class of partial neutral functional differential equation modeled in the form

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \quad t > 0, \quad (1)$$

$$x_0 = \varphi \in \mathcal{D}, \quad (2)$$

where A is the infinitesimal generator of a compact analytic semigroup of linear operators, $(T(t))_{t \geq 0}$, on a Banach space X ; the history x_t , $x_t(\theta) = x(t + \theta)$, belongs to an appropriate phase space \mathcal{D} and $G, F : \mathbb{R} \times \mathcal{D} \rightarrow X$ are continuous functions.

The Massera Theorem [10] is a pioneer work in the study of the relations between the boundedness of solutions and the existence of periodic solutions. In Massera [10], this relation was explained for a two dimensional periodic ordinary differential equation. Subsequently, several authors considered similar relations, see for example Yoshizawa for a n -dimensional differential equation; Lopes and Hale for n -dimensional ordinary and functional equations with delay and Yong [14], for functional differential equations.

Recently Ezzinbi [1], using a Massera type criterion, showed the existence of a periodic solution for a partial functional differential equation modeled in the form

$$\dot{x}(t) = Ax(t) + F(t, x_t),$$
$$x_0 = \varphi \in \mathcal{C} = C([-r, 0] : X),$$

where A is the infinitesimal generator of a compact semigroup of bounded linear operators on a Banach space.

Our purpose in this paper is to establish similar existence results at those in Ezzinby [1], now for the case of partial neutral functional differential equation modelled as (1)-(2).

Neutral differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature of neutral functional differential equations with finite delay is the Hale book [6] and the references therein. The work in partial neutral functional differential equations with infinite delay was initiated by Hernández & Henríquez in [8, 7]. In these papers, Hernández & Henríquez proved the existence of mild, strong and periodic solutions for a neutral equation modeled in the form

$$\begin{cases} \frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \\ x_0 = \varphi \in \mathcal{B}. \end{cases}$$

where A is the infinitesimal generator of an analytic semigroup of linear operators on a Banach space and \mathcal{B} is a phase space defined axiomatically. In general, the results were obtained using the semigroup theory and the Sadovskii fixed point Theorem.

Throughout this paper, X will be a Banach space provided with norm $\|\cdot\|$ and $A : D(A) \rightarrow X$ will be the infinitesimal generator of a compact analytic semigroup, $(T(t))_{t \geq 0}$, of linear operators on X . For the

theory of C_0 semigroup, refer to Pazy [11]. We will point out here some notations and properties that will be used in this work. It is well known that there exist $\tilde{M} \geq 1$ and $\rho \in \mathbb{R}$ such that $\|T(t)\| \leq \tilde{M}e^{\rho t}$, for every $t \geq 0$. If $(T(t))_{t \geq 0}$ is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power

$(-A)^\alpha$, for $\alpha \in (0, 1]$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X and the expression $\|x\|_\alpha = \|(-A)^\alpha x\|$, $x \in D(-A)^\alpha$, defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$, then the following properties are well known:

LEMMA 1.1. *If the previous conditions hold:*

1. Let $0 < \alpha \leq 1$. Then X_α is a Banach space.
2. If $0 < \beta < \alpha \leq 1$ then $X_\alpha \hookrightarrow X_\beta$ and the imbedding is compact whenever the resolvent operator of A is compact.
3. For every $0 < \alpha \leq 1$ there exists $C_\alpha > 0$ such that

$$\|(-A)^\alpha T(t)\| \leq \frac{C_\alpha}{t^\alpha}, \quad t > 0.$$

In what follows, we suppose that $0 \in \rho(A)$ and that for $0 < \vartheta \leq 1$

$$\|T(t)\| \leq \tilde{M}, \quad t \geq 0, \quad \text{and} \quad \|(-A)^\vartheta T(t)\| \leq \frac{C_\vartheta}{t^\vartheta}, \quad t > 0, \quad (3)$$

for some positive constant C_ϑ .

In this paper, $0 < \beta \leq 1$ and $\omega > 0$ are fixed numbers, $G, F : \mathbb{R} \times \mathcal{D} \rightarrow X$ are continuous and we use the notation \mathbf{H}_i for the following conditions

H₁ The function G is X_β -valued and $(-A)^\beta G$ is continuous.

H₂ $G(t, \psi) = V(t, \psi) + h(t)$ where V, h are X_β -valued; $(-A)^\beta V, (-A)^\beta h$ are continuous; $(-A)^\beta V(\cdot, \psi), (-A)^\beta h$ are ω -periodic and $(-A)^\beta V(t, \cdot)$ is linear.

H₃ $G(t, \psi) = V(t, \psi) + G_1(t, \psi)$ where the functions V, G_1 are X_β -valued; $(-A)^\beta V, (-A)^\beta G_1$ are continuous; $(-A)^\beta V(\cdot, \psi), (-A)^\beta G_1(\cdot, \psi)$ are ω -periodic and $(-A)^\beta V(t, \cdot)$ is linear.

H₄ $F(t, \psi) = L(t, \psi) + f(t)$ where L, f are continuous; $L(\cdot, \psi), f$ are ω -periodic and $L(t, \cdot)$ is linear.

H₅ $F(t, \psi) = L(t, \psi) + F_1(t, \psi)$ where L, F_1 are continuous; $L(\cdot, \psi), F_1(\cdot, \psi)$ are ω -periodic and $L(t, \cdot)$ is linear.

H₆ For every $R > 0$ and all $T > 0$, the set of functions

$$\{s \rightarrow G(s, x_s) : x \in C([-r, T] : X), \sup_{\theta \in [-r, T]} \|x(\theta)\| \leq R\}$$

is equicontinuous on $[0, T]$.

H₇ For every $R > 0$ and all $T > 0$, the set of functions

$$\{s \rightarrow G(s, x_s) : x \in C_b((-\infty, T] : X), \sup_{\theta \in [-\infty, T]} \|x(\theta)\| \leq R\}$$

is equicontinuous on $[0, T]$.

The paper has four sections. In section 2, we discuss the existence of a periodic solution for a partial functional neutral differential equation defined on $\mathbb{R} \times C([-r, 0] : X)$. In section 3, by employing the results in section 2, we consider the existence of a periodic solution for a neutral equation with unbounded delay modeled on $\mathbb{R} \times \mathcal{B}$, where \mathcal{B} is a phase space defined axiomatically as in Hale and Kato [3]. The section 4 is reserved for examples. Our results are based on the properties of analytic semigroups and the ideas and techniques in Hernández & Henríquez [8, 7] and Ezzinbi [1].

Throughout this paper, $x(\cdot, \varphi)$ denotes a solution of (1)- (2). In addition, $B_r(x : Z)$, $(B_r[x : Z])$ will be the open (the closed) ball in a metric space Z with center at x and radius r . For a bounded function $\xi : [a, b] \rightarrow [0, \infty)$ and $a \leq t \leq b$ we will employ the notation $\xi_{a,t}$ for $\xi_{a,t} = \sup\{\xi(s) : s \in [a, t]\}$. If \mathcal{D} is a Banach phase space, the norm in \mathcal{D} will be denoted by $\|\cdot\|_{\mathcal{D}}$. Finally, we remark that for the proofs of our results we will use the following results.

THEOREM 1.1. [3] *Let Y be a Banach space and $\Gamma := \Gamma_1 + y$ where $\Gamma_1 : Y \rightarrow Y$ is a bounded linear operator and $y \in Y$. If there exist $x_0 \in Y$ such that the set $\{\Gamma^n(x_0) : n \in \mathbb{N}\}$ is relatively compact in Y , then Γ has a fixed point in Y .*

THEOREM 1.2. [2] *Let X be a Banach space and M be a nonempty convex subset of X . If $\Gamma : M \rightarrow 2^X$ is a multivalued map such that*

- (i) For every $x \in M$, the set $\Gamma(x)$ is nonempty, convex and closed,
- (ii) The set $\Gamma(M) = \bigcup_{x \in M} \Gamma x$ is relatively compact,
- (iii) Γ is upper semi-continuous,

then Γ has a fixed point in M .

2. A PERIODIC SOLUTION FOR A PARTIAL NEUTRAL DIFFERENTIAL EQUATION WITH BOUNDED DELAY

In this section, we prove the existence of a periodic solution of the initial value problem

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \quad (1)$$

$$x_0 = \varphi \in \mathcal{C} = C([-r, 0] : X). \quad (2)$$

DEFINITION 2.1. A function $x : [-r, T] \rightarrow X$ is a mild solution of the abstract Cauchy problem (1)-(2) if: $x_0 = \varphi$; the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous; for each $0 \leq t < T$ the function $AT(t-s)G(s, x_s)$, $s \in [0, t]$, is integrable and

$$\begin{aligned} x(t) = & T(t)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t AT(t-s)G(s, x_s)ds \\ & + \int_0^t T(t-s)F(s, x_s)ds, \quad t \in [0, T]. \end{aligned} \quad (3)$$

The existence of mild solutions for the abstract Cauchy problem (1)-(2) follows from Theorems 2.1 and 2.2 in [7]; for this reason, we choose to omit the proof of the next two results.

THEOREM 2.1. Let $\varphi \in \mathcal{C}$, $T > 0$ and assume that the following conditions hold:

- (a) There exist constants $\beta \in (0, 1)$ and $L \geq 0$

such that the function G is X_β -valued,

$L \| (-A)^{-\beta} \| < 1$ and

$$\| (-A)^\beta G(t, \psi_1) - (-A)^\beta G(s, \psi_2) \| \leq L(|t-s| + \| \psi_1 - \psi_2 \|_c), \quad (4)$$

for every $0 \leq s, t \leq T$ and $\psi_1, \psi_2 \in \mathcal{C}$.

- (b) The function F is continuous and takes bounded sets into the bounded sets.

Then there exists a mild solution $x(\cdot, \varphi)$ of the abstract Cauchy problem (1)-(2) defined on $[-r, b]$, for some $0 < b \leq T$.

THEOREM 2.2. Let $\varphi \in \mathcal{C}$ and $T > 0$. Assume that condition **(a)** of the previous Theorem holds and that there exists $N > 0$ such that

$$\| F(t, \varphi) - F(t, \psi) \| \leq N \| \varphi - \psi \|_{\mathcal{C}}, \tag{5}$$

for all $0 \leq t \leq T$ and every $\varphi, \psi \in \mathcal{C}$. Then there exists a unique mild solution $x(\cdot, \varphi)$ of (1)-(2) defined on $[-r, b]$ for some $0 < b \leq T$. Moreover, b can be chosen as $\min\{T, b_0\}$, where b_0 is a positive constant independent of φ .

To prove the main result of this section, it is fundamental the next result.

THEOREM 2.3. Let $T > r$ and assume that assumption **H₁**, **H₆** hold. Suppose, furthermore, that the following conditions hold.

(a) For every $\varphi \in \mathcal{C}$ the set $X(\varphi) = \{x \in C([-r, T] : X) : x \text{ is solution of (1)-(2)}\}$ is nonempty.

(b) For every $R > 0$, the set

$$\{(-A)^\beta G(s, x_s), F(s, x_s) : s \in [0, T], x \in X(\varphi) \text{ and } \|\varphi\|_{\mathcal{C}} \leq R\},$$

is bounded.

Then the multivalued map $\Upsilon : \mathcal{C} \rightarrow 2^{\mathcal{C}}; \varphi \rightarrow X_T(\varphi) = \{x_T : x \in X(\varphi)\}$ is compact, that is, for every $R > 0$ the set $\mathcal{U}_{R,T} = \bigcup_{\|\varphi\|_{\mathcal{C}} \leq R} X_T(\varphi)$ is relatively compact in \mathcal{C} .

Proof: Let $R > 0$ and $\mathcal{U}_R = \bigcup_{\|\varphi\|_{\mathcal{C}} \leq R} X(\varphi)$. From **(b)**, we fix $N > 0$ such that $\|(-A)^\beta G(s, x_s)\| \leq N$ and $\|F(s, x_s)\| \leq N$ for every $x \in \mathcal{U}_R$ and every $s \in [0, T]$. In order to use the Ascoli Theorem, we divide the proof in two steps.

Step 1 The set $\mathcal{U}_R(t) = \{x(t) : x \in \mathcal{U}_R\}$ is relatively compact for $t \in (0, T]$.

Let $0 < \epsilon < t \leq T$. Since $(T(t))_{t \geq 0}$ is analytic, the operator function $s \rightarrow AT(s)$ is continuous in the uniform operator topology on $(0, T]$, which by the estimate

$$\|(-A)^{1-\beta} T(t-s)(-A)^\beta G(s, x_s)\| \leq \frac{C_{1-\beta} N}{(t-s)^{1-\beta}}, \quad s \in [0, t], x \in \mathcal{U}_R \tag{6}$$

implies that the function $s \rightarrow \| AT(t-s)G(s, x_s) \|$ is integrable on $[0, t]$ for every $x \in \mathcal{U}_R$. Under the previous conditions, for $x \in \mathcal{U}_R$ we get

$$\begin{aligned}
x(t) &= T(\epsilon)T(t-\epsilon)(\varphi(0) + G(0, \varphi)) + (-A)^{-\beta}(-A)^\beta G(t, x_t) \\
&\quad + T(\epsilon) \int_0^{t-\epsilon} (-A)^{1-\beta} T(t-s-\epsilon) (-A)^\beta G(s, x_s) ds \\
&\quad + \int_{t-\epsilon}^t (-A)^{1-\beta} T(t-s) (-A)^\beta G(s, x_s) ds \\
&\quad + T(\epsilon) \int_0^{t-\epsilon} T(t-s-\epsilon) F(s, x_s) ds + \int_{t-\epsilon}^t T(t-s) F(s, x_s) ds, \\
x(t) &\in T(\epsilon) \tilde{M} B_{R+N}[0, X] + (-A)^{-\beta} B_N[0, X] + W_t^1 + C_\epsilon^1 + W_t^2 + C_\epsilon^2,
\end{aligned}$$

where each W_t^i is compact, $\text{diam}(C_\epsilon^1) \leq 2C_{1-\beta} N \frac{\epsilon^\beta}{\beta}$ and $\text{diam}(C_\epsilon^2) \leq 2\tilde{M}N\epsilon$. Since $(-A)^{-\beta}$ is compact, these remarks imply that $\mathcal{U}_R(t)$ is relatively compact in X .

Step 2 \mathcal{U}_R is equicontinuous on $(0, T]$.

Let $0 < \epsilon < t_0 < t \leq T$. The strong continuity of $(T(t))_{t \geq 0}$ implies that the set of functions $\{s \rightarrow T(s)x : x \in T(\epsilon)B_{R+N}(0, X)\}$ is equicontinuous on $[0, T]$. Let $0 < \delta < \epsilon$ be such that

$$\begin{aligned}
\| T(s)x - T(s')x \| &< \epsilon, \quad x \in T(\epsilon)B_{R+N}(0, X), \\
\| G(t, u_t) - G(t_0, u_{t_0}) \| &< \epsilon, \quad u \in C([-r, T] : X), \quad \| u \|_{-r, T} \leq R + N,
\end{aligned}$$

when $|s - s'| < \delta$, $0 \leq s, s' \leq T$ and $0 \leq t - t_0 < \delta$.

Under the previous conditions, for $x \in \mathcal{U}_R$ and $0 \leq t - t_0 < \delta$ we get

$$\begin{aligned} & \| x(t_0) - x(t) \| \\ & \leq \| (T(t_0 - \epsilon) - T(t - \epsilon))T(\epsilon)(\varphi(0) + G(0, \varphi)) \| \\ & \quad + \| G(t_0, x_{t_0}) - G(t, x_t) \| \\ & \quad + \int_0^{t_0 - \epsilon} \| (-A)^{1-\beta}T(t_0 - s - \epsilon)(I - T(t - t_0))T(\epsilon)(-A)^\beta G(s, x_s) \| ds \\ & \quad + \int_{t_0 - \epsilon}^{t_0} \| I - T(t - t_0) \| \| (-A)^{1-\beta}T(t_0 - s)(-A)^\beta G(s, x_s) \| ds \\ & \quad + \int_{t_0}^t \| (-A)^{1-\beta}T(t - s)(-A)^\beta G(s, x_s) \| ds \\ & \quad + \int_0^{t_0 - \epsilon} \| T(t_0 - s - \epsilon)(I - T(t - t_0))T(\epsilon)F(s, x_s) \| ds \\ & \quad + \int_{t_0 - \epsilon}^{t_0} \| (T(t_0 - s) - T(t - s))F(s, x_s) \| ds + \int_{t_0}^t \| T(t - s)F(s, x_s) \| ds \\ & \leq 2\epsilon + \epsilon C_{1-\beta} \frac{(t_0 - \epsilon)^\beta}{\beta} + 2\tilde{M}C_{1-\beta}N \frac{\epsilon^\beta}{\beta} + NC_{1-\beta} \frac{(t - t_0)^\beta}{\beta} + \epsilon\tilde{M}(t_0 - \epsilon) \\ & \quad + 2\tilde{M}N\epsilon + \tilde{M}N(t - t_0), \end{aligned}$$

and hence

$$\| x(t_0) - x(t) \| \leq c_1\epsilon + c_2\epsilon^\beta + c_3\delta^\beta + c_4\delta,$$

where the constants c_i are independent of $x(\cdot)$. Thus, \mathcal{U}_R is equicontinuous from the right side at $t_0 > 0$. The equicontinuity in $t_0 > 0$ is proved in similar form, we omit details. Thus, \mathcal{U}_R is equicontinuous on $(0, T]$.

From the steps 1 and 2, it follow that $\{x|_{[\mu, T]} : x \in \mathcal{U}_R\}$ is relatively compact in $C([\mu, T] : X)$ for every $\mu > 0$, which in turn implies that $\mathcal{U}_{R,T}$ is relatively compact in $C([-r, 0] : X)$. This completes the proof.

COROLLARY 2.1. *Assume that the hypothesis in Theorems 2.2 and 2.3 are fulfilled. Then the map $\varphi \rightarrow x_T(\cdot, \varphi)$ is a completely continuous function.*

Proof: The assertion follows from (6), the Lebesgue dominated convergence Theorem and Theorem 2.3. We omit details.

DEFINITION 2.2. A function $x : \mathbb{R} \rightarrow X$ is an ω -periodic solution of equation (1) if: $x(\cdot)$ is a mild solution of (1) and $x(t + \omega) = x(t)$ for every $t \in \mathbb{R}$.

Using the ideas and techniques in [8], it is possible to establish sufficient conditions for the existence of global solutions of (1). In what follows, we always assume that the mild solutions are defined on $[0, \infty)$.

THEOREM 2.4. *Let conditions \mathbf{H}_2 , \mathbf{H}_4 and \mathbf{H}_6 be satisfied. If the equation (1) has a bounded mild solution, then there exists an ω -periodic solution of (1).*

Proof: For a mild solution $x(\cdot) = x(\cdot, \varphi)$, we introduce the decomposition $x(\cdot) = v(\cdot) + z(\cdot)$ where $v(\cdot)$ is the mild solution of

$$\begin{aligned} \frac{d}{dt}(u(t) + V(t, u_t)) &= Au(t) + L(t, u_t), \\ u_0 &= \varphi, \end{aligned}$$

and $z(\cdot)$ is the mild solution of

$$\begin{aligned} \frac{d}{dt}(u(t) + V(t, u_t) + h(t)) &= Au(t) + L(t, u_t) + f(t), \\ u_0 &= 0. \end{aligned}$$

Let $y : [-r, \infty) \rightarrow X$ be a bounded mild solution of (1) and $\Gamma : \mathcal{C} \rightarrow \mathcal{C}$ be the map $\Gamma(\varphi) := \Gamma_1(\varphi) + z_\omega := v_\omega + z_\omega$. Since Γ_1 is a bounded linear operator and $\bigcup_{n \geq 1} \Gamma^n(y_0) = \{y_{n\omega} : n \in \mathbb{N}\}$ is relatively compact in \mathcal{C} , see Theorem 2.3, it follows from Theorem 1.1 that Γ has a fixed point in \mathcal{C} . This fixed point give a periodic solution. The proof is complete.

Remark 2. 1. In what follow CP is the space $CP = \{u : \mathbb{R} \rightarrow X : u \text{ is } \omega\text{-periodic}\}$ endowed with the uniform convergence topology.

Now we prove the main result of this work.

THEOREM 2.5. *Let conditions \mathbf{H}_3 , \mathbf{H}_5 and \mathbf{H}_6 be satisfied and assume that the following conditions are fulfilled.*

(a) *The functions $(-A)^\beta V$, $(-A)^\beta G_1$, L and F_1 takes bounded sets into the bounded sets.*

(b) *There is $\rho > 0$ such that for every $v \in B_\rho[0, CP]$ the neutral*

equation

$$\frac{d}{dt}(x(t) + V(t, x_t) + G_1(t, v_t)) = Ax(t) + L(t, x_t) + F_1(t, v_t),$$

has an ω -periodic solution $u(\cdot, v) \in B_\rho[0, CP]$.

Then the equation (1) has an ω -periodic solution.

Proof: On $B_\rho = B_\rho[0, CP]$, we define the multivalued map $\Gamma : B_\rho \rightarrow 2^{B_\rho}$ by: $x \in \Gamma(v)$ if, and only if,

$$\begin{aligned} x(t) = & T(t-s)(x(s) + V(s, x_s) + G_1(s, v_s)) - V(t, x_t) - G_1(t, v_t) \\ & - \int_s^t AT(t-\tau)(V(\tau, x_\tau) + G_1(\tau, v_\tau))d\tau \\ & + \int_s^t T(t-\tau)(L(\tau, x_\tau) + F_1(\tau, v_\tau))d\tau, \quad t > s. \end{aligned}$$

Next we prove that Γ verifies the conditions (i)-(iii) of Theorem 1.2. Clearly assumption (i) holds. The condition (ii) follows using the steps in the proof of Theorem 2.3. In relation to (iii), we observe that from (ii) is sufficient to show that Γ is closed. If $(v^n)_{n \in \mathbb{N}}$ and $(x^n)_{n \in \mathbb{N}}$ are convergent sequences in CP to points v, x then

$$\begin{aligned} (-A)^\beta(V(\tau, x_\tau^n) + G_1(\tau, v_\tau^n)) & \rightarrow (-A)^\beta(V(\tau, x_\tau) + G_1(\tau, v_\tau)), \\ T(t)(L(\tau, x_\tau^n) + F_1(\tau, v_\tau^n)) & \rightarrow T(t)(L(\tau, x_\tau) + F_1(\tau, v_\tau)), \end{aligned}$$

for $\tau \in \mathbb{R}$ and $t \geq s$. From the Lebesgue dominated convergence Theorem, assumption (a) and the estimate

$$\| AT(t-s)(V(\tau, x_\tau^n) + G_1(\tau, v_\tau^n)) \| \leq C_{1-\beta} \frac{\| (-A)^\beta(V(\tau, x_\tau^n) + G_1(\tau, v_\tau^n)) \|}{(t-s)^{1-\beta}},$$

we conclude that

$$\begin{aligned} x(t) = & T(t-s)(x(s) + V(s, x_s) + G_1(s, v_s)) - V(t, x_t) - G_1(t, v_t) \\ & - \int_s^t AT(t-\tau)(V(\tau, x_\tau) + G_1(\tau, v_\tau))d\tau \\ & + \int_s^t T(t-\tau)(L(\tau, x_\tau) + F_1(\tau, v_\tau))d\tau, \quad t > s, \end{aligned}$$

which proves that $x \in \Gamma v$. Thus, Γ is closed and consequently upper semi-continuous.

From Theorem 1.1 the operator Γ has a fixed point. This fixed point is an ω -periodic solution of (1). The proof is finished.

2.1. A periodic solution for a partial neutral differential equation with unbounded delay

In this section, we discuss the existence of an ω -periodic solution for a partial functional neutral differential equation with unbounded delay modeled in the form

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + F(t, x_t), \tag{7}$$

$$x_\sigma = \varphi \in \mathcal{B}, \tag{8}$$

where the history $x_t : (-\infty, 0] \rightarrow X$, $x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space \mathcal{B} defined axiomatically and $F, G : \mathbb{R} \times \mathcal{B} \rightarrow X$ are appropriate continuous functions.

In the rest of this paper, \mathcal{B} will be an abstract phase space defined axiomatically as in Hale and Kato [4]. To establish the axioms of the space \mathcal{B} , we follow the terminology used in Hino-Murakami-Naito [9], and thus, \mathcal{B} will be a linear space of functions mapping $(-\infty, 0]$ into X , endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$. We will assume that \mathcal{B} satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \rightarrow X$, $a > 0$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$, the following conditions hold:

- i) x_t is in \mathcal{B} .
- ii) $\|x(t)\| \leq H \|x_t\|_{\mathcal{B}}$.
- iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma) \|x_\sigma\|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \rightarrow [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), x_t is a \mathcal{B} -valued continuous function on $[\sigma, \sigma + a)$.

(B) The space \mathcal{B} is complete.

(C2) If a uniformly bounded sequence $(\varphi^n)_n$ in C_{00} converges to a function φ in the compact-open topology, then $\varphi \in \mathcal{B}$ and

$$\|\varphi^n - \varphi\|_{\mathcal{B}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

EXAMPLE 2.1.

We consider the phase space $\mathcal{B} := C_r \times L^p(g; X)$, $r \geq 0$, $1 \leq p < \infty$, see [9], which consists of all classes of functions $\varphi : (-\infty, 0] \rightarrow X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable and $g|\varphi(\cdot)|^p$ is Lebesgue integrable on $(-\infty, -r)$, where $g : (-\infty, -r) \rightarrow \mathbb{R}$ is a positive Lebesgue integrable function. The seminorm in $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\varphi\|_{\mathcal{B}} := \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} g(\theta) \|\varphi(\theta)\|^p d\theta \right)^{1/p}.$$

We will assume that g satisfies conditions (g-6) and (g-7) in the terminology of [9]. This means that g is integrable on $(-\infty, -r)$ and that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that

$$g(\xi + \theta) \leq \gamma(\xi) g(\theta),$$

for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. In this case, \mathcal{B} is a phase space which verifies axioms (A), (A1), (B) and (C2) (see [9], Theorem 1.3.8).

DEFINITION 2.3. A function $x : (-\infty, T] \rightarrow X$ is a mild solution of the abstract Cauchy problem (7)-(8) if: $x_0 = \varphi$; the restriction of $x(\cdot)$ to the interval $[0, T]$ is continuous; for each $0 \leq t < T$ the function $AT(t-s)G(s, x_s)$, $s \in [0, t)$, is integrable and

$$x(t) = T(t)(\varphi(0) + G(0, \varphi)) - G(t, x_t) - \int_0^t AT(t-s)G(s, x_s)ds + \int_0^t T(t-s)F(s, x_s)ds, \quad t \in [0, T]. \tag{9}$$

LEMMA 2.1. Let Assumption $\mathbf{H}_1, \mathbf{H}_7$ be satisfied. If $x : (-\infty, T] \rightarrow X$ is a bounded mild solution of (7), then the set $U = \{x_{n\omega} : n \in \mathbb{N}\}$ is relatively compact in \mathcal{B} .

Proof: Let $(x_{n_k\omega})_{k \in \mathbb{N}}$ be a sequence in U . For $n > 0$, we define the set $U(n, r) = \{x_{n_k\omega}|_{[-r, 0]} : n_k \geq n\}$. Using the same arguments in the proof of Theorem 2.3, it follows that $U(n, r)$ is relatively compact in $C([-r, 0]; X)$ when $n\omega > r$. Now we can choose a subsequence of $(x_{n_k\omega})_{k \in \mathbb{N}}$; which is indicated by the same index, that converges uniformly on compact subsets of $(-\infty, 0]$ to some function $x \in C_b((-\infty, 0] : X)$. Since axiom \mathbf{C}_2 hold, it follow that $x \in \mathcal{B}$ and that $x_{n_k\omega} \rightarrow x$ in \mathcal{B} . Thus, U is relatively compact in \mathcal{B} .

DEFINITION 2.4. A function $x : \mathbb{R} \rightarrow X$ is an ω -periodic solution of equation (7) if: $x(\cdot)$ is a mild solution of (7) and $x(t + \omega) = x(t)$ for every $t \in \mathbb{R}$.

The proofs of the following results are similar to the proofs of Theorems 2.4 and 2.5. We only remark that the continuity of $\varphi \rightarrow x_\omega(\cdot, \varphi)$ is discussed in [8].

THEOREM 2.6. Let conditions $\mathbf{H}_2, \mathbf{H}_4$ and \mathbf{H}_7 be satisfied. If the equation (7) has a bounded mild solution then there exists an ω -periodic solution of (7).

THEOREM 2.7. Let conditions $\mathbf{H}_3, \mathbf{H}_5$ and \mathbf{H}_7 be satisfied and assume that the following conditions are fulfilled.

- (a) The functions $(-A)^\beta V, (-A)^\beta G_1, L$ and F_1 takes bounded sets into the bounded sets.
- (b) There is $\rho > 0$ such that for every $v \in B_\rho[0, CP]$ the neutral equation

$$\frac{d}{dt}(x(t) + V(t, x_t) + G_1(t, v_t)) = Ax(t) + L(t, x_t) + F_1(t, v_t),$$

has an ω -periodic solution $u(\cdot, v) \in B_\rho[0, CP]$.

Then the neutral equation (7) has an ω -periodic solution.

3. APPLICATIONS

In this section, we illustrate some of the results in this work. Along this section, $X = L^2([0, \pi])$ and A is the operator $Ax = x''$ with domain

$$D(A) := \{f(\cdot) \in L^2([0, \pi]) : f''(\cdot) \in L^2([0, \pi]), f(0) = f(\pi) = 0\}.$$

It's well known that A is the infinitesimal generator of a C_0 semigroup, $(T(t))_{t \geq 0}$, on X , which is compact, analytic and self-adjoint. Moreover, A has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := \left(\frac{2}{\pi}\right)^{1/2} \sin(n\xi)$ and the following properties hold:

- (a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X .
- (b) If $f \in D(A)$ then $A(f) = -\sum_{n=1}^{\infty} n^2 \langle f, z_n \rangle z_n$.
- (c) For every $f \in X$, $(-A)^{-\frac{1}{2}} f = \sum_{n=1}^{\infty} \frac{1}{n} \langle f, z_n \rangle z_n$. In particular, $\|(-A)^{-1/2}\| = 1$.
- (d) The operator $(-A)^{\frac{1}{2}}$ is given by $(-A)^{\frac{1}{2}} f = \sum_{n=1}^{\infty} n \langle f, z_n \rangle z_n$ on the space

$$D((-A)^{\frac{1}{2}}) = \left\{ f \in X : \sum_{n=1}^{\infty} n \langle f, z_n \rangle z_n \in X \right\}.$$

(e) For every $f \in X$, $T(t)f = \sum_{n=1}^{\infty} e^{-n^2 t} \langle f, z_n \rangle z_n$. Moreover, it follows from this expression that

$$\|T(t)\| \leq e^{-t}, \quad t \geq 0, \quad \text{and that} \quad \|(-A)^{\frac{1}{2}} T(t)\| \leq \frac{1}{\sqrt{2}} e^{-\frac{t}{2}} t^{-\frac{1}{2}}, \quad \text{for } t > 0.$$

3.1. A neutral equation with bounded Delay

Considering the example in Ezzimby [1], in this section we study the neutral equation

$$\begin{aligned} \frac{d}{dt} [u(t, \xi) + \int_{-r}^0 \int_0^{\pi} a_0(t) b(s, \eta, \xi) u(t+s, \eta) d\eta ds] &= \frac{\partial^2}{\partial \xi^2} u(t, \xi) \\ &+ a_1(t) x(t-r, \xi) + a_2(t) p(t, x(t-r, \xi)) + q(t, \xi), \end{aligned} \quad (1)$$

$$u(t, 0) = u(t, \pi) = 0, \quad t \geq 0, \quad (2)$$

$$u(\tau, \xi) = \varphi(\tau, \xi), \quad \tau \in [-r, 0], \quad 0 \leq \xi \leq \pi, \quad (3)$$

where

(i) The function $b(\cdot)$ is measurable and

$$\sup_{t \in [-r, \infty)} \int_0^{\pi} \int_0^{\pi} b^2(t, \eta, \xi) d\eta d\xi < \infty.$$

(ii) The function $\frac{\partial}{\partial \zeta} b(\tau, \eta, \zeta)$ is measurable; $b(\tau, \eta, \pi) = 0$; $b(\tau, \eta, 0) = 0$ and $N_1 r < 1$ where

$$N_1 := \int_0^\pi \int_{-r}^0 \int_0^\pi \left(\frac{\partial}{\partial \zeta} b(\tau, \eta, \zeta) \right)^2 d\eta d\tau d\zeta.$$

(iii) The functions $p, q : \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous and ω -periodic in the first variable.

(iv) The substitution operators $g : \mathbb{R} \times X \rightarrow X$, $f : \mathbb{R} \rightarrow X$ defined by $g(t, x)(\xi) = p(t, x(\xi))$ and $f(t)(\xi) = q(t, \xi)$ are ω -periodic, continuous and there exists $k > 0$ such that

$$\|g(t, x)\| \leq k \|x\|, \quad (t, x) \in \mathbb{R} \times X.$$

(v) The functions $a_1, a_2 : \mathbb{R} \rightarrow \mathbb{R}$ are continuous, ω -periodic and there exists a constant l such that

$$-1 + |a_1(t)| + |a_2(t)| k \leq -l, \quad t \geq 0.$$

(vi) The function $a_0 : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, nondecreasing, ω -periodic and $0 \leq a_0(t) \leq (1 - e^{-t})$, $t \geq 0$.

On the space $\mathbb{R} \times \mathcal{C}$, we define the maps G, F, F_1 by

$$\begin{aligned} G(t, \psi)(\xi) &:= a_0(t)V(t, \psi), \\ V(t, \psi)(\xi) &:= \int_{-r}^0 \int_0^\pi b(s, \eta, \xi)\psi(s, \eta)d\eta ds \\ L(t, \psi)(\xi) &:= a_1(t)\psi(-r)(\xi), \\ F_1(t, \psi)(\xi) &:= a_2(t)g(t, \psi(-r))(\xi) + f(t)(\xi). \end{aligned}$$

With the previous notations, the initial boundary value problem (1)-(3) can be modeled as the abstract Cauchy problem

$$\frac{d}{dt}(x(t) + G(t, x_t)) = Ax(t) + L(t, x_t) + F_1(t, x_t), \quad t \geq 0. \tag{4}$$

$$x_0 = \varphi \in \mathcal{C} = C([-r, 0] : X). \tag{5}$$

A straightforward estimation using (i)-(iv) shows that the functions G, L and F_1 are continuous. Moreover, from (d) and (ii), it follows that G is a bounded linear operator with values in $X_{\frac{1}{2}}$ and that $\|(-A)^{1/2}G(t, \cdot)\| \leq a_0(t)(N_1 r)^{\frac{1}{2}}$ for every $t \in \mathbb{R}$.

Next we prove that G satisfies \mathbf{H}_6 . Considering the condition (\mathbf{vi}) , we only proof that V verifies \mathbf{H}_6 . Let $R > 0$ and $x \in C([-r, T]; X)$ such that $\|x\|_{-r, T} \leq R$. For $t > 0$ we get

$$\begin{aligned} & V(t+h, x_{t+h})(\xi) - V(t, x_t)(\xi) \\ &= \int_{-r+t+h}^{-r+t} \int_0^\pi b(\theta-t-h, \eta, \xi) x(\theta, \eta) d\eta d\theta \\ &+ \int_{-r+t}^t \int_0^\pi (b(\theta-t-h, \eta, \xi) - b(\theta-t, \eta, \xi)) x(\theta, \eta) d\eta d\theta \\ &+ \int_t^{t+h} \int_0^\pi b(\theta-t-h, \eta, \xi) x(\theta, \eta) d\eta d\theta \end{aligned}$$

and hence

$$\begin{aligned} \left\| \frac{d}{dt} V(t, x_t) \right\|_X^2 &\leq 4 \|x(-r+t)\|^2 \int_0^\pi \int_0^\pi b^2(-r, \eta, \xi)^2 d\eta d\xi \\ &+ 4 \|x\|_\infty^2 r \int_0^\pi \int_{-r+t}^t \int_0^\pi \left(\frac{\partial b}{\partial \theta}(\theta-t, \eta, \xi) \right)^2 d\eta d\theta d\xi \\ &+ 4 \|x\|_\infty^2 \int_0^\pi \int_0^\pi b^2(0, \eta, \xi) d\eta d\xi. \end{aligned}$$

Thus,

$$\left\| \frac{d}{dt} V(t, x_t) \right\|^2 \leq C\left(\frac{\partial b}{\partial \theta}, b\right), \quad (6)$$

where $C(\frac{\partial b}{\partial \theta}, b) > 0$ is independent of $t > 0$ and x with $\|x\|_{-r, T} \leq R$. This implies that the set

$$\mathcal{U} = \{s \rightarrow V(s, x_s) : x \in C([-r, T]; X), \sup_{\theta \in [-r, T]} \|x(\theta)\| \leq R\}$$

is equicontinuous from the right side at $t > 0$. The equicontinuity of \mathcal{U} on \mathbb{R} is proved in similar form. Thus, V verifies condition \mathbf{H}_6 .

PROPOSITION 3.1. *Assume that the previous conditions hold and that*

$$l > (N_1 r)^{\frac{1}{2}} \rho \left(1 + \frac{1}{\sqrt{2}} (2e^{\frac{-1}{2}} + 2)\right), \quad (7)$$

where $\rho = 1 + \frac{\|f\|_\infty}{l}$. Then there exists an ω -periodic solution of (1)-(3).

Proof: Let $v \in B_\rho(0, CP)$ and $\varphi \in B_\rho(0, \mathcal{C})$. From Theorem 2.1 and the condition $\|(-A)^{1/2} G(t, \cdot)\| \leq (N_1 r)^{\frac{1}{2}} < 1$, we know that there exist a local mild solution, $x(\cdot, \varphi)$, of

$$\frac{d}{dt}(y(t) + G(t, y_t)) = Ay(t) + L(t, y_t) + F_1(t, v_t), \quad t \geq 0, \quad (8)$$

$$y_0 = \varphi. \quad (9)$$

We affirm that $x(\cdot, \varphi)$ is bounded by ρ on $[0, a_\varphi)$, where $[0, a_\varphi)$ is the maximal interval of definition of $x(\cdot, \varphi)$. Assume that the affirmation is false and let $t_0 = \inf\{t > 0 : \|x(t)\| > \rho\}$. Clearly, $x(t_0) = \rho$. If $t_0 > 1$, by employing the estimates in Ezzinbi [1], pp. 227, we have that

$$\begin{aligned} \|x(t_0)\| &\leq \rho - l(1 - e^{-t_0}) + a_0(t_0) \|V(t_0, x_{t_0})\| + a_0(t_0) \int_0^{t_0} \|AT(t_0 - s)V(s, x_s)\| ds \\ &\leq \rho - l(1 - e^{-t_0}) + a_0(t_0)(N_1 r)^{\frac{1}{2}} \rho + a_0(t_0) \left(\frac{N_1 r}{2}\right)^{\frac{1}{2}} \int_0^{t_0-1} e^{-\frac{(t_0-s)}{2}} \|x_s\|_C ds \\ &\quad + a_0(t_0) \left(\frac{N_1 r}{2}\right)^{\frac{1}{2}} \int_{t_0-1}^{t_0} \frac{e^{-\frac{(t_0-s)}{2}}}{(t_0-s)^{\frac{1}{2}}} \|x_s\|_C ds \\ &\leq \rho - l(1 - e^{-t_0}) + a_0(t_0)(N_1 r)^{\frac{1}{2}} \rho + a_0(t_0) \left(\frac{N_1 r}{2}\right)^{\frac{1}{2}} \rho(2e^{-\frac{1}{2}} + 2), \end{aligned}$$

thus,

$$\|x(t_0)\| \leq \rho - \left(l - (N_1 r)^{\frac{1}{2}} \rho \left(1 + \frac{2}{\sqrt{2}} (e^{-\frac{1}{2}} + 1) \right) \right) (1 - e^{-t_0}). \tag{10}$$

Similarly, if $t_0 \in (0, 1]$

$$\|x(t_0)\| \leq \rho - \left(l - (N_1 r)^{\frac{1}{2}} \rho \left(1 + \frac{2}{\sqrt{2}} \right) \right) (1 - e^{-t_0}). \tag{11}$$

From (7), (10) and (11), it follows that $\|x(t_0)\| < \rho$, which is an absurd.

Now we prove that $a_\varphi = \infty$. Assume that $a_\varphi < \infty$ and let N_2 be the number $N_2 = \rho(|a_1|_\infty + |a_2|_\infty k) + \|f\|_\infty$. For $\epsilon > 0$, we fix $0 < \delta < \frac{a_\varphi}{2}$ such that

$$\|T(s)x - T(s')x\| < \epsilon, \quad x \in T(\epsilon)B_{2\rho+N_2}[0, X],$$

when $|s - s'| < \delta$ and $s, s' \in [0, a_\varphi]$. Let $\frac{a_\varphi}{2} < t < t_0 < a_\varphi$. Using that $a_0(t) \leq 1$ and the estimate in step 2 of the proof of Theorem 2.3, we have

$$\begin{aligned} \|x(t_0) - x(t)\| &\leq \epsilon + \|G(t_0, x_{t_0}) - G(t, x_t)\| + 2\epsilon \frac{1}{\sqrt{2}} (t_0 - \epsilon)^{\frac{1}{2}} \\ &\quad + 4\left(\frac{N_1 r}{\sqrt{2}}\right)^{\frac{1}{2}} \rho \epsilon^{\frac{1}{2}} + 2\left(\frac{N_1 r}{\sqrt{2}}\right)^{\frac{1}{2}} \rho (t - t_0)^{\frac{1}{2}} + \epsilon(t_0 - \epsilon) \\ &\quad + 2N_2 \epsilon + N_2(t - t_0) \end{aligned}$$

which from (6), permit to conclude that $x(\cdot)$ is uniformly continuous on $[\frac{a_\varphi}{2}, a_\varphi)$. Let $\tilde{x} : [-r, a_\varphi] \rightarrow X$ be the unique continuous extension of $x(\cdot)$. From Theorem 2.2,

there exists a mild solution, $y(\cdot)$, of (4) with initial condition $y_{a_\varphi} = \tilde{x}_{a_\varphi}$. This solution give an extension of $x(\cdot, \varphi)$, which is a contradiction. Thus $x(\cdot, \varphi)$ is defined on \mathbb{R} .

From Theorem 2.4, we infer that for every $v \in B_\rho[0, CP]$ there exists an ω -periodic, $u(\cdot, v)$, of (8) and that $u(\cdot, v) \in B_\rho[0, CP]$. Finally, the existence of an ω -periodic solution of (4) follows from Theorem 2.5. The proof is complete.

3.2. A neutral equation with unbounded delay

Next we consider the following boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}[u(t, \xi) + \int_{-\infty}^t \int_0^\pi b(s-t, \eta, \xi) u(s, \eta) d\eta ds] &= \frac{\partial^2}{\partial \xi^2} u(t, \xi) + a_0(\xi) u(t, \xi) \\ &+ \int_{-\infty}^t a(s-t) u(s, \xi) ds + a_1(t, \xi), \end{aligned} \quad (12)$$

$$u(t, 0) = u(t, \pi) = 0, \quad t \geq 0, \quad (13)$$

$$u(\tau, \xi) = \varphi(\tau, \xi), \quad \tau \leq 0, \quad 0 \leq \xi \leq \pi, \quad (14)$$

which is studied by Hernández and Henriquez in [8].

Let $\mathcal{B} := C_r \times L^2(g; X)$, $r = 0$, be the phase space studied in Example 2.1. In this case, $H = 1$; $M(t) = \gamma(-t)^{\frac{1}{2}}$ and $K(t) = 1 + \left(\int_{-t}^0 g(\tau) d\tau\right)^{\frac{1}{2}}$ for all $t \geq 0$. Assuming the conditions (i)-(iii) of Example 3.1 in [8], this problem can be modeled in the form

$$\begin{aligned} \frac{d}{dt}(x(t) + G(t, x_t)) &= Ax(t) + F(t, x_t) + f(t), \\ x_0 &= \varphi \in \mathcal{B}. \end{aligned}$$

where

$$\begin{aligned} G(t, \psi)(\xi) &:= \int_{-\infty}^0 \int_0^\pi b(s, \eta, \xi) \psi(s, \eta) d\eta ds, \\ F(t, \psi)(\xi) &:= a_0(\xi) \psi(0, \xi) + \int_{-\infty}^0 a(s) \psi(s, \xi) ds, \\ f(t) &:= a_1(t, \cdot). \end{aligned}$$

Moreover, $F(t, \cdot), G(t, \cdot)$ are bounded linear operator, the range of G is contained in $X_{\frac{1}{2}}$, $\|(-A)^{1/2} G(t, \cdot)\| \leq N_1^{\frac{1}{2}}$ and $\|F(t, \cdot)\| \leq N_2$ where

$$\begin{aligned} N_1 &:= \int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{g(s)} \left(\frac{\partial}{\partial \zeta} b(s, \eta, \zeta) \right)^2 d\eta ds d\zeta, \\ N_2 &:= \max\left\{ \|a_0\|_\infty, \left(\int_{-\infty}^0 \frac{a^2(\theta)}{g(\theta)} d\theta \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

Next we will assume that the function $g(\cdot)$ verifies the following conditions:

- (g_i) $\ln(g)$ is uniformly continuous,
- (g_{ii}) $k_1 = \int_{-\infty}^0 g(\theta)d\theta < \infty$,
- (g_{iii}) the function $\gamma(\cdot)$ is bounded on $(-\infty, 0]$.

In these conditions, the functions $K(\cdot), M(\cdot)$ are bounded. In the following result we use the symbol K for $\sup_{s \geq 0} K(s)$.

THEOREM 3.1. *Assume that the previous conditions hold and that*

$$K \left[N_1 + \frac{2N_1}{\sqrt{2}} e^{-\frac{1}{2}} + \frac{N_1}{\sqrt{2}} \int_0^1 e^{-\frac{s}{2}} s^{-\frac{1}{2}} ds + N_2 \right] < 1.$$

If f is continuous and ω -periodic, then there exists an ω -periodic solution of (11).

Proof: Using similar estimates that those in the subsection 3.1, it follows that condition **H₇** holds. From Lemmas 3.1, 3.2 and Proposition 3.4 in [8] we know that each mild solution of (11) is bounded on $[0, \infty)$. The existence of an ω -periodic solution for (11)-(13)-(14) is now consequence of Theorem 2.6. The proof is finished.

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