CADERNOS DE MATEMÁTICA **03**, 123–147 April (2002)
ARTIGO NÚMERO SMA#140

# d-Simple rings and maximal ideals of the Weyl algebra

Daniel Levcovitz*

*Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil*
E-mail: lev@icmsc.sc.usp.br

José Carlos de Souza Junior †

*Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil*
E-mail: jcjr@icmc.sc.usp.br

The differential operators $S = \frac{\partial}{\partial_x} + \beta\frac{\partial}{\partial_y} + \gamma$ of $A_2(\mathbb{C})$ such that $S$ generates a maximal left ideal of $A_2(\mathbb{C})$ were characterized in [4] by Bratti and Takagi. We generalize their work to $A_n(K)$, $n \geq 2$, $K$ a field of characteristic zero. We also show that if $S = \frac{\partial}{\partial_{x_1}} + \alpha_2\frac{\partial}{\partial_{x_2}} + \ldots + \alpha_n\frac{\partial}{\partial_{x_n}} + \gamma$ generates a maximal left ideal of $A_n(K)$, then the polynomial ring $K[x_1, \ldots, x_n]$ is $d$-simple with respect to the derivation $d = S - \gamma = \frac{\partial}{\partial_{x_1}} + \alpha_2\frac{\partial}{\partial_{x_2}} + \ldots + \alpha_n\frac{\partial}{\partial_{x_n}}$. April, 2002
ICMC-USP

## 1. INTRODUCTION

Let $A_n(K) = K[x_1, \ldots, x_n] < \partial_1, \ldots, \partial_n >$ be the $n$-th Weyl algebra over a field $K$ of characteristic zero (here $\partial_n$ denotes the usual derivation $\frac{\partial}{\partial_{x_n}}$). Lately, there has been a lot of research about the principal maximal left (or right) ideals of $A_n(K)$, that is, ideals of the form $A_n(K)S$ (or $SA_n(K)$), $S \in A_n(K)$.

The first author to address this problem was Stafford who exhibited explicitely, for the first time, a family of principal maximal right ideals of $A_n(\mathbb{C})$. He was also able to show that given $\lambda, \mu \in \mathbb{C}, \lambda \notin \mathbb{Q}, \mu \notin \mathbb{Z}$,the following specific operator of $A_2(\mathbb{C})$:

$$S = x_2 + x_1\partial_1\partial_2 + \lambda(\partial_1^2 x_1 + \mu\partial_1) + x_1,$$

generates a maximal right ideal of $A_2(\mathbb{C})$ and is not a member of the previous family. (See [1]). In this way he gave the first counter-examples to the conjecture that every simple module over $A_n(\mathbb{C})$ should be holonomic. (Since $\frac{A_n(\mathbb{C})}{A_n(\mathbb{C})S}$ is simple but not holonomic if

* Partially supported by a research grant 300407/89-0,CNPq, Brazil
† Supported by FAPESP grant 99/00240-2, Brazil



Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

$n \geq 2$.)(See [1]). Later on, Berstein and Lunts proved that, in a certain sense, the generic operator of $A_n(\mathbb{C})$ generates a maximal left ideal (See [7]). Nevertheless the examples discovered by Stafford were of a different kind of the generic ones of Berstein and Lunts.

Stafford's examples were generalized by Coutinho in [2]. Starting with a derivation $d$ of $K[x_1, \ldots, x_n]$ that makes this ring into a $d$-simple ring, he was able to give a suitable pertubation of $d$, say $d + \gamma$, $\gamma \in K[x_1, \ldots, x_n]$, such that the left ideal $A_n(K)(d + \gamma)$ is maximal. Here, for the first time, we see that there is a straight connection between $d$-simplicity of $K[x_1, \ldots, x_n]$ and principal maximal ideals of $A_n(K)$.

More recently, Bratti and Takagi gave another characterization of principal maximal ideals of $A_2(\mathbb{C})$ ([4], Theorem 2.2). As an application they were able to relate the problem of finding an operator $S \in A_2(\mathbb{C})$ such that $A_2(\mathbb{C})S$ is maximal to the problem of finding (or not) certain rational solutions of a system of PDE's ([4],Theorem 2.4). In doing so, they use both algebraic and analytical methods.

The first objective of this paper is to generalize the results of Bratti and Takagi to an arbitrary number $n \geq 2$ of variables and to an arbitrary base field $K$ of characteristic zero. In other words, we characterize the principal maximal left ideals of $A_n(K)$, $n \geq 2$. To do so we use only algebraic methods (see Theorem 2.1 ).

The second objective is to establish, for certain derivations $d$ of $K[x_1, \ldots, x_n]$, a relation between the $d$-simplicity of $K[x_1, \ldots, x_n]$ and the existence of a suitable pertubation of $d$, say $d+\gamma$, $\gamma \in K[x_1, \ldots, x_n]$, such that $A_n(K)(d+\gamma)$ is a maximal left ideal of $A_n(K)$. In this way, using Stafford's results, we could give an example of a derivation $d$ that makes the ring $K[x_1, \ldots, x_n]$ into a $d$-simple ring and is not of Shamsuddin's type (see definition 3.2).

Finally we give some connections between our work and the paper "Differential simplicity of polynomial rings and algebraic independence of power series" of P.Brumatti, Y.Lequain and the first author.

This paper is divided as follows: In Section 2 we characterize the principal maximal ideals of the Weyl algebra $A_n(K)$ (Theorem 2.1) and relate this maximal ideals to the existence (or not) of rationals solutions to certain PDE's with polynomials coefficients. The results are similar to the ones in $A_2(\mathbb{C})$ but the techniques of the proofs are much more involved. In Section 3 we apply the results of Section 2 to the theory of $d$-simplicity of the polynomial ring $K[x_1, \ldots, x_n]$. We give a connection between $d$-simplicity and principal maximal ideals of $A_n(K)$. We also obtain new examples of derivations that make the ring $K[x_1, \ldots, x_n]$ into a $d$-simple ring.

## 2. PRINCIPAL MAXIMAL IDEALS OF THE WEYL ALGEBRA AND PARTIAL DIFFERENTIAL EQUATIONS

Definition 2.1. Let $K$ be a field of characteristic zero and let $A_n = A_n(K) = K[x_1, \ldots, x_n] < \partial_1, \ldots, \partial_n >$ be the Weyl algebra in n variables, over the field $K$. Observe that $A_n(K)$ has generators $\partial_i, x_j$, for $1 \leq i, j \leq n$, satisfying the relations $[\partial_i, x_j] := \partial_i x_j - x_j \partial_i = \delta_{ij}$ and other commutators being zero.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Definition 2.2. Let $A_{n-1}$ be the $K$-subalgebra of $A_n(K)$ generated by $x_i$ and $\partial_i$, for $2 \leq i \leq n$. Thus, throughout this work, we set $A_{n-1}[x_1] := K[x_1, \ldots, x_n] < \partial_2, \ldots, \partial_n >$.

Definition 2.3. A *multi-index* $\alpha$ is an element of $\mathbb{N}^n$; say $\alpha = (\alpha_1, \ldots, \alpha_n)$. Now by $\partial^\alpha$ we mean the monomial ${\partial_1}^{\alpha_1} \ldots {\partial_n}^{\alpha_n}$. The *order* of this monomial is the *length* $|\alpha|$ of the multi-index $\alpha$, namely $|\alpha| = \alpha_1 + \ldots + \alpha_n$. An element $d \in A_n(K)$ may be written in the form $d = \sum_\alpha q_\alpha \partial^\alpha$, where $q_\alpha \in K[x_1, \ldots, x_n]$. The *order* of $d$, denoted by ord$(d)$, is the largest $|\alpha|$ for which $q_\alpha \neq 0$. We use the convention that the zero polynomial has order $-\infty$. An example will suffice: the order of ${x_1}^3\partial_2 + {x_1}^7 x_2 {\partial_1}^3 {\partial_2}^2$ is 5.

We begin with some technical lemmas that will prepare for the proof of Theorem 2.1. These lemmas are well known for the specialist, but we prove them here for a lack of a good reference.

Lemma 2.1. *Let $R$ be an element of $A_{n-1}[x_1]$. Then $\partial_1 R = R\partial_1 + \widetilde{R}$, where $\widetilde{R} \in A_{n-1}[x_1]$.*

**Proof:** It is enough to prove when $R$ is a monomial.

Suppose that $R = c_{\alpha,\beta}{x_1}^{\alpha_1} \ldots {x_n}^{\alpha_n}{\partial_2}^{\beta_2} \ldots {\partial_n}^{\beta_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (0, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$, with $c_{\alpha,\beta} \in K$, then

$$\begin{aligned}
\partial_1 R &= \partial_1 c_{\alpha,\beta}{x_1}^{\alpha_1} \ldots {x_n}^{\alpha_n}{\partial_2}^{\beta_2} \ldots {\partial_n}^{\beta_n} \\
&= c_{\alpha,\beta}{x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n}(\partial_1 {x_1}^{\alpha_1}){\partial_2}^{\beta_2} \ldots {\partial_n}^{\beta_n} \\
&= c_{\alpha,\beta}{x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n}({x_1}^{\alpha_1}\partial_1 + \alpha_1 {x_1}^{\alpha_1 - 1}){\partial_2}^{\beta_2} \ldots {\partial_n}^{\beta_n} \\
&= c_{\alpha,\beta}{x_1}^{\alpha_1}{x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n}{\partial_2}^{\beta_2} \ldots {\partial_n}^{\beta_n}\partial_1 \\
&\qquad\qquad + \alpha_1 c_{\alpha,\beta}{x_1}^{\alpha_1 - 1}{x_2}^{\alpha_2} \ldots {x_n}^{\alpha_n}{\partial_2}^{\beta_2} \ldots {\partial_n}^{\beta_n} \\
&= R\partial_1 + \widetilde{R}, \quad \text{where } \widetilde{R} \in A_{n-1}[x_1].
\end{aligned}$$

□

Definition 2.4. Let $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma$ be an element in $A_n$, where $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$. Let $P \in A_n$. The element $R \in A_{n-1}[x_1]$ such that $P = QS + R$ is called the residue of $P$ with respect to $S$.

Lemma 2.2. *(Division algorithm). Let $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma$ be an element in $A_n$, where $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$. Given $P \in A_n$, we have $P = QS + R$, for some $Q \in A_n$ and $R \in A_{n-1}[x_1]$. Moreover, $R$ and $Q$ are uniquely determined.*

**Proof:** We will first prove, by induction on $n$, that $\partial_1^n = QS + R$, for some $Q \in A_n$ and $R \in A_{n-1}[x_1]$.

Note that $\partial_1 = 1S + R$, where $R = -\alpha_2\partial_2 - \ldots - \alpha_n\partial_n - \gamma \in A_{n-1}[x_1]$. Suppose now that the result is true for $i = n$. Then

$$\partial_1^{n+1} = \partial_1\partial_1^n = \partial_1(AS + B) = \partial_1 AS + \partial_1 B,\ A \in A_n, B \in A_{n-1}[x_1].$$

By lemma 2.1, we have

$$\partial_1^{n+1} = \partial_1 AS + B\partial_1 + \widetilde{B},\ \widetilde{B} \in A_{n-1}[x_1].$$

Since $\partial_1 = S + R$,

$$\begin{aligned}
&\partial_1^{n+1} = \partial_1 AS + B(S + R) + \widetilde{B} \\
&= (\partial_1 A + B)S + BR + \widetilde{B} \\
&= Q'S + R', \quad \text{where } Q' \in A_n \text{ and } R' = BR + \widetilde{B} \in A_{n-1}[x_1].
\end{aligned}$$

This completes the induction.

Now, if $P \in A_n$, we can write $P$ in the form $P = E_n\partial_1^n + \ldots + E_1\partial_1 + E_0$, where $E_i \in A_{n-1}[x_1]$.

Thus, $P = E_n(H_nS + B_n) + \ldots + E_1(H_1S + B_1) + E_0$, where $H_1, \ldots, H_n \in A_n$ and $B_1, \ldots B_n \in A_{n-1}[x_1]$. Then,

$$P = (E_nH_n + \ldots + E_1H_1)S + \underbrace{(E_nB_n + \ldots + E_1B_1 + E_0)}_{\in\ A_{n-1}[x_1]}.$$

Therefore, any element of $A_n$ is congruent, modulo $A_nS$, to an element in $A_{n-1}[x_1]$. Moreover, the residue is unique. In fact, let $R, R' \in A_{n-1}[x_1]$ be residues of some $P \in A_n$. So, $R - R' = aS$, for some $a \in A_n$. Writting $a = Q\partial_1 + A$ and $S = \partial_1 + B$, where $Q \in A_n$, $A, B \in A_{n-1}[x_1]$, we have

$$aS = Q{\partial_1}^2 + Q\partial_1 B + A\partial_1 + AB.$$

Since $R, R' \in A_{n-1}[x_1]$, then so does $aS$. Hence, looking at $Q$ as a polynomial in $\partial_1$ with coefficients in $A_{n-1}[x_1]$ we conclude that $Q = 0$. So, $aS = A\partial_1 + AB$. By the same way, we conclude that $A = 0$. Then $a = 0$, $R = R'$ and $Q = Q'$.

□

LEMMA 2.3. *Let $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma$ be an element in $A_n$, where $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$. If $R \in A_{n-1}[x_1]$, then*

$$[S, R] = SR - RS \in A_{n-1}[x_1].$$

**Proof:**

$$\begin{aligned}
[S, R] &= [\partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma, R] \\
&= [\partial_1, R] + \underbrace{[\alpha_2\partial_2, R]}_{(1)} + \underbrace{\ldots}_{(2)} + \underbrace{[\alpha_n\partial_n, R]}_{(3)} + \underbrace{[\gamma, R]}_{(4)}.
\end{aligned}$$

Note that the terms $(1),(2),(3)$ and $(4)$ are in $A_{n-1}[x_1]$.
By lemma 2.1, we have that $[\partial_1, R] = \widetilde{R} \in A_{n-1}[x_1]$.

□

LEMMA 2.4. *Let $A_nS$ be a left ideal of $A_n$, where $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma \in A_n$, $\alpha_2, \ldots, \alpha_n$ and $\gamma \in K[x_1, \ldots, x_n]$. Then $A_nS$ is a maximal left ideal of $A_n$ if, and only if, $A_nS + A_nR = A_n$, for every $0 \neq R \in A_{n-1}[x_1]$.*

**Proof:** $(\Rightarrow)$ If $R \in A_{n-1}[x_1]$ then $R \notin A_nS$.
So, $A_nS + A_nR = A_n$, because $A_nS$ is a maximal left ideal of $A_n$.
$(\Leftarrow)$ We know that $A_nS$ is a maximal left ideal of $A_n$ if, and only if, $A_nS + A_nP = A_n$, for all $P \notin A_nS$.
But, if $P \notin A_nS$, by lemma 2.2, we have that $P = QS + R$, for some $Q \in A_n$ and $R \in A_{n-1}[x_1]$, $R \neq 0$ .
Thus,

$$A_nS + A_nP = A_nS + A_n(QS + R) = A_nS + A_nR = A_n, \text{ by hypothesis.}$$

□

LEMMA 2.5. *Let $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma \in A_n$, where $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$ and $R \in A_{n-1}[x_1]$, with $ord(R) > 0$. Suppose that $\mu[S, R] = \eta R$ for some $\mu \in K[x_1, \ldots, x_n]$, $\mu \neq 0$, $\eta \in K[x_1, \ldots, x_n]$. Then, there exists $h \in K[x_1, \ldots, x_n]$ and $\widetilde{R} \in A_{n-1}[x_1]$, with $ord(\widetilde{R}) > 0$, such that $[S, \widetilde{R}] = h\widetilde{R}$.*

**Proof:** Suppose that

$$\boxed{\mu[S, R] = \eta R} \tag{1}$$

with $\mu, \eta, S$ and $R$ satisfying the hypothesis. We can write $R$ in the form

$$\sum_{i_2+\ldots+i_n=0}^{N} p_{i_2,\ldots,i_n} {\partial_2}^{i_2} \ldots {\partial_n}^{i_n}, \text{ where } p_{i_2,\ldots,i_n} \in K[x_1, \ldots, x_n].$$

Write $R = \alpha_0\widetilde{R}$, where $\alpha_0$ is the greatest commom divisor of the elements $p_{i_2,\ldots,i_n}$.
Since $\mu \in K[x_1, \ldots, x_n] \setminus \{0\}$, by (1), $\mu$ divides $\eta\alpha_0$, say $\eta\alpha_0 = \mu\zeta$, for some $\zeta \in K[x_1, \ldots, x_n]$. Hence,

$$[S, \alpha_0\widetilde{R}] = \zeta\widetilde{R}.$$

But, $[S, \alpha_0\widetilde{R}] = [S, \alpha_0]\widetilde{R} + \alpha_0[S, \widetilde{R}]$. Then,

$$\boxed{\alpha_0[S, \widetilde{R}] = \underbrace{(\zeta - [S, \alpha_0])}_{\lambda \in K[x_1,\ldots,x_n]} \widetilde{R} = \lambda\widetilde{R}, \text{ where } \lambda \in K[x_1, \ldots, x_n]} \tag{2}$$

By the definition of $\alpha_0$, it follows from (2), that $\lambda = h\alpha_0$, for some $h \in K[x_1, \ldots, x_n]$. Thus

$$[S, \widetilde{R}] = h\widetilde{R}.$$

□

The proof of the following lemma is completely analogous of the previous one. It will be omitted.

LEMMA 2.6. *Let $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma$ be in $A_n$, where $\alpha_2, \ldots, \alpha_n$ and $\gamma \in K[x_1, \ldots, x_n]$ and let $p$ be an element in $K[x_1, \ldots, x_n]\backslash K$. Suppose that $g[S, p] = hp$ for some $h \in K[x_1, \ldots, x_n]$ and $g \in K[x_1, \ldots, x_{n-1}]\backslash\{0\}$. Then, there exists $\zeta \in K[x_1, \ldots, x_n]$ and $\widetilde{p} \in K[x_1, \ldots, x_n] \setminus K$ such that*

$$[S, \widetilde{p}] = \zeta\widetilde{p}.$$

*Remark 2. 1.* The next result generalizes theorem 2.2, of [4].

THEOREM 2.1.

**(a)** *Suppose that $A_nS$ is a maximal left ideal of $A_n$, where$S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma$, with $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$. Then, for every $R \in A_{n-1}[x_1]$, where $R$ is not a constant, we have*

$$[S, R] \notin K[x_1, \ldots, x_n] \cdot R.$$

**(b)** *Let $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma$ be in $A_n$, where $\alpha_2 \in K[x_1, x_2], \ldots, \alpha_n \in K[x_1, x_n]$ and $\gamma \in K[x_1, \ldots, x_n]$. Suppose that for every $R \in A_{n-1}[x_1]$, where $R$ is not a constant, we have $[S, R] \notin K[x_1, \ldots, x_n] \cdot R$. Then $A_nS$ is a maximal left ideal of $A_n$.*

**Proof:** (**a**) Let $\lambda, \mu$ be elements in $A_n$ such that $\lambda S + \mu R = 1$. If $deg_{\partial_1}(\lambda) = m$, then $deg_{\partial_1}(\mu) = m + 1$.

Let us write $\lambda$ and $\mu$ in the form:

$$\lambda = \widetilde{B}_m{\partial_1}^m + \ldots + \widetilde{B}_1\partial_1 + \widetilde{B}_0,$$
$$\mu = \widetilde{C}_{m+1}{\partial_1}^{m+1} + \ldots + \widetilde{C}_1\partial_1 + \widetilde{C}_0,$$

where $\widetilde{B}_i, \widetilde{C}_j \in A_{n-1}[x_1]$, $\widetilde{B}_m \neq 0$, $\widetilde{C}_{m+1} \neq 0$ .

Using that $\partial_1 = S - (\alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma)$ and lemma 2.3, it follows that we can rewrite $\lambda$ and $\mu$ in the form:

$$\lambda = B_mS^m + \ldots + B_1S + B_0,$$
$$\mu = C_{m+1}S^{m+1} + \ldots + C_1S + C_0,$$

where $B_i, C_j \in A_{n-1}[x_1]$, $B_m \neq 0$, $C_{m+1} \neq 0$. So,

$$\lambda S + \mu R = \sum_{k=0}^{m} B_k S^{k+1} + \sum_{k=0}^{m+1} C_k S^k R = 1.$$

Hence,

$$\boxed{0 = [x_1, \lambda S + \mu R] = -\sum_{k=0}^{m}(k+1)B_k S^k - \sum_{k=1}^{m+1} kC_k S^{k-1} R}$$

Commuting with $x_1$ more $m$ times, we obtain

$$\boxed{B_m + C_{m+1} R = 0} \tag{3}$$

Now, if $[S, R] = \eta R$, for some $\eta \in K[x_1, \ldots, x_n]$, then we would have

$$\begin{aligned}
\lambda S + \mu R &= \sum_{k=0}^{m} B_k S^{k+1} + \sum_{k=0}^{m+1} C_k S^k R \\
&= \sum_{k=0}^{m-1} B_k S^{k+1} + \sum_{k=0}^{m} C_k S^k R + B_m S^{m+1} + C_{m+1} S^{m+1} R \\
&\overset{(3)}{=} \sum_{k=0}^{m-1} B_k S^{k+1} + \sum_{k=0}^{m} C_k S^k R - C_{m+1} R S^{m+1} + C_{m+1} S^{m+1} R \\
&= \sum_{k=0}^{m-1} B_k S^{k+1} + \sum_{k=0}^{m} C_k S^k R - C_{m+1}(R S^{m+1} - S^{m+1} R) \\
&= \sum_{k=0}^{m-1} B_k S^{k+1} + \sum_{k=0}^{m} C_k S^k R - C_{m+1}[R, S^{m+1}].
\end{aligned} \tag{4}$$

Since $[S, R] = \eta R$, we have

$$\begin{aligned}
& S^{m+1}R = S^m(SR) = S^m(RS + \eta R) = S^m RS + S^m \eta R = \\
& = S^{m-1}(SR)S + S^m \eta R = S^{m-1}(RS + \eta R)S + S^m \eta R = \\
& = S^{m-1}RS^2 + S^{m-1}(\eta R)S + S^m \eta R = \\
& \vdots \\
& = RS^{m+1} + (\varphi_{m-1}S^{m-1} + \varphi_{m-2}S^{m-2} + \ldots + \varphi_1 S + \varphi_0)S + \\
& \qquad +(\xi_m S^m + \xi_{m-1}S^{m-1} + \ldots + \xi_1 S + \xi_0)R
\end{aligned}$$

for some $\varphi_i, \xi_j \in A_{n-1}[x_1],\ 0 \leq i \leq m-1,\ 0 \leq j \leq m$.

Then, we have

$$\lambda S+\mu R=\sum_{k=0}^{m-1}\widetilde{E}_k S^{k+1}+\sum_{k=0}^{m}\widetilde{D}_k S^k R=1,$$

for some $\widetilde{E}_k, \widetilde{D}_k \in A_{n-1}[x_1]$.

Commutating with $x_1$ again and repeating the argument, we obtain

$$\boxed{\lambda S+\mu R=E_0S+D_0R+D_1SR=1} \tag{5}$$

for some $E_0, D_0, D_1 \in A_{n-1}[x_1]$.

Note that $E_0=-D_1R$. In fact,

$$\begin{aligned} 0 &= [x_1,\lambda S+\mu R]=[x_1,E_0S+D_0R+D_1SR] \\ &= E_0[x_1,S]+D_1[x_1,S]R \\ &= -E_0-D_1R \Rightarrow E_0=-D_1R. \end{aligned}$$

Thus, from (5) follows that

$$\begin{aligned} -D_1RS+D_0R+D_1SR&=1 \\ D_1[S,R]+D_0R&=1 \\ D_1(\eta R)+D_0R&=1 \\ (D_1\eta+D_0)R&=1. \end{aligned}$$

Therefore, $deg(D_1\eta+D_0)+deg(R)=0$, and we get a contradiction, since $deg(R)>0$ ($R$ is not a constant ).

Now we prove (**b**).

Let $R$ be in $A_{n-1}[x_1]$, such that $\mathrm{ord}(R)=N>0$. We can write $R$ in the form:

$$R=\sum_{i_2+\ldots+i_n=0}^{N} p_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}, \text{ where } p_{i_2,\ldots,i_n}\in K[x_1,\ldots,x_n].$$

Since $\mathrm{ord}([S,R])=\mathrm{ord}(S)+\mathrm{ord}(R)-1$, then $\mathrm{ord}([S,R])=N$. Therefore we can also write $[S,R]$ in the form

$$[S,R]=\sum_{i_2+\ldots+i_n=0}^{N} q_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}, \text{ where } q_{i_2,\ldots,i_n}\in K[x_1,\ldots,x_n].$$

As $\mathrm{ord}(R)=N$, there exists $p_{i_{2_0},\ldots,i_{n_0}}\neq 0$, such that

$$i_{2_0}+\ldots+i_{n_0}=N.$$

By hypothesis and by lemma 2.5, we have that

$$\boxed{p_{i_{2_0},\dots,i_{n_0}}[S,R] - q_{i_{2_0},\dots,i_{n_0}}R \neq 0}$$

Note that, from this equation, we have

$$0 \leq \ \text{ord}(p_{i_{2_0},\dots,i_{n_0}}[S,R] - q_{i_{2_0},\dots,i_{n_0}}R) \leq N.$$

Moreover,

$$\widetilde{R} := p_{i_{2_0},\dots,i_{n_0}}[S,R] - q_{i_{2_0},\dots,i_{n_0}}R = \dots + 0\ {\partial_2}^{i_{2_0}} \dots {\partial_n}^{i_{n_0}} + \dots$$

Now, $\widetilde{R} \in (A_nS + A_nR) \setminus \{0\}$ is such that $0 \leq \text{ord}(\widetilde{R}) \leq N$, and the term with order $N$ involving ${\partial_2}^{i_{2_0}} \dots {\partial_n}^{i_{n_0}}$ does not appear in $\widetilde{R}$.

*Claim 2.1.* If we repeat this process with $\widetilde{R}$, then the term with order $N$ that we eliminated previously won't be back.

Let's assume, for a while, claim 2.1.

In this way, if $\widetilde{R}$ has another term with order $N$, we can repeat the process and eliminate it too. Therefore, after a finite number of steps, we have a new $\widetilde{R} \in (A_nS + A_nR) \setminus \{0\}$, with $0 \leq \text{ord}(\widetilde{R}) \leq N - 1$.

Proceeding in this way, we obtain

$$(A_nS + A_nR) \cap (K[x_1, \dots, x_n] \setminus \{0\}) \neq \emptyset.$$

Let $p = \sum_{k=0}^{m} r_k {x_n}^k$, where $r_k \in K[x_1, \dots, x_{n-1}]$, be a polynomial contained in $(A_nS + A_nR) \cap (K[x_1, \dots, x_n] \setminus \{0\})$ with the least degree in $x_n$.

If $m$ were strictly greater than zero, then, by the Euclidean Algorithm (applied to $[S,p]$ and $p$ considered as elements in $K(x_1, \dots, x_{n-1})[x_n]$), there exists $d \in K[x_1, \dots, x_{n-1}] \setminus \{0\}$ such that

$$d[S,p] = \eta p + r, \ \text{for some } \eta, r \in K[x_1, \dots, x_n] \text{ where } deg_{x_n}(r) < deg_{x_n}(p) \text{ or } r = 0.$$

But this implies that $r = 0$, by the choice of $p$.

Hence, $d[S,p] = \eta p$, which is a contradiction, by lemma 2.6. So $m = 0$ and $(A_nS + A_nR) \cap (K[x_1, \dots, x_{n-1}] \setminus \{0\}) \neq \emptyset$.

Proceeding in this way, we obtain

$$(A_nS + A_nR) \cap (K[x_1] \setminus \{0\}) \neq \emptyset.$$

Let $p = a_l {x_1}^l + \dots + a_0$, with $a_i \in K$, be in $(A_nS + A_nR) \cap (K[x_1] \setminus \{0\})$. Note that

$$[S,p] = la_l {x_1}^{l-1} + \dots + a_1 \in (A_nS + A_nR) \cap (K[x_1] \setminus \{0\}).$$

Repeating this process $l$ times, we have that

$$l!a_l \in (A_nS + A_nR) \cap (K \setminus \{0\}), \text{then } A_nS + A_nR = A_n.$$

By lemma 2.4, it follows that $A_nS$ is a maximal left ideal of $A_n$.

To finish the prove, we have to show claim 2.1.

**Proof of the claim 2.1:** For a while, let us suppose that $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma \in A_n$, where $\alpha_2, \ldots, \alpha_n, \gamma \in K[x_1, \ldots, x_n]$ and $R = \sum_{i_2+\ldots+i_n=0}^{N} P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}$, where $P_{i_2,\ldots,i_n} \in K[x_1, \ldots, x_n]$, with $i_2 + \ldots + i_n = 0, \ldots, N$.

Then,

$$[S, R] = \left[S, \sum_{i_2+\ldots+i_n=0}^{N} P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}\right] = \sum_{i_2+\ldots+i_n=0}^{N} [S, P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}].$$

Note that: $[\partial_1, P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}] = \partial_1(P_{i_2,\ldots,i_n})\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}$, and

$$\begin{aligned}
&[\alpha_2\partial_2, P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}] = \alpha_2\partial_2 P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n} - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}\alpha_2\partial_2\\
&= \alpha_2(P_{i_2,\ldots,i_n}\partial_2 + \partial_2(P_{i_2,\ldots,i_n}))\partial_2{}^{i_2}\ldots\partial_n{}^{i_n} - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}\alpha_2\partial_2\\
&= \underbrace{\alpha_2 P_{i_2,\ldots,i_n}\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\partial_n{}^{i_n} + \alpha_2\partial_2(P_{i_2,\ldots,i_n})\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}}_{(\triangle)} - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}\alpha_2\partial_2\\
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-1}{}^{i_{n-1}}\big(\alpha_2\partial_n{}^{i_n} +\\
&\qquad\qquad + i_n\partial_n(\alpha_2)\partial_n{}^{i_n-1} + \text{ terms with lower order}\big)\,\partial_2\\
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-1}{}^{i_{n-1}}\alpha_2\partial_n{}^{i_n}\partial_2 +\\
&\qquad\qquad - i_n P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-1}{}^{i_{n-1}}\partial_n(\alpha_2)\partial_n{}^{i_n-1}\partial_2 + \ldots\\
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-1}{}^{i_{n-1}}\alpha_2\partial_n{}^{i_n}\partial_2 +\\
&\qquad\qquad - i_n P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-2}{}^{i_{n-2}}(\partial_n(\alpha_2)\partial_{n-1}{}^{i_{n-1}} +\\
&\qquad\qquad\qquad + \text{terms with lower order})\partial_n{}^{i_n-1}\partial_2 + \ldots\\
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-1}{}^{i_{n-1}}\alpha_2\partial_n{}^{i_n}\partial_2 +\\
&\qquad\qquad - i_n P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-2}{}^{i_{n-2}}\partial_n(\alpha_2)\partial_{n-1}{}^{i_{n-1}}\partial_n{}^{i_n-1}\partial_2 +\\
&\vdots \qquad\qquad\qquad + \text{ terms with lower order}
\end{aligned}$$

$$\begin{aligned}
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-1}{}^{i_{n-1}}\alpha_2\partial_n{}^{i_n}\partial_2 + \\
&\qquad - \underbrace{i_n\partial_n(\alpha_2)P_{i_2,\ldots,i_n}\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\partial_n{}^{i_{n-1}}}_{\square} + \\
&\qquad + \text{ terms with lower order} \\
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-2}{}^{i_{n-2}}(\alpha_2\partial_{n-1}{}^{i_{n-1}} + \\
&\qquad + i_{n-1}\partial_{n-1}(\alpha_2)\partial_{n-1}{}^{i_{n-1}-1} + \ldots)\partial_n{}^{i_n}\partial_2 + \\
&\qquad - \square + \text{ terms with lower order} \\
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-2}{}^{i_{n-2}}\alpha_2\partial_{n-1}{}^{i_{n-1}}\partial_n{}^{i_n}\partial_2 + \\
&\qquad - i_{n-1}P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-2}{}^{i_{n-2}}\partial_{n-1}(\alpha_2)\partial_{n-1}{}^{i_{n-1}-1}\partial_n{}^{i_n}\partial_2 + \\
&\qquad - \square + \text{ terms with lower order} \\
&\vdots \\
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_{n-2}{}^{i_{n-2}}\alpha_2\partial_{n-1}{}^{i_{n-1}}\partial_n{}^{i_n}\partial_2 + \\
&\qquad - i_{n-1}P_{i_2,\ldots,i_n}\partial_{n-1}(\alpha_2)\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\partial_{n-1}{}^{i_{n-1}-1}\partial_n{}^{i_n} + \\
&\qquad - \square + \text{ terms with lower order} \\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&= \triangle - P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\alpha_2\partial_3{}^{i_3}\ldots\partial_n{}^{i_n}\partial_2 - i_3P_{i_2,\ldots,i_n}\partial_3(\alpha_2)\partial_2{}^{i_2+1}\partial_3{}^{i_3-1}\partial_4{}^{i_4}\ldots\partial_n{}^{i_n} + \\
&\qquad + \ldots - i_nP_{i_2,\ldots,i_n}\partial_n(\alpha_2)\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\partial_n{}^{i_n-1} + \\
&\qquad + \text{ terms with lower order} \\
&= \triangle - P_{i_2,\ldots,i_n}(\alpha_2\partial_2{}^{i_2} + i_2\partial_2(\alpha_2)\partial_2{}^{i_2-1} + \text{ terms with lower order})\partial_3{}^{i_3}\ldots\partial_n{}^{i_n}\partial_2 + \\
&\qquad - i_3P_{i_2,\ldots,i_n}\partial_3(\alpha_2)\partial_2{}^{i_2+1}\partial_3{}^{i_3-1}\partial_4{}^{i_4}\ldots\partial_n{}^{i_n} + \ldots + \\
&\qquad - i_nP_{i_2,\ldots,i_n}\partial_n(\alpha_2)\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\partial_n{}^{i_n-1} + \\
&\qquad + \text{ terms with lower order} \\
&= \triangle - P_{i_2,\ldots,i_n}\alpha_2\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\partial_n{}^{i_n} - i_2P_{i_2,\ldots,i_n}\partial_2(\alpha_2)\partial_2{}^{i_2}\ldots\partial_n{}^{i_n} + \\
&\qquad - i_3P_{i_2,\ldots,i_n}\partial_3(\alpha_2)\partial_2{}^{i_2+1}\partial_3{}^{i_3-1}\partial_4{}^{i_4}\ldots\partial_n{}^{i_n} + \ldots + \\
&\qquad - i_nP_{i_2,\ldots,i_n}\partial_n(\alpha_2)\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\partial_n{}^{i_n-1} + \\
&\qquad + \text{terms with lower order} \\
&= [\alpha_2\partial_2(P_{i_2,\ldots,i_n}) - i_2P_{i_2,\ldots,i_n}\partial_2(\alpha_2)]\partial_2{}^{i_2}\ldots\partial_n{}^{i_n} + \\
&\qquad + [-i_3P_{i_2,\ldots,i_n}\partial_3(\alpha_2)]\partial_2{}^{i_2+1}\partial_3{}^{i_3-1}\partial_4{}^{i_4}\ldots\partial_n{}^{i_n} + \ldots + \\
&\qquad + [-i_nP_{i_2,\ldots,i_n}\partial_n(\alpha_2)]\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\partial_n{}^{i_n-1} + \\
&\qquad + \text{terms with lower order.}
\end{aligned}$$

Hence, the terms with order $N$ in $\sum_{i_2+\ldots+i_n=0}^{N}[\alpha_2\partial_2, P_{i_2,\ldots,i_n}\partial_2{}^{i_2}\ldots\partial_n{}^{i_n}]$ are:

$$\begin{aligned}
&\textstyle\sum_{i_2+\ldots i_n=N}\{[\alpha_2\partial_2(P_{i_2,\ldots,i_n}) - i_2P_{i_2,\ldots,i_n}\partial_2(\alpha_2)]\partial_2{}^{i_2}\ldots\partial_n{}^{i_n} + \\
&\qquad + [-i_3P_{i_2,\ldots,i_n}\partial_3(\alpha_2)]\partial_2{}^{i_2+1}\underbrace{\partial_3{}^{i_3-1}}_{i_3\geq 1}\partial_4{}^{i_4}\ldots\partial_n{}^{i_n} + \ldots \\
&\qquad + [-i_nP_{i_2,\ldots,i_n}\partial_n(\alpha_2)]\partial_2{}^{i_2+1}\partial_3{}^{i_3}\ldots\underbrace{\partial_n{}^{i_n-1}}_{i_n\geq 1}\}.
\end{aligned}$$

Similarly, the terms with order $N$ in $\sum_{i_2+\ldots+i_n=0}^{N}[\alpha_j\partial_j, P_{i_2,\ldots,i_n}{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}]$, where $j = 2,\ldots,n$; are:

$$\begin{array}{l}\sum_{i_2+\ldots i_n=N}\{[\alpha_j\partial_j(P_{i_2,\ldots,i_n}) - i_jP_{i_2,\ldots,i_n}\partial_j(\alpha_j)]{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}+\\ \sum_{\substack{k=2\\k\neq j}}^{n}[-i_kP_{i_2,\ldots,i_n}\partial_k(\alpha_j)]{\partial_j}^{i_j+1}\underbrace{{\partial_k}^{i_k-1}}_{i_k\geq 1}\ldots{\partial_{j-1}}^{i_{j-1}}{\partial_{j+1}}^{i_{j+1}}\ldots{\partial_{k-1}}^{i_{k-1}}{\partial_{k+1}}^{i_{k+1}}\}.\end{array}$$

Since the $\mathrm{ord}([\gamma, R]) = N - 1$, then the terms with order $N$ in $[S, R]$ are:

$$\begin{array}{l}\sum_{i_2+\ldots+i_n=N}\{[\partial_1(P_{i_2,\ldots,i_n}) + \sum_{j=2}^{n}[\alpha_j\partial_j(P_{i_2,\ldots,i_n}) - i_jP_{i_2,\ldots,i_n}\partial_j(\alpha_j)]]{\partial_2}^{i_2}\ldots{\partial_n}^{i_n}+\\ +\sum_{j=2}^{n}\sum_{\substack{k=2\\k\neq j}}^{n}[-i_kP_{i_2,\ldots,i_n}\partial_k(\alpha_j)]{\partial_j}^{i_j+1}\underbrace{{\partial_k}^{i_k-1}}_{i_k\geq 1}\ldots{\partial_{j-1}}^{i_{j-1}}{\partial_{j+1}}^{i_{j+1}}\ldots{\partial_{k-1}}^{i_{k-1}}{\partial_{k+1}}^{i_{k+1}}\}.\end{array}$$

Now, observe that if $P_{i_2,\ldots,i_n}$ is the coefficient of ${\partial_2}^{i_2}\ldots{\partial_n}^{i_n}$ in $R$, with $i_2+\ldots+i_n = N$, then the corresponding coefficient in $[S, R]$ is:

$$\begin{array}{rl}q_{i_2,\ldots,i_n} := & [\partial_1(P_{i_2,\ldots,i_n}) + \sum_{j=2}^{n}[\alpha_j\partial_j(P_{i_2,\ldots,i_n}) - i_jP_{i_2,\ldots,i_n}\partial_j(\alpha_j)]+\\ & \sum_{j=2}^{n}\sum_{\substack{k=2\\k\neq j}}^{n}[-(i_k+1)P_{i_2,\ldots,i_{j-1},i_j-1,i_{j+1},\ldots,i_{k-1},i_k+1,i_{k+1},\ldots,i_n}\partial_k(\alpha_j)]].\end{array}$$

Note that, unfortunately $q_{i_2,\ldots,i_n}$ does not depend only on $P_{i_2,\ldots,i_n}$. But, under the hypothesis that $\alpha_i \in K[x_1, x_i], i = 2,\ldots,n$ we have, $\partial_k(\alpha_j) = 0$ for all $j, k = 2,\ldots,n; j \neq k$.

In this case, $q_{i_2,\ldots,i_n}$ depends only on $P_{i_2,\ldots,i_n}$. Therefore, if $P_{i_2,\ldots,i_n} = 0$, then $q_{i_2,\ldots,i_n} = 0$ and we have that the coefficient of ${\partial_2}^{i_2}\ldots{\partial_n}^{i_n}$ in $\zeta[S, R] - \eta R$ is zero, for all $\eta, \zeta \in K[x_1,\ldots,x_n]$. □

Now we establish a connection between principal maximal ideals of the Weyl algebra and certain partial differential equations over the polynomial ring $K[x_1,\cdots,x_n]$. This connection was originally discovered by Bratti and Takagi when $n = 2$([4]). We generalize their theorem for an arbitrary number of variables. We begin with some calculations that will be useful in the proof of the next theorem.

Let $S = \partial_1 + \alpha_2\partial_2 + \alpha_3\partial_3 + \gamma$ be an element in $A_3(K)$, where $\alpha_2 \in K[x_1, x_2], \alpha_3 \in K[x_1, x_3]$ and $\gamma \in K[x_1, x_2, x_3]$. Let

$$R = \sum_{i+j=0}^{n} P_{i,j}{\partial_2}^{i}{\partial_3}^{j} \in K[x_1, x_2, x_3] < \partial_2, \partial_3 >,$$

$n \geq 1$ and $P_{i,j} \in K[x_1, x_2, x_3]$. Then, a simple calculation gives the following expression of $[S, R]$:

$$
\begin{aligned}
[S,R] \;=\; & \sum_{i+j=0}^{n} \{[\partial_1(P_{i,j}) + \alpha_2\partial_2(P_{i,j}) - iP_{i,j}\partial_2(\alpha_2) + \\
+ \; & \alpha_3\partial_3(P_{i,j}) - jP_{i,j}\partial_3(\alpha_3)]{\partial_2}^i{\partial_3}^j + \\
+ \; & [-\binom{i}{2}P_{i,j}{\partial_2}^2(\alpha_2) - iP_{i,j}\partial_2(\gamma)]{\partial_2}^{i-1}{\partial_3}^j + \\
+ \; & [-\binom{j}{2}P_{i,j}{\partial_3}^2(\alpha_3) - jP_{i,j}\partial_3(\gamma)]{\partial_2}^i{\partial_3}^{j-1} + \\
+ \; & \text{terms with order lower than } i+j-1\}
\end{aligned} \tag{6}
$$

Note that
$\lambda := \partial_1(P_{k,n-k}) + \alpha_2\partial_2(P_{k,n-k}) - kP_{k,n-k}\partial_2(\alpha_2) + \alpha_3\partial_3(P_{k,n-k}) - (n-k)P_{k,n-k}\partial_3(\alpha_3)$ is the coefficient of the monomial ${\partial_2}^k{\partial_3}^{n-k}$ in $[S,R]$. Now, if $P_{k,n-k} \neq 0$, then in the expression

$$P_{k,n-k}[S,R] - \lambda R$$

the coefficient of $\partial_2^k\partial_3^{n-k}$ is zero while the coefficient of ${\partial_2}^{k-1}{\partial_3}^{n-k}$, say $q_{k-1,n-k} \in K[x_1,x_2,x_3]$, is the following:

$$
\begin{aligned}
q_{k-1,n-k} \;=\; & P^2_{k,n-k}\left[\partial_1\left(\frac{P_{k-1,n-k}}{P_{k,n-k}}\right) + \alpha_2\partial_2\left(\frac{P_{k-1,n-k}}{P_{k,n-k}}\right) + \alpha_3\partial_3\left(\frac{P_{k-1,n-k}}{P_{k,n-k}}\right) + \right. \\
+ \; & \left. \partial_2(\alpha_2)\left(\frac{P_{k-1,n-k}}{P_{k,n-k}}\right) - k\partial_2(\gamma) - \binom{k}{2}{\partial_2}^2(\alpha_2)\right], \\
& \text{where } 1 \leq k \leq n, \text{ and } \binom{k}{2} = 0 \text{ if } k < 2.
\end{aligned} \tag{7}
$$

Similarly, the coefficient $q_{k,n-k-1} \in K[x_1,x_2,x_3]$ of ${\partial_2}^k{\partial_3}^{n-k-1}$ in the expression

$$P_{k,n-k}[S,R] - \lambda R$$

is the following:

$$
\begin{aligned}
q_{k,n-k-1} \;=\; & P^2_{k,n-k}\left[\partial_1\left(\frac{P_{k,n-k-1}}{P_{k,n-k}}\right) + \alpha_2\partial_2\left(\frac{P_{k,n-k-1}}{P_{k,n-k}}\right) + \alpha_3\partial_3\left(\frac{P_{k,n-k-1}}{P_{k,n-k}}\right) + \right. \\
+ \; & \left. \partial_3(\alpha_3)\left(\frac{P_{k,n-k-1}}{P_{k,n-k}}\right) - (n-k)\partial_3(\gamma) - \binom{n-k}{2}{\partial_3}^2(\alpha_3)\right], \\
& \text{where } 0 \leq k \leq n-1, \text{ and } \binom{n-k}{2} = 0 \text{ if } n-k < 2.
\end{aligned} \tag{8}
$$

Theorem 2.2. *Let* $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma \in A_n(K)$*, where* $\alpha_i(x_1, x_i) = \alpha_{i0}(x_1) + \alpha_{i1}(x_1)x_i \in K[x_1, x_i], i = 2, \ldots, n$ *and* $\gamma \in K[x_1, \ldots, x_n]$*. Then the following statements are equivalent:*

**(i)** $A_nS$ *is a left maximal ideal of* $A_n(K)$*.*
**(ii)** *The equations*

$$\partial_1\left(\frac{p}{q}\right) + \alpha_2\partial_2\left(\frac{p}{q}\right) + \ldots + \alpha_n\partial_n\left(\frac{p}{q}\right) + \partial_i(\alpha_i)\cdot\left(\frac{p}{q}\right) = \partial_i(\gamma),\ 2 \le i \le n \qquad (9)$$

*do not have any solution in* $K(x_1, \ldots, x_n)$ *and the equations*

$$\partial_1\left(\frac{r}{s}\right) - \alpha_{i1}\cdot\left(\frac{r}{s}\right) = -\alpha_{i0},\ i = 2, \ldots, n \qquad (10)$$

*do not have any solution in* $K(x_1)$*.*

**Proof:** We will prove the theorem only in the case $n = 3$ to avoid a heavy notation. The proof in the general case is analogous.

If $n = 3$, the equations (9) become :

$$\partial_1\left(\frac{p}{q}\right) + \alpha_2\partial_2\left(\frac{p}{q}\right) + \alpha_3\partial_3\left(\frac{p}{q}\right) + \partial_2(\alpha_2)\cdot\left(\frac{p}{q}\right) = \partial_2(\gamma) \qquad (11)$$

$$\partial_1\left(\frac{p}{q}\right) + \alpha_2\partial_2\left(\frac{p}{q}\right) + \alpha_3\partial_3\left(\frac{p}{q}\right) + \partial_3(\alpha_3)\cdot\left(\frac{p}{q}\right) = \partial_3(\gamma) \qquad (12)$$

and the equations (10) become:

$$\partial_1\left(\frac{r}{s}\right) - \alpha_{21}\cdot\left(\frac{r}{s}\right) = -\alpha_{20} \qquad (13)$$

$$\partial_1\left(\frac{r}{s}\right) - \alpha_{31}\cdot\left(\frac{r}{s}\right) = -\alpha_{30} \qquad (14)$$

First, we prove that (**i**) implies (**ii**).

Suppose that $\frac{p}{q} \in K(x_1, x_2, x_3)$ is a solution of (11). Let $R = q\partial_2 + p$. Then,

$$\begin{aligned}[S, R] &= (\partial_1(q) + \alpha_2\partial_2(q) - q\partial_2(\alpha_2) + \alpha_3\partial_3(q))\partial_2 + \\ &+ (\partial_1(p) + \alpha_2\partial_2(p) + \alpha_3\partial_3(p) - q\partial_2(\gamma)).\end{aligned}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Put $\lambda := \partial_1(q) + \alpha_2\partial_2(q) - q\partial_2(\alpha_2) + \alpha_3\partial_3(q)$, the coefficient on $\partial_2$ in $[S,R]$. Then,

$$q[S,R] - \lambda R = q^2 \left[ \underbrace{\partial_1\left(\frac{p}{q}\right) + \alpha_2\partial_2\left(\frac{p}{q}\right) + \alpha_3\partial_3\left(\frac{p}{q}\right) + \partial_2(\alpha_2)\cdot\left(\frac{p}{q}\right) - \partial_2(\gamma)}_{\frac{p}{q}\text{ is a solution of (11)}} \right] = 0.$$

Therefore, $q[S,R] = \lambda R$, which is contrary to the theorem 2.1 ( using lemma 2.5), since $A_3S$ is a maximal ideal.

Thus, the equation (11) does not have any solution in $K(x_1,x_2,x_3)$.

Analogously, if $\frac{p}{q}$ is a solution of (12), taking $R = q\partial_3 + p$ we obtain a contradiction. So, equation (12) does not have any solution in $K(x_1,x_2,x_3)$ either.

Now, if $\frac{r}{s} \in K(x_1)$ is a solution of (13), let $P = r(x_1) + s(x_1)x_2$. Then,

$$[S,P] = r' + s'x_2 + \alpha_2 s, \text{ where } r' \text{ and } s' \text{ denote the derivative of } r \text{ and } s.$$

Since $\alpha_2(x_1,x_2) = \alpha_{20}(x_1) + \alpha_{21}(x_1)x_2$, we have

$$[S,P] = (r' + \alpha_{20}s) + (s' + \alpha_{21}s)x_2.$$

Then,

$$s[S,P] - (s' + \alpha_{21}s)P = s^2 \left( \underbrace{\partial_1\left(\frac{r}{s}\right) - \alpha_{21}\cdot\left(\frac{r}{s}\right) + \alpha_{20}}_{\frac{r}{s}\text{ is a sol. of (13)}} \right) = 0.$$

Therefore, $s[S,P] = (s' + \alpha_{21}s)P$, which is contrary to the theorem 2.1 ( using lemma 2.6), since $A_3S$ is a maximal ideal.

Thus, the equation (13) does not have any solution in $K(x_1)$.

Analogously, if $\frac{r}{s} \in K(x_1)$ is a solution of (14), taking $P = r(x_1) + s(x_1)x_3$ we obtain a contradiction. So, equation (14) does not have any solution in $K(x_1)$ either.

Conversely, let us assume (**ii**).

Let $R = \sum_{i+j=0}^{n} P_{i,j}{\partial_2}^i{\partial_3}^j$ be in $K[x_1,x_2,x_3] < \partial_2, \partial_3 >$ an element with order $n > 0$. So, there exists $P_{k,n-k} \neq 0$ for some $k = 0,\ldots,n$.

Observe that in the case that $\alpha_2(x_1,x_2) = \alpha_{20}(x_1) + \alpha_{21}(x_1)x_2$ and $\alpha_3(x_1,x_3) = \alpha_{30}(x_1) + \alpha_{31}(x_1)x_3$, the equations (7) and (8) simplify:

$$\begin{aligned} q_{k-1,n-k} &= P_{k,n-k}^2 \left[\partial_1\left(\frac{P_{k-1,n-k}}{P_{k,n-k}}\right) + \alpha_2\partial_2\left(\frac{P_{k-1,n-k}}{P_{k,n-k}}\right) + \alpha_3\partial_3\left(\frac{P_{k-1,n-k}}{P_{k,n-k}}\right) + \right. \\ &+ \left. \partial_2(\alpha_2)\left(\frac{P_{k-1,n-k}}{P_{k,n-k}}\right) - k\partial_2(\gamma)\right] \end{aligned} \tag{15}$$

$$
\begin{aligned}
q_{k,n-k-1} &= P_{k,n-k}^2\left[\partial_1\left(\frac{P_{k,n-k-1}}{P_{k,n-k}}\right)+\alpha_2\partial_2\left(\frac{P_{k,n-k-1}}{P_{k,n-k}}\right)+\alpha_3\partial_3\left(\frac{P_{k,n-k-1}}{P_{k,n-k}}\right)+\right.\\
&+ \left.\partial_3(\alpha_3)\left(\frac{P_{k,n-k-1}}{P_{k,n-k}}\right)-(n-k)\partial_3(\gamma)\right] \qquad (16)
\end{aligned}
$$

Note that, if the equations (11) and (12) do not have any solution in $K(x_1,x_2,x_3)$, neither do have the equations (15) and (16). Hence, the ideal $A_3S + A_3R$ contains an element of the form:

$$R_1 := \ldots + 0{\partial_2}^k{\partial_3}^{n-k} + \ldots + q_{k-1,n-k}{\partial_2}^{k-1}{\partial_3}^{n-k} + q_{k,n-k-1}{\partial_2}^k{\partial_3}^{n-k-1} + \ldots$$

with $q_{k,n-k-1} \neq 0$ or $q_{k-1,n-k} \neq 0$; (actually if $1 \leq k \leq n-1$, then both coefficients are non zero).

So, $n-1 \leq \operatorname{ord}(R_1) \leq n$. Now, if we calculate $[S, R_1]$, the term with order $n$ eliminated before does not appear again. Hence, if $R_1$ has another term with order n, we use the same process to eliminate it and we get a new $R_2 \in A_3S + A_3R, R_2 \neq 0$, such that $n-1 \leq \operatorname{ord}(R_2) \leq n$.

Proceeding in this way, we obtain a non-zero element

$$\widetilde{R} \in A_3S + A_3R,\ \text{where } \operatorname{ord}(\widetilde{R}) = n-1.$$

Repeating this process, we see that

$$(A_3S + A_3R) \cap (K[x_1,x_2,x_3] \setminus \{0\}) \neq \emptyset.$$

Now, let $P = \sum_{i+j=0}^{N} P_{i,j}{x_2}^i{x_3}^j$ be in $(A_3S + A_3R) \cap (K[x_1,x_2,x_3] \setminus \{0\})$, where $P_{i,j} \in K[x_1]$. Similarly, since the equations (13) and (14) do not have any solution in $K(x_1)$, we conclude that

$$(A_3S + A_3R) \cap (K[x_1] \setminus \{0\}) \neq \emptyset.$$

Therefore, $A_3S + A_3R = A_3$. By lemma 2.4, $A_3S$ is a left maximal ideal of $A_3(K)$. □

Example 2.1. *Let $S = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n + \gamma \in A_n(K)$, where $\alpha_i(x_1,x_i) = \alpha_{i0}(x_1) + \alpha_{i1}(x_1)x_i \in K[x_1,x_i], i = 2,\ldots,n$. Observe that if $\gamma \in K[x_1]$, then the equations given by (9) have $u = 0$ as a solution. Therefore, $A_nS$ is never a maximal ideal of $A_n$.*

## 3. APPLICATIONS TO THE $D$-SIMPLICITY OF THE RING $K[X_1, \cdots, X_N]$

In this section we apply the previous section to study the $d$-simplicity of the ring $K[x_1, \cdots, x_n]$.

DEFINITION 3.1. Let $d$ be a derivation of a commutative ring $\mathcal{A}$. An ideal $I$ of $\mathcal{A}$ is a $d$-ideal if $d(I) \subset I$. We will say that $\mathcal{A}$ is $d$-simple if it does not contain any proper non-zero $d$-ideals.

If $\mathcal{A}$ is a Noetherian ring and $d$ is a derivation of $\mathcal{A}$, it is well known that $\mathcal{A}$ is $d$-simple if and only if no prime ideal of $\mathcal{A}$ is a $d$-ideal. As a corollary of our next theorem we obtain that for certain type of derivations of $K[x_1, \cdots, x_n]$ we need to check only if the prime ideals of height one are $d$-ideals.

THEOREM 3.1. *Let $d = \partial_1 + \alpha_2\partial_2 + \ldots + \alpha_n\partial_n$ be a derivation of $K[x_1, \ldots, x_n]$ where $\alpha_2, \ldots, \alpha_n \in K[x_1, \ldots, x_n]$. Then the following statements are equivalent:*

**(i)** *For every $P \in K[x_1, \ldots, x_n] \setminus K$ we have*

$$[d, P] \notin K[x_1, \ldots, x_n]P.$$

**(ii)** *$K[x_1, \ldots, x_n]$ is $d$-simple.*
**(iii)** *No principal ideal of $K[x_1, \ldots, x_n]$ is a $d$-ideal.*

**Proof: (i) $\Leftrightarrow$ (ii)**: Suppose that $K[x_1, \ldots, x_n]$ is not $d$-simple. Let $I$ be a proper non-zero $d$-ideal of $K[x_1, \ldots, x_n]$. Let $P = \sum_{k=0}^{l} r_k {x_n}^k$, where $r_k \in K[x_1, \ldots, x_{n-1}]$, be a non-zero polynomial contained in $I$ with the least degree in $x_n$.

If $l > 0$, since $d(P) \in I$, then $t := deg_{x_n}(d(P)) \geq l$. But,

$$d(P) = [d, P] \in I.$$

Then, $deg_{x_n}([d, P]) = t \geq l$.

By the Euclidean Algorithm (applied to $[S, P]$ and $P$ considered as elements in $K(x_1, \ldots, x_{n-1})[x_n]$), there exists an $g \in K[x_1, \ldots, x_{n-1}] \setminus \{0\}$ such that

$$g[S, P] = \eta P + r, \text{ for some } \eta, r \in K[x_1, \ldots, x_n], \text{ where } deg_{x_n}(r) < deg_{x_n}(P) \text{ or } r = 0.$$

But this implies that $r = 0$, by the choice of $P$. Thus, $g[d, P] = \eta P$. By lemma 2.6, there exist $\xi \in K[x_1, \ldots, x_n]$ and $\widetilde{P} \in K[x_1, \ldots, x_n] \setminus K$ such that

$$[d, \widetilde{P}] = \xi \widetilde{P}.$$

Conversely, suppose that there exists $P \in K[x_1, \ldots, x_n] \setminus K$ such that $[d, P] = \xi P$, for some $\xi \in K[x_1, \ldots, x_n]$.

Note that $[d, P] = d(P)$. Consider the ideal $I = (P) \subset K[x_1, \ldots, x_n]$. Let $f \in I$, then $f = \eta P$, for some $\eta \in K[x_1, \ldots, x_n]$. Hence,

$$d(f) = d(\eta P) = \underbrace{d(\eta)P}_{\in I} + \underbrace{\eta\xi P}_{\in I} \in I.$$

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Therefore, I is a $d$-ideal.
**(iii)**: It is equivalent to **(i)** since $[d,P]=d(P)$.

□

Corollary 3.1. *Let $d=\partial_1+\alpha_2\partial_2+\ldots+\alpha_n\partial_n$ be a derivation of $K[x_1,\ldots,x_n]$ where $\alpha_2,\ldots,\alpha_n\in K[x_1,\ldots,x_n]$. Then, $K[x_1,\ldots,x_n]$ is $d$-simple if and only if no prime ideal of height one is a $d$-ideal.*

**Proof:** By Theorem 3.1 $K[x_1,\ldots,x_n]$ is $d$-simple if and only if no principal ideal is a $d$-ideal. But if an ideal $I$ is a $d$-ideal, then every minimal prime of $I$ is also a $d$-ideal. By Krull's Principal Ideal Theorem, every minimal prime ideal of a principal ideal has height one.

□

Examples of derivations $d$ that make the polynomial ring $K[x_1,\ldots,x_n]$ into a $d$-simple ring are not easy to find out. A family of them were discovered by Coutinho in [2] (generalizing an example of Stafford) and are based in a result of Shamsuddin (See [6]).

Definition 3.2. A derivation $d$ of the polynomial ring $K[x_1,\ldots,x_n]$ is called a Shamsuddin's derivation if it has the following form:

$$d=\partial_1+\alpha_2(x_1,x_2)\partial_2+\ldots+\alpha_n(x_1,x_n)\partial_n$$

where $\alpha_j(x_1,x_j)=a_j(x_1)+b_j(x_1)x_j$ are polynomials in $x_1,x_j$ of degree one in $x_j$, $j=2,\ldots,n$.

Let $S$ be the element in $A_2(\mathbb{C})$, given by

$$S=\partial_1+(x_1x_2+\lambda {x_2}^2+1)\partial_2+\lambda(2-\mu)x_2,\ \lambda\notin\mathbb{Q}\text{ and }\mu\notin\mathbb{Z}.$$

Stafford proved in [1] (proposition 2.2) that$A_2S$ is a maximal left ideal of $A_2(\mathbb{C})$ (actually, our operator is obtained from Stafford's after a transposition and a change of indices).

Consider the derivation $d=\partial_1+(x_1x_2+\lambda {x_2}^2+1)\partial_2$ of $\mathbb{C}[x_1,x_2]$ extracted from S. Is $\mathbb{C}[x_1,x_2]$ $d$-simple?

The answer is yes and is provided by the next theorem. Note that in this case, we get an example where $\mathbb{C}[x_1,x_2]$ is $d$-simple and $d$ is not a Shamsuddin's derivation.

Theorem 3.2. *Let $S=d+\gamma$ be an element in $A_n(K)$, where $d=\partial_1+\alpha_2\partial_2+\ldots+\alpha_n\partial_n$ is a derivation of $K[x_1,\ldots,x_n]$, with $\alpha_2,\ldots,\alpha_n,\gamma\in K[x_1,\ldots,x_n]$. If $A_nS$ is a maximal left ideal of $A_n$, then $K[x_1,\ldots,x_n]$ is $d$-simple.*

**Proof:** Suppose that $K[x_1,\ldots,x_n]$ is not $d$-simple. Let $I$ be a proper non-zero $d$-ideal of $K[x_1,\ldots,x_n]$. Let $p=\sum_{k=0}^{l} r_k {x_n}^k$, where $r_k\in K[x_1,\ldots,x_{n-1}]$, be a non-zero polynomial contained in $I$ with the least degree in $x_n$.

If $l>0$, since $d(p)\in I$, then $t:=deg_{x_n}(d(p))\geq l$. But,

$$d(p)=[d,p]=[d+\gamma,p]=[S,p]\in I.$$

Then $deg_{x_n}([S,p]) = t \geq l$. By the Euclidean Algorithm (applied to $[S,p]$ and $p$ considered as elements in $K(x_1,\ldots,x_{n-1})[x_n]$), there exists an $g \in K[x_1,\ldots,x_{n-1}] \setminus \{0\}$ such that

$$g[S,p] = \eta p + r, \text{ for some } \eta, r \in K[x_1,\ldots,x_n] \text{ where } deg_{x_n}(r) < deg_{x_n}(p) \text{ or } r = 0.$$

But this implies that $r = 0$, by the choice of $p$. So, $g[S,p] = \eta p$ and this contradicts the fact of $A_n S$ be a maximal ideal, by lemma 2.6 and theorem 2.1(a).

Then $l = 0$ and $p \in K[x_1,\ldots,x_{n-1}]$. Hence, $I \cap K[x_1,\ldots,x_{n-1}] \neq (0)$. Proceeding in this way, we obtain that $I \cap K[x_1] \neq (0)$. Let $p \in I \cap K[x_1],\ p \neq 0$. We can write $p$ in the form:

$$p = a_m {x_1}^m + \ldots + a_0,\ a_i \in K, a_m \neq 0.$$

Since $p \in I$ and $I$ is a $d$-ideal, then $d(p) \in I$. So,

$$d(p) = [S,p] = m a_m {x_1}^{m-1} + \ldots + a_1 \in I.$$

Proceeding in this way, we obtain

$$m! a_m \in I, \text{ so } I = K[x_1,\ldots,x_n] \text{ (Contradiction!)}$$

Therefore, $K[x_1,\ldots,x_n]$ is $d$-simple.

□

The following lemma is useful to check if the equations 10 of Theorem 2.2 do have a solution.

Lemma 3.1. *Let $\alpha_1(x_1), \alpha_0(x_1) \in K[x_1]$. If the equation*

$$\partial_1(u) - \alpha_1(x_1) \cdot u = -\alpha_0(x_1) \tag{17}$$

*has a solution $\xi \in K(x_1)$, then $\xi \in K[x_1]$.*

**Proof:** Let $\xi = \frac{r}{s} \in K(x_1)$ be a solution of the equation (17), with $\gcd(r,s) = 1$. Thus,

$$\boxed{\partial_1\left(\frac{r}{s}\right) - \alpha_1 \cdot \left(\frac{r}{s}\right) = -\alpha_0} \tag{18}$$

We can write $s$ in the form:

$$s = a_0 + \ldots + a_n {x_1}^n, \text{ where } a_i \in K.$$

If $s(0) \neq 0$, then $a_0 \neq 0$. Therefore,

$$\frac{r}{s} = \frac{r}{a_0} \cdot \left( \underbrace{\frac{1}{1 + b_1 x_1 + \ldots + b_n {x_1}^n}}_{\text{it's a unit in } K[[x_1]]} \right).$$

So, $\frac{r}{s}$ is a solution of (18) that belongs to $K[[x_1]]$. Since $\xi \in K(x_1) \cap K[[x_1]]$, we have that $\xi$ is not transcendental over $K(x_1)$. Then, by proposition 2.3(b) of [5], we have that $\xi \in K[x_1]$.

We claim that $s(0) \neq 0$. In fact, note that the derivation of order $n$ of a product is given by

$$(f \cdot g)^n = \sum_{i=0}^{n} \binom{n}{i} f^{(i)} g^{(n-i)}, \text{ where } \begin{cases} f^{(0)} = f; f^{(i)} = \frac{\partial^i f}{\partial {x_1}^i}, \ i \geq 1 \\ g^{(0)} = g; g^{(i)} = \frac{\partial^i g}{\partial {x_1}^i}, \ i \geq 1 \end{cases}$$

Since $\gcd(r, s) = 1$, if $s(0) = 0$ then $r(0) \neq 0$.

From the equation (18), it follows that

$$\boxed{sr' - rs' - \alpha_1 rs + \alpha_0 s^2 = 0} \tag{19}$$

Evaluating at 0, we obtain that

$$\underbrace{s(0)}_{=0} r'(0) - r(0)s'(0) - \alpha_1(0)r(0)\underbrace{s(0)}_{=0} + \alpha_0(0)\underbrace{(s(0))^2}_{=0} = 0.$$

Then, $\underbrace{r(0)}_{\neq 0} s'(0) = 0 \Rightarrow s'(0) = 0.$

Now, suppose by induction that $s'(0) = s^{(2)}(0) = \ldots = s^{(n)}(0) = 0$. It follows from (19) that:

$$(sr' - rs' - \alpha_1 rs + \alpha_0 s^2)^{(n)} = 0$$
$$\sum_{i=0}^{n} \binom{n}{i} \{(s)^{(i)}(r')^{(n-i)} - (r)^{(i)}(s')^{(n-i)} - (\alpha_1 r)^{(i)}(s)^{(n-i)} + (\alpha_0)^{(i)}(s^2)^{(n-i)}\} = 0.$$

By induction hypothesis, we have that $\underbrace{r(0)}_{\neq 0} s^{(n+1)}(0) = 0 \Rightarrow s^{(n+1)}(0) = 0.$

Hence, $s^{(n)}(0) = 0, \forall\, n \in \mathbb{N}$, then $s \equiv 0$.This is a contradiction.

Therefore, $s(0) \neq 0$.

□

*Remark 3. 1.* In [5] (Theorem 4.1 a) Brumatti, Lequain and Levcovitz give a characterization of the $d$-simplicity of the ring $K[X, Y]$, where $d$ is a Shamsuddin derivation, in terms of the existence of a polynomial solution of a certain ordinary differential equation. The following theorem generalizes this result for an arbitrary number of variables.

THEOREM 3.3.*Let $d = \partial_1 + \alpha_2 \partial_2 + \ldots + \alpha_n \partial_n$ be a Shamsuddin derivation of $K[x_1, \ldots, x_n]$. Then the following statements are equivalent:*

**(i)**$K[x_1, \ldots, x_n]$ *is $d$-simple.*

**(ii)** *For every $P \in K[x_1,\dots,x_n]$, where $P$ is not a constant, we have*

$$[d,P] \notin K[x_1,\dots,x_n]P.$$

**(iii)** *The equations*

$$\partial_1(u) - \alpha_{i_1}\cdot u = -\alpha_{i_0},\ i = 2,\dots,n$$

*do not have any solution in $K(x_1)$.*

**(iv)** *The equations*

$$\partial_1(u) - \alpha_{i_1}\cdot u = -\alpha_{i_0},\ i = 2,\dots,n$$

*do not have any solution in $K[x_1]$.*

**Proof:** (**i**) $\Leftrightarrow$ (**ii**). This is a consequence of theorem 3.1.

(**ii**) $\Rightarrow$ (**iii**). Suppose that $\frac{r}{s} \in K(x_1)$ is a solution of the equation $\partial_1(u) - \alpha_{i_1}\cdot u = -\alpha_{i_0}$, for some $i \in \{2,\dots,n\}$. Let $P = r(x_1) + s(x_1)x_i$. Then, $[d,P] = r' + s'x_i + \alpha_i s$, where $r'$ and $s'$ denote the derivative of $r$ and $s$.

Since $\alpha_i(x_1,x_i) = \alpha_{i_0}(x_1) + \alpha_{i_1}(x_1)x_i$, we have

$$[d,P] = (r' + \alpha_{i_0}s) + (s' + \alpha_{i_1}s)x_i.$$

Then,

$$s[d,P] - (s' + \alpha_{i_1}s)P = s^2\left(\underbrace{\partial_1\left(\frac{r}{s}\right) - \alpha_{i_1}\left(\frac{r}{s}\right) + \alpha_{i_0}}_{=0}\right) = 0.$$

Therefore, $s[d,P] = (s' + \alpha_{i_1}s)P$. By lemma 2.6, there exists $\xi \in K[x_1,\dots,x_n]$ and $\widetilde{P} \in K[x_1,\dots,x_n] \setminus K$ such that $[d,\widetilde{P}] = \xi\widetilde{P}$.

(**iii**) $\Rightarrow$ (**iv**). It is obvious.

(**iv**) $\Rightarrow$ (**i**). Suppose that the equations

$$\partial_1(u) - \alpha_{i_1}\cdot u = -\alpha_{i_0},\ i = 2,\dots,n$$

do not have any solution in $K[x_1]$. By lemma 3.1, we can conclude that these equations also do not have any solution in $K(x_1)$.

Now, let $I$ be a non-zero $d$-ideal of $K[x_1,\dots,x_n]$ and let $P$ be a non-zero polynomial contained in $I$.

We will show that $I = K[x_1,\dots,x_n]$. So, $K[x_1,\dots,x_n]$ is $d$-simple.

If $P$ is a non-zero constant, then there is nothing to be done. So, we shall suppose from now on that $P$ is not a constant.

We can write $P$ in the form:

$$P = \sum_{i_2+\dots+i_n=0}^{N} P_{i_2,\dots,i_n} {x_2}^{i_2}\dots {x_n}^{i_n},\ \text{where } P_{i_2,\dots,i_n} \in K[x_1].$$

If $N=0$ then $P\in K[x_1]\setminus K$ and we are done.

If $N>0$, consider the degree function as the degree in the variables $x_2,\ldots,x_n$ ( that is $deg(P)=N>0$) we have that

$$d(P)=[d,P]=\sum_{i_2+\ldots+i_n=0}^{N}\left[d,P_{i_2,\ldots,i_n}{x_2}^{i_2}\ldots{x_n}^{i_n}\right].$$

A simple calculation shows that

$$\begin{array}{l} d(P)=\sum_{i_2+\ldots+i_n=0}^{N}\{[\partial_1(P_{i_2,\ldots,i_n})+P_{i_2,\ldots,i_n}(i_2\alpha_{2_1}+\ldots+i_n\alpha_{n_1})]{x_2}^{i_2}\ldots{x_n}^{i_n}\\ +[P_{i_2,\ldots,i_n}i_2\alpha_{2_0}]{x_2}^{i_2-1}x_3^{i_3}\ldots{x_n}^{i_n}+\ldots+[P_{i_2,\ldots,i_n}i_n\alpha_{n_0}]{x_2}^{i_2}\ldots{x_n}^{i_n-1}\\ +\text{terms with degree lower than } i_2+\ldots+i_n-1\}. \end{array}$$

As $deg(P)=N>0$, we have that there exists $P_{i_{2_0},\ldots,i_{n_0}}\neq 0$, such that

$$i_{2_0}+\ldots+i_{n_0}=N.$$

Thus,

$$\begin{array}{rcl} d(P) & = & \ldots+[\partial_1(P_{i_{2_0},\ldots,i_{n_0}})+P_{i_{2_0},\ldots,i_{n_0}}(i_{2_0}\alpha_{2_1}+\ldots+i_{n_0}\alpha_{n_1})]{x_2}^{i_{2_0}}\ldots{x_n}^{i_{n_0}}+\ldots\\ & + & [\partial_1(P_{i_{2_0}-1,\ldots,i_{n_0}})+P_{i_{2_0}-1,i_{3_0},\ldots,i_{n_0}}((i_{2_0}-1)\alpha_{2_1}+\ldots+i_{n_0}\alpha_{n_1})+\\ & + & P_{i_{2_0},\ldots,i_{n_0}}i_{2_0}\alpha_{2_0}]{x_2}^{i_{2_0}-1}x_3^{i_{3_0}}\ldots{x_n}^{i_{n_0}}+\ldots+[\partial_1(P_{i_{2_0},\ldots,i_{n_0}-1})+\\ & + & P_{i_{2_0},\ldots,i_{n_0}-1}(i_{2_0}\alpha_{2_1}+\ldots+(i_{n_0}-1)\alpha_{n_1})+\\ & + & P_{i_{2_0},\ldots,i_{n_0}}i_{n_0}\alpha_{n_0}]{x_2}^{i_{2_0}}\ldots{x_n}^{i_{n_0}-1}+\ldots \end{array}$$

Now, put $\lambda:=\partial_1(P_{i_{2_0},\ldots,i_{n_0}})+P_{i_{2_0},\ldots,i_{n_0}}(i_{2_0}\alpha_{2_1}+\ldots+i_{n_0}\alpha_{n_1})$ the coefficient ${x_2}^{i_{2_0}}\ldots{x_n}^{i_{n_0}}$ in $d(P)$.

Then,

$$\begin{array}{l} \underbrace{P_{i_{2_0},\ldots,i_{n_0}}d(P)-\lambda P}_{\in I}=\ldots+0{x_2}^{i_{2_0}}\ldots{x_n}^{i_{n_0}}+\ldots+\\ +P^2_{i_{2_0},\ldots,i_{n_0}}[\partial_1(\frac{P_{i_{2_0}-1,i_{3_0},\ldots,i_{n_0}}}{P_{i_{2_0},\ldots,i_{n_0}}})-\alpha_{2_1}(\frac{P_{i_{2_0}-1,i_{3_0},\ldots,i_{n_0}}}{P_{i_{2_0},\ldots,i_{n_0}}})+i_{2_0}\alpha_{2_0}]{x_2}^{i_{2_0}-1}x_3^{i_{3_0}}\ldots{x_n}^{i_{n_0}}+\ldots\\ \vdots\\ +P^2_{i_{2_0},\ldots,i_{n_0}}[\partial_1(\frac{P_{i_{2_0},\ldots,i_{n_0}-1}}{P_{i_{2_0},\ldots,i_{n_0}}})-\alpha_{n_1}(\frac{P_{i_{2_0},\ldots,i_{n_0}-1}}{P_{i_{2_0},\ldots,i_{n_0}}})+i_{n_0}\alpha_{n_0}]{x_2}^{i_{2_0}}\ldots{x_n}^{i_{n_0}-1}+\ldots \end{array}$$

Hence, it is easy to see that if the equations $\partial_1(u)-\alpha_{i_1}\cdot u=-\alpha_{i_0},\ i=2,\ldots,n$ do not have any solution in $K(x_1)$, neither do have the equations

$$\partial_1(u)-\alpha_{j_1}\cdot u=-i_{j_0}\alpha_{j_0},\ j=2,\ldots,n,\ \text{whith } i_{j_0}\neq 0.$$

But since $i_{2_0} + \ldots + i_{n_0} = N > 0$ and $i_{2_0}, \ldots, i_{n_0} \in \mathbb{N}$, it follows that there exists some $i_{j_0} \neq 0,\ j \in \{2, \ldots, n\}$.

Thus, the ideal $I$ contains a non zero element of the form:

$$\begin{array}{l} P_1 = \ldots + 0{x_2}^{i_{2_0}} \ldots {x_n}^{i_{n_0}} + \ldots \\ + \underbrace{q_{i_{2_0}-1, i_{3_0}, \ldots, i_{n_0}}}_{\in K[x_1]} {x_2}^{i_{2_0}-1} {x_3}^{i_{3_0}} \ldots {x_n}^{i_{n_0}} + \ldots \\ \vdots \\ \underbrace{q_{i_{2_0}, \ldots, i_{n_0}-1}}_{\in K[x_1]} {x_2}^{i_{2_0}} \ldots {x_n}^{i_{n_0}-1} + \ldots \end{array}$$

such that $N - 1 \leq deg(P_1) \leq N$.

*Claim 3.1.* If we repeat this process with $P_1$, the term with degree $N$ eliminated before does not appear again.

Let's assume, for a while, claim 3.1.

In this way, if $P_1$ has another term with degree $N$, we use the same process to eliminate it and we get a new $P_2 \in I, P_2 \neq 0$, such that $N - 1 \leq deg(P_2) \leq N$.

Proceeding in this way, we obtain a non-zero element $\widetilde{P} \in I$, where $deg(\widetilde{P}) = N - 1$.

Repeating this process, we see that

$$I \cap (K[x_1] \setminus \{0\}) \neq \emptyset.$$

Therefore, $I = K[x_1, \ldots, x_n]$.

To finish the prove, we have to show claim 3.1.

**Proof of the claim 3.1:** Let $P = \sum_{i_2+\ldots+i_n=0}^{N} P_{i_2,\ldots,i_n} {x_2}^{i_2} \ldots {x_n}^{i_n}$, with $P_{i_2,\ldots,i_n} \in K[x_1]$, where $P$ is not a constant.

We know that

$$\begin{array}{l} d(P) = \sum_{i_2+\ldots+i_n=0}^{N} \{[\partial_1(P_{i_2,\ldots,i_n}) + P_{i_2,\ldots,i_n}(i_2\alpha_{2_1} + \ldots + i_n\alpha_{n_1})]{x_2}^{i_2} \ldots {x_n}^{i_n} \\ +[P_{i_2,\ldots,i_n} i_2\alpha_{2_0}]{x_2}^{i_2-1}{x_3}^{i_3} \ldots {x_n}^{i_n} + \ldots + [P_{i_2,\ldots,i_n} i_n\alpha_{n_0}]{x_2}^{i_2} \ldots {x_n}^{i_n-1} \\ +\text{terms with degree lower than } i_2 + \ldots + i_n - 1\}. \end{array}$$

Hence, the terms with degree $N$ in $d(P)$ are:

$$\sum_{i_2+\ldots+i_n=N} \{[\partial_1(P_{i_2,\ldots,i_n}) + P_{i_2,\ldots,i_n}(i_2\alpha_{2_1} + \ldots + i_n\alpha_{n_1})]{x_2}^{i_2} \ldots {x_n}^{i_n}$$

Now, observe that if $P_{i_{2_0},\ldots,i_{n_0}}$ is the coefficient of ${x_2}^{i_{2_0}} \ldots {x_n}^{i_{n_0}}$ in $P$, with $i_{2_0}+\ldots+i_{n_0} = N$, then the corresponding coefficient in d(P) is:

$$q_{i_{2_0},\ldots,i_{n_0}} := \partial_1(P_{i_{2_0},\ldots,i_{n_0}}) + P_{i_{2_0},\ldots,i_{n_0}}(i_{2_0}\alpha_{2_1} + \ldots + i_{n_0}\alpha_{n_1}).$$

Note that, $q_{i_{2_0},\ldots,i_{n_0}}$ depends only on $P_{i_{2_0},\ldots,i_{n_0}}$. Therefore, if $P_{i_{2_0},\ldots,i_{n_0}} = 0$, then $q_{i_{2_0},\ldots,i_{n_0}} = 0$ and we have that the coefficient of ${x_2}^{i_{2_0}} \ldots {x_n}^{i_{n_0}}$ in $\xi d(P) - \eta P$ is zero, for all $\xi, \eta \in K[x_1]$.

□

We will now use our theorem 3.3 to recover theorem 3.3 of [2]. He considers, for $2 \leq i \leq n$, $a_i, b_i$ non-zero polynomials in $K[x_1]$ such that:

1. $a_2, \ldots, a_n$ are linearly independent over $\mathbb{Q}$ and
2. $deg(a_i) > deg(b_i)$ for $i = 2, \ldots, n$.

He shows that $K[x_1, \ldots, x_n]$ is $d$-simple with respect to the derivation

$$d = \partial_1 + \sum_{i=2}^{n} (x_i a_i + b_i)\partial_i.$$

One advantage of our approach is that we don't need the polynomials $a_2, \ldots, a_n$ to be linearly independent over $\mathbb{Q}$.

Example 3.2. *Consider, for $2 \leq i \leq n$, $a_i, b_i$ non-zero polynomials in $K[x_1]$ such that $deg(a_i) > deg(b_i)$ for $i = 2, \ldots, n$. Then, the derivation*

$$d = \partial_1 + \sum_{i=2}^{n} (x_i a_i + b_i)\partial_i$$

*makes the ring $K[x_1, \ldots, x_n]$ into a d-simple ring.*

*In fact, we must analyze if the equations*

$$\partial_1(u) - a_i \cdot u = -b_i, \; i = 2, \ldots, n$$

*do not have any solution in $K[x_1]$.*

*Observe that if $u \in K[x_1]$ is a solution, then,*

$$\underbrace{\partial_1(u)}_{deg(u)-1} - \underbrace{a_i \cdot u}_{deg(a_i)+deg(u)} = - \underbrace{b_i}_{deg(b_i)}, \; i = 2, \ldots, n$$

*Since $deg(a_i) > deg(b_i), i = 2, \ldots, n$, these equations do not have any solution in $K[x_1]$. By theorem 3.3, it follows that $K[x_1, \ldots, x_n]$ is d-simple.*

We give now a different family of Shamsuddin's derivations that make $K[x_1, \ldots, x_n]$ into a $d$-simple ring.

Example 3.3. *For $2 \leq i \leq n$, let $a_i, b_i$ be monic polynomials in $K[x_1]$ such that $deg(a_i) = deg(b_i) = k_i$. Then $K[x_1, \ldots, x_n]$ is d-simple with respect to the derivation*

$$d = \partial_1 + ({x_1}^2 b_2 + x_1 a_1 x_2)\partial_2 + \ldots + ({x_1}^2 b_n + x_1 a_n x_n)\partial_n.$$

*By theorem 3.3 we must analyze if the equations*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

$$\partial_1(u) - x_1 a_i u = -{x_1}^2 b_i,\ \ i = 2, \ldots, n \tag{20}$$

*do not have any solution in $K[x_1]$.*
*Observe that if $u \in K[x_1]$ is a solution of an equation given in (20), then*

$$\underbrace{\partial_1(u)}_{deg(u)-1} - \underbrace{x_1 a_i u}_{deg(u)+k_i+1} = \underbrace{-{x_1}^2 b_i}_{k_i+2} \Rightarrow deg(u) = 1.$$

*We can write $a_i, b_i$ and $u$ in the form:*

$$\begin{aligned} a_i &= {x_1}^{k_i} + {}_i a_{k_i-1} {x_1}^{k_i-1} + \ldots + {}_i a_0 \\ b_i &= {x_1}^{k_i} + {}_i b_{k_i-1} {x_1}^{k_i-1} + \ldots + {}_i b_0 \\ u &= c x_1 + d \end{aligned}$$

*Thus, from (20) it follows that the polynomial $(-c+1){x_1}^{k_i+2} + \ldots + c \equiv 0$, which is a contradiction.*

*Therefore, the equations given by (20) do not have any solution in $K[x_1]$, so $K[x_1, \ldots, x_n]$ is d-simple.*

## REFERENCES


1. J. T. Stafford, *Non-holonomic modules over Weyl Algebras and Enveloping Algebras*, Invent. Math. 79 (1985), pp. 619-638.
2. S. C. Coutinho, *d-Simple Rings and Simple D-modules*, Math. Proc. Caamb. Phil. Soc. 125(1999), pp. 405-415.
3. S. C. Coutinho, *A primer of algebraic D-modules*. London Mathematical Society Student Texts 33 (Cambridge University Press, 1995).
4. G. Bratti & M. Takagi, *Differential equations and maximal ideals on the Weyl Algebra* $A_2(\mathbb{C})$ (Preprint).
5. Paulo Brumatti, Yves Lequain & Daniel Levcovitz, *Differential simplicity of polynomial rings and algebraic independence of power series* (Preprint).
6. A. Shamsuddin, *Ph.D. Thesis*, University of Leeds (1977).
7. J. Bernstein and V. Lunts, *On non-holonomic irreducible D-modules*. Invent. Math. 94 (1988), pp. 233-243.


Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC