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A structure theorem for foliations on non-compact two-manifolds

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We consider singular orientable foliations, which admit nontrivial recurrent leaves, on two-manifolds of finite or infinite genus. We give a structure theorem for this foliations. This one is similar to Gutierrez's structure theorem [Gu1] for flows on compact surfaces. May, 2002 ICMC-USP

1. INTRODUCTION

In this paper we study the structure of non-trivial recurrence leaves. The obtained results have been shown to be very useful in the smoothing of the continuous singular foliations on arbitrary genus two-manifolds [Lo1]. From the famous example of Denjoy [De], it is clear that it is very important to understand the dynamical structure of non-trivial recurrence. For compact two-manifolds, the study of non-trivial recurrence together with the smoothing problems for orientable singular foliations was carried out by C. Gutierrez [Gu1]. On the other hand, recent work shows that infinite genus surfaces can support orientable foliations whose recurrent leaves have dynamics which can not appear on compact surfaces (see [Gu-He-Lo]). Besides, the existence of minimal foliations on those surfaces was already proven by J. Beniere (see [Be]). Our main theorem generalize, to orientable singular foliations on two-manifolds of infinite genus, C. Gutierrez's structure theorem [Gu1]. To state the theorem, we need some definitions.

Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a map whose domain of definition ($\mathrm{Dom}(T)$) and image ($\mathrm{Im}(T)$) are open and dense subsets of $\mathbb{R}/\mathbb{Z}$. We say that T is a *generalized interval exchange transformation*(or shortly a GIET) if T takes homeomorphically each connected component of its domain of definition onto a connected component of its image. A GIET is said to be *affine* (resp. *isometric*) if it restricted to every connected component of its domain of definition, is affine (resp. isometric). If T is an isometric GIET such that $\mathbb{R}/\mathbb{Z} \setminus \mathrm{Dom}(T)$ is at most finite, then we shall say that T is a *standard interval exchange transformation* (standard IET).

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Let $F, G : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be two transformations whose domains of definitions are open subsets of $\mathbb{R}/\mathbb{Z}$. We shall say that F is *semi-conjugate* to G if there exists a continuous map $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ preserving orientation such that $h \circ F(x) = G \circ h(x)$ for all point $x \in \mathrm{Dom}(F)$ satisfying $h(x) \in \mathrm{Dom}(G)$. If h is a homeomorphism, then we shall say that the transformation F is *conjugate* to G. Notice that our definition of semi-conjugation and conjugation are not exactly as the usual ones because the involved maps are not defined everywhere. Let M be a foliated manifold. A leaf L of a foliation of M will be called *non-trivial recurrent leaf* if $L \subset \overline{L} \setminus L$, and *exceptional leaf* if $\overline{L}$ have the local structure of a Cantor set. Here $\overline{L}$ denotes the topological closure of L in M.

A non-compact two-manifold M is said to be of *finite genus* if there exists a compact bordered manifold $N \subset M$ such that $M \setminus N$ is homeomorphic to a subset of the plane $\mathbb{R}^2$. Otherwise, M is said to be of *infinite genus.* The *genus* of a compact bordered two-manifold N is defined to be the genus of the compact manifold obtained by attaching a disc to each boundary component of N.

Our main result is the following:

THEOREM. *Let $\mathcal{F}$ be an orientable singular foliation on a smooth Hausdorff two-manifold M of finite or infinite genus with a countable base. Then there is a collection, at most denumerable of circles $\{C_i\}_{i\in\Delta}$ transversal to $\mathcal{F}$ such that the following conditions are satisfied:*

(1) *The saturated of the circles C_i by $\mathcal{F}$ are pairwise disjoint and each non-trivial recurrent leaf of $\mathcal{F}$ either intersects some C_i or belongs to the closure of the union of the saturated of the circles C_i.*

(2) *If T_i denotes the forward Poincaré map in C_i induced by $\mathcal{F}$, then each circle C_i is of one of the following different types:*

Finite type: The saturated of C_i by $\mathcal{F}$ is contained in a submanifold of M (possibly with boundary) of finite genus and there exists a non-trivial recurrent leaf L such that either:

(2.1) *$\overline{L \cap C_i} = C_i$ and T_i is conjugate to a standard IET, or*

(2.2) *$\overline{L \cap C_i}$ is a Cantor set and T_i is semi-conjugate to a standard IET.*

Infinite type: The circle C_i is not of finite type and it satisfies one of the following properties:

(2.3) *There is a recurrent leaf intersecting C_i densely and T_i is conjugate to an affine GIET.*

(2.4) *The points in C_i belonging to recurrent leaves form a residual dense subset of C_i and every recurrent leaf intersecting C_i is exceptional. The map T_i is conjugate to an affine GIET.*

(2.5) *The circle C_i contains a point belonging to an exceptional leaf L. The union of the wandering sets of C_i with the interior of the set of points belonging to compact leaves form an open, dense subset of C_i. Besides, T_i is semi-conjugate to an affine GIET. In the* *particular case that $\overline{L}$ is a minimal compact set, then the wandering points is open and dense in C_i and T_i is semi-conjugate to a standard IET.*

We do not know if the map T_i, from items (2.3) and (2.4) (resp. (2.5)) is conjugate (resp. semi-conjugate) to an isometric GIET.

Now we wish to mention other results related to the structure of non-trivial recurrence. Gardiner [Ga] showed that flows on a compact orientable two-manifolds admit transversal circles which permit a decomposition of the manifolds into components in such a way that the closure of each nontrivial recurrent trajectories is contained in some of the referred components. Similar results for singular foliations, with saddle type singularities, was proved by G. Levitt [Le]. The structure theorem of Aranson and Zhuzhoma [Ar-Zh] gives an estimative of the number of sets determined by closure of non-trivial recurrent half-leaves, to orientable (and a 'class' of non-orientable) foliations on compact surfaces with finite singularities.

The paper is organized as follows. In §3 the manifold is turned compact by their ends. The structure of non-trivial recurrent leaves at the region, invariant by foliation, of finite or infinite genus, is studied separately by introducing the notion of point of finite type. In §4 we generalize the equivalence relation introduced by C. Gutierrez in [Gu1] and in the last part of this section we give a precise description of the local structure of the non-trivial recurrent leaves which intersect no circle C_i of theorem. In §5 we prove a fundamental proposition: Every continuous injective map with a dense set of non-trivial recurrent points is conjugate to an affine GIET.

2. PRELIMINARIES

Let M be a two-manifold and $\mathcal{F}$ an orientable foliation of M with a closed set of singularities $\text{sing}(\mathcal{F}) \subset M$. A leaf of $\mathcal{F}$ that is homeomorphic to a circle is called a compact leaf. The leaf of $\mathcal{F}$ passing through the point $p \in M \setminus \text{sing}(\mathcal{F})$ will be denoted by $L(p)$. Let L be a non-compact leaf, a point $p \in L$ divides L into two half-leaves. Thus fixed an orientation of $\mathcal{F}$, $L^+(p)$ (resp. $L^-(p)$) denotes the component of $L \setminus \{p\}$ which keeps the positive (resp. negative) orientation of $\mathcal{F}$ beginning in p. If $q \in L^+(p)$, then $L(p, q)$ denotes the segment of the half-leaf in $L^+(p)$ with endpoints p, q. Let $L^+(p)$ (resp. $L^-(p)$) be a half-leaf and let $\gamma : L^+(p) \to [0, +\infty)$ (resp. $\gamma : L^-(p) \to (-\infty, 0]$) be a parametrization of $L^+(p)$ (resp. $L^-(p)$) with $\gamma(0) = p$ which is a homeomorphism in the intrinsic topology of $L^+(p)$ (resp. $L^-(p)$). The *ω-limit points* of $L^+(p)$ (resp. *α-limit points* of $L^-(p)$) are the set of points $q \in M$ such that each neighborhood of q contains points of $L^+(p)$ (resp. $L^-(p)$) corresponding to arbitrarily large values of the parameter. The set of *ω-limit points* (resp. *α-limit points*) of $L^+(p)$ (resp. $L^-(p)$) will be denoted by $\omega(L^+(p))$ (resp. $\alpha(L^-(p))$). Such a set is closed, invariant and non-empty when M is a compact two-manifold. We say that the leaf $L(p)$ is *ω-recurrent* (resp. *α-recurrent*) if $L(p) \subset \omega(L^+(p))$ (resp. $L(p) \subset \alpha(L^-(p))$) and we say that $L(p)$ is recurrent if it is ω- and α- recurrent. Thus $L(p)$ is a non-trivial recurrent leaf if $L(p)$ is not a compact leaf. We say that $L(p)$ is a proper leaf when $\overline{L(p)} \setminus L(p)$ is closed in M, locally dense if $\overline{L(p)}$ has non-empty interior, and exceptional if $L(p)$ is non-proper and nowhere dense. Let B be a subset of M. The

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saturated of B by the foliations $\mathcal{F}$ (or shortly $\operatorname{Sat}(B)$) is the set formed by the union of leaves $L(p)$, with $p \in B \setminus \sin(\mathcal{F})$, and the set $\operatorname{sing}(\mathcal{F}) \cap B$. A set $B \subset M$ which does not contain singularities of $\mathcal{F}$ is a *minimal set* of $\mathcal{F}$ if it is a closed, non-empty, and invariant set (i.e., $\operatorname{Sat}(B) = B$), which is minimal (in the sense of inclusion) for these properties.

A segment $\Sigma \subset M \setminus \operatorname{sing}(\mathcal{F})$ is called a segment transversal to the foliation $\mathcal{F}$ if every point $p \in \Sigma$ has a neighborhood U with local coordinates $(x; y) : U \to \mathbb{R}^2$ such that the leaves in U are determined by the equation $y =$const and the intersection $U \cap \Sigma$ is given by the equation $x =$const. A closed curve $C \subset M \setminus \operatorname{sing}(\mathcal{F})$ is called a transversal circle if each segment in C is transversal to $\mathcal{F}$. Let C_1 (resp. C_2) be a segment or a circle transversal to $\mathcal{F}$. The *forward Poincaré map* (resp. backward Poincaré map) induced by $\mathcal{F}$ is the mapping which associates with each point $p \in C_1$ (resp. $q \in C_2$), the first point where $L^+(p)$ (resp. $L^-(q)$) intersects C_2 (resp. C_1).

Let C be a circle transversal to the foliation $\mathcal{F}$. Fix an orientation "$<$" in C compatible with the orientation of $\mathbb{R}/\mathbb{Z}$. Given $a, b \in C$, *we define the open interval* (a, b) in C as $(a, b) = \{x \in C \setminus \{a, b\}; a < x \text{ and } x < b\}$. The interval in C closed in the left and open in the right $[a, b)$ is defined as the set $(a, b) \cup \{a\}$. Analogously $[a, b] = (a, b) \cup \{a, b\}$.

A segment Σ which is an open interval transversal to the foliation $\mathcal{F}$ will be said to be *wandering* if every leaf of $\mathcal{F}$ intersects Σ at most once. A point $p \in M \setminus \operatorname{sing}(\mathcal{F})$ is said to be wandering if there is a wandering transversal open interval containing p. For convention, every singularity of $\tilde{\mathcal{F}}$ will be considered as a non-wandering point. Let C be a topological circle and let $T : I \to I$ be a map defined in $I \in \{\mathbb{R}/\mathbb{Z}, C\}$. The positive (resp. negative) orbit $O^+(p)$ (resp. $O^-(p)$) of T through p will be the set $\{T^n(p); n \in \mathbb{Z}^+(\text{resp.}\mathbb{Z}^-) \text{ and } p \in \operatorname{Dom}(T^n)\}$. The orbit $O(p)$ of T through p will be the set $O^-(p) \cup \{p\} \cup O^+(p)$. The point $p \in I$ will be said *recurrent* (*T-recurrent*) if it is an accumulation point of its orbit $O(p)$. A point p is *non-trivial recurrent* if it is a recurrent point such that $O(p) \subset \overline{\overline{O(p)} \setminus O(p)}$.

3. THE COMPACTIFICATION

We begin this section announcing a fundamental theorem, proved by I. Richard [Ri], which permits to give a concrete representation of a non-compact two-manifold.

THEOREM 3.1. ([*Ri*]) *Every two-manifold M is homeomorphic to a surface formed from a sphere S^2 by first removing a closed totally disconnected set $\beta(M)$ from S^2, then removing the interior of a finite or infinite sequence $D_1, D_2, \ldots$ of nonoverlapping closed discs in $S^2 - \beta(M)$, and finally suitably identifying the boundaries of these discs in pairs (it may be necessary to identify the boundary of one disc with itself to produce an add "cross cap"). The sequence $D_1, D_2, \ldots$ "approaches $\beta(M)$" in the sense that, for any open set U in S^2 containing $\beta(M)$, all but a finite number of the D_i are contained in U.*

The space $\overline{M} = M \cup \beta(M)$ endowed with the usual topology is a compact, connected Hausdorff space with countable base. Notice that if M has infinite genus then $\overline{M}$ is no more a surface. The set $\beta(M)$ is called the *space of ends* of M. The elements of $\beta(M) \subset \overline{M}$ belonging to the accumulation set of the sequence D_i, $i = 1, 2, \ldots$ will be called *non-flat ends*. If $\mathcal{F}$ is a foliation of M then the compactification of M by its ends induces a foliation on $\overline{M} \setminus \beta(M)$ that we will denote by $\tilde{\mathcal{F}}$. The union of the closed totally disconnected set $\beta(M)$ with the singularities of $\mathcal{F}$ will be called singularities of $\tilde{\mathcal{F}}$ and we will denote it by sing($\tilde{\mathcal{F}}$). We say that $p \in \overline{M}$ is a *flat (resp. not-flat) singularity* of $\tilde{\mathcal{F}}$ if p is a flat (resp. not-flat) end of M.

DEFINITION. *A point $p \in M \setminus sing(\mathcal{F})$ is said to be of finite type if and only if there exists neighborhoods of p whose saturated by $\mathcal{F}$ is contained in a sub-manifold of M (possibly with boundary) of finite genus.*

Here and subsequently, we will use the following notation: $\Omega(\tilde{\mathcal{F}})$ will denote the non-wandering point set of $\overline{M}$, $\Omega_1(\tilde{\mathcal{F}})$ are the elements of $\Omega(\tilde{\mathcal{F}})$ of finite type, and $\Omega_2(\tilde{\mathcal{F}})$ is the set $\Omega(\tilde{\mathcal{F}}) \setminus \Omega_1(\tilde{\mathcal{F}})$. Notice that it follows from the definition that $\Omega_2(\tilde{\mathcal{F}})$ is a closed subset of $\overline{M}$ clearly invariant by $\tilde{\mathcal{F}}$.

DEFINITION. *A minimal subset of the foliation $\mathcal{F}$ in the manifold M is of limited type, if there exists a compact region of M containing it.*

Let $\Delta \subset M$ be a minimal subset of $\mathcal{F}$. If Δ is compact in M then it is immediate that it is a minimal set of limited type. On the other hand if Δ is not of limited type then from the compactification of M by its ends it follows that the closure of Δ in $\overline{M}$ contains at least one singularity of $\tilde{\mathcal{F}}$, planar or not planar. If M is compact, then every minimal set of $\mathcal{F}$ is of limited type.

The following lemma is a consequence of M. Peixoto [Pe] and the foliation theory (see [Ca-Ne]).

LEMMA 3.1. *For every point p belonging to a non-trivial recurrent leaf of $\tilde{\mathcal{F}}$ and for every open neighborhood V of p, there exists circle C transversal to $\tilde{\mathcal{F}}$ passing through p such that:*

(1)*$Sat(V) \supset Sat(C)$;*
(2)*If $L(p)$ is the leaf passing through p, then either $\overline{L(p) \cap C} = C$ or $\overline{L(p) \cap C}$ is a Cantor set.*

4. AN EQUIVALENCE RELATION.

Let C be a circle transversal to $\tilde{\mathcal{F}}$ and $T : C \to C$ the forward Poincaré map induced by the oriented foliation $\tilde{\mathcal{F}}$. Given $x, y \in C$, we say that *x is equivalent to y* and we will denote it by $x \sim y$ if, and only if, we have the following two conditions:

• there is a closed sub-interval $[p, q]$ of C with endpoints $p, q \in$Dom(T) such that $x, y \in [p, q]$;

• If $D_{p,q}$ denotes the region of $\overline{M}$ whose boundary is equal to $[p, q] \cup L(p, T(p)) \cup L(q, T(q)) \cup [r, s]$, where $[r, s]$ is $[T(p), T(q)]$ if $T(p) < T(q)$ or $[T(q), T(p)]$ if $T(q) < T(p)$, then $S(D_{p,q}) := \text{int}(\overline{\text{Sat}(C) \cap D_{p,q}})$ is an open disc punctured by a countable set of disc whose boundaries are formed by separatrices going from a singularity to another one.

Is not hard to see that $\sim$ determines an equivalence relation on C. We will call the equivalence relation $\sim$, the relation associated to T.

We remark that the relation defined above is 'larger' than the one defined by Gutierrez in [Gu1], i.e. if $x, y \in C$ are equivalent according to the definition of Gutierrez, then they are equivalent according to the relation $\sim$ defined above. Clearly the converse is false.

4.1. The map T_C.

Let C be a circle transversal to $\tilde{\mathcal{F}}$ and define $Rec(C)$ to be the set of elements of C that belong to non-trivial recurrent leaves of $\tilde{\mathcal{F}}$. If $Rec(C) \neq \emptyset$ consider the set $\mathcal{A}(C)$ made up of all the closed sub-intervals $[a, b]$ of C that satisfy one of the following two conditions:

- $[a, b]$ is the closure of a connected component of $C \setminus \overline{Rec(C)}$
- $a = b$ and a does not belong to the closure of any connected component of $C \setminus \overline{Rec(C)}$

Let $T : C \to C$ be the forward Poincaré map on C induced by the oriented foliation $\tilde{\mathcal{F}}$. We define the mapping $T_C : \mathcal{A}(C) \to \mathcal{A}(C)$ as follows: if $[a, b], [c, d] \in \mathcal{A}(C)$ we will write $T_C([a, b]) = [c, d]$ if, and only if, we have one of the following conditions:

- $a = b \in Rec(C)$ and $[c, d] = \{T(a)\}$
- $[a, b] \notin Rec(C)$ and there are sequences $\{p_n\}, \{q_n\}$ in $\mathrm{Dom}(T) \cap Rec(C)$ with $a \sim b \sim p_n \sim q_n$ such that $a = \lim p_n$, $b = \lim q_n$, and $\{c, d\} = \{\lim T(p_n), \lim T(q_n)\}$

Here $\sim$ is the relation associated to T. On the other hand, as no point in $Rec(C)$ is isolated, the elements of $\mathcal{A}(C)$ are disjoint, closed subsets of C, and therefore $\mathcal{A}(C)$ is a partition of C. In what follows, we will consider $\mathcal{A}(C)$ with the quotient topology.

As an immediate consequence of the introduced definition we have the following lemma.

Lemma 4.1.

(1) *The equivalence class on C determined by the relation $\sim$ associated to T are open, connected subsets of C.*

(2) *$\mathcal{A}(C)$ is homeomorphic to C and T_C is a continuous and injective map.*

Lemma 4.2. *If C is a circle transversal to $\tilde{\mathcal{F}}$ such that every point in C is of finite type then the relation $\sim$ associated to T (resp. the set $\mathcal{A}(C) \setminus Dom(T_C)$) has finitely many equivalence classes (resp. finitely many elements).*

Proof. If every point in C is of finite type, then the closure of the saturate of C by $\tilde{\mathcal{F}}$ is a topological submanifold (possibly with boundary) of finite genus. In spite of the relation $\sim$ to be 'larger' than the one defined by C. Gutierrez, the proof of the assertion follows in the same manner as in lemma 3.4 of [Gu1]. ∎

The following theorem is a consequence of lemma 3.1 and Gutierrez's structure theorems [Gu1].

Theorem 4.1. *For M as in the beginning, there is a set, at most denumerable, of circles $\{C_i\}_{i \in \Delta_1}$ transversal to $\tilde{\mathcal{F}}$ such that:*

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(1)*Every point in each C_i is of finite type and $Sat(C_i) \cap Sat(C_j) = \emptyset$, $\forall i \neq j$*

(2)*Each non-trivial recurrent leaf $L(p)$ passing through some point p of finite type intersects some C_i. Besides, the circle C_i can be taken so that either:*

(2.2)$\overline{L(p) \cap C_i} = C_i$, *or*

(2.3)$\overline{L(p) \cap C_i}$ *is a Cantor set.*

(3)*If $p, q \in C_i$ with $L(p)$ and $L(q)$ non-trivial recurrent leaves passing through p and q resp., then $\overline{L(p)} = \overline{L(q)}$*

(4)*If C_i is as in* (2.1) *(resp.* (2.2)*), the forward Poincaré map $T_i : C_i \to C_i$ induced by $\tilde{\mathcal{F}}$ is conjugate (resp. semi-conjugate) to a standard IET.*

Proof. Let $L(p_1)$ be a non-trivial recurrent leaf passing through a point p_1 of finite type. The condition on p_1 and lemma 3.1 permits to affirm that there exists an open neighborhood $U(p_1)$ of p_1 in $\overline{M} \setminus \operatorname{sing}(\tilde{\mathcal{F}})$ such that each point in $U(p_1)$ is of finite type. Notice that this implies that $\overline{\operatorname{Sat}(U(p_1))}$ is contained in a topological two-manifold (possibly with boundary) of finite genus. From the Gutierrez's structure theorem, for $L(p_1)$ as above there exists a circle C_1 transversal to $\tilde{\mathcal{F}}$ such that (1), either (2.1) or (2.2), (3), and (4) are verified. On the other hand, if there exists a non-trivial recurrent leaf $L(p_2)$ of $\tilde{\mathcal{F}}$ with p_2 of finite type such that $L(p_2) \cap C_1 = \emptyset$, then $L(p_2) \cap \overline{\operatorname{Sat}(C_1)} = \emptyset$. Therefore there exists an open neighborhood of p_2 in $\overline{M} \setminus \overline{\operatorname{Sat}(C_1)}$. That, together with lemma 3.1, the structure theorem by Gutierrez and the hypothesis on p_2, implies that there exists a circle C_2 transversal to $\tilde{\mathcal{F}}$ intersecting $L(p_2)$ and satisfying (1), either (2.1) or (2.2), (3) and (4) of theorem. Repeating this argument, we obtain a family $\{C_i\}$ of circles transversal to $\tilde{\mathcal{F}}$ satisfying these conditions. As $\{\operatorname{Sat}(C_i)\}$ forms a family of open sets pairwise disjoint in the space $\overline{M}$, which has countable base, this family is at most denumerable. Now we will see that there exists such a family satisfying the theorem. Let $D = \{\bigcup_{i \in I_n} \operatorname{Sat}(C_i); I_n \subset \mathbb{N}\}$, that is, D is the set formed by all possible countable unions of the saturated of the circles obtained above. Define in D the partial relation of order '$\subseteq$' determined by the usual inclusion of sets. As there is only a countable set of circles C_i as above, D, with this order relation, is inductively ordered. Therefore, from Zorn's lemma, D has a maximal element $\bigcup_{i \in \Delta_1} \operatorname{Sat}(C_i)$.

Now we claim that each non-trivial recurrent leaf $L(p)$, with p of type finite, meets some circle of the countable family $\{C_i\}_{i \in \mathbb{N}}$ that determines the element maximal of D. Suppose that the assertion is false. It is easy to check that for every circle C_i as above we have that $L(p) \cap \overline{\operatorname{Sat}(C_i)} = \emptyset$. On the other hand, if $L(p) \cap \overline{\bigcup_{i \in \Delta_1} \operatorname{Sat}(C_i)} \neq \emptyset$, then this last equality implies that p is of infinite type, but this contradicts our assertion. Therefore $L(p) \cap \overline{\bigcup_{i \in \Delta_1} \operatorname{Sat}(C_i)} = \emptyset$. This last permits to consider an open neighborhood of p in $\overline{M}$ disjointed from $\bigcup_{i \in \Delta_1} \operatorname{Sat}(C_i)$. Then, as above we can obtain a circle $\tilde{C}$ transversal to $\tilde{\mathcal{F}}$ such that $\operatorname{Sat}(\tilde{C}) \cap \operatorname{Sat}(C_i) \neq \emptyset$, $\forall i \in \Delta_1$ with $\operatorname{Sat}(\tilde{C}) \cup \bigcup_{i \in \Delta_1} \operatorname{Sat}(C_i)$ belonging to D, which contradicts the maximality of $\bigcup_{i \in \Delta_1} \operatorname{Sat}(C_i)$ in D. Therefore the proof is completed. ∎

The following result of Cherry [Ch] is one of the first theorems concerning the structure to recurrences of orientable foliations on surfaces. Aranson and Zhuzhoma obtain a similar theorem to nonorientable foliations [Ar-Zh]

Theorem 4.2. *([Ch]) Let C be a circle transversal to $\tilde{\mathcal{F}}$. If $L^+(p)$ (resp. $L^-(p)$) is a non-trivial recurrent half-leaf ω-(resp. α-) of $\tilde{\mathcal{F}}$ passing through $p \in C$, then $\overline{L^+(p)}$ (resp. $\overline{L^-(p)}$ contains a continuous of non-compact recurrent leaves, each one of which is everywhere dense in $\overline{L^+(p)}$ (resp. $\overline{L^-(p)}$).*

Now we intend to study the invariant region by $\mathcal{F}$ of 'infinite genus'.
We begin supposing that $\Omega_2(\tilde{\mathcal{F}})$ is a set with non-empty interior.

Lemma 4.3. *If $\Omega_2(\tilde{\mathcal{F}})$ has non-empty interior then:*

(1) *There are no compact leaves contained in $\overline{int(\Omega_2(\tilde{\mathcal{F}}))}$.*
(2) *If C is a circle transversal to $\tilde{\mathcal{F}}$ contained in the interior of $\Omega_2(\tilde{\mathcal{F}})$, then the points of C which have a singularity of $\tilde{\mathcal{F}}$ as a ω- or α-limit set, form a dense subset of C.*

Proof. (1) Let L be a compact leaf (i.e. homeomorphic to a circle) of $\tilde{\mathcal{F}}$. taking $V \subset \overline{M} \setminus \text{sing}(\tilde{\mathcal{F}})$ to neighborhoods of L homeomorphic to a cylinder (resp. Moebiüs Band) if M is orientable (resp. non-orientable). Consider a neighborhood $W \subset V$ of L sufficiently small. From the structure of the foliation restricted to W, it is easy to see that if $p \in W$ is not a wandering point, then p belongs to a compact leaf, but by definition of $\Omega_2(\tilde{\mathcal{F}})$, this is impossible in the closure of interior of $\Omega_2(\tilde{\mathcal{F}})$. Therefore the first assertion in lemma follows.

(2) Suppose the contrary, i.e., suppose that there exists an interval I in C such that no point in I has a singularity of $\tilde{\mathcal{F}}$ as ω- or α-limit set. From lemma 3.1, there exists a circle $\tilde{C}$ transversal to $\tilde{\mathcal{F}}$ such that $\text{Sat}(\tilde{C}) \subset \text{Sat}(I)$. i.e., no point in $\tilde{C}$ has as ω- or α-limit set a singularity. If we denote by T the forward Poincaré map in $\tilde{C}$ induced by $\tilde{\mathcal{F}}$, then T is defined in all $\tilde{C}$. Thus $\text{Sat}(\tilde{C}) = \bigcup_{p\in\tilde{C}} L(p, T(p))$ is a compact connected submanifold without boundary of M, which is clearly impossible since $\text{int}(\Omega_2(\tilde{\mathcal{F}})) \neq \emptyset$ implies that M is a manifold of infinite genus. Therefore (2) follows and the proof of the lemma is completed. ∎

Lemma 4.4. *If C is a circle transversal to $\tilde{\mathcal{F}}$ contained in the interior of $\Omega_2(\tilde{\mathcal{F}})$, then the points of C belonging to ω- or α-recurrent leaves form a residual dense subset of C.*

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable basin of open intervals in C. For each $n \in \mathbb{N}$ define:

$$A_n = \{p \in \overline{U}_n; \#\{L(p) \cap U_n\} < n\}$$

where, as before, $L(p)$ denotes the leaf passing through p and $\#\{L(p) \cap U_n\}$ denotes the number of intersections of $L(p)$ with the interval U_n. We claim that for each n, A_n is

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a closed set with empty interior. In fact, the first assertion follows immediately from the theorem on continuous dependence of leaves on initial conditions. We now will assume that the second assertion is false. Then we can find an open interval $I \subset C$ such that $I \subset \overline{U_n}$ and $\#\{L(x) \cap U_n\} < n$ for all $x \in I$. On the other hand, as $C \subset$int$(\Omega_2(\tilde{\mathcal{F}}))$, we have that there exists a leaf (non-compact by lemma 4.3) intersecting I in at least two points. Let L be a such leaf. Let $p \in I \cap L$ and $q \in C \cap L^+(p)$ be such that $\#\{L(p,q) \cap U_n\} = \#\{L(p) \cap U_n\} < n$. Again, by the theorem on the continuous dependence of leaves on initial conditions, there exists a neighborhood $V(L(p,q)) \subset \overline{M} \setminus \mathrm{sing}(\tilde{\mathcal{F}})$ of the segment of the leaf $L(p,q)$ with the following property: for all point $p_2 \in I$ sufficiently close to p, there exists $q_2 \in C \cap L_2^+(p_2)$ such that $L_2(p_2,q_2) \subset V(L(p,q))$ and

$$\#\{L_2(p_2,q_2) \cap U_n\} = \#\{L(p,q) \cap U_n\} < n$$

As $I \subset \mathrm{int}(\Omega_2(\tilde{\mathcal{F}}))$, for every open interval $J \subset V(L(p,q)) \cap I$ containing p_2, we have that there exists a leaf L (non-compact) intersecting J in at least two points. But a piece of the compact segment of L is contained in $V(L(p,q))$, therefore we can take J so that:

$$\#\{L_2(p_2,q_2) \cap U_n\} < \#\{L \cap U_n\}$$

Repeated the previous arguments with the point p, now in $L \cap J$, and the leaf $L(p)$ as L, we have that is possible to find a leaf $L_k(p_k)$ passing through of $p_k \in I$ such that:

$$n < \#\{L_k(p_k) \cap U_n\}$$

Which contradicts the fact that $I \subset A_n$. Therefore int$(A_n)= \emptyset$.

As C is a Baire space, then $R(C) = \bigcap_{n=1}^{\infty}(C \setminus A_n)$ is a residual set dense in C. We now claim that every element of $R(C)$ belongs to a ω- or α-recurrent leaf of $\tilde{\mathcal{F}}$. In fact, let $x \in R(C)$ and let $U \subset C$ be an arbitrary open interval of x. Of course, there is an increasing sequence of integers $\{n_k\}_{k=1}$ such that $x \in U_{n_k} \subset U$ where $\{U_{n_k}\}_{k\in\mathbb{N}}$ is a nested subsequence of elements of the base $\{U_n\}_{n\in\mathbb{N}}$. Let $L(x)$ be the leaf (non-compact by lemma 4.3) passing through x. From the definition of $R(C)$ we have that for all k

$$\#\{L(x) \cap U_{n_k}\} \geq n_k$$

It is easy to check that this last inequality and the theorem on the continuous dependence of leaves on initial conditions imply that either $L(x) \subset \omega(L^+(x))$ or $L(x) \subset \alpha(L^-(x))$, which completes the proof. ▌

The previous lemma and theorem 4.2 imply immediately the following corollary:

Corollary 4.1. *If C is as in lemma 4.4, then the points of C belonging to non-trivial recurrent leaves form a residual dense subset of C.*

Theorem 4.3. *If the non-wandering set of infinite type of $\tilde{\mathcal{F}}$ has interior non-empty ($int(\Omega_2(\tilde{\mathcal{F}})) \neq \emptyset$) then there is a set , at most denumerable, of circles $\{C_i\}_{i\in\Delta_2}$ transversal to $\tilde{\mathcal{F}}$ such that:*

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(1)*$C_i \subset int(\Omega_2(\tilde{\mathcal{F}}))$, $\forall i \in \Delta_2$*
(2)*$Sat(C_i) \cap Sat(C_j) = \emptyset$, $\forall i \neq j$*
(3)*For all $i \in \mathbb{N}$ the set of elements $p \in C_i$ such that $\omega(L^+(p))$ or $\alpha(L^-(p))$ is one singularity, is dense in C_i*
(4)*The points in C_i belonging to non-trivial recurrent leaves form a residual dense subset of C_i. Besides, given $i \in \mathbb{N}$, we have one of the following alternatives. Either:*

(4.1)*there is a recurrent leaf intersecting C_i densely , or*

(4.2)*any recurrent leaf intersecting C_i is exceptional.*

(5)*$int(\Omega_2(\tilde{\mathcal{F}})) \subset \overline{\bigcup_{i\in\Delta_2} Sat(C_i)}$*

Proof. We begin assuming that there exists a leaf $L(p)$ locally dense with $p \in \text{int}(\Omega_2(\tilde{\mathcal{F}}))$. Notice that by corollary 4.1, there is a non-trivial recurrent leaf in $\text{int}(\Omega_2(\tilde{\mathcal{F}}))$. From lemma 3.1, it follows that there is a circle C_1 transversal to $\tilde{\mathcal{F}}$ such that:

- $\overline{L(p) \cap C_1} = C_1$
- $\text{Sat}(C_1) \subset \text{int}(\Omega_2(\tilde{\mathcal{F}}))$

Now, if $\text{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\text{Sat}(C_1)}$ is an empty set, then we have from lemma 4.3 and corollary 4.1 that the proof is complete. On the other hand, if this difference is not null, then we will proceed as follows:

(i) Consider in $\text{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\text{Sat}(C_1)}$ a point p_2 such that the leaf $L(p_2)$ passing through p_2 is locally dense (otherwise, go to step (iv)). Proceeding analogously as in the beginning, we can claim that there is circle C_2 transversal to $\tilde{\mathcal{F}}$ such that:

- $\overline{L(p_2) \cap C_2} = C_2$
- $C_2 \subset \text{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\text{Sat}(C_1)}$

Notice that this last sentence implies that $\text{Sat}(C_1) \cap \text{Sat}(C_2) = \emptyset$. On the other hand, if $\text{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\text{Sat}(C_1) \cup \text{Sat}(C_2)} = \emptyset$, then by the above, by lemma 4.3, and by corollary 4.1, we have that the proof is complete. Contrary case we repeat inductively the previous argument to obtain a family $\{C_i\}$ of circles transversal to $\tilde{\mathcal{F}}$ satisfying (1)-(3) and (4.1) of the theorem. On the other hand, of course, $\{\text{Sat}(C_i)\}_i$ is a family in $\overline{M}$ of open sets pairwise disjoint. Therefore this family is at most denumerable, i.e,

(ii) There is a family $C_i, i = 1, 2, \ldots$, at most denumerable, of circle transversal to $\tilde{\mathcal{F}}$ verifying (1)-(3) and (4.1) of theorem.

Let $D = \{\bigcup_{i\in I_n} \text{Sat}(C_i); I_n \subset \mathbb{N}\}$. Define in D the relation of partial order determined by the usual inclusion of set '$\subseteq$'. We can now proceed analogously as in the proof of the theorem 4.1 and to conclude:

(iii) The set D has a maximal element $\bigcup_{i\in\Delta_d} \text{Sat}(C_i)$.

Notice that if $\text{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\bigcup_{i\in\Delta_d} \text{Sat}(C_i)} = \emptyset$, the proof is complete. Therefore we now suppose the contrary.

(iv) Let $q_1 \in \text{int}(\Omega_2(\tilde{\mathcal{F}})) \setminus \overline{\bigcup_{i\in\Delta_d} \text{Sat}(C_i)}$. From lemma 3.1 there exists a circle C'_i transversal to $\tilde{\mathcal{F}}$ contained q_1 such that $\text{Sat}(C'_i) \subset \text{int}(\Omega_2(\tilde{\mathcal{F}}) \setminus \overline{\bigcup_{i\in\Delta_d} \text{Sat}(C_i)})$. From corollary 4.1, there is a non-trivial recurrent leaf passing through C_i'. Now using similar arguments applied in the proof of theorem 4.1 we can verify that $L(q_1)$ is an exceptional recurrent leaf.

Proceeding analogously as in the case of the locally dense leaves, we prove that there exists a family (at most denumerable) $\{C_i'\}_i$ of circles traversal to $\tilde{\mathcal{F}}$ satisfying (1)-(3) and (4.2) of the theorem. Considering these circles, define $E = \{\bigcup_{i\in I_n} \mathrm{Sat}(C_i'); I_n \subset \mathbb{N}\}$. Analogously, as for the set D defined, it is proved that:

(v) There exists a maximal element $\bigcup_{i\in\Delta_e} \mathrm{Sat}(C_i')$ of E with respect to the relation of inclusion of set '$\subseteq$'.

Finally, considering $\Delta_2 = \Delta_d \cup \Delta_e$, the circles transversal $\bigcup_{i\in\Delta_d} \mathrm{Sat}(C_i)$ and $\bigcup_{i\in\Delta_e} \mathrm{Sat}(C_i')$ obtained in (iii) and (v) respectively, we have that under a suitable reordering: $\mathrm{int}(\Omega_2(\tilde{\mathcal{F}})) \subset \overline{\bigcup_{i\in\Delta_2} \mathrm{Sat}(C_i) \cup \mathrm{Sat}(C_i')}$, which completes the proof. ▮

Examples of two-manifolds supporting foliations with the non-wandering set of infinite type having interior non-empty are found in [Be, Gu-He-Lo]. In both cases, there exists a global circle transversal to the foliations. In [Be] the circle satisfies the item (4.1) of theorem and in [Gu-He-Lo], the item (4.2).

Let C a circle transversal to $\tilde{\mathcal{F}}$ and $T : C \to C$ the forward Poincaré map induced by $\tilde{\mathcal{F}}$. Let $\mathcal{A}(C)$ and T_C be as defined in section 4. If C is the circle obtained in the previous theorem, then is immediate that $\mathcal{A}(C) = C$, and for all $x \in \mathrm{Dom}(T)$, $T_C(x) = T(x)$. Therefore the next corollary follows immediately from 4.1 or 4.2 of the previous theorem.

Corollary 4.2. *If C is one of the circles obtained in the previous theorem, then we have that:*

(1) *The forward Poincaré map in C is conjugate to T_C.*

(2) *$\mathcal{A}(C) \setminus Dom(T_C)$ is a compact totally disconnected set with an infinite number of elements. Besides T_C admit a dense subset of non-trivial recurrent points.*

In the following theorem $Per(C)$ will denote the set formed by the points of C belonging to compact leaves.

Theorem 4.4. *Let $\bigcup_{i\in\Delta_3} Sat(C_i)$ be the union of the saturated by $\tilde{\mathcal{F}}$ of the circles obtained in the theorems 4.1 and 4.3. If there is a non-trivial recurrent leaf disjointed from $\overline{\bigcup_{i\in\Delta_3} Sat(C_i)}$ then, at most, there is a countable set $\{\tilde{C}_j\}_{j\in\Delta_4}$ of circles transversal to $\tilde{\mathcal{F}}$ such that:*

(1) *Each circle $\tilde{C}_j$ contains a point p of infinite type belonging to a non-trivial recurrent leaf $L(p)$ so that $\overline{L(p) \cap \tilde{C}_j}$ is a Cantor set and the wandering points in C_j together with the interior of $Per(\tilde{C}_j)$ form an open, dense subset of $\tilde{C}_j$.*

(2) *For each $j \in \mathbb{N}$ the forward Poincaré map on $\tilde{C}_j$ induced by $\tilde{\mathcal{F}}$ is semi-conjugate to $T_{\tilde{C}_j}$.*

(3) *$\mathcal{A}(\tilde{C}_j) \setminus Dom(T_{\tilde{C}_j})$ is a compact, totally disconnected set with an infinite number of elements and $T_{\tilde{C}_j}$ admits a dense subset of non-trivial recurrent points.*

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(4)*If* $L(q)$ *is a non-trivial recurrent leaf such that* $q \notin \overline{\bigcup_{i\in\Delta_3} Sat(C_i)}$ *and* $\overline{L(q)}$ *is a minimal subset of limited type in* M *then* $q \in \tilde{C}_j$ *for some* $j \in \Delta_4$*, the wandering point set is open and dense in* $\tilde{C}_j$*,* $\mathcal{A}(\tilde{C}_j) \setminus Dom(T_{\tilde{C}_j})$ *is a finite set and* $T_{\tilde{C}_j}$ *has a dense orbit.*

(5)$\tilde{C}_j \subset \overline{M} \setminus \overline{\bigcup_{i\in\Delta_3} Sat(C_i)}$, $\forall j \in \Delta_4$, $Sat(\tilde{C}_j) \cap Sat(\tilde{C}_k) = \emptyset$, $\forall j \neq k$ *and any non-trivial recurrent leaf disjointed of* $\overline{\bigcup_{i\in\Delta_3} Sat(C_i)}$ *is contained in* $\overline{\bigcup_{j\in\Delta_4} Sat(\tilde{C}_j)}$

Proof. (1) If $L(p)$ is a non-trivial recurrent leaf of $\tilde{\mathcal{F}}$ with $p \in \Omega(\tilde{\mathcal{F}}) \setminus \overline{\bigcup_{i\in\Delta_3} \mathrm{Sat}(C_i)}$ then there exists an open neighborhood of p which contains wandering points and is disjointed from any point belonging to some locally dense leaf. From lemma 3.1, there exists a circle $\tilde{C} \subset \overline{M} \setminus \overline{\bigcup_{i\in\Delta_3} \mathrm{Sat}(C_i)}$ transversal to $\tilde{\mathcal{F}}$ containing the point p and wandering points such that $\overline{L(p) \cap \tilde{C}}$ is a Cantor set. Notice that from theorem 4.1, p is of infinite type. On the other hand, is clear that if $Per(\tilde{C})$ has non-empty interior $(\mathrm{int}(Per(\tilde{C})) \neq \emptyset)$ then $\mathrm{int}(Per(\tilde{C}))$ is formed by an union of open sub-intervals of $\tilde{C}$ pairwise disjoint where all element of oneself connected component has the same T-period. Let I be one of these components and let $n_I > 0$ be the smaller integer such that $T^{n_I}(I) = I$. Of course, the boundary of the set $\{L(q); q \in I\}$ is a subset of the union of compact leaves, singularities and separatrices jointing the singularities. From arguments similar to those ones used in the lemma 4.3, we can prove that $\mathrm{int}(Per(\tilde{C}))$ together with the wandering points of $\tilde{C}$ is an open, dense set of $\tilde{C}$.

(2) Let $\tilde{C}_j$ be a circle obtained in (1) and let $h : \tilde{C}_j \to \mathcal{A}(\tilde{C}_j)$ be the natural quotient map. The definitions of h and $T_{\tilde{C}_j}$ imply that the following diagram:

$$\begin{array}{ccc} \tilde{C}_j & \xrightarrow{T} & \tilde{C}_j \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{A}(\tilde{C}_j) & \xrightarrow[T_{\tilde{C}_j}]{} & \mathcal{A}(\tilde{C}_j) \end{array}$$

is commutative, which proves this item. We remark that if I is a connected component of $\mathrm{int}(Per(C))$ of T-period equal to n_I then, $T_{\tilde{C}_j}$ is not defined in $h(T^i(I)) = [x_i] \in \mathcal{A}(\tilde{C}_j)$ for some $0 < i \leqslant n_I$.

(3) This assertion is followed from the diagram above and the fact that the closure of any non-trivial recurrent leaf $\overline{L(p)}$ contains a continuous set of non-trivial recurrent leaves which are dense in $\overline{L(p)}$ (see theorem 4.2).

(4) Notice that the circles $\tilde{C}_j$ defined above, can be obtained firstly considering the point p such that $\overline{L(p)}$ is a minimal set of M of limited type. Considered this remark, $\tilde{C}_j$ and $\overline{L(p)}$ are contained in a compact region of M. In spite of $\overline{\mathrm{Sat}(\tilde{C}_j)}$ to be contained in a topological two-manifold (possibly with boundary) of infinite genus (p only can be of infinite type) the proof of this item is similar to the compact case.

(5) Following arguments similar to those ones used in theorems 4.1 and 4.3, it is possible to prove that at most there is a countable set of circles satisfying (1)-(5). ∎

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As an immediate consequence of theorems 4.1, 4.3 and 4.4, we have the following corollary.

Corollary 4.3. *If $L(p)$ is a non-trivial recurrent leaf of $\tilde{\mathcal{F}}$ (passing through p) which does not intersect any of the circles C_i obtained in the theorems 4.1, 4.3 and 4.4, then it is an exceptional leaf, p is of infinite type and there exists a section $\Sigma(p)$ transversal to $\tilde{\mathcal{F}}$ containing p such that either the set $\pitchfork(\Sigma(p))$ of recurrent points in $\Sigma(p)$ belonging to leaves that intersect some of these circles C_i is open and dense in $\Sigma(p)$ or the wandering point set is open and dense in the set $\Sigma(p)\setminus \pitchfork(\Sigma(p))$.*

5. PROOF OF THE MAIN THEOREM.

From theorems 4.1, 4.3, 4.4 and corollary 4.2 we only need to prove that the mapping T_C is conjugate topological for an affine GIET (resp. for a standard IET), where the circle C is given by theorem 4.3 or (3) of theorem 4.4 (resp. (4) of theorem 4.4). In the case of item (4), the assertion follows from lemma 5 of [Gu2]. The two first cases are consequence of the following fundamental proposition.

Proposition 5.1. *Let $C \in \{S^1, \mathbb{R}/\mathbb{Z}\}$ and let $T : C \to C$ be a continuous injective map defined everywhere except in a compact totally disconnected set of points. If T admits a dense subset of non-trivial recurrent points, then T is conjugate to an affine GIET.*

We will divide the proof into a sequence of lemmas and claims.

Claim 5.1. If $T : C \to C$ is as in the previous proposition, then in C there exists a dense subset of recurrent points, each one of which contains in its ω- and α-limit, points where T is not defined.

In fact, it is clear that we only need to prove that in C exists a dense subset of recurrent points such that each one contain in its ω- or α-limit, points where T is not defined. Similarly, as the proof of item (4) of theorem 4.4, if we suppose this last assertion to be false, then we can find in C a compact minimal exceptional set and wandering points, which is impossible under the conditions on T. Thus the claim follows.

Remark that given a compact totally disconnected subset K of C, an easy computation shows that for each value $\varepsilon > 0$, K admits a finite covering of open intervals pairwise disjoint such that each interval has length less than ε.

Let $\{\varepsilon_n; n = 1, 2, \dots\}$ a decreasing sequence in $(0, 1)$ converging to 0. Let K be the set of points where the map T is not defined. For each n, let t_n be a positive integer such that $\{\mathcal{U}_i(\varepsilon_n)\}_{i=1}^{t_n}$ denotes a finite covering of K formed by open intervals pairwise disjoints of length less than ε_n.

Under the previous notation we have the following claim.

Claim 5.2. For each $n \in \mathbb{Z}^+$ there exists a set $\Theta(n)$ made up of a finite collection of finite trajectories of T such that if $q \in \Theta(n)$ and $T(q) \notin \Theta(n)$ (resp. $q \in \Theta(n)$ and $T^{-1}(q) \notin$

$\Theta(n)$), then $q \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$ (resp. $T^{-1}(q) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$). Besides $\Theta = \bigcup_{n=1} \Theta(n)$ is dense in C.

If fact, we have only two cases: either there is a dense orbit in C or not. Suppose the first case. Let p be a point in $C \setminus \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1)$ such that the orbit of T through p is dense in C. In this case, for each n, we fix the smallest integers positives m_n, n_n such that $m_n > m_{n-1}, n_n > n_{n-1}$ with m_0, n_0 large enough so that the set $\Theta(n)$ defined by $\{T^i(p); i = -m_n, -m_n+1, \ldots, n_n-1, n_n\}$ intersects each connected component of $C \setminus \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$ and satisfies the claim. For the second case, we will define $\Theta(n)$ proceeding inductively on n.

Let $n=1$. Of course, $C \setminus \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1)$ is a finite union of closed intervals pairwise disjoint where each component is contained in the domain of continuity of T i.e., $C \setminus \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1) = \bigcup_{k=0}^{k_1} J_k^{(1)}$ for some integer $k_1 > 0$. For each $k < k_1$, split $J_k^{(1)}$ into a finite union of intervals $J_{k,s}^{(1)}$, $s = 1, \ldots, s_k^{(1)}$ of equal lengths. Here $s_k^{(1)}$ is an integer sufficiently large. For hypothesis on T, for each integer k and each $s \in \{1, \ldots, s_k^{(1)}\}$, we can take in the interior of the interval $J_{k,s}^{(1)}$, a T-recurrent point $p_{k,s}$ such that for each $k \in \{0, 1, \ldots, k_1\}$, $p_{k,r} \notin \{T^i(p_{k,t})\}_{i \in \mathbb{Z}}$ for different elements r, t in $\{1, \ldots, s_k^{(1)}\}$, $O^+(p_{k,s}) \cap \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1) \neq \emptyset$, and $O^-(p_{k,s}) \cap \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1) \neq \emptyset$ (seem claim 5.1).
Let $n_1^+ := n_1^+(k,s), n_1^- := n_1^-(k,s)$ be positive integers such that:

$$n_1^+ = \min_{r \geq 1}\{ \{T^i(p_{k,s})\}_{i=1}^r \cap J_{k,s}^{(1)} \neq \emptyset \;, T^r(p_{k,s}) \in \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1)\}$$

$$n_1^- = \min_{r \geq 1}\{ \{T^{-i}(p_{k,s})\}_{i=1}^r \cap J_{k,s}^{(1)} \neq \emptyset \;, \; T^{-r}(p_{k,s}) \in \bigcup_{i=1}^{t_1} \mathcal{U}_i(\varepsilon_1)\}$$

With these notations, we define the finite orbit set $\Theta(1)$ as follows:

$$\Theta(1) = \{T^i(p_{k,s});\; k = 1, \ldots, k_1;\; s = 1, \ldots, s_k^{(1)};\; i = -n_1^- + 1, \ldots, n_1^+\}$$

Let us suppose defined the set $\Theta(n-1)$ $\forall n \geq 2$. Then we define $\Theta(n)$ considering the following two steps:

First step. Given $\varepsilon_n < \varepsilon_{n-1}$, for each $n \geq 2$ let

$$\widetilde{\varepsilon}_n = \min_{x \neq y}\{d(x,y);\; (x,y) \in \Theta(n-1) \times (\Theta(n-1) \cup \bigcup_{i=1}^{t_n} \partial\{\mathcal{U}_i(\varepsilon_n)\})\},$$

where $d(.,.)$ is the usual distance in C and $\partial\{.\}$ denotes the boundary of the set $\{.\}$.

Now, for each n denote by:

- $\widetilde{J}^n(q)$ the intervals centered in points $q \in \Theta(n-1)$ of radius less than $\widetilde{\varepsilon}_n/2$.

• J_k^n the intervals which are connected component of $C \setminus [\bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n) \cup \{\widetilde{J}^n(q); q \in \Theta(n-1)\}]$ and denote by k_n the number of these intervals.

Similar to the case $n = 1$, for all $n \geq 2$ we split up each interval J_k^n into a finite union of $s_k^{(n)}$ intervals of equal lengths such that $s_k^{(n-1)} < s_k^{(n)}$. Denote by $J_{k,s}^{(n)}$, $s = 1, 2, \dots, s_k^{(n)}$ these intervals.

Second step. For hypothesis, for each $k \in \{1, \dots, k_n\}$, $s \in \{1, \dots, s_k^{(n)}\}$ we can take points $p_{k,s} \in J_{k,s}^{(n)}$ such that the T-orbit $O(p_{k,s})$ by $p_{k,s}$ is recurrent with $O^+(p_{k,s}) \cap \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n) \neq \emptyset$ and $O^-(p_{k,s}) \cap \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n) \neq \emptyset$ (see claim 5.1). On the other hand, for each $k \in \{1, \dots, k_n\}$ and $s \in \{1, \dots, s_k^{(n)}\}$ define $n_n^+ := n_n^+(k,s)$ and $n_n^- := n_n^-(k,s)$ by the following values:

$$n_n^+ = \min_{r \geq 1} \{ \ \{T^i(p_{k,s})\}_{i=1}^r \cap J_{k,s}^{(n)} \neq \emptyset \ , \ T^r(p_{k,s}) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)\}$$

$$n_n^- = \min_{r \geq 1} \{ \ \{T^{-i}(p_{k,s})\}_{i=1}^r \cap J_{k,s}^{(n)} \neq \emptyset \ , \ T^{-r}(p_{k,s}) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)\}$$

For construction, $\Theta(n-1)$ is made up of a finite number of finite orbits. Then let $q \in C$ be an element which generates one of these finite orbit, and consider the least positive value j_n^+ (resp. j_n^-) of the index m such that:

- $\{T^i(q)\}_{i=1}^m \cap \tilde{J}^n(q) \neq \emptyset$ (resp. $\{T^{-i}(q)\}_{i=1}^m \cap \tilde{J}^n(q) \neq \emptyset$)
- $T^m(q) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$ (resp. $T^{-m}(q) \in \bigcup_{i=1}^{t_n} \mathcal{U}_i(\varepsilon_n)$)
- $\{T^i(q); i = 0, 1, \dots, m\}$ (resp. $\{T^{-i}(q); i = 0, 1, \dots, m\}$), intersect $\tilde{J}^n(x) \setminus \{x\}$ for each $x \in O(q) \cap \Theta(n-1)$.

Finally, define the set $\Theta(n)$ by:

$\Theta(n) = \{T^j(q); -j_n^- \leq j \leq j_n^+\} \bigcup \{T^j(p_{k,s}); -n_n^- + 1 \leq j \leq n_n^+, 1 \leq k \leq k_n, 1 \leq s \leq s_k^{(n)}\}$.

where q varies in each element that generates each finite orbit that forms $\Theta(n-1)$.

For construction, $\Theta(n) \subset \Theta(n+1)$ $\forall n \in \mathbb{Z}^+$. On the other hand, if for each $k \in \mathbb{Z}^+$ we take $s_k^n \to +\infty$ when $n \to +\infty$ then this property together with the fact that $\varepsilon_n \to 0$ (therefore $\tilde{\varepsilon}_n \to 0$) implies that $\Theta = \bigcup_{n=1} \Theta(n)$ is a dense subset of C. Therefore the claim is complete.

A infinite partition $\mathcal{P}$ on C. We will begin defining, by recurrence on n, a sequence $\{\mathcal{P}(n); n = 1, 2, \dots\}$ of finite partitions of C. Given $n \in \mathbb{Z}^+$, for each $x \in \bigcup_{i=1}^{t_{n+1}} \partial\{\mathcal{U}_i(\varepsilon_{n+1})\}$ denote by y_x^j the point closest to x in $\{\Theta(j) \cap (C \setminus \bigcup_{i=1}^{t_{n+1}} \mathcal{U}_i(\varepsilon_{n+1}))$. For all $n \in \mathbb{Z}^+$ we define $\mathcal{P}(n+1)$ to be the union of $\mathcal{P}(n)$ with the set $\{y_x^j;\ x \in \bigcup_{i=1}^{t_{n+1}} \partial\{\mathcal{U}_i(\varepsilon_{n+1})\}$ and $1 \leq j \leq n+1\}$, where $\mathcal{P}(1) = \{y_x^1;\ x \in \bigcup_{i=1}^{t_1} \partial\{\mathcal{U}_i(\varepsilon_1)\}\}$.

Convention. Given a finite partition $0 = x_0 < x_1 < \dots < x_n = 1$ of C, we will say that A is an *interval determined by the partition* or it is an *element of the partition* if, and only if, $A = [x_i, x_{i+1})$ for some $0 \leq i \leq n-1$.

Notice that for each $n \in \mathbb{Z}^+$, $\mathcal{P}(n+1)$ is a refinement of $\mathcal{P}(n)$. Moreover, if A is an element determined by the partition $\mathcal{P}(n+1)$ and A is not determinate by $\mathcal{P}(n)$, then necessarily it is contained in some element of $\mathcal{P}(n)$ that contains points where T is not defined.

The *Infinite partition* $\mathcal{P}$ in C is defined as follows. A sub-interval of C is an element determined by $\mathcal{P}$ if, and only if it, for some n_0, is an element determined by the partition $\mathcal{P}(n_0)$ that does not contain points where T it is not defined.

Notice that this definition implies that the element determined by $\mathcal{P}$ is also an element determined by $\mathcal{P}(n)$ for all integer n sufficiently large.

Claim 5.3. If $[a,b)$ is an interval contained in some element of the partition $\mathcal{P}$ of C, then there exists $n_0 > 0$ such that $\#\{\Theta(n) \cap [a,b)\} = \#\{\Theta(n) \cap T([a,b))\}$ $\forall n > n_0$.

In fact, let A be an element of $\mathcal{P}$ containing $[a,b)$. Let n_0 be the least positive integer such that A is fixed by $\mathcal{P}(n_0)$. As $[a,b)$ is contained in the domain of continuity of T, we only consider the case $T(a) < T(b)$ i.e. $T([a,b)) = [T(a),T(b))$. Therefore, if $x \in \Theta(n) \cap [a,b)$ for $n \geq n_0$ then $T(x) \in \Theta(n) \cap [T(a),T(b))$ if, and only if, $T(x) \in \Theta(n)$, i.e. if x is not one of the last elements of any of the finite orbits which define $\Theta(n)$. By the construction of the partitions $\mathcal{P}(n)$ and from claim 5.2, these elements do not belong to the element of $\mathcal{P}(n)$ included in a component of continuity of T. Therefore as $\Theta(n)$ is made up of a finite number of finite orbit, it follows from the condition on $[a,b)$ that: $\#\{\Theta(n) \cap [a,b)\} \leq \#\{\Theta(n) \cap [T(a),T(b))\}$ for all $n \geq n_0$. Similarly, it is verified that $\#\{\Theta(n) \cap [T(a),T(b))\} \leq \#\{\Theta(n) \cap [a,b)\}$ for all $n \geq n_0$. Therefore, the claim is proved.

Give $n \in \mathbb{Z}^+$, let $\Theta(n)$ be as above. Now we will define a sequence $\{\mu_n; n = 1,2\ldots\}$ of measures of probability in C such that for each n, μ_n is supported in $\Theta(n)$.

• To $n=1$, define $\mu_1(\{p\}) = 1/m_1$ for each $p \in \Theta(1)$. Here m_1 is the number of elements of $\Theta(1)$.

• To $n \geq 2$, the measure of probability is defined as follows: if $s \in \Theta(n)$, then $\mu_n(\{s\}) = \mu_{n-1}([p,q))/m_n$, where $[p,q)$ is the smallest sub-interval of C with $p \leq s < q$ and $p,q \in \Theta(n-1)$. Here m_n is the number of elements of $\Theta(n) \cap [p,q)$.

As C is compact, the sequence of probability $\{\mu_n; n = 1,2\ldots\}$ admits a subsequence (that for simplicity we continue denoting by $\{\mu_n; n = 1,2\ldots\}$) which converges in the weak topology to a probability μ in C, i.e.

$$\int_C \varphi d\mu_n \to \int_C \varphi d\mu, \ \forall \varphi \in \mathrm{C}^0(C)$$

Notice that as C is a space metric, μ and μ_n will be regular measures.

Lemma 5.1. *Given $p,q \in \Theta = \bigcup_{n=1}^{\infty} \Theta(n)$ with $p < q$ in C, we have*

(1)$\lim_{n\to\infty} \mu_n(\{p\}) = 0$

Besides, there exists $n_0 \in \mathbb{Z}^+$ such that:

(2)$\mu_n([p,q)) = \mu_{n_0}([p,q))$ *for all* $n \geq n_0$*;*
(3)$\mu([p,q)) = \mu_{n_0}([p,q))$*, and* μ *is positive in every open set of* C*.*

Proof. Let n_0 be the least positive integer such that $p \in \Theta(n_0)$. An easy computation shows that for all $k \geq 1$; $\mu_{n_0+k}(\{p\}) = \frac{\mu_{n_0+k-1}([p,q_k))}{m_{n_0+k}} = \frac{\mu_{n_0}(\{p\})}{m_{n_0+1}m_{n_0+2}\dots m_{n_0+k}}$, where $p < q_k$ in C with $q_k \in \Theta(n_0+k-1)$ is such that $[p,q_k) \cap \Theta(n_0+k-1) = \{p\}$. Here m_{n_0+k} denotes the number of elements of the set $[p,q_k) \cap \Theta(n_0+k)$. Therefore (1) follows since $m_n \geq 2$ for all n. Now let n_0 be the smallest positive integer such that $p,q \in \Theta(n_0)$. Suppose first that $p,q \in \Theta(n_0) \setminus \Theta(n_0-1)$ with $[p,q) \cap \Theta(j) = \emptyset$ for all $1 \leq j \leq n_0-1$. In this case, it is proved by induction on n that; $\mu_n([p,q)) = \mu_{n_0}([p,q)) = \#\{\Theta(n_0) \cap [p,q)\}\mu_{n_0}(\{p\})$ for all $n \geq n_0$. Now suppose the second case; $q \in \Theta(n_0)$, and $p \in \Theta(n_0-1)$ with $p \notin \Theta(j)$ for all $1 \leq j \leq n_0-2$ and $[p,q) \cap \Theta(n_0) = \{p\}$. In this case, it is easy to see that it follows from the definition of μ_n that for all $n \geq n_0$, $\mu_n([p,q)) = \mu_{n_0}([p,r))/m_0$, where $[p,r)$ is the smallest interval containing $[p,q)$ with $r \in \Theta(n_0-1)$ and m_0 is the number of elements of $\Theta(n_0) \cap [p,r)$. In order to prove (2) in the general case, it is enough to split the interval $[p,q)$ into a finite union of sub-intervals that satisfy the previous two cases. Now we will prove (3). As μ_n converges to μ in the weak topology, for the open and closed sub-intervals $(p,q),[p,q]$ of C respectively, we have that $\mu((p,q)) \leq \liminf_{n\to\infty} \mu_n((p,q))$ and $\limsup_{n\to\infty} \mu_n([p,q]) \leq \mu([p,q])$. On the other hand, from definition of the measure μ_n, it follows that the bounded sequences $\{\mu_n((p,q)); n = 1,2,\dots\}$ and $\{\mu_n([p,q]); n = 1,2,\dots\}$ are increasing and decreasing monotonically respectively. Thus we have that

$$\mu((p,q)) \leq \lim_{n\to\infty} \mu_n((p,q)) \quad \text{and} \quad \lim_{n\to\infty} \mu_n([p,q]) \leq \mu([p,q]) \tag{§}$$

On the other hand, let $\{(p_j,q_j); j = 1,2,\dots\}$ be a sequence of open sub-intervals of C with endpoints in $\Theta = \bigcup_{n=1} \Theta(n)$ such that $[p,q] \subset (p_j,q_j) \subset (p_{j-1},q_{j-1})$ for all j, and $\bigcap_{j=1}(p_j,q_j) = [p,q]$. As the measure μ is regular, we have that $\mu([p,q]) = \lim_{j\to\infty} \mu((p_j,q_j))$. An easy computation shows that this last equality together with the first inequality of (§), items (1), (2) of lemma, and the fact that μ_n is also a regular measure imply that there exists $n_0 \in \mathbb{Z}^+$ such that $\mu([p,q]) \leq \lim_{j,n\to\infty} \mu_n((p_j,q_j)) = \mu_{n_0}([p,q))$. On the other hand, from the second inequality of (§), (1) and (2) of lemma, we have that $\mu_{n_0}([p,q)) \leq \mu([p,q])$. Thus $\mu([p,q]) = \mu_{n_0}([p,q))$. Considering a crescent sequence $\{[p_j,q_j]; j = 1,2,\dots\}$ of closed intervals in C with endpoints in Θ in such a way that for all j, $[p_j,q_j] \subset [p_{j+1},q_{j+1}] \subset [p,q]$ and $\bigcup_{j=1}[p_j,q_j] = (p,q)$. A similar proof shows that $\mu((p,q)) = \mu_{n_0}([p,q))$. Therefore, this last inequality and the previous one prove (3). Therefore the proof is completed. ∎

Lemma 5.2. *Let* $T : C \to C$ *be as in the hypotheses of proposition 5.1, then on* C *there exists an infinite partition* $\mathcal{P}(C)$ *and a probability, positive in open sets, such that for each interval* A *determined by* $\mathcal{P}(C)$*, there exists a constant* $\lambda_A > 0$ *with the following property: for each sub-interval* $[p,q)$ *of interval* A *we have that*

$$\mu(T([p,q))) = \lambda_A \mu([p,q))$$

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Proof. As before, we only consider the case when $T|_A$ is increasing. Let $j \in \mathbb{Z}^+$ be the smallest integer such that A is fixed by $\Theta(j)$. From the density of $\Theta = \bigcup_{n=1} \Theta(n)$ in C and continuity of T in A, we can assume that $p, q \in \Theta$. First we see the following particular case.

Particular case. $q \in \Theta(i)$, $p \in \Theta(i+1) \setminus \Theta(i)$ with $[p,q] \cap \Theta(i) = q$ or $p \in \Theta(i)$, $q \in \Theta(i+1) \setminus \Theta(i)$ with $[p,q] \cap \Theta(i) = p$.
We remark that from construction necessarily $i \geq j$. Now we check the lemma in the first particular case. The other is similar.

(1) Let $I_{q,i}$ be the smallest interval in C closed in the left and open in the right such that it has as endpoints q and an element of $\Theta(i)$.

(2) Let $\alpha(q,i)$ (resp. $\tilde{\alpha}(\overline{pq}, i)$) be the number of elements of $\Theta(i+1) \cap I_{q,i}$ (resp. of $\Theta(i+1) \cap [p,q)$).
From the claim 5.3, we have that:

(3) The number of elements of $\Theta(i+1) \cap [T(p), T(q))$ is $\tilde{\alpha}(\overline{pq}, i)$.

(4) Let $I_{T(q),i}$ be the smallest sub-interval of C closed in the left and open in the right contained the interval $[T(p), T(q))$ such that it has as endpoints $T(q)$ and an element of $\Theta(i)$.

(5) Let $\alpha(T(q), i)$ denote the number of elements of $\Theta(i) \cap I_{T(q),i}$.
Let $\{\mu_k; k = 1, 2, \ldots\}$ be the sequence of probability in C defined as above such that it converges in the weak topology to the probability μ. From (3) of lemma 5.1, we have that there exists an integer k_0 (that we can assume greater than $i+1$) such that $\mu(I_{q,i}) = \mu_k(I_{q,i}) = \mu_{k_0}(I_{q,i})$ and $\mu(I_{T(q),i}) = \mu_k(I_{T(q),i}) = \mu_{k_0}(I_{T(q),i})$, for all $k \geq k_0$. From (1), construction of μ_k and the conditions on the particular case, we have that:

(6) $\mu([p,q)) = \mu_{k_0}([p,q)) = (\mu(I_{q,i})/\alpha(q,i))\tilde{\alpha}(\overline{pq}, i)$.
Notice that $\mu(I_{q,i})/\alpha(q,i)$ is the measure μ_{k_0} of each element of $\Theta(i+1) \cap I_{q,i}$.
From (3), (5) and similar considerations we have

(7) $\mu([T(p), T(q))) = \mu_{k_0}([T(p), T(q))) = (\mu(I_{T(q),i})/\alpha(T(q), i))\tilde{\alpha}(\overline{pq}, i)$
On the other hand, in this particular case there only exist two possibilities; either $I_{q,i} \subset A$ or $A \subset I_{q,i}$. In the first possibility, we have that $\mu(I_{q,i}) = [\mu(A)/(\delta_j \delta_{j+1} \ldots \delta_i \delta_{i+1})]\eta_{i+1}$, $\mu(I_{T(q),i}) = [\mu(T(A))/(\delta_j \delta_{j+1} \ldots \delta_i \delta_{i+1})]\eta_{i+1}$, and $\alpha(q,i) = \alpha(T(q), i)$. Here $\eta_{i+1} = \#\{\Theta(i+1) \cap [p,q)\}$, $\delta_j = \#\{\Theta(j) \cap A\}$ and $\forall k \in \{j+1, \ldots, i+1\}$, $\delta_k = \max\{\tilde{\delta}_k, 1\}$, where $\tilde{\delta}_k$ denotes the number of elements of $\Theta(k)$ in the smallest interval containing $[p,q)$, closed in the left and open in the right, with endpoints belonging to $\Theta(k-1)$. Notice that in this particular case, the second possibility implies that $\mu(A) = \mu([p,q)) = (\mu(I_{q,i})/\alpha(q,i))\tilde{\alpha}(\overline{pq}, i)$ and $\mu(T(A)) = \mu([T(p), T(q)))$ which is equal to $(\mu(I_{T(q),i})/\alpha(T(q), i))\ \tilde{\alpha}(\overline{pq}, i)$. Therefore in both possibilities, from (7) we have that:

(8) For all interval $[p,q)$ as in the particular case:

$$\mu([T(p), T(q))) = (\mu(T(A))/\mu(A))\mu([p,q))$$

Now we will prove the lemma in the general case. Let j be the smallest positive integer such that A is fixed by $\Theta(j)$ and let i be the smallest positive integer such that $p, q \in \Theta(i)$ (which necessarily will have to be greater than j). It follows from the definition of the

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infinite partition $\mathcal{P}$ that the two endpoints of A belong to $\Theta(j)$ or one of them belongs to $\Theta(j-1)$ and the other one to $\Theta(j)$. By commodity, we suppose that the left-point belongs to $\Theta(j)$ and the right-point to $\Theta(j-1)$, the other cases is similar. Let l_j be the smallest positive integer with $j \leq l_j \leq i$ such that $[p,q) \cap \Theta(l_j) \neq \emptyset$. Split the interval $[p,q)$ (whenever possible) into the union of intervals determined by $\Theta(l_j)$, i.e:

(9) $\mu([p,q)) = \mu([p, p^1_{l_j})) + \sum_{k=1}^{\tilde{\alpha_1}(l_j)-1} \mu([p^k_{l_j}, p^{k+1}_{l_j})) + \mu([p^{\tilde{\alpha}_1(l_j)}_{l_j}, q)),$

where $p^k_{l_j} \in \Theta(l_j), \forall k \in \{1, \ldots, \tilde{\alpha_1}(l_j)\}$ and

(10) $\tilde{\alpha_1}(l_j)$ is the number of elements of $\Theta(l_j) \cap [p,q)$.

From the claim 5.3, we have that:

(11) The number of elements of $\Theta(l_j) \cap [T(p), T(q))$ and $\Theta(l_j) \cap [p,q)$ are equal.

Therefore from (9), (10) and (11) we have that:

(12) $\mu([T(p), T(q)) = \mu([T(p), T(p^1_{l_j})) + \sum_{k=1}^{\tilde{\alpha}_1(l_j)-1} \mu([T(p^k_{l_j}), T(p^{k+1}_{l_j})))+$

$+\mu([T(p^{\tilde{\alpha}_1(l_j)}_{l_j}), T(q))).$

On the other hand, proceed as in case 1.

(13) Let $I_{\overline{pq},l_j}$ (resp. $I_{T(\overline{pq}),l_j}$) be the smallest sub-interval of C containing $[p,q)$ (resp. $[T(p), T(q))$) such that it is closed in the left and open in the right with endpoint elements in $\Theta(l_j)$ (therefore contained in A).

(14) From (11) and the claim 5.3, the number of elements of $\Theta(l_j) \cap I_{\overline{pq},l_j}$ and $\Theta(l_j) \cap T(I_{\overline{pq},l_j})$ are equal to $\tilde{\alpha}_1(l_j)+1$.

From lemma 5.1 we have that there exists $k_0 \in \mathbb{Z}^+$ (that we can assume larger than $i+1$) such that $\forall k \geq k_0$

$\mu(I_{\overline{pq},l_j}) = \mu_{k_0}(I_{\overline{pq},l_j}) = \mu_k(I_{\overline{pq},l_j})$ and

$\mu(T(I_{\overline{pq},l_j})) = \mu_{k_0}(I_{T(\overline{pq}),l_j}) = \mu_k(I_{T(\overline{pq}),l_j})$

In a similar way (and with the same notations) as in (8)

$\mu(I_{\overline{pq},l_j}) = [\mu(A)/\delta_j\delta_{j+1}\ldots\delta_{l_j}](\tilde{\alpha}_1(l_j)+1)$ and

$\mu((I_{T(\overline{pq}),l_j})) = [\mu(T(A))/\delta_j\delta_{j+1}\ldots\delta_{l_j}](\tilde{\alpha}_1(l_j)+1)$

Considering k_0 large enough, from the construction of the probabilities μ_k we have that for all $k \in \{1, \ldots, \tilde{\alpha_1}(l_j)\}$:

$\mu([p^k_{l_j}, p^{k+1}_{l_j})) = \mu(I_{\overline{pq},l_j})/(\tilde{\alpha}_1(l_j)+1) = \mu(A)/\delta_j\delta_{j+1}\ldots\delta_{l_j}.$

$\mu([T(p^k_{l_j}), T(p^{k+1}_{l_j}))) = \mu((I_{T(\overline{pq}),l_j}))/(\tilde{\alpha}_1(l_j)) = \mu(T(A))/\delta_j\delta_{j+1}\ldots\delta_{l_j}.$

Replacing these last equations in (9) and (12), we have that:

$$\begin{aligned}\mu([p,q)) &= \mu([p, p^1_{l_j+1})) + (\mu(A)/\delta_j\delta_{j+1}\ldots\delta_i\delta_{l_j})(\tilde{\alpha_1}(l_j)-1)\\ &+\mu([p^{\tilde{\alpha}_1(l_j)}_{l_j}, q))\end{aligned}$$

$$\begin{aligned}\mu([T(p), T(q))) &= \mu([T(p), T(p^1_{l_j+1}))) + \mu([T(p^{\tilde{\alpha}_1(l_j)}_{l_j}), T(q)))\\ &+(\mu(T(A))/\delta_j\delta_{j+1}\ldots\delta_{l_j})(\tilde{\alpha_1}(l_j)-1).\end{aligned}$$

On the other hand, as for all $n \in \mathbb{Z}^+$, $\Theta(n+1) \subset \Theta(n)$ and each $\Theta(n)$ made up of a finite number of elements, the intervals $[p, p^1_{l_j+1}), [p^{\tilde{\alpha}_1(i)}_{l_j}, q)$, and their respective images by T, can be split into a finite union of intervals that satisfy the particular case.

Therefore in the general case it follows from (9) and the two previous equalities that:

$$\mu([T(p), T(q))) = (\mu(T(A))/\mu(A))\mu([p, q)).$$

Proving thus the lemma. ▮

Proof of proposition. Let μ be the probability in C obtained in the lemma 5.2. Fixed $x_0 \in C$. Define the map $g : C \to \mathbb{R}/\mathbb{Z}$ as $x \mapsto \mu([x_0, x))$. As μ is positive in each open set then, the map g is increasing. We claim that g is a continuous map. If fact, suppose the contrary. Let $x \in C$ be a discontinuity point of g and let $\{r_n\}_{n=1}$ and $\{s_n\}_{n=1}$ be a sequence of elements belonging to Θ converging to x such that: 1) For each n, r_n and s_n belongs to $\Theta(n) \setminus \Theta(n-1)$ (remark that this is always possible from the characteristics of the definition of $\Theta(n)$). 2) $r_1 < r_2 < \ldots < x < \ldots < s_2 < s_1$ and 3) $\Theta(n) \cap [r_n, s_n) = \{r_n\}$. It follows from the monotonicity and discontinuity of g in x that given $\epsilon_0 > 0$ we have $|g(s_n) - g(r_n)| \geq \epsilon_0$ for all n large enough. On the other side, if $\{\mu_k; k = 1, 2 \ldots\}$ is the sequence of probability that converges to μ (in the weak topology), we have that there exists $k_n \in \mathbb{Z}^+$ such that $|g(s_n) - g(r_n)| = \mu([r_n, s_n)) = \mu_{k_n}([r_n, s_n))$. By selection of r_n and s_n, $\mu_{k_n+1}([r_{n+1}, s_{n+1})) = \mu_{k_n}([r_n, s_n))/m_n$, where $m_n > 2$ is the number of elements of $\Theta(n+1) \cap [r_n, s_n)$. By induction on n is proved that, given a positive integer n_0, $\mu_{k_n}([r_n, s_n)) = \mu_{k_{n_0}}([r_{n_0}, s_{n_0}))/(m_n m_{n-1} \ldots m_{n_0-1})$ for all inter $n \geq n_0$. But in the limit, this equality contradicts $|g(s_n) - g(r_n)| \geq \epsilon_0$. Therefore g is a continuous and monotone crescent map. Then it is a homeomorphism in C. Now, let $[r, s)$ be an arbitrary interval in C contained in an interval A determined by the infinite partition $\mathcal{P}$. We only consider the case where the map T is increasing in the component of continuity that contains the interval A. It follows from the previous proposition that there exists a constant $\lambda_A > 0$ such that: $|(g \circ T)(r) - (g \circ T)(s)| = |\mu([T(s) - T(r))| = \lambda_A \mu([s, r)) = \lambda_A |g(r) - g(s)|$. On the other hand, let $E = g \circ T \circ g^{-1} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, and let $g(\mathcal{P})$ be the infinite partition in $\mathbb{R}/\mathbb{Z}$ induced by $\mathcal{P}$ through the map g. If $[\tilde{r}, \tilde{s})$ is an arbitrary interval contained in some interval $g(A)$ obtained from the partition $g(\mathcal{P})$, then it follows from the previous equality that: $|E(\tilde{r}) - E(\tilde{s})| = \lambda_A |\tilde{r} - \tilde{s}|$. Therefore, the proof follows from deleting of $\mathbb{R}/\mathbb{Z}$ first the points which determine the partition $g(\mathcal{P})$ and second its respective points of accumulation. ▮

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