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Remarks on Immersions in the Metastable Range Dimension

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Let $f : M \to N$ be a continuous map between two closed n-manifolds such that $f_* : H_*(M, \mathbb{Z}_2) \to H_*(N, \mathbb{Z}_2)$ is an isomorphism. Suppose that N immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$. Then M also immerses in $\mathbb{R}^{n+k}$. We use techniques of normal bordism theory to prove this result which complements the work Biasi et al.[BGL]. We also present conditions for a map to be homotopic to an immersion. May, 2002 ICMC-USP

Key Words: bordism, normal bordism, immersion of manifold,localization

1. INTRODUCTION

This paper is concerned about conditions which on there exist immersions in the metastable range dimension, which were already considered in [BGL]. For the two problems we have considered, we give conditions on the induced maps in homology groups with $\mathbb{Z}_2$ coefficients.

For the first problem, let us consider $h : M^n \to X^{n+k}$ a continuous map from a closed smooth connected n-manifold into a smooth connected $(n + k)$-manifold, $5 \leq n < 2k$, bordant to an immersion, in the sense of Conner and Floyd. When is h homotopic to an immersion? By exploiting the works of Koschorke ([K1],[K2], [K3]) and Salomonsen ([S]) we obtain an answer in terms of the induced homology groups maps.

For the second problem, let $f : M \to N$ be a continuous map between two closed smooth connected n-dimensional manifolds. Suppose that N immerses in $\mathbb{R}^{n+k}$, for some k, with $5 \leq n < 2k$. Under which conditions on f does M immerses in $\mathbb{R}^{n+k}$? The case when it is supposed that M immerses in $\mathbb{R}^{n+k}$ and it is looking for conditions on f such that N also immerses in $\mathbb{R}^{n+k}$ has been considered in the work of Biasi et al. ([BGL]) and Glover et al. ([GH1], [GH2],[GM]).

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We will use a normal bordism approach to investigate these problems. We prove the following main results:

Theorem A: *Let $h : M^n \to X^{n+k}$ be a continuous map from a closed smooth connected n-manifold into a smooth connected $(n+k)$-manifold, $5 \leq n < 2k$, and let $g : M \to BO(q)$, for q large, be the classifying map of ν_h, the stable normal bundle of h. Suppose that*
$(h, g) : M \to X \times BO(q)$, q large, induces
$(h, g)_* : H_i(M, \mathbb{Z}_2) \to H_i(X \times BO(q), \mathbb{Z}_2)$ which is an isomorphism for $i < n - k$ and an epimorphism, for $i = n - k$.
Then if h is bordant to an immersion, h is homotopic to an immersion.

Theorem B: *Let M and N be a closed connected n-manifolds and let $f : M \to N$ be a continuous map such that*

$$f_* : H_i(M, \mathbb{Z}_2) \longrightarrow H_i(N, \mathbb{Z}_2)$$

is an isomorphism, for $i \geq 0$.
Then if N immerses in $\mathbb{R}^{n+k}$ for $5 \leq n < 2k$, so does M .

The work is divided in four sections. In Section 2, we present two exact sequences of bordism groups. One of them is a generalization of the exact sequence normal bordism group given by Salomonsen [S], by using identifications of some normal bordism groups.

In Section 3, we prove Theorem A and B and in Section 4, we present an application using a non standard obstruction theory.

Notations and conventions: In this work, $\mathcal{C}$ will denote the class of all torsion groups where the torsion is odd.

2. EXACT SEQUENCES OF BORDISM GROUPS

Given a topological space X and a virtual bundle ϕ, $\Omega_i(X, \phi)$ denotes the i^{th} normal bordism group of X with coefficient ϕ. For the definition and more details about normal bordism see [K1] or [S]. We adopt the Salomonsen convention.

We get a generalization of the exact sequence normal bordism group given by Salomonsen [S], by using identifications of some normal bordism groups.

For each q, let $\varphi^q = \varepsilon^{p+q} - (\alpha^p \times \gamma^q)$ and $\psi^q = \gamma^q - \varepsilon^q$ be virtual bundles over $X \times BO(q)$, where γ^q denotes the universal vector bundle over $BO(q)$. We can construct a fibre bundle $\widetilde{V}_k(\psi^q) \xrightarrow{\pi} X$ with $(k-1)$-connected fibre and we define a k-dimensional vector bundle μ^k over $\widetilde{V}_k(\psi^q)$ such that $\mu^k \oplus \varepsilon^q \simeq \varepsilon^k \oplus \gamma^q$. ([S]).

Let us consider $\theta' : \widetilde{V}_k(\psi^q) \to BO(k)$ the classifying map of the vector bundle μ^k , which is a high homotopy equivalence, for k large enough.

The following diagram is commutative:

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$$\begin{array}{ccc} \Omega_n(\widetilde{V}_k(\psi^k),\varphi^k) & \longrightarrow & \Omega_n(\widetilde{V}_k(\psi^q),\varphi^q) \\ \downarrow{\scriptstyle \pi_*} & \swarrow{\scriptstyle \theta_*} & \downarrow{\scriptstyle \pi_*} \\ \Omega_n(X\times BO(k),\varphi^k) & \longrightarrow & \Omega_n(X\times BO(q),\varphi^q) \end{array} \qquad \text{(I)}$$

where θ_*, induced by θ', is an isomorphism for q large.

Also, for q large, $\Omega_n(X\times BO(q),\varphi^q)\simeq \pi^S_{n+q+p}(T(\alpha)\wedge MO(q))$, where $T(\alpha)$ is the Thom space ([K1]). Since $T(\alpha)$ is $(p-1)$-connected we can conclude that $\eta_n(X)\simeq\Omega_n(X\times BO(q),\varphi^q)$ and this normal bordism group does not depend of α^p.

Let us denote by $I_n(X)$ the bordism group of continuous maps $h: M^n\to X^{n+k}$, which are homotopic to an immersion and let $\mathcal{F}: I_n(X)\to\eta_n(X)$ be the forgetful map.

Let us consider X a $(n+k)$-manifold and let ν^p_X be the stable normal bundle of X. If $\varphi^k=\varepsilon^{p+k}-\nu^p_X\times\gamma^k$, for p large, an element of $\Omega_n(X\times BO(k),\varphi^k)$ can be considered as $[M^n,(h,g),H]$ where $(h,g): M^n\to X\times BO(k)$ is a continuous map,

$$H: TM\oplus h^*(\nu^p_X)\oplus g^*(\gamma^k)\to\varepsilon^{p+k}\oplus\varepsilon^n$$

is a stable isomorphism and g is the classifying map of the stable normal bundle ν^S_h. Therefore, $\Omega_n(X\times BO(k),\varphi^k)$ can be identified with $I_n(X)$.

By using these identifications and diagram (I), in Salomonsen sequence ([S]) we get the following exact sequence, for q large and $n\le 2k+2$.

$$\begin{aligned} \text{(II)}\ \longrightarrow\ &\Omega_{n-k}(X\times BO(q)\times P^\infty,\Gamma_k)\longrightarrow I_n(X)\xrightarrow{\mathcal{F}}\eta_n(X)\xrightarrow{\widetilde{\gamma}_{k-1}} \\ \longrightarrow\ &\Omega_{n-k-1}(X\times BO(q)\times P^\infty,\Gamma_{k-1})\longrightarrow\cdots \end{aligned}$$

where $\Gamma_k=\nu^p_X\times\gamma^q\oplus(\varepsilon^{q-n+k}-\gamma^q)\otimes\lambda-\varepsilon^{p+q+n-k}$ and λ is the canonical line bundle over the real projective space P^∞.

Let us consider now ψ a virtual vector bundle over M.

From the exact sequence of Salomonsen, for $5\le n<2k$, we have the following exact sequence:

$$\text{(III)}\longrightarrow\Omega_n(\widetilde{\nu}_k(\psi),TM^0)\xrightarrow{\pi_{M_*}}\Omega_n(M,TM^0)\xrightarrow{\gamma_M}\Omega_{n-k-1}(M\times P^\infty,\Phi)\longrightarrow\cdots$$

where $\Phi=-(n-k-1)\lambda-\lambda\otimes\psi+TM^0$ and γ_M is defined by the construction of the sequence.

We recall that if $\psi=h^*TX-\varepsilon^k\oplus TM$, then γ_M $([M])$ is the invariant $\omega_k(\nu_h)$ defined by Koschorke ([K2]), ([K3]), which is an obstruction to the existence of a monomorphism from $M\times\mathbb{R}^\ell$ into γ_h.

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Here, $[M] = [M, 1_M, t_M] \in \Omega_n(M, TM^0)$ is the fundamental class of M where $t_M : TM \oplus \varepsilon^n \to \varepsilon^n \oplus TM$ is the isomorphism which interchange factors.

3. PROOFS OF THEOREMS A AND B

Proof of Theorem A:
Let $h : M \to X$ be a continuous map from a closed connected smooth n-dimensional manifold M into a smooth connected $(n+k)$-dimensional manifold X.

Let us consider now the following commutative diagram, where the left hand vertical sequence is (III) and the right hand vertical sequence is (II) and $(h,g)_*$ and $((h,g)\times Id)_*$ are induced maps of (h,g) in convenient normal bordism groups.

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\Omega_n(M, TM^0) & \xrightarrow{(h,g)_*} & \eta_n(X) \\
\downarrow{\scriptstyle \gamma_M} & & \downarrow{\scriptstyle \widetilde{\gamma}_{k-1}} \\
\Omega_{n-k-1}(M \times P^\infty, \Phi) & \xrightarrow{((h,g)\times Id)_*} & \Omega_{n-k-1}(X \times BO(q) \times P^\infty, \Gamma_{k-1}) \\
\downarrow & & \downarrow
\end{array}$$

Suppose that h is bordant to an immersion. Then

$$0 = \widetilde{\gamma}_{k-1}([M,h]) = ((h,g)\times Id)_*\left(\gamma_M([M])\right) .$$

Since $(h,g) : M \to X \times BO(q)$, q large, induces

$$(h,g)_* : H_i(M, \mathbb{Z}_2) \to H_i(X \times BO(q), \mathbb{Z}_2)$$

which is an isomorphism for $i < n-k$ and an epimorphism for $i = n-k$, we conclude that $((h,g)\times Id)_*$ is a $\mathcal{C}$-isomorphism for $i = n-k-1$ and then $\ker((h,g)\times Id)_* \in \mathcal{C}$.

We recall that the elements of image of γ_M have order a potency of 2. Therefore $\gamma_M([M,h]) = 0$ and h is homotopic to an immersion. □

Proof of Theorem B:
We recall that under hypotheses of Theorem B,

$$f_* : \Omega_n(M, f^*TN^0) \to \Omega_n(N, TN^0)$$

is a $\mathcal{C}$-isomorphism and $f^*(\beta_2) = \alpha_2$, where $\alpha = \nu_M$ and $\beta = \nu_N$ are the stable normal bundles of M and N and α_2 and β_2, the respectively 2-localization ([BGL]).

Let us consider the following commutative diagram

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$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
\Omega_n(\widetilde{V}_k(\psi_M), f^*TN^0) & \xrightarrow{G_*} & \Omega_n(\widetilde{V}_k(\psi_N), TN^0) \\
\downarrow (\pi'_M)_* & & \downarrow (\pi_N)_* \\
\Omega_n(M, f^*TN^0) & \xrightarrow{f_*} & \Omega_n(N, TN^0) \\
\downarrow \gamma'_M & & \downarrow \gamma_N \\
\Omega_{n-k-1}(M \times P^\infty, \phi_M) & \xrightarrow{F_*} & \Omega_{n-k-1}(N \times P^\infty, \phi_N)
\end{array}
$$

where the vertical sequences are obtained from (III), G_* and F_* are given in [S], $\psi_M = \varepsilon^{n+k} - TM \oplus \varepsilon^k$ and $\psi_N = \varepsilon^{n+k} - TN \oplus \varepsilon^k$.

We observe that $(\pi'_M)_*$ is the induced map of π_M in normal bordism groups with virtual bundle f^*TN^0.

If N immerses in $\mathbb{R}^{n+k}$, $(\pi_N)_*$ is sobrejective and since $f_* : H_i(M, \mathbb{Z}_2) \to H_i(N, \mathbb{Z}_2)$ is an isomorphism for $i \geq 0$, then F_* is a $\mathcal{C}$-monomorphism. Therefore $(\pi'_M)_*$ is a $\mathcal{C}$-epimorphism and since every element of the image of γ'_M has order a potency of 2 ([S]), we conclude that $(\pi'_M)_*$ is an epimorphism.

Let us consider now the commutative diagram

$$
\begin{array}{ccc}
\pi^s_{n+p}(T\widehat{\alpha}) & \longrightarrow & \pi^s_{n+p}(Tf^*(\widehat{\beta}) \\
\downarrow (\pi_M)_* & & \downarrow (\pi'_M)_* \\
\pi^s_{n+p}(T\alpha) & \longrightarrow & \pi^s_{n+p}(Tf^*\alpha)
\end{array}
$$

where $\widehat{\beta}$ and $\widehat{\alpha}$ denote the pull back of β and α by π_N and π_M, respectively; then two horizontal maps are $\mathcal{C}$-isomorphisms.

We conclude that $(\pi_M)_*$ is a $\mathcal{C}$-epimorphism and since the elements of the image of γ_M has order a potency of 2, the result follows. □

4. APPLICATIONS

Let M and N be closed smooth manifolds of dimension n and $(n+k)$, respectively and let $f : M \to N$ be a continuous map. Define $U_f \in H^k(N, \mathbb{Z}_2)$ to be the image of the fundamental class $[M] \in H_n(M,\mathbb{Z}_2)$ by the composite

$$H_n(M,\mathbb{Z}_2) \xrightarrow{f_*} H_n(N,\mathbb{Z}_2) \xrightarrow{D_N^{-1}} H^k(N,\mathbb{Z}_2)$$

where D_N denotes the Poincaré duality isomorphism.

We also consider the following commutative diagram:

$$\begin{array}{ccc} H^p(N,\mathbb{Z}_2) & \xrightarrow{\ \cup\, U_f\ } & H^{p+k}(N,\mathbb{Z}_2) \\ \Big\downarrow D_M \circ f^* & & \Big\downarrow D_N \\ H_{n-p}(M,\mathbb{Z}_2) & \xrightarrow{\ f_*\ } & H_{n-p}(N,\mathbb{Z}_2) \end{array}$$

Theorem 1: *Let M and N be closed smooth manifolds of dimension n . Suppose that*

$$H_i(M,\mathbb{Z}_2) \simeq H_i(N,\mathbb{Z}_2)\,, \ \textit{for all} \ \ i \geq 0$$

and there exists $f : M \to N$ with $deg_2 f = 1$. Then $f_* : H_i(M,\mathbb{Z}_2) \to H_i(N,\mathbb{Z}_2)$ is an isomorphism, for $i \geq 0$.

Proof: Since M and N have dimension n , we have that $U_f \in H^0(N,\mathbb{Z}_2)$ and $U_f = \deg_2 f$.

Therefore $\cup U_f$ is a multiple of $\deg_2 f$ and since $\deg_2 f = 1$, we have that

$$\cup U_f : H^p(N,\mathbb{Z}_2) \to H^p(N,\mathbb{Z}_2) \ \text{ is the identity}$$

map, for $p \geq 0$ and

$$f_* : H_{n-p}(M,\mathbb{Z}_2) \to H_{n-p}(N,\mathbb{Z}_2) \ \text{ is onto}\,,$$

for all $p \geq 0$. But $H_i(M,\mathbb{Z}_2) \simeq H_i(N,\mathbb{Z}_2)$, $i \geq 0$, and the result follows. □

Corollary 2: *Let M and N be closed smooth n-manifolds with isomorphic homology groups. Suppose that there exists $f : M \to N$ with $deg_2 f = 1$. Then M immerses in $\mathbb{R}^{n+k}$, $5 \leq n < 2k$, if and only if N immerses.* □

Let M and N be closed smooth n-manifolds. Given $x_0 \in M^n$ and $y_0 \in N^n$, let us take D_1^n and D_2^n disks containing x_0 and y_0, respectively, for which there exists a homeomorphism $h : D_1^n \to D_2^n$ with $h(x_0) = y_0$.

We denote by $A = \partial\, D_1$, $M_{n-1} = M^{(n-1)} \cup A$, where $M^{(n-1)}$ is the $(n-1)$-skeleton of M , $Y = N - h(\overset{o}{D}_1)$, $f_0 = h|_A$ and

$$\chi_n^{n-1} : H^n(M, A, \pi_{n-1}(Y)) \to H^n(M, A, H_{n-1}(Y))$$

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is the homomorphism induced in cohomology by the Hurewicz homomorphism.

Let us suppose that f_0 extends to M_{n-1}, Y is $(n-1)$-simple and $H_{n-1}(A, \mathbb{Z})$ is a free group.

Theorem 3: *Suppose that M^n and N^n are such that $H_*(M, \mathbb{Z}_2) \simeq H_*(N, \mathbb{Z}_2)$.*

If χ_n^{n-1} is a monomorphism and there exists a homomorphism $\psi : H_n(M, \mathbb{Z}) \to H_n(N, \mathbb{Z})$ such that $(f_0)_* = \psi \circ i_*$, with $i_* : H_n(A, \mathbb{Z}) \to H_n(M, \mathbb{Z})$ induced by the inclusion, then there exists $f : M \to N$ with $deg_2 f = 1$.

Proof: In these conditions f_0 extends to $f : M \to N$ (see[B]) with $f(M - \overset{o}{D}_1) = N - f(\overset{o}{D}_1)$. By excision $H_n(M, \mathbb{Z}_2)$,(resp. $H_n(N, \mathbb{Z}_2)$) is isomorphic to $H_n(M, M - x_0, \mathbb{Z}_2)$, (resp. $H_n(N, N - y_0, \mathbb{Z}_2)$), which is isomorphic to $H_n(D_1, D_1 - x_0, \mathbb{Z}_2)$, (resp. $H_n(f(D_1), f(D_1) - y_0, \mathbb{Z}_2)$) and the result follows. □

REFERENCES

A. M.F.Atiyah. Thom Complexes.*Proc. London Math.* (3) 11 (1961) 291-310.

B. C.Biasi. Teoria da obstrução e aplicações. *Notas do Instituto de Ciências Matemáticas de S.Carlos-USP.* (1986)

BGL. C. Biasi, D.L. Gonçalves and A.K.M. Libardi. Immersions in the metastable dimension range via the normal bordism approach. *Topology and its Applications* (to appear).

GH1. H. Glover and W. Homer. Immersing manifolds and 2-equivalence. *Lectures Notes in Math.* 657 (1978) 194-197. Springer Verlag.

GH2. ——————————. Metastable immersion, span and the two-type of a manifold. *Canadian Math. Bull,* 29 (1986) 20-24.

GM. H. Glover and G. Mislin. Immersion in the metastable range and 2-localization. *Proceedings of the Am. Math. Soc.* 43 (1974) 443-448.

H. M. Hirsch. Immersions of manifolds. *Trans. AMS.* 93 (1959) 242-276.

HW. P.J. Hilton and S. Wylie. *Homology Theory An Introdution to Algebraic Topology.* Cambridge University Press (1960).

K1. U. Koschorke. Vector Fields and Other Vector Bundle Morphism - A Singularity Approach. *Lecture Notes in Mathematics* 847 (1981) - Springer Verlag.

K2. ————————-. The Singularity Method and Immersions of m-manifolds into Manifolds of dimensions $2m-2$, $2m-3$ and $2m-4$. *Lecture Notes in Mathematics* 1350 (1988) - Springer Verlag.

K3. ————————-. *Nonstable and stable monomorphisms of Vector Bundles.* Topology and its Applications. 75 (1997) 261-286.

M. S. MacLane. *Homology.* Grundlehren Math. Wiss. 114, Springer Verlag, Berlin-Heidelberg-New York (1963).

S. H.A. Salomonsen. Bordism and Geometric Dimension. *Math. Scand.* 32 (1973) 87-111.

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