

## A Second Order Differential Equation with Nonlocal Conditions

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In this paper we give some existence results for a partial second order non-local Cauchy problem. May, 2002 ICMC-USP

### 1. INTRODUCTION

This paper is concern with the second order nonlocal Cauchy problem

$$\begin{cases} \ddot{u}(t) = Au(t) + f(t, u(t), \dot{u}(t)), & t \in (-T_0, T_1), \\ u(0) = x_0 + q(t_1, t_2, \dots, t_n : u(\cdot), \dot{u}(\cdot)), \\ \dot{u}(0) = y_0 + p(t_1, t_2, \dots, t_n : u(\cdot), \dot{u}(\cdot)), \end{cases}$$

where  $A$  is the infinitesimal generator of a strongly continuous cosine function of linear operators,  $(C(t))_{t \in \mathbb{R}}$ , on a Banach space  $X$ ;  $-T_0 < 0 < t_1 < t_2 < \dots < t_n < T_1$ ;  $f : \mathbb{R} \times X \times X \rightarrow X$ ,  $q, p : [-T_0, T_1]^n \times X^2 \rightarrow X$  are appropriated functions and the symbol  $h(t_1, t_2, t_3, \dots, t_n, u(\cdot), \dot{u}(\cdot))$  is used in the sense that “ $\cdot$ ” can be substitute only for the points  $t_i$ , for instance  $h(t_1, t_2, t_3, \dots, t_n, u(\cdot), \dot{u}(\cdot)) = \sum_{i=1}^n (\alpha_i u(t_i) + \beta \dot{u}(t_i))$ .

Motivated for numerous applications Byszewski studied in [1], the first order evolution nonlocal problem

$$\begin{cases} \dot{u}(t) = Au(t) + f(t, u(t)), & t \in [0, T], \\ u(0) = x_0 + q(t_1, t_2, \dots, t_n : u(\cdot)), \end{cases}$$

where  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of linear operators on a Banach space  $X$ . In the cited paper, Byszewski prove the existence of mild, strong and classical solutions for (1), employing usual assumptions on  $f$  and the semigroup theory. For a more complete reference for differential equations with nonlocal conditions refer to [1]-[4].

On the other hand, some second order nonlocal initial value problems are studied by Ntouyas & Tsamatos in [6, 5, 1]. In these works the authors discuss the existence of solu-

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tions for a type of second order delay integrodifferential equations with nonlocal conditions described in the form

$$x''(t) = Ax(t) + f(t, x(\sigma_1(t)), \int_0^t k(t-s)h(s, x(\sigma_2(s)), \dot{x}(\sigma_3(s)))ds, x(\sigma_4(t))), \quad t \in (0, T),$$

$$x(0) = g(x) + x_0, \quad \dot{x}(0) = \eta,$$

where  $A$  is the generator of a strongly continuous cosine family of bounded linear operators on  $X$ ;  $x_0, \eta \in X$  and  $g : C(I : X) \rightarrow X$ ,  $f, h : I \times X^3 \rightarrow X$  are appropriated functions. In general the result are obtained using the Leray-Schauder Alternative and the strong assumption that the cosine family  $(C(t))_{t \in \mathbb{R}}$  is compact, which in turn implies that  $\dim(X) < \infty$ .

Throughout this paper,  $X$  will be a Banach space provided with a norm  $\| \cdot \|$ ;  $A : D(A) \rightarrow X$  will be the infinitesimal generator of strongly continuous cosine function of linear operators,  $C = (C(t))_{t \in \mathbb{R}}$ , on  $X$  and  $S = (S(t))_{t \in \mathbb{R}}$  will be the associated sine function, that is  $S(t)x = \int_0^t C(s)x ds$ ,  $x \in X$ . For the theory of cosine functions, refer Fatorinni [7] and Travis [8]. We will point out here some notations and properties that will be essential for us. Along of this paper we denote by  $D(A)$  and  $E$  the sets;

$$D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable}\},$$

$$E = \{x \in X : C(t)x \text{ is once continuously differentiable}\}.$$

The existence of solutions of the second order abstract Cauchy problem

$$\begin{aligned} \ddot{x}(s) &= Ax(s) + h(s), \quad 0 \leq s \leq a, \\ x(0) &= x_0, \quad \dot{x}(0) = x_1, \end{aligned}$$

where  $h : [0, a] \rightarrow X$  is an integrable function, has been discussed in [8]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [2]. We only mention here that the function  $x(\cdot)$  given by

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s) ds, \quad 0 \leq t \leq a, \quad (1)$$

is called a mild solution of (1). In the case in which  $x_0 \in E$  then  $x(\cdot)$  is continuously differentiable and

$$x(t) = AS(t)x_0 + C(t)x_1 + \int_0^t C(t-s)h(s) ds. \quad (2)$$

The properties in the next result are well know ( [8, pp.77] ).

LEMMA 1.1. *In the previous condition, the next properties hold.*

1. If  $x \in E$  then  $S(t)x \in D(A)$ ,  $\frac{d}{dt}C(t)x = AS(t)x$  and  $\frac{d^2}{dt^2}S(t)x = AS(t)x$ .
2. If  $x \in D(A)$  then  $C(t)x \in D(A)$  and  $\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax$ .
3. If  $x \in D(A)$  then  $S(t)x \in D(A)$  and  $\frac{d^2}{dt^2}S(t)x = AS(t)x = S(t)Ax$ .

This work contain two sections. In section 2 we discuss the existence of solution for the abstract nonlocal problem (1). Firstly, we introduce the concept of mild and classical solution and subsequently employing the contraction mapping principle and the ideas in [8], we prove the existence of mild and classical solutions.

The terminology and notations are those generally used in operator theory. In particular, if  $Z$  is a Banach space, we indicate by  $\mathcal{L}(Z)$  the Banach space of the bounded linear operators from  $Z$  into  $Z$ . In addition  $B_r[x : Z]$  will denote the closed ball in  $Z$  with center at  $x$  and radius  $r$ . For a nonnegative bounded function  $\xi : (-T_0, T_1) \rightarrow \mathbb{R}$  we will employ the notation  $\xi_t$ ,  $t \in (-T_0, T_1)$ , for  $\xi_t = \sup\{\xi(\theta) : \theta \in (-T_0, t]\}$

*Remark 1. 1.* In the rest of this work, we use the notations  $q(t_i : x, y)$  and  $p(t_i : x, y)$  for  $q(t_1, t_2, \dots, t_n : x, y)$  and  $p(t_1, t_2, \dots, t_n : x, y)$  respectively.

## 2. EXISTENCE RESULTS

In this section we discuss the existence of mild and classical solutions for the abstract nonlocal Cauchy problem (1). By comparison with Travis [8], we introduce the following definitions:

DEFINITION 2.1. A function  $u \in C^1((-T_0, T_1) : X)$  is a mild of the nonlocal Cauchy problem (1), if

$$u(t) = C(t)(x_0 + q(t_i : u, \dot{u})) + S(t)(y_0 + p(t_i : u, \dot{u})) + \int_0^t S(t-s)f(s, u(s), \dot{u}(s))ds \quad \forall t \in (-T_0, T_1).$$

DEFINITION 2.2. A function  $u(\cdot) \in C^2((-T_0, T_1) : X)$  is a classical solution of the nonlocal Cauchy problem (1), if  $u(\cdot)$  is solution of (1).

Next, we show the existence and uniqueness of mild solutions using the contraction mapping principle, for this reason we introduce the following technical assumptions.

**Assumptions :** In the following assumptions  $r_1, r_2$  are fixed positive numbers.

(**A<sub>1</sub>**) The function  $f$  is continuous and there exist positive constants  $l^1(f)$  and  $l^2(f)$  such that

$$\| f(t, x, y) - f(t, x', y') \| \leq l^1(f) \| x - x' \| + l^2(f) \| y - y' \|,$$

for every  $x, x' \in B_{r_1}[0, X]$  and  $y, y' \in B_{r_2}[0, X]$ .

(**A<sub>2</sub>**) The functions  $q(t_i : \cdot), p(t_i : \cdot) : [-T_0, T_1]^n \times X^2 \rightarrow X$  are continuous and there exist positive constants  $l^i(p), l^i(q)$  such that

$$\begin{aligned} \| q(t_i : u, \tilde{u}) - q(t_i : v, \tilde{v}) \| &\leq l^1(q) \| u - \tilde{u} \|_{T_1} + l^2(q) \| v - \tilde{v} \|_{T_1}, \\ \| p(t_i : u, \tilde{u}) - p(t_i : v, \tilde{v}) \| &\leq l^1(p) \| u - \tilde{u} \|_{T_1} + l^2(p) \| v - \tilde{v} \|_{T_1}, \end{aligned}$$

for every  $(u, \tilde{u}), (v, \tilde{v}) \in C((-T_0, T_1) : B_{r_1}[0, X] \times B_{r_2}[0, X])$

(**A<sub>3</sub>**) There exist functions  $B : [-T_0, T_1] \rightarrow \mathcal{L}(X)$  and  $g : [-T_0, T_1]^n \times X^2 \rightarrow X$  such that

i)  $B(0) = 0$ ,  $B(\cdot)$  is strongly continuous and  $\frac{d}{dt}C(t)q(t_i : x, y) = B(t)g(t_i : x, y)$  for every  $x, y \in X$ ,

ii)  $g$  is continuous and there exist positive constants  $l^i(g)$  such that

$$\| g(t_i : u, v) - g(t_i : \tilde{u}, \tilde{v}) \| \leq l^1(g) \| u - \tilde{u} \|_{T_1} + l^2(g) \| v - \tilde{v} \|_{T_1},$$

for every  $(u, \tilde{u}), (v, \tilde{v}) \in C((-T_0, T_1) : B_{r_1}[0, X] \times B_{r_2}[0, X])$ .

*Remark 2. 1.* In the proof of the following theorem we will employ the notations  $N_m$ ,  $m = p, q, g$ , for

$$N_m = \sup\{m(t_i : u, v) : (u, v) \in C((-T_0, T_1) : B_{r_1}[0, X] \times B_{r_2}[0, X])\}$$

and  $N_f$  for the sup of the function  $f$  on  $[-T_0, T_1] \times B_{r_1}[0, X] \times B_{r_2}[0, X]$ .

**THEOREM 2.1.** *Let  $x_0 \in E \cap B_{r_1}(0, X)$ ,  $y_0 \in B_{r_2}(0, X)$  and assume that assumption **A<sub>1</sub>-A<sub>3</sub>** holds. If*

$$\| C \|_{T_1} (\| x_0 \| + N_q) + \| S \|_{T_1} (\| y_0 \| + N_p + N_f(T_0 + T_1)) \leq r_1, \quad (1)$$

$$\| AS(\cdot)x_0 \|_{T_1} + \| B \|_{T_1} N_g + \| C \|_{T_1} (\| y_0 \| + N_p + N_f(T_0 + T_1)) \leq r_2. \quad (2)$$

and  $\Theta = \max_{i \in \{1, 2\}} \{\Lambda_i\} < 1$ , where

$$\Lambda_i = \| C \|_{T_1} (l^i(q) + l^i(p) + (T_0 + T_1)l^i(f)) + \| S \|_{T_1} (T_0 + T_1)l^i(f) + l^i(p) + \| B \|_{T_1} l^i(g)$$

then there exists a unique mild solution of (1).

**Proof:** Let  $Z = \{(u, v) : u, v \in C((-T_0, T_1) : X), \dot{u} = v\}$  endowed of the norm  $\|(u, v)\|_1 = \|u\|_{T_1} + \|v\|_{T_1}$  and  $Y = \{(u, v) \in Z : \|u\|_{T_1} \leq r_1, \|v\|_{T_1} \leq r_2\}$ . Clearly

$Z$  is a Banach space and  $Y$  a closed subset of  $Z$ . On the space  $Y$  we define the mapping  $\Phi : (u, v) \rightarrow (\Phi_1(u), \Phi_2(u))$  by

$$\begin{aligned}\Phi_1(u)(t) &= C(t)(x_0 + q(t_i : u, \dot{u})) + S(t)(y_0 + p(t_i : u, \dot{u})) \\ &\quad + \int_0^t S(t-s)f(s, u(s), \dot{u}(s))ds, \\ \Phi_2(u)(t) &= AS(t)x_0 + B(t)g(t_i : u, \dot{u}) + C(t)(y_0 + p(t_i : u, \dot{u})) \\ &\quad + \int_0^t C(t-s)f(s, u(s), \dot{u}(s))ds.\end{aligned}$$

Next we'll prove that  $\Phi$  is a contraction mapping on  $Y$ . A simple estimate show that for  $(u, v) \in Y$

$$\| \Phi_1(u)(t) \| \leq \| C \|_{T_1} (\| x_0 \| + N_q) + \| S \|_{T_1} (\| y_0 \| + N_p + N_f(T_0 + T_1)), \quad (3)$$

$$\| \Phi_2(u)(t) \| \leq \| ASx_0 \|_{T_1} + \| B \|_{T_1} N_g + \| C \|_{T_1} (\| y_0 \| + N_p + N_f(T_0 + T_1)), \quad (4)$$

thus, from (1)-(2) and (3)-(4) follow that  $\Phi(u, v) \in Y$ .

To prove that  $\Phi$  is a contraction, we take  $(u, w), (v, z)$  in  $Y$ . Then for  $t \in (-T_0, T_1)$  we see that can be compute

$$\begin{aligned}\| \Phi_1(u)(t) - \Phi_1(v)(t) \| &\leq \| C \|_{T_1} (l^1(q) \| u - v \|_{T_1} + l^2(q) \| \dot{u} - \dot{v} \|_{T_1}) \\ &\quad + \| S \|_{T_1} (l^1(p) \| u - v \|_{T_1} + l^2(p) \| \dot{u} - \dot{v} \|_{T_1}) \\ &\quad + \int_0^t \| S \|_{T_1} (l^1(f) \| u - v \|_{\theta} + l^2(f) \| \dot{u} - \dot{v} \|_{\theta}) d\theta,\end{aligned}$$

thus

$$\begin{aligned}\| \Phi_1(u) - \Phi_1(v) \|_{T_1} &\leq (\| C \|_{T_1} l^1(q) + \| S \|_{T_1} (l^1(p) + (T_0 + T_1)l^1(f))) \| u - v \|_{T_1} \\ &\quad + (\| C \|_{T_1} l^2(q) + \| S \|_{T_1} (l^2(p) + (T_0 + T_1)l^2(f))) \| \dot{u} - \dot{v} \|_{T_1}.\end{aligned} \quad (5)$$

Similarly

$$\begin{aligned}\| \Phi_2(u) - \Phi_2(v) \|_{T_1} &\leq (\| B \|_{T_1} l^1(g) + \| C \|_{T_1} (l^1(p) + (T_0 + T_1)l^1(f))) \| u - v \|_{T_1} \\ &\quad + (\| B \|_{T_1} l^2(g) + \| C \|_{T_1} (l^2(p) + (T_0 + T_1)l^2(f))) \| \dot{u} - \dot{v} \|_{T_1}.\end{aligned} \quad (6)$$

From (5)-(6)

$$\| \Phi(u, w) - \Phi(v, z) \|_1 \leq \Theta \| (u, w) - (v, z) \|_1,$$

and thus,  $\Phi$  is a contraction. Clearly the unique fix point of  $\Phi$  is the unique mild solution of (1). The proof is complete.

Now we consider, briefly, some simple examples of functions satisfying  $(\mathbf{A}_3)$ . In the next examples, we said that  $h(t_i : \cdot) : [0, T]^n \times Z \rightarrow W$  satisfies the condition  $\mathbf{H}(\mathbf{Z}, \mathbf{W})$  if exist positive constants  $l^i(h)$  such that

$$\| h(t_i : u, \tilde{u}) - h(t_i : v, \tilde{v}) \|_W \leq l^1(h) \| u - v \|_Z + l^2(h) \| \tilde{u} - \tilde{v} \|_Z,$$

for every  $(u, \tilde{u}), (v, \tilde{v})$  in  $C((-T_0, T_1) : Z \times Z)$ .

1) Let  $[D(A)]$  be the space  $D(A)$  endowed with the graph norm. If  $h(t_i : \cdot)$  verifies the condition  $\mathbf{H}(\mathbf{X}, [\mathbf{D}(\mathbf{A})])$  then  $h(t_i : \cdot)$  verifies  $(\mathbf{A}_3)$  since,  $\frac{d}{dt}C(t)h(t_i : x, y) = AS(t)h(t_i : x, y) = S(t)Ah(t_i : x, y)$ , for each  $x, y \in X$ .

2) Assume that  $(C(t))_{t \in \mathbb{R}}$  satisfies condition  $F$  ( see [8, pp.81] for details ). If  $[D(B)]$  is the space  $D(B)$  endowed with the graph norm and  $h(t_i : \cdot)$  verifies the condition  $\mathbf{H}(\mathbf{X}, [\mathbf{D}(\mathbf{B})])$  then  $h(t_i : \cdot)$  verifies  $(\mathbf{A}_3)$ . The assertion is clear since  $BS(\cdot)$  is uniformly bounded on  $[-T_0, T_1]$  and  $\frac{d}{dt}C(t)h(t_i : x, y) = AS(t)h(t_i : x, y) = BS(t)Bh(t_i : x, y)$  for every  $x, y \in X$ .

3) Let  $r \in \mathbb{R}$  and  $h(t_i : \cdot)$  verifying condition  $\mathbf{H}(\mathbf{X}, \mathbf{X})$ . From Proposition 2.2 in [8] we have that,  $\frac{d}{dt}C(t)S(r)h(t_i : x, y) = AS(t)S(r)h(t_i : x, y) = \frac{1}{2}(C(r+t) - C(r-t))h(t_i : x, y)$  for each  $x, y \in X$ , thus  $q(t_i : x, y) = S(r)h(t_i : x, y)$  verifies  $\mathbf{A}_3$ .

## 2.1. Classical Solutions

The proof of the following theorem is similar to the proof of Proposition 3.2 in Travis [8] and for this reason we only consider the principal steps of the proof by completeness.

**THEOREM 2.2.** *(Classical solution) Let assumptions in Theorem 2.1 be satisfied. Assume that  $f$  is continuously differentiable and that exists  $l^3(f) > 0$  such that*

$$\max\{\|f_{x_j}(t, x, y) - f_{x_j}(t, x, \tilde{y})\| : j = 1, 2, 3\} \leq l^3(f) \|y - \tilde{y}\|, \quad (x, y) \in B_{r_1}[0, X] \times X.$$

*Let  $u(\cdot)$  be the mild solution of (1). If  $u(0) \in D(A)$  and  $\dot{u}(0) \in E$  then  $u(\cdot)$  is a classical solution.*

**Proof:** Since  $f_{x_3}(s, u(s), \dot{u}(s))$  is bounded on  $(-T_0, T_1)$ , from the contraction mapping principle there exists a unique solution,  $v \in C((-T_0, T_1) : X)$ , of the integral equation

$$\begin{aligned} v(t) &= C(t)(Au(0) + f(0, u(0), \dot{u}(0))) + AS(t)(\dot{u}(0)) \\ &+ \int_0^t C(t-s)(f_{x_1}(s, u(s), \dot{u}(s)) + f_{x_2}(s, u(s), \dot{u}(s))\dot{u}(s) + f_{x_3}(s, u(s), \dot{u}(s))v(s)) ds \end{aligned}$$

From the steps in the proof of Proposition 3.3 in [8], follow that  $z(t) = \dot{u}(0) + \int_0^t v(s)ds = \dot{u}(t)$ , thus  $s \rightarrow f(s, u(s), \dot{u}(s))$  is once continuously differentiable on  $(-T_0, T_1)$ . From Proposition 2.4 in Travis [8], there exist a unique classical solution,  $w : (-T_0, T_1) \rightarrow X$ , of

$$\begin{aligned} \ddot{w}(t) &= Aw(t) + f(t, u(t), \dot{u}(t)), & t \in (-T_0, T_1), \\ w(0) &= x_0 + q(t_i : u, \dot{u}) = u(0), \\ \dot{w}(0) &= y_0 + p(t_i : u, \dot{u}) = \dot{u}(0), \end{aligned} \tag{7}$$

moreover

$$w(t) = C(t)(x_0 + q(t_i : u, \dot{u})) + S(t)(y_0 + p(t_i : u, \dot{u})) + \int_0^t S(t-s)f(s, u(s), \dot{u}(s))ds.$$

Obviously  $u = w$  and therefore  $u$  is a classical solution. The proof is complete.

The methods used in the proofs of the previous theorems can be used to show similar existence results for

$$\begin{cases} \ddot{u}(t) = Au(t) + f(t, u(t)), & t \in (-T_0, T_1), \\ u(0) = x_0 + q(t_i : u), \\ \dot{u}(0) = y_0 + p(t_i : u). \end{cases}$$

DEFINITION 2.3. A function  $u \in C^1((-T_0, T_1) : X)$  is a mild solution of (8) if

$$u(t) = C(t)(x_0 + q(t_i : u)) + S(t)(y_0 + p(t_i : u)) + \int_0^t S(t-s)f(s, u(s))ds, \quad \forall t \in (-T_0, T_1).$$

DEFINITION 2.4. A function  $u \in C^2((-T_0, T_1) : X)$  is a classical solution of the nonlocal Cauchy problem (8), if  $u$  is solution of (8).

**Assumption :** In the following assumption,  $r_1$  is a fixed positive constant.

(A<sub>4</sub>) There exist positive constants  $l(f), l(q), l(p)$  such that

$$\begin{aligned} \|f(t, x) - f(t, y)\| &\leq l(f) \|x - y\|, \\ \|q(t_i : u) - q(t_i : v)\| &\leq l(q) \|u - v\|_{T_1}, \\ \|p(t_i : u) - p(t_i : v)\| &\leq l(p) \|u - v\|_{T_1}, \end{aligned}$$

for every  $(x, y) \in B_{r_1}[0, X] \times B_{r_1}[0, X]$ ,  $(u, v) \in C((-T_0, T_1) : B_{r_1}[0, X] \times B_{r_1}[0, X])$ .

THEOREM 2.3. Assume that assumption (A<sub>4</sub>) hold and that

$$\begin{aligned} \|C\|_{T_1} (\|x_0\| + N_q) + \|S\|_{T_1} (\|y_0\| + N_p + N_f(T_0 + T_1)) &\leq r_1, \\ \|C\|_{T_1} l(q) + \|S\|_{T_1} (l(f)(T_0 + T_1) + l(p)) &< 1. \end{aligned}$$

Then there exist a unique mild solution of the nonlocal Cauchy problem (8).

THEOREM 2.4. Assume that assumption in Theorem 2.3 hold and that  $f$  is continuously differentiable. If  $u(\cdot)$  is the mild solution of (8) and  $(u(0), \dot{u}(0)) \in D(A) \times E$  then  $u(\cdot)$  is a classical solution.

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