Bilipschitz determinacy of quasihomogeneous germs

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We obtain estimates for the degree of bilipschitz determinacy of quasihomogeneous function-germs. April, 2002 ICMC-USP

1. INTRODUCTION

A basic problem in Singularity Theory is the local classification of mappings module diffeomorphisms. In 1965, H. Whitney justified the rigidity of the classification problem by $C^1$-diffeomorphism giving the following example:

$$F_t(x, y) = xy(x - y)(x - ty); \quad 0 < t < 1$$

which presents the following phenomenon: for any $t \neq s$ in $I = (0, 1)$ it is not possible to construct a $C^1$-diffeomorphism $\phi : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ such that $F_t = F_s \circ \phi$. This motivated the classification of mappings by “isomorphisms” weaker than diffeomorphisms.

There is an extensive literature related to $C^r$-equivalence ($1 \leq r < \infty$) of map-germs, among them [5], [4] and [1] which are more closely related to this work. However, only few

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recent works deal with the problem of bilipschitz classification of map-germs. This work is inspired in a recent paper by J.-P. Henry and A. Parusinski [2], where they show that the bilipschitz equivalence of analytic function-germs admits continuous moduli. We obtain estimates for the degree of bilipschitz determinacy of quasihomogeneous function-germs. Examples are given to show that the estimates are sharp.

2. BILIPSCHITZ EQUIVALENCE

Let $\lambda \in \mathbb{R}$ be a positive number. A mapping $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^p$ is called $\lambda$-Lipschitz, or simply Lipschitz if it satisfies:

$$\|\phi(x) - \phi(y)\| \leq \lambda \|x - y\| \quad \forall \ x, y \in U.$$ 

When $n = p$ and $\phi$ has a Lipschitz inverse, we say that $\phi$ is bilipschitz.

Two germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ are called bilipschitz equivalent if there exists a bilipschitz map-germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f = g \circ \phi$.

Example 2.2.1. Let $f, g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be given by $f(x) = x, g(x) = x^3$. It is easy to show that $f$ and $g$ are not bilipschitz equivalent. On the other hand, there is a homeomorphism $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $f = \phi \circ g$.

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be the germ of an analytic function,

$$f(x) = f_m(x) + f_{m+1}(x) + \cdots,$$

with $f_i$ a homogeneous form of degree $i$, and $f_m \neq 0$. We denote by $m_f := m$, the multiplicity of $f$. We say that $f$ has non-degenerate tangent cone if $0 \in \mathbb{R}^n$ is the only point in $\mathbb{R}^n$ in which

$$\frac{\partial f_m}{\partial x_1} = \cdots = \frac{\partial f_m}{\partial x_1} = 0.$$

Proposition 2.2.2. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be the germ of an analytic function. Then

$$m_f = \text{ord}_r \left[ \sup_{B(0,r)} |f| \right],$$

where $B(0, r)$ denote the ball centered at the origin with radius $r$.

Proof. Let $\alpha = \text{ord}_r \left[ \sup_{B(0,r)} |f| \right]$. Write

$$f(x) = f_m(x) + f_{m+1}(x) + \cdots$$
with \( f_i \) a homogeneous form of degree \( i \), and \( f_m \neq 0 \). Let \( x = (x_1, \ldots, x_n) \) be such that \( f_m(x) \neq 0 \). Then, given \( r > 0 \) we have

\[
|f(rx)| = r^m|f_m(x) + rf_{m+1}(x) + \cdots| \\
\geq K r^m
\]

for some constant \( K > 0 \), hence \( m \geq \alpha \).

On the other hand, from the Curve Selection Lemma, there exists an analytic arc \( \gamma : [0, \epsilon) \to \mathbb{R}^n \), \( \gamma(0) = 0 \), such that

\[
\alpha = \text{ord}_r |f(\gamma(r))|
\]

and \( |\gamma(r)| \leq r \) for each \( r > 0 \). Since \( \gamma(0) = 0 \), we can write \( \gamma(r) = r\tilde{\gamma}(r) \) with \( \lim_{r \to 0} \tilde{\gamma}(r) < \infty \). Therefore,

\[
|f(\gamma(r))| = r^m|f_m(\tilde{\gamma}(r)) + rf_{m+1}(\tilde{\gamma}(r)) + \cdots| \\
\leq L r^m
\]

for some constant \( L > 0 \). Hence, \( m \leq \alpha \).

**Corollary 2.2.3.** Let \( f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) be germs of analytic functions. If \( f \) and \( g \) are bilipschitz equivalent, then \( m_f = m_g \).

The corollary above in the complex case was proved by J.-J. Risler and D. Trotman in [3]. It is obvious that the converse statement is false, but we can prove the following result

**Proposition 2.2.4.** Let \( F_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) be a smooth family of smooth function-germs. If \( m_{F_t} \) is constant and \( F_t \) has non-degenerate tangent cone for each \( t \), then for each \( t \neq s \), \( F_t \) and \( F_s \) are bilipschitz equivalent.

The result above will follow as consequence of Theorem 3.3.3.

**Corollary 2.2.5.** The family (1) satisfies: \( F_t \) and \( F_s \) are bilipschitz equivalent \( \forall \ t, s \in (0, 1) \).

It is valuable to observe that the Proposition 2.2.4 does not guarantee the non-rigidity of the bilipschitz classification problem for analytic functions. In fact, J.-P. Henry and A. Parusinski ([2]) presented the family \( F_t : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) given by \( F_t(x, y) = x^3 - 3t^2xy^2 + y^6 \) which satisfies: for any \( t \neq s \in (0, \frac{1}{2}) \) there is no bilipschitz map-germ \( \phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0) \) such that \( F_t = F_s \circ \phi \). The proof is based on the analysis of the expansion of the germs of the family along each arc of their polar curves. The argument in [2] also holds in the real case, that is, the following holds:
The family $F_t : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ given by $F_t(x, y) = x^3 - 3t^2 xy^2 + y^6$ satisfies: for any $t \neq s \in (0, \frac{1}{2})$ there is no bilipschitz map $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ such that $F_t = F_s \circ \phi$.

Note that $F_t$ is a deformation of the quasihomogeneous germ $f = x^3 + y^6$ which has an isolated singularity at origin. Therefore, it is natural to ask for which $\theta(x, y)$ the family $f + t\theta$ is bilipschitz trivial.

### 3. Bilipschitz Determinacy of Quasihomogeneous Germs

Let $f_t : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, $t \in I$ (an interval in $\mathbb{R}$), be a smooth family of smooth function-germs. That is, there is a neighborhood $U$ of 0 in $\mathbb{R}^n$ and a smooth function $F : U \times I \to \mathbb{R}$ such that $F(0, t) = 0$ and $f_t(x) = F(x, t)$ $\forall t \in I, \forall x \in U$. We call $f_t$ strongly bilipschitz trivial when there is a continuous family of $\lambda$-Lipschitz map-germs (vector field) $v_t : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that

$$\frac{\partial f_t}{\partial t}(x) = (df_t)_x(v_t(x))$$

$\forall t \in \mathbb{R}$ and $\forall x$ near 0 in $\mathbb{R}^n$.

**Theorem 3.3.1.** If $f_t$ is bilipschitz trivial, then for each $t \neq s \in I$ there is a bilipschitz map-germ $\phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f_t = f_s \circ \phi$.

The above theorem is known as a result of Thom-Levine type and its proof is immediate, since the flow of a Lipschitz vector field is bilipschitz.

Let $E_n$ be the space of smooth function-germs $(\mathbb{R}^n, 0) \to \mathbb{R}$. Given $f \in E_n$, we denote $Nf(x) = \sum \left[\frac{\partial f}{\partial x_i}(x)\right]^2$. We say that $Nf(x)$ satisfies a Lojasiewicz condition if there exist constants $c > 0$ and $\alpha > 0$ such that $Nf(x) \geq c\|x\|^\alpha$.

Fix the weights $(r_1, \ldots, r_n)$. We recall that a function $f$ is called quasihomogeneous with respect to the given weights if there is a number $d$ such that $f$ satisfies the following equation:

$$f(\lambda \cdot x) = \lambda^d(x_1, \ldots, x_n)$$

$\forall \lambda \in \mathbb{R}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where $\lambda \cdot x = (\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n)$. With respect to the given weights, for each monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, we define $\text{fil}(x^\alpha) = \sum_{i=1}^{n} \alpha_i r_i$. We define a filtration in the ring $E_n$ via the function $\text{fil}(f) = \min\{\text{fil}(x^\alpha) : (\frac{\partial f}{\partial x_i})(0) \neq 0\}$, for each $f \in E_n$. We can extend this definition to $E_{n+1}$, the ring of 1-parameter families of smooth function-germs in $E_n$, by defining $\text{fil}(x^\alpha t^\beta) = \text{fil}(x^\alpha)$.

Let $(r_1, \ldots, r_n; 2k)$ be fixed. The standard control function of type $(r_1, \ldots, r_n; 2k)$ is $\rho(x) = x^{2\alpha_1} + \cdots + x^{2\alpha_n}$, where the $\alpha_i$ are chosen such that the function $\rho$ is quasihomogeneous of type $(r_1, \ldots, r_n; 2k)$.
Lemma 3.3.2. Let \( h(x) \) be a quasihomogeneous polynomial function of type \((r_1, \ldots, r_n; 2k)\), with \( r_1 \leq \cdots \leq r_n \), \( \rho \) the standard control function of same type that \( h \) and \( h_t(x) \) a deformation of \( h \) such that:

\[
\text{fil}(h_t) \geq 2k + r_n, \quad t \in [0, 1].
\]

Then the function \( \frac{h_t(x)}{\rho(x)} \) is \( c \)-Lipschitz, with \( c \) independent of \( t \).

Proof. Without loss of generality, we can suppose that \( h_t(x) \) is quasihomogeneous of type \((r_1, \ldots, r_n; d)\) where \( d \geq 2k + r_n \). We consider \( G_t(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) given by \( G_t(x, y) = |\rho(y)h_t(x) - \rho(x)h_t(y)| \), \( m_t(x, y) = \|x - y\|\rho(x)\rho(y)\) and \( M = \{(x, y, t) : G_t(x, y) = 1\} \). Since \( M \) is closed, the number \( c = \inf\{m_t(x, y) : (x, y, t) \in M\} \) is positive.

Now, let \( x, y \in \mathbb{R}^n \) be sufficiently near the origin \( x \neq 0 \), \( y \neq 0 \) and \( x \neq y \). Let \( \lambda > 0 \) be such that \( G_t(\lambda \cdot x, \lambda \cdot y) = 1 \), that is,

\[
G_t(x, y) = \frac{1}{\lambda^{2k+d}}
\]

On the other hand, we use that \( \lambda > 1 \) to obtain:

\[
m_t(\lambda \cdot x, \lambda \cdot y) = \lambda^{4k}\|\lambda \cdot x - \lambda \cdot y\|\rho(x)\rho(y)
\leq \lambda^{4k+r_n}\|x - y\|\rho(x)\rho(y)
\]

\[
= \lambda^{4k+r_n}m_t(x, y)
\]

\[
\therefore
\]

\[
m_t(x, y) \geq \frac{1}{\lambda^{4k+r_n}} c.
\]

Now, we use that \( \lambda > 1 \), \( d \geq 2k + r_n \), (3), (3) and we obtain the following inequality \( m_t(x, y) \geq cG_t(x, y) \), that is,

\[
\|\frac{h_t(x)}{\rho(x)} - \frac{h_t(y)}{\rho(y)}\| \leq c^{-1}\|x - y\|.
\]

Theorem 3.3.3. Let \( f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0 \) be the germ of a quasihomogeneous polynomial function of type \((r_1, \ldots, r_n; d)\), \( r_1 \leq \cdots \leq r_n \) with isolated singularity. Let \( f_t(x) = f(x) + t\Theta(x, t), t \in [0, 1], \) be a smooth deformation of \( f \). If \( \text{fil}(\Theta) \geq d + r_n - r_1 \), then \( f_t \) admits a strong bilipschitz trivialization along \( I = [0, 1] \).

Proof. We can see that for each \( i \) there exists a \( s_i \) such that \( \frac{\partial f}{\partial x_i} \) is quasihomogeneous of the type \((r_1, \ldots, r_n; s_i)\), \( s_i = d - r_i \).
Let $N^*f$ be defined by

$$N^*f = \sum \left[ \frac{\partial f}{\partial x_i} \right]^{2\alpha_i},$$

where $\alpha_i = \frac{k}{s_i}$ and $k = \text{l.c.m.}(s_i)$. Therefore $N^*f$ is a quasihomogeneous control function of the type $(r_1, \ldots, r_n; 2k)$.

The lemma below is proved in [4].

**Lemma 3.3.4.** There exist constants $0 < c_2 < c_1$ such that

$$c_2 \rho(x) \leq N^*f_t(x) \leq c_1 \rho(x).$$

We have the following equation:

$$\frac{\partial f_t}{\partial t} N^*f_t = df_t(W),$$

where $W$ is given by

$$W = \sum W_i \frac{\partial}{\partial x_i} \text{ where } W_i = \frac{\partial f_t}{\partial t} \left[ \frac{\partial f}{\partial x_i} \right]^{2\alpha_i-1}.$$ 

Since $\text{fil} \left( \frac{\partial f_t}{\partial t} \right) \geq d + r_n - r_1$ and

$$\text{fil} \left( \left[ \frac{\partial f_t}{\partial x_i} \right]^{2\alpha_i-1} \right) = (2\alpha_i - 1)\text{fil} \left( \frac{\partial f_t}{\partial x_i} \right) = (2\alpha_i - 1)(d - r_i) = 2K - d + r_i \geq 2k - d + r_1$$

we have that $\text{min} \text{fil}(W_i) \geq \text{fil}(\Theta) + 2k - d + r_1 \geq 2k + r_n$.

Let $v: \mathbb{R}^n \times \mathbb{R}, 0 \to \mathbb{R}^n \times \mathbb{R}, 0$ be the vector field given by $\frac{W}{\sqrt{N^*f_t}}$. From Lemma 3.3.2, it follows that $v$ is a Lipschitz vector field.

Finally the equation $(\frac{\partial f_t}{\partial t})(x) = (df_t)(v(x,t))$ gives the strong bilipschitz triviality of the family $f_t(x)$ along a small open interval around $t = 0$. Since the same argument is true for each $t \in I$, the proof is complete.

The following result shows that the estimate given in Theorem 3.3.3 is sharp.

**Proposition 3.3.5.** Let $f_t: (\mathbb{R}^2, 0) \to (\mathbb{R}, 0); \ t \in I = (-\delta, \delta) \subset \mathbb{R}$ be given by

$$f_t(x, y) = \frac{1}{3}x^3 - \frac{1}{2}x^2y + y^3.$$
Then $f_t$ is not strongly bilipschitz trivial.

**Remark 3.3.6.** Let $f(x, y) = \frac{1}{3}x^3 + y^3$. Note that $f$ is quasihomogeneous of type $(n, 1; 3n)$. From Theorem 3.3.3 it follows that $f + t\theta$ is strongly bilipschitz trivial for each $\theta(x, t)$ such that $fil(\theta) \geq 4n - 1$.

**Proof (of the Proposition 3.3.5).** Let $m = 3n - 2$. Here we repeat the argument-proof from Theorem 1.1 in [2]. Suppose that $v(x, y, t) = v_1(x, y, t)\frac{\partial}{\partial x} + v_2(x, y, t)\frac{\partial}{\partial y}$ is a vector field such that:

$$\left(\frac{\partial f_t}{\partial t}\right)(x, y) = (df)_x(v(x, y, t))$$

The polar curve of $f_t \{ (x, y) \in \mathbb{R}^2 : \frac{\partial f_t}{\partial x}(x, y) = 0 \}$ is equal to the set $\{ (x, y) \in \mathbb{R}^2 : x^2 = t^2y^m \}$. Thus, $v_1(ty^{m/2}, y, t)$ and $v_2(-ty^{m/2}, y, t)$ satisfy:

$$v_1(ty^{m/2}, y, t)\frac{\partial f_t}{\partial y}(ty^{m/2}, y, t) = -\frac{\partial f_t}{\partial t}(ty^{m/2}, y, t)$$ (5)

$$v_2(-ty^{m/2}, y, t)\frac{\partial f_t}{\partial y}(-ty^{m/2}, y, t) = -\frac{\partial f_t}{\partial t}(-ty^{m/2}, y, t).$$ (6)

From equations (5) and (6) we have:

$$v_1(ty^{m/2}, y, t) = \frac{2t^2y^{m/2-1}}{mt^4y^{m/2-2} + 3n}$$

$$v_2(-ty^{m/2}, y, t) = \frac{-2t^2y^{m/2-1}}{mt^4y^{m/2-2} + 3n}$$

Thus,

$$v_1(ty^{m/2}, y, t) - v_2(-ty^{m/2}, y, t) \sim y^{m/2-1}$$ (7)

On the other hand,

$$\|(ty^{m/2}, y, t) - (-ty^{m/2}, y, t)\| \sim y^{m/2}$$ (8)

But, (7) and (8) show that $v_2$ is not Lipschitz. Hence $f$ is not strongly bilipschitz trivial.

The invariant for bilipschitz equivalence $\text{Inv}(f_t)$ presented in [2] is independent of $t$, hence does not distinguish the element of the given family $f_t$. 

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