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Deformations with constant Milnor number and multiplicity of non-degenerate complex hypersurfaces

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We investigate the constancy of the Milnor number of one parameter deformations of holomorphic germs of functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity, in terms of some Newton polyhedra associate to such germs.

We show that if the Jacobian ideals $J(f_t) = \langle \partial f_t/\partial x_1 \ldots, \partial f_t/\partial x_n \rangle$ of a deformation $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ are non-degenerate on some fixed Newton polyhedron Γ_+, then this family is μ-constant for small values of t, if and only if all germs g_s have non-decreasing Γ-order with respect to f.

For the μ-constant families of germs obtained as a consequence of these results we have a positive answer for the Zariski's question: *Whether for a hypersurface singularity the multiplicity is an invariant of the topological type?*
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1. μ-CONSTANT DEFORMATIONS AND MULTIPLICITY

The determination of conditions for a family of isolated singularity germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ to have constant Milnor number is one of the most interesting questions in singularity theory. This question is being investigated by several authors, Varchenko gives in [13] a full answer for this question for the case of weighted homogeneous germs with isolated singularity.

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Theorem 1.1. *[13] Let $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ be a deformation of a weighted homogeneous polynomial germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity at 0, where $\delta_s : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and $g_s : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic germs of functions and $\delta_s \neq 0$. Then the family $f_t(x) = F(x,t)$ is μ-constant for small values of t, if and only if all monomials of each germ g_s have weighted degree higher or equal than the weighted degree of f.*

Another number that is associated to germs of functions is the multiplicity, and Zariski asked in [15] the following question: *Whether for a hypersurface singularity the multiplicity is an invariant of the topological type?*

A positive answer for Zariski's question was previously known for the case of plane curves and for homogeneous surfaces. In the case of families of semi quasi homogeneous germs, a positive answer for Zariski's question is showed by Greuel in [4] and independently by D. O'Shea in [8]. Both authors applied the theorem 1.1 of Varchenko, but used different methods.

Theorem 1.2. *[4], [8] Let $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ be a μ-constant deformation of a weighted homogeneous polynomial germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity at 0. Then the family $F(x,t) = f_t(x)$ is m-constant (m denotes the multiplicity).*

In this article we investigate the relationship between these questions and some Newton polyhedra associate to the germ f.

The use of the Newton polyhedron of a germ f to give sufficient conditions for the μ-constancy was done by Kouchnirenko in [5]. Yoshinaga in [14] and Damon-Gaffney in [3] also dealt with this Newton polyhedron to show sufficient conditions for the topological triviality.

Here we first show necessary and sufficient conditions for the μ-constancy in terms of the Newton polyhedron defined by the Jacobian ideal of f. Then we show necessary and sufficient conditions for the μ-constancy of families of germs which are non-degenerate on some fixed Newton polyhedron Γ_+ (see 4.4), we show that the family is μ-constant for small values of t, if and only if all germs g_s have non-decreasing Γ-order with respect to f.

In the final section we apply these results to show that these μ-constant families also have constant multiplicity, giving a positive answer for Zariski's question for this kind of germs.

2. μ-CONSTANT DEFORMATIONS AND INTEGRAL CLOSURE

We fix a system of local coordinates x of $\mathbb{C}^n$, consider the ring $\mathcal{O}_n$ of holomorphic germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ and denote by m_n its maximal ideal. Due to the identification between $\mathcal{O}_n$ and the ring of convergent power series $\mathbb{C}\{x_1, \ldots, x_n\}$ we identify a germ $f \in \mathcal{O}_n$ with its power series $f(x) = \sum a_\alpha x^\alpha$, with $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

The Milnor number of a germ f, denoted by $\mu(f)$, is algebraically defined as the $\dim_{\mathbb{C}} \mathcal{O}_n / J(f)$, where $J(f)$ denotes the ideal generated by the partial derivatives $\{\partial f/\partial x_1,$

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$\ldots, \partial f/\partial x_n\}$. A deformation $F\colon (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ of f is μ-constant if $\mu(f_t) = \mu(f)$ for small values of t. We denote by $J(F) = \langle \partial F/\partial x_1, \ldots, \partial F/\partial x_n\rangle$, the ideal generated by the partial derivatives of F with respect to the variables $x_1, \ldots, x_n$.

Greuel gives in [4] a full characterization of μ-constant deformations of f in terms of the integral closure of the Jacobian ideal of $J(F)$.

The **integral closure** of an ideal I in a ring R, is the ideal $\overline{I}$, of the elements $h \in R$ which satisfies a relation $h^k + a_1 h^{k-1} + \cdots + a_{k-1}h + a_k = 0$, with $a_i \in I^i$.

Teissier gave in [10, p.288] the following characterization for the integral closure of ideals in $R = \mathcal{O}_n$.

Proposition 2.1. *If I is an ideal in $\mathcal{O}_n$, the following statements are equivalent:*

1. $h \in \overline{I}$;

2. *for each system of generators $h_1, \ldots, h_r$ of I there exists a neighbourhood U of 0 and a constant $C > 0$ such that $|h(x)| \leq C \sup\{|h_1(x)|, \ldots, |h_r(x)|\}$, for all $x \in U$;*

3. *for each analytic curve $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, $h \circ \varphi$ lies in $(\varphi^*(I))\,\mathcal{O}_1$.*

The item 3 of this proposition is called *valuative criterion* since it is equivalent to the condition $\nu(h \circ \varphi) \geq \inf\{\nu(h_1 \circ \varphi), \ldots, \nu(h_r \circ \varphi)\}$, where ν denotes the usual valuation of a complex curve. In this case, the valuation is the multiplicity of the curve, see section 5 for the definition of multiplicity.

Theorem 2.1. *[4, p.161] Let $F\colon (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a one parameter deformation of a holomorphic germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity. The following statements are equivalent*

1. F is a μ-constant deformation of f;
2. $\frac{\partial F}{\partial t} \in \overline{J(F)}$;
3. $\frac{\partial F}{\partial t} \in \sqrt{J(F)}$, here $\sqrt{J(F)}$ denotes the radical of $J(F)$;
4. the polar curve of F with respect to $\{t = 0\}$ does not split, i.e.,

$$\left\{(x,t) \in \mathbb{C}^n \times \mathbb{C} \mid \frac{\partial F}{\partial x_i}(x,t) = 0, \forall\ i = 1, \ldots, n\right\} = \{0\} \times \mathbb{C} \text{ near } (0,0).$$

3. μ-CONSTANT DEFORMATIONS AND NEWTON POLYHEDRA

We show here necessary and sufficient conditions for a deformation $F(x,t)$ to be μ-constant. These conditions will be given in terms of some suitable Newton polyhedra associate to the germ f.

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For a germ $f(x) = \sum a_k x^k$, we define supp $f = \{k \in \mathbb{Z}^n \ : \ a_k \neq 0\}$. For an ideal I in $\mathcal{O}_n$, we call supp $I = \cup \{supp\ g \ : \ g \in I\}$.

Definition 3.1. The Newton polyhedron of I, denoted by $\Gamma_+(I)$, is the convex hull in $\mathbb{R}^n_+$ of the set

$$\cup \left\{k + v \ : \ k \in supp\ I, \ v \in \mathbb{R}^n_+\right\}.$$

$\Gamma(I)$ denotes the union of all compact faces of $\Gamma_+(I)$.

A germ g is of **non-decreasing Newton order** with respect to $\Gamma_+(I)$ if $\Gamma_+(g) \subseteq \Gamma_+(I)$.

From the integral closure of the Jacobian ideal $J(f)$ we define the polyhedron $T(f)$, which is a key tool to study the μ-constancy.

Definition 3.2. $T(f)$ is the convex hull in $\mathbb{R}^n_+$ of $\cup \left\{m + v \colon x^m \in \overline{J(f)} \text{ and } v \in \mathbb{R}^n_+\right\}$.

In the next lemma we show a necessary condition for the μ-constancy of families defined by first order deformations.

Lemma 3.1. *Let $F(x,t) = f(x) + tg(x)$ be a first order deformation of a complex germ f with isolated singularity. A necessary condition for the μ-constancy of the family f_t is $\Gamma_+(g) \subseteq T(f)$.*

Proof: If $\Gamma_+(g) \not\subseteq T(f)$, it follows from the valuative criterion that there exists an holomorphic curve $\gamma \colon (\mathbb{C},0) \to (\mathbb{C}^n,0)$, such that

$$\nu\left(g \circ \gamma\right) < \inf\left\{\nu\left(\frac{\partial f}{\partial x_i} \circ \gamma\right), \ \forall i = 1,\dots,n\right\}.$$

We define the curve $\psi \colon (\mathbb{C},0) \to (\mathbb{C}^n \times \mathbb{C},0)$ as $\psi = (\gamma,0)$. Since $\dfrac{\partial F}{\partial x_i} = \dfrac{\partial f}{\partial x_i} + t\dfrac{\partial g}{\partial x_i}$, we obtain $\dfrac{\partial F}{\partial x_i} \circ \psi = \dfrac{\partial f}{\partial x_i} \circ \gamma$ and the result follows from item 3 of the Proposition 2.1 and the Theorem 2.1. ▮

We describe now sufficient conditions for the μ constancy.

Yoshinaga in [14] gave conditions for the topological triviality of families of type $F(x,t) = f(x) + tg(x)$ in terms of the gradient polyhedron $\Lambda_+(f)$, defined as the convex hull of the set

$$\bigcup\left\{m + v, : v \in +\mathbb{R}^n_+ \text{ and } \left|x_1\frac{\partial f}{\partial x_1}\right| + \ldots + \left|x_n\frac{\partial f}{\partial x_n}\right| \geq \ \mathcal{E}\,|x^m|\right\},$$

for a positive $\mathcal{E}(m)$ in a neighbourhood of the origin in $\mathbb{C}^n$. In the theorem 1.6 of [14] it is shown that if $\Gamma_+(g) \ \subset \ \Lambda_+(f)$, then $F(x,t) = f(x) + tg(x)$ is topologically trivial for sufficiently small values of t. Damon-Gaffney in [3] also gave similar results for the topological triviality.

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We remark that if a germ g satisfies the condition $\Gamma_+(g) \subset \Lambda_+(f)$, it is equivalent to say that g is in integral closure of the ideal generated by the system $\left\{x_1\frac{\partial f}{\partial x_1}, \dots, x_n\frac{\partial f}{\partial x_n}\right\}$. Since this ideal is in the integral closure of the ideal $J(F)$, we conclude from the results of Yoshinaga, Damon-Gaffney and the Theorem 2.1, the following

PROPOSITION 3.1. *Let $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$, be a deformation of f with isolated singularity at* 0*. If $\Gamma_+(g_s) \subset \Lambda_+(f)$ for all $s = 1, \dots, \ell$, then $F(x,t)$ is μ-constant for sufficiently small values of t.*

Next we show a sufficient condition for the μ-constancy in terms of the polyhedron $T(f)$.

PROPOSITION 3.2. *Let $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ be a deformation of a complex germ f with isolated singularity. If $g_s(x) \in m_n$ and $\Gamma_+(J(g_s)) \subseteq T(f)$ for all $s = 1, \dots, \ell$, then $F(x,t)$ is μ-constant for small values of t.*

Proof: Suppose that $\Gamma_+(J(g_s)) \subseteq T(f)$, from the item 2 of the Proposition 2.1, we obtain that for each $i = 1, \dots, n$ and $s = 1, \dots, \ell$ there exist a neighbourhood $U_{i,s}$ of 0 and a constant $C_{i,s} > 0$, such that $\left|\frac{\partial g_s}{\partial x_i}\right| \leq C_{i,s} \sup\left\{\left|\frac{\partial f}{\partial x_1}\right|, \dots, \left|\frac{\partial f}{\partial x_n}\right|\right\}$, then

$$\sum_{s=1}^{\ell} |\delta_s(t)| \left|\frac{\partial g_s}{\partial x_i}\right| \leq \sum_{s=1}^{\ell} |\delta_s(t)|\ C_{i,s}\ \sup\left\{\left|\frac{\partial f}{\partial x_1}\right|, \dots, \left|\frac{\partial f}{\partial x_n}\right|\right\}.$$

Therefore, for all x in a neighbourhood $U \subset U_{i,s}$ for all $s = 1, \dots, \ell$ and $i = 1, \dots, n$,

$$\begin{aligned}\sup_i \left|\frac{\partial F}{\partial x_i}\right| = \sup_i \left|\frac{\partial f}{\partial x_i} + \sum_{s=1}^{\ell} \delta_s(t) \cdot \frac{\partial g_s}{\partial x_i}\right| &\geq \sup_i \left|\frac{\partial f}{\partial x_i}\right| - \sup_i \sum_{s=1}^{\ell} |\delta_s(t)| \left|\frac{\partial g}{\partial x_i}\right| \\ &\geq \sup_i \left|\frac{\partial f}{\partial x_i}\right| - \sum_{s=1}^{\ell} |\delta_s(t)| C_{i,s} \sup_i \left\{\left|\frac{\partial f}{\partial x_i}\right|\right\} \\ &\geq (1 - \sum_{s=1}^{\ell} |\delta_s(t)| C_{i,s}) \sup_i \left|\frac{\partial f}{\partial x_i}\right| \\ &\geq (1-\alpha) \sup_i \left|\frac{\partial f}{\partial x_i}\right|,\end{aligned}$$

for some $0 < \alpha < 1$ with $\sum_{s=1}^{l} |\delta_s(t) C_{i,s}| \leq \alpha$ for all $i = 1, \dots, n$ and $s = 1, \dots, \ell$.

This inequality implies that $\left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \subset \left\langle \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right\rangle$.

Now we show that for each analytic curve $\psi : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$,

$$\nu\left(\frac{\partial F}{\partial t} \circ \psi\right) \geq \min_i \left\{\nu\left(\frac{\partial F}{\partial x_i} \circ \psi\right)\right\}.$$

We write $\psi = (\varphi, \lambda)$, with $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ and $\lambda : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, hence

$$\nu\left(\frac{\partial F}{\partial t} \circ \psi\right) \geq \min_s \left\{\nu\left(\delta_s' \circ \lambda\right) + \nu\left(g_s \circ \varphi\right)\right\} \geq \min_s \left\{\nu\left(g_s \circ \varphi\right)\right\}.$$

From the hypothesis that $g_s(x) \in m_n$ and $\Gamma_+(J(g_s)) \subseteq T(f)$, we obtain that $\Gamma_+(g_s) \in T(f)$, hence $g_s \in \overline{\left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle}$.

Therefore $\nu(g_s \circ \varphi) \geq \min_i \left\{ \nu\left(\frac{\partial f}{\partial x_i} \circ \varphi \right) \right\} \geq \min_i \left\{ \nu\left(\frac{\partial F}{\partial x_i} \circ \psi \right) \right\}$, and

$$\nu\left(\frac{\partial F}{\partial t} \circ \psi\right) \geq \min_s \{\nu(g_s \circ \varphi)\} \geq \min_i \left\{ \nu\left(\frac{\partial F}{\partial x_i} \circ \psi \right) \right\}$$

and the result follows from the Proposition 2.1. ▌

We see in the example below that the conditions shown in the Propositions 3.1 and 3.2 gives rise to different classes of μ-constant deformations $F(x,t) = f(x) + tg(x)$.

EXAMPLE 3.1. Let $f(x,y) = y^7 + x^4 y + x^9$.

From the theorem 1.3 of Yoshinaga (see the Corollary 4.1), we know that $\Lambda_+(f) = \Gamma_+(f)$ is the polygon with vertices $(0,7)$, $(4,1)$ and $(9,0)$. From the Corollary 4.1 we see that the polygon $T(f)$ has vertices $(0,6)$, $(3,1)$ and $(4,0)$.

If we consider $g(x,y) = x^2 y^4$, $\Gamma_+(g) \subset \Lambda_+(f)$, hence the deformation $F(x,y,t) = y^7 + x^4 y + x^9 + tx^2 y^4$ is μ-constant from the Proposition 3.1, but $\Gamma_+(J(g)) \not\subset T(f)$.

We also see in this example that the condition $\Gamma_+(J(g)) \subset T(f)$ does not imply that $\Gamma_+(g) \subset \Lambda_+(f)$, for this, consider $g(x,y) = x^6$. Here we conclude that the deformation $F(x,y,t) = y^7 + x^4 y + x^9 + tx^6$ is μ-constant from the Proposition 3.2, and $\Gamma_+(g) \not\subset \Lambda_+(f)$.

4. THE NON-DEGENERATE CASE

In this section we describe how to obtain μ-constant deformations in terms of non-degeneracy conditions given by some Newton polyhedra associate to the germ f.

Let I be an ideal of finite codimension in $\mathcal{O}_n$, i.e, $\dim_{\mathbb{C}} \mathcal{O}_n / I < \infty$. For each face $\Delta \subseteq \Gamma(I)$ we denote by $C(\Delta)$ the cone given by the union of half-rays emanating from the origin and passing through Δ. We call $\mathcal{A}_\Delta$ the subring with unity of $\mathcal{O}_n$ given by $\mathcal{A}_\Delta = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq C(\Delta) \cap \mathbb{Z}^k\}$.

If D is a fixed subset of $\Gamma_+(I)$ and $g = \sum_k a_k x^k$, we set $g_D = \sum_{k \in D} a_k x^k$.

DEFINITION 4.1. A subset $\Delta \subset \Gamma_+(I)$ is Newton non-degenerate if the ideal generated by $g_{1_\Delta}, g_{2_\Delta}, \dots, g_{s_\Delta}$ has finite codimension in $\mathcal{A}_\Delta$.

DEFINITION 4.2. An ideal I is *Newton non-degenerate* if there exists a system of generators $\{g_1, \dots, g_s\}$ of I such that, for each compact face $\Delta \subseteq \Gamma(I)$, the ideal I_Δ generated by $\{g_{1_\Delta}, \dots, g_{s_\Delta}\}$ has finite codimension in $\mathcal{A}_\Delta$.

We denote by $C(\overline{I})$ the convex hull in $\mathbb{R}^n_+$ of the set $\cup\{m : x^m \in \overline{I}\}$.

We remark that if I is the ideal generated by $\left\{ x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n} \right\}$, then $C(\overline{I}) = \Lambda_+(f)$ and $T(f) = C(\overline{J(f)})$, where $J(f)$ is the Jacobian ideal of f.

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A germ $g = \sum_k a_k x^k$ is *Newton non-degenerate*, if the ideal generated by the system $\{x_1 \partial g/\partial x_1, \ldots, x_n \partial g/\partial x_n\}$ is of finite codimension in $\mathcal{O}_n$ and is Newton non-degenerate.

When the germ f is Newton non-degenerate we have the following

Corollary 4.1. *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a Newton non-degenerate germ with an isolated singularity. If each g_s has non-decreasing Newton order with respect to $\Gamma_+(f)$, the deformation $f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ is μ-constant for sufficiently small values of t.*

When the Jacobian ideal $J(f)$ is Newton non-degenerate we also have the following:

Corollary 4.2. *Suppose that the system $\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\}$ of generators of the Jacobian ideal $J(f)$ is Newton non-degenerate and $\Gamma_+(J(g_s)) \subseteq \Gamma_+(J(f))$ for all s. Then $F(x, t)$ is μ-constant for small values of t.*

The proof of these results is a direct consequence of the Propositions 3.1 (or 3.2) and the following

Theorem 4.1. *[9] Let I be an ideal with finite codimension in $\mathcal{O}_n$. Then $C(\overline{I}) \subseteq \Gamma_+(I)$. The equality holds if and only if I is Newton non-degenerate.*

The next example shows that these two conditions are not enough to give all μ-constant deformations of any germ with isolated singularity.

Example 4.1. Briançon and Speder showed in [1] that the family of hypersurfaces $X_t \in \mathbb{C}^3$ defined by the equations $f_t(x, y, z) = z^5 + y^7 x + x^{15} + t y^6 z = 0$ has constant topological type, but the monomial $y^6 z$ does not satisfies the condition of the Corollary 4.2. We can not apply the Corollary 4.1 either, since f is not Newton non-degenerate because the ideal generated by the system $\{x\partial f/\partial x, y\partial f/\partial y, z\partial f/\partial z\}$ does not have finite codimension in $\mathcal{O}_n$.

The germ $f(x, y, z) = z^5 + y^7 x + x^{15}$ has isolated singularity at 0 and is weighted homogeneous of weights $(1, 2, 3)$ and degree $\mathrm{d}(f) = 15$, hence the μ-constancy of this family follows from the Theorem 1.1 of Varchenko.

In order to generalize these results for a bigger class of germs that contains the Newton non-degenerate germs and also the class of semi weighted homogeneous germs, we apply the results of Bivia-Fukui-Saia, given in [2] to show a necessary and sufficient condition for the μ-constancy of families defined by germs which are non-degenerate on some Newton polyhedron. We recover here the basic results for this definition.

A subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a *Newton polyhedron* if there exist some $k_1, \ldots, k_r \in \mathbb{Q}^n_+$ such that Γ_+ is the convex hull in $\mathbb{R}^n_+$ of the set $\{k_i + v : v \in \mathbb{R}^n_+,\ i = 1, \ldots, r\}$ and Γ_+ intersects all the coordinate axis. We call Γ the union of the compact faces of Γ_+, Γ_- the set $\mathbb{R}^n - \Gamma_+$ and $\mathrm{V}_n(\Gamma_-)$ denotes the n-dimensional volume of Γ_-. From the boundary of a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n$ we construct a piecewise-linear function $\phi_\Gamma : \mathbb{R}^n \to \mathbb{R}$ with the following properties:

1. ϕ_Γ is linear on each cone $C(\Delta)$, where Δ is a compact face of Γ;
2. ϕ_Γ takes positive integer values on the lattice points of $\mathbb{R}^n_+ - \{0\}$;
3. there exists a positive integer M such that $\phi_\Gamma(k) = M$, for all $k \in \Gamma$.

This map ϕ_Γ induces a *Newton filtration* of $\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots$, by the ideals $\mathcal{A}_q = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq \phi_\Gamma^{-1}(q + \mathbb{N})\}$, for all $q \in \mathbb{N}$, see [5](2.1). For any compact face Δ of Γ, this filtration induces a filtration on $\mathcal{A}_\Delta$ in a natural way.

Definition 4.3. The Γ-*order* of a germ $g = \sum_k a_k x^k$, with respect to the filtration induced by Γ is defined as

$$\mathrm{d}(g) = \min\{\phi_\Gamma(k) : k \in \operatorname{supp} g\} = \max\{q : g \in \mathcal{A}_q\}.$$

The *principal part of* g is the polynomial $\mathrm{in}(g) = \sum a_k x^k$ such that $\phi_\Gamma(k) = \mathrm{d}(g)$. Given a face Δ of Γ, *the principal part of* g *over* Δ, denoted by $\mathrm{in}_\Delta(g)$, is the polynomial

$$\mathrm{in}_\Delta(g) = \sum \big\{a_k x^k : k \in \operatorname{supp} g \cap C(\Delta) \text{ and } \phi_\Gamma(k) = \mathrm{d}(g)\big\}.$$

Definition 4.4. A system of generators $g_1, \dots, g_s$ of an ideal I is *non-degenerate on* Γ_+ if, for each compact face $\Delta \subseteq \Gamma$, the ideal of $\mathcal{A}_\Delta$ generated by $\mathrm{in}_\Delta(g_1), \dots, \mathrm{in}_\Delta(g_s)$ has finite codimension in $\mathcal{A}_\Delta$. When the system $g_1, \dots, g_s$ does not satisfy the above definition, we say that this system is *degenerate* on Γ_+.

We remark that this definition depends of the system of generators, while the definition of Newton non-degeneracy does not depend of the system on generators of the ideal.

The next theorem is essential to prove the main results of this section.

Theorem 4.2. *([2], Theorem 3.3.) Let $g_1, \dots, g_n$ be a system of generators of an ideal I with finite codimension in $\mathcal{O}_n$ and let $\Gamma_+ \subseteq \mathbb{R}^n$ be a Newton polyhedron. If M is the value on Γ of the filtration induced by Γ_+ and $d_1 = \mathrm{d}(g_1), \dots, d_n = \mathrm{d}(g_n)$ are the levels of the given set of generators of I with respect to this filtration, then*

1. $\dim_{\mathbb{C}} \mathcal{O}_n/I \geq \dfrac{d_1 \dots d_n}{M^n} n! \, \mathrm{V}_n(\Gamma_-)$;
2. *the equality holds if, and only if the system $g_1, \dots, g_n$ is non-degenerate on Γ_+.*

In the sequel we consider a deformation $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$, of a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity at 0 and fix a Newton polyhedron Γ_+.

We denote by D_i, D_i^s, D_{ti}, the Γ-order of the partial derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial g_s}{\partial x_i}$, $\frac{\partial f_t}{\partial x_i}$ respectively.

Theorem 4.3. *Suppose that the system of generators $\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\}$ of the Jacobian ideal $J(f)$ is non-degenerate on Γ_+. If $D_i^s \geq D_i$ for all $i = 1, \ldots, n$ and all $s = 1, \ldots, \ell$, then, for small values of t, the system of generators $\{\partial f_t/\partial x_1, \ldots, \partial f_t/\partial x_n\}$ of each Jacobian ideal $J(f_t)$ is non-degenerate on Γ_+. Moreover, $\mu(f_t)$ is constant.*

Proof: Suppose first that $D_i^s > D_i$ for all $i = 1, \ldots, n$ and all $s = 1, \ldots, \ell$, hence, $D_{ti} = D_i$ for all $i = 1, \ldots, n$ and, for all t the principal part of each $\partial f_t/\partial x_i$ is equal to the principal part of $\partial f/\partial x_i$. From the non-degeneracy of $J(f)$ on Γ_+, we obtain the non-degeneracy of each Jacobian ideal $J(f_t)$ and the equality $\mu(f_t) = \mu(f)$ follows from the item 2 of the Theorem 4.2.

If there exist $D_i^s = D_i$, we conclude that each Jacobian ideal $J(f_t)$ in non-degenerate for small values of t from the fact that the set of non-degenerate ideals on some Newton polyhedron is open for the Zariski topology.

Hence the equality $\mu(f_t) = \mu(f)$ follows from the item 2 of the Theorem 4.2. ▮

The following example illustrates this result.

Example 4.2. Let $f(x, y) = x^{12} + y^8$. Fix the Newton polyhedron $\Gamma_+ = \Gamma_+(f)$, since f is weighted homogeneous with respect to the weight $w = (2, 3)$, we obtain that Γ_+ has only one compact face with vertices $(12, 0)$ and $(0, 8)$. The jacobian ideal $J(f) = \langle f_x, f_y\rangle$ is non-degenerate on Γ_+, with $\mathrm{d}(f_x) = 22$ and $\mathrm{d}(f_y) = 21$.

Consider the family $f_t(x, y) = x^{12} + y^8 - 2tx^6y^4$. Here $\mathrm{d}(f_{tx}) = 22 = \mathrm{d}(f_x)$ and $\mathrm{d}(f_{ty}) = 21 = \mathrm{d}(f_y)$. The principal part of the generators $\{f_{tx}, f_{ty}\}$ of the Jacobian ideal $J(f_t)$ is different from the principal part of the generators $\{f_x, f_y\}$ of the Jacobian ideal $J(f)$, but for each $0 \leq t < 1$, the system of generators $\{f_{tx}, f_{ty}\}$ is also non-degenerate, hence $\mu(f) = \mu(f_t)$ for all $0 \leq t < 1$. For $t = 1$, we see that the germ $f_1 = (x^6 - y^4)^2$ has not isolated singularity, hence $\mu(f_1) = \infty$.

We see in the example below that it is possible to find families f_t such that the germ f is non-degenerate, there exists a $t \neq 0$ such that f_t is degenerate, but the family f_t is also μ-constant.

Example 4.3. Let $f(x, y, z) = x^2 + y^2 + xz + z^2$. The germ f is homogeneous and Newton non-degenerate. Consider the family $f_t = x^2 + y^2 + xz + z^2 + 2txy$. From the Theorem 4.3 we see that for $0 \leq t < 1$, $\mu(f_t) = \mu(f)$. When $t = 1$, the Jacobian ideal $J(f_1)$ is not Newton non-degenerate (see [5], pag. 08), but $\mu(f_1) = \mu(f)$.

On the other side, we can also prove the following

Theorem 4.4. *Suppose that the system of generators $\{\partial f/\partial x_1, \ldots, \partial f/\partial x_n\}$ of the Jacobian ideal $J(f)$ is non-degenerate on Γ_+. If $\mu(f_t)$ is constant for small values of t and the system of generators $\{\partial f_t/\partial x_1, \ldots, \partial f_t/\partial x_n\}$ is non-degenerate on Γ_+, then $D_i^s \geq D_i$ for all $i = 1, \ldots, n$ and all $s = 1, \ldots, \ell$.*

Proof: If $\mu(f_t) = \mu(f)$, since $J(f_t)$ is non-degenerate on Γ_+, it follows from the item 2 of the Theorem 4.2 that

$$\mu(f_t) = \frac{D_{t1}\dots D_{tn}}{M^n} n!\,\mathrm{V}_n(\Gamma_-) \text{ and } \mu(f) = \frac{D_1\dots D_n}{M^n} n!\,\mathrm{V}_n(\Gamma_-).$$

From the hypothesis we have

$$\frac{D_{t1}\dots D_{tn}}{M^n} n!\,\mathrm{V}_n(\Gamma_-) = \frac{D_1\dots D_n}{M^n} n!\,\mathrm{V}_n(\Gamma_-).$$

Hence we conclude that $D_{t1}\dots D_{tn} = D_1 \dots D_n$, but $D_{ti} = \min.\{D_i, D_i^s\}$ for small values of t and all $i = 1\dots, n$, $s = 1, \dots, \ell$.

Therefore we obtain $D_{ti} = D_i$ for all $i = 1\dots, n$ and $D_i^s \geq D_i$ for all $i = 1\dots, n$, $s = 1, \dots, \ell$. ▮

Example 4.4. **Generalized Briançon-Speder Example**

We shall consider here families of type $f_t(x, y, z) = z^5 + y^7x + x^{15} + tx^ay^bz^c$ for a fixed monomial $x^ay^bz^c$. In order to study the μ-constancy of this family, we first check when $x^ay^bz^c$ satisfies the necessary condition $\Gamma_+(x^ay^bz^c) \subseteq T(f)$ given in the Lemma 3.1.

The Jacobian ideal $J(f) = \langle 15x^{14} + y^7, 7y^6x, 5z^4\rangle$, is Newton non-degenerate, hence $T(f) = \Gamma_+(J(f))$, whose vertices are $(14, 0, 0)$, $(0, 7, 0)$, $(1, 6, 0)$ and $(0, 0, 4)$ in $\mathbb{R}^3_+$. Therefore, we consider the monomials $x^ay^bz^c$ such that $(a, b, c) \in \Gamma_+(J(f))$.

For instance, if we fix the monomial $x^ay^bz^c = y^6z$, we consider the Newton polyhedron Γ_+ with vertices $(3, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$, which has one 2-dimensional compact face associate to the weights $(1, 2, 3)$. An easy calculation shows that the Γ-order of the partial derivatives of f are: $\mathrm{d}(15x^{14} + y^7) = 14$, $\mathrm{d}(7y^6x) = 13$, $\mathrm{d}(5z^4) = 12$ and $J(f)$ is non-degenerate on Γ_+.

The Γ-order of the partial derivatives of the monomial y^6z are: $\mathrm{d}(\partial(y^6z)/\partial x) = \infty$, $\mathrm{d}(6y^5z) = 13$ and $\mathrm{d}(y^6) = 12$, hence we apply the Theorem 4.3 to conclude that the system of generators $\left\{15x^{14} + y^7, 7y^6x + t6y^5z, 5z^4 + ty^6\right\}$ is non-degenerate on Γ_+ and the family f_t is μ-constant for small values of t.

Analogous argument can be applied to any family $f_t(x) = f + \sum_{s=1}^{\ell} \delta_s(t)g_s$ satisfying the conditions of the Theorem 4.3 for this Newton polyhedron.

But there are some monomials $x^ay^bz^c$ satisfying the condition $(a, b, c) \in \Gamma_+(J(f))$ which do not satisfy the conditions of the Theorem 4.3 for this Newton polyhedron. For example, if we consider $x^ay^bz^c = yz^4$, the Γ-order of $\mathrm{d}(\partial(yz^4)/\partial y) = 11$ and we can not apply the Theorem 4.3.

On the other side we apply the Theorem 4.4 to show that the family $f_t(x, y, z) = z^5 + y^7x + x^{15} + tyz^4$ is not μ-constant. For this we fix the Newton polyhedron $\Gamma_+ = \Gamma_+(J(f_t))$. Here $J(f_t)$ is Newton non-degenerate and $J(f)$ is degenerate on this polyhedron, hence $\mu(f_t) < \mu(f)$ for all $t \neq 0$.

Analogous argument can be applied to any family $f_t(x) = f + \sum_{s=1}^{\ell} \delta_s(t)g_s$ such that $J(f_t)$ is Newton non-degenerate, the germs g_s satisfies the condition $\Gamma_+(g_s) \in \Gamma_+(J(f))$, but do not satisfies the condition of the Theorem 4.3.

5. μ-CONSTANT DEFORMATIONS AND THE MULTIPLICITY

The *multiplicity* of a germ $f(x) = \sum a_k x^k$, is defined as the lowest degree in the power series of $f(x)$.

Zariski proposed in [15] the following question:

Whether for a hypersurface singularity the multiplicity is an invariant of the topological type?

It is well known that this question has a positive answer in the case of plane curves and for quasi homogeneous complex hypersurfaces, as it is shown by Greuel in [4].

For the case of first order deformations $F(x,t) = f(x) + tg(x)$ of a complex germ f with isolated singularity, Trotman gives in [12] a positive answer for Zariski's question.

From the results given in the section 4 we obtain a positive answer for Zariski's question for families $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(t)$ of germs which are non-degenerate on some Newton polyhedron Γ_+.

Corollary 5.1. *Let $F(x,t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(t)$ be a deformation of a germ f with isolated singularity. Suppose that for small values of t, the system of generators $\left\{ \frac{\partial f_t}{\partial x_1}, \dots, \frac{\partial f_t}{\partial x_n} \right\}$ of each Jacobian ideal $J(f_t)$ is non-degenerate on Γ_+. If $\mu(f_t)$ is constant, then $F(x,t)$ is m-constant.*

Proof: From the hypothesis and the Theorem 4.4 we conclude that $D_i^s \geq D_i$ for all $i = 1, \dots, n$ and all $s = 1, \dots, \ell$, therefore $\Gamma_+(g_s) \subset \Gamma_+(f)$ for all $s = 1, \dots, \ell$, hence the equality $m(f_t) = m(f)$ follows. ▮

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