CADERNOS DE MATEMÁTICA **02**, 57–72 March (2001)
ARTIGO NÚMERO SMA#91

Topological triviality of families of functions on analytic varieties

Maria Aparecida Soares Ruas*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 15560-970, São Carlos SP, Brazil
E-mail: maasruas@icmc.sc.usp.br

João Nivaldo Tomazella

Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, 13560-905, São Carlos, SP, Brazil.
E-mail: tomazela@dm.ufscar.br

We present sufficient conditions for the topological triviality of families of germs of functions defined on an analytic variety V. The main result is an infinitesimal criterion using the integral closure of a convenient ideal as the tangent space to a subset of the set of topologically trivial deformations of a given germ. Results of M. Saia [20] on the determination of the integral closure of an ideal in terms of its Newton Polyhedron are used to describe the topological triviality of Newton non degenerate families of map germs. Applications to the problem of equisingularity of families of sections of V are also discussed. March, 2001 ICMC-USP

1. INTRODUCTION

Let $V, 0$ be the germ of an analytic subvariety of k^n ($k = \mathbb{R}\, or\, \mathbb{C}$) and $\mathcal{R}_V$ (respectively C^0-$\mathcal{R}_V$) be the group of germs of diffeomorphisms (respectively homeomorphisms) preserving $V, 0$. In this paper we introduce a sufficient condition for the C^0-$\mathcal{R}_V$- triviality of families of map germs $h : k^n \times k, 0 \to k^p, 0$, based on the integral closure of $T\mathcal{R}_V(h)$, the tangent space to the orbit of h under the action of the group $\mathcal{R}_V$. Our main result establishes that if $\frac{\partial h}{\partial t} \in \overline{T\mathcal{R}_V(h)}$, then h is topologically $\mathcal{R}_V$-trivial.

We are specially concerned with the case $p = 1$, that is, with families $h : k^n \times k, 0 \to k, 0$. In this case $h^{-1}(0)$ defines a family of sections of the analytic variety $V, 0$.

Using results of M. Saia on the determination of the integral closure of an ideal in terms of its Newton Polyhedron we describe a method for the topological triviality of Newton non degenerate families of sections of $V, 0$. As a corollary of the methods, we obtain sharp results when the analytic variety is weighted homogeneous and the family of sections is

*Research of the first author was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazil, grant # 300066/88-0 and by FAPESP, grant # 97/10735-3

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

a deformation of a weighted homogeneous map germ h_0 (consistent with V) by terms of filtration higher than or equal to the filtration of h_0. This result was previously proved by Damon in [8]. In the final section, we introduce a notion of V-equisingularity of the family of sections and we show that the hypothesis of the main theorem implies this geometric condition. For other results related to the subject discussed in this paper, see for instance [2], [8], [23].

2. BASIC RESULTS

Let $\mathcal{O}_n$ be the ring of the germs of analytic functions $h : k^n, 0 \to k$, $k = \mathbb{R}$ *or* $\mathbb{C}$.

A germ of a subset $V, 0 \subset k^n, 0$ is the germ of an analytic variety if there exist germs of analytic functions $f_1, ..., f_r$ such that $V = \{x : f_1(x) = \cdots = f_r(x) = 0\}$.

Our aim is to study map germs $h : k^n, 0 \to k^p, 0$ under the equivalence relation that preserves the analytic variety $V, 0$. We say that two germs h_1 and $h_2 : k^n, 0 \to k^p, 0$ are $\mathcal{R}_V$-equivalent (respectively C^0-$\mathcal{R}_V$-equivalent) if there exists germ of diffeomorphism (respectively homeomorphism) $\phi : k^n, 0 \to k^n, 0$ with $\phi(V) = V$ e $h_1 \circ \phi = h_2$. That is,

$$\mathcal{R}_V = \{\phi \in \mathcal{R} : \phi(V) = V\},$$

where $\mathcal{R}$ is the group of germs of diffeomorphisms in $k^n, 0$.

A one parameter deformation $h : k^n \times k, 0 \to k^p, 0$ of $h_0 : k^n, 0 \to k^p, 0$ is topologically $\mathcal{R}_V$-trivial (or C^0-$\mathcal{R}_V$-trivial) if there exists homeomorphism $H : k^n \times k, 0 \to k^n \times k, 0$, $H(x,t) = (\bar{h}(x,t), t)$, such that $g \circ H(x,t) = g_0(x)$ and $H(V \times k) = V \times k$.

We denote by θ_n the set of germs of tangent vector fields in $k^n, 0$; θ_n is a free $\mathcal{O}_n$ module of rank n. Let $I(V)$ be the ideal of $\mathcal{O}_n$ consisting of germs of analytic functions vanishing on V. We denote by $\Theta_V = \{\eta \in \theta_n : \eta(I(V)) \subseteq I(V)\}$, the submodule of germs of vector fields tangent to V (see [2] for more details).

The tangent space to the action of the group $\mathcal{R}_V$ is $T\mathcal{R}_V(h) = dh(\Theta_V^0)$, where Θ_V^0 is the submodule of Θ_V given by the vector fields that are zero at zero.

The group $\mathcal{R}_V$ is a geometric subgroup of the contact group, as defined by J.Damon [5], [6], hence the infinitesimal criterion for $\mathcal{R}_V$-determinacy holds (see [2] for a proof).

THEOREM 2.1. *The germ h is $\mathcal{R}_V$-finitely determined if and only if there exists a positive integer k such that $T\mathcal{R}_V(h) \supset \mathcal{M}_n^k$.*

The following theorem is the geometric criterion for the $\mathcal{R}_V$-finite determinacy([2]).

THEOREM 2.2. *Let $V, 0 \subseteq \mathbb{C}^n, 0$ be the germ of an analytic variety and $h : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ be the germ of an analytic function. Let*

$$V(h) = \{x \in \mathbb{C}^n : \xi h(x) = 0 \textit{ for all } \xi \in \Theta_V\}$$

Then h is $\mathcal{R}_V$-finitely determined if and only if $V(h) = \{0\}$ or $\emptyset$.

As a consequence of this result, it follows that if h is $\mathcal{R}_V$-finitely determined, then $h^{-1}(c)$ is transverse to V away from 0, for sufficiently small values of c.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

In the real case, the necessary condition remains true, that is, if h is $\mathcal{R}_V$-finitely determined then the set $\{x \in \mathbb{R}^n : \xi h(x) = 0 \text{ for all } \xi \in \Theta_V\}$ is $\{0\}$ or $\emptyset$.

3. BASIC FACTS ON INTEGRAL CLOSURE OF IDEALS

Let I be an ideal in a ring A. An element $h \in A$ is said to be integral over I if it satisfies an integral dependence relation $h^n + a_1 h^{n-1} + ... + a_n = 0$ with $a_i \in I^i$. The set of such elements form an ideal $\overline{I}$ in A, called the integral closure of I.

When $A = \mathcal{O}_{X,x_0}$, the local ring of a complex analytic set, Teissier gives in [22] various notions equivalent to the above concept.

THEOREM 3.3. *([11], Proposition 1.2) Let I be an ideal in $\mathcal{O}_{X,x_0}$, where X is a complex analytic space. The following statements are equivalent:*
(a) $h \in \overline{I}$
(b) For each choice of generators $\{g_i\}$ of I there exist a neighbourhood U of x_0 and a constant $C > 0$ such that for all $x \in U$:

$$|h(x)| \leq C sup_i |g_i(x)|$$

(c) For each analytic curve $\varphi : \mathbb{C}, 0 \to X, x_0$, $h \circ \varphi$ lies in $(\varphi^*(I))\mathcal{O}_1$.
(d) There exists a faithful $\mathcal{O}_{X,x_0}$ module L of finite type such that $h.L \subset I.L$.

In the real case, the above algebraic definition of integral closure is not appropriate. But, one can use condition (c) above as a definition. More precisely,

DEFINITION 3.4. *Let I be an ideal of the ring $\mathcal{O}_{X,x_0}$, where X is a real analytic set. The real integral closure of I is the set of h such that for all analytic $\varphi : \mathbb{R}, 0 \to X, x_0$, we have $h \circ \varphi \in (\varphi^*(I))\mathcal{O}_1$.*

Gaffney ([11], p. 30) shows that $h \in \overline{I}$ if and only if for each choice of generators $\{g_i\}$ of I there exists a neighbourhood U of x_0 and a constant $C > 0$ such that for all $x \in U$:

$$|h(x)| \leq C sup_i |g_i(x)|$$

4. THE MAIN RESULT

Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ of an analytic function and $h : k^n \times k, 0 \to k, 0$ an analytic deformation of h_0. The property of being $\mathcal{R}_V$-finitely determined is open in the sense that the germ $\{x \in k^n : dh_t \xi(x) = 0, \forall \xi \in \Theta_V\}$ at 0 is $\{0\}$ or empty for sufficiently small values of the parameters ([2]). However, this does not guarantee the existence of a neighbourhood U of 0 in $k^n, 0$ and an open ε-ball, B_ε, centered at the origin in k such that the above condition holds, $\forall x \in U$ and $\forall t \in B_\varepsilon$. We then need the following definition:

DEFINITION 4.5. *Let $h_0 : k^n, 0 \to k, 0$ be a $\mathcal{R}_V$-finitely determined germ. We say that a deformation $h : k^n \times k, 0 \to k, 0$ of h_0 is a* good deformation *if $V(h) \subseteq \{0\} \times k$, where $V(h) = \{(x,t) \in k^n \times k, 0\, ; \, dh_t(x)\xi(x) = 0 \, \forall \xi \in \Theta_V\}$.*

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Example 4.4.1. *Let V be the x-axis in k^2; Θ_V is generated by $(1,0)$ and $(0,y)$. The germ $h_0(x,y)=x^2+y^3$ is $\mathcal{R}_V$-finitely determined. The deformation $h_t(x,y)=x^2+y^3+ty^2$ of h_0 has the property that h_t is $\mathcal{R}_V$-finitely determined for each fixed t, but we cannot find $\varepsilon>0$ such that the above condition holds for all $t\in B_\varepsilon$.*

Our main result is the following theorem:

Theorem 4.6. *Let $h_0 : k^n,0\to k,0$ be a $\mathcal{R}_V$-finitely determined germ and $h : k^n\times k,0\to k,0$ be a good deformation of h_0. If $\frac{\partial h}{\partial t}\in\overline{dh_t(\Theta_V^0)}$, then h is C^0-$\mathcal{R}_V$-trivial.*

The proof of the theorem is a consequence of the following results.

In what follows we can assume that $dh_t\xi(0)=0,\forall\,\xi\in\Theta_V$. In fact, if $\xi\in\Theta_V$, then $dh_t\xi.\frac{\partial h}{\partial t}=dh_t(\frac{\partial h}{\partial t}.\xi)$. If $dh_t\xi_0(0)\neq 0$ for some ξ_0, then

$$\frac{\partial h}{\partial t}=dh_t(\frac{\frac{\partial h}{\partial t}.\xi_0}{dh_t\xi_0})$$

and hence the deformation is C^ω-$\mathcal{R}_V$-trivial (analytically trivial) (notice that $\frac{\frac{\partial h}{\partial t}.\xi_0}{dh_t\xi_0}\in\Theta_V^0$).

Lemma 4.7. *Let I and J be ideals in $\mathcal{O}_n$ with $\mathcal{M}_nI\subseteq J\subseteq I$ and $\mathcal{V}(I)=\{0\}$, where $\mathcal{V}(I)$ is the variety of the ideal I. Then $\mathcal{V}(J)=\{0\}$.*

Proof : From the hypothesis, $\mathcal{V}(\mathcal{M}_nI)\supseteq\mathcal{V}(J)\supseteq\mathcal{V}(I)$. Since $\mathcal{V}(\mathcal{M}_nI)=\mathcal{V}(\mathcal{M}_n)\cup\mathcal{V}(I)=\{0\}\cup\{0\}$, we then get $\mathcal{V}(J)=\{0\}$. ♢

Let $h_0 : k^n,0\to k,0$ be a $\mathcal{R}_V$-finitely determined germ and $h : k^n\times k,0\to k,0$ be a good deformation of h_0. Let $\{\xi_1,...,\xi_r\}$ be generators of Θ_V and $s_i=dh_t\xi_i$. If I is the ideal of $\mathcal{O}_{n+1}$ generated by $\{s_1,...,s_r\}$ then $\mathcal{V}(I)=\{0\}\times k$, since h is a good deformation of h_0. Let $\{\alpha_1,...,\alpha_m\}$ be the generators of Θ_V^0, $dh_t\alpha_i=\rho_i$ and J be the ideal generated by $\{\rho_1,...,\rho_m\}$. Then $\mathcal{M}_nI\subset J\subset I$ and from Lemma 4.7, we get that $\mathcal{V}(J)=\{0\}\times k$.

Let $\rho(x,t)=\sum_{i=1}^m|\rho_i|^2$. The condition $\mathcal{V}(J)=\{0\}\times k$ implies that $\rho\geq 0$ and $\rho_t(x)=0\Leftrightarrow x=0$. Then, the following result holds.

Lemma 4.8. *Let $h_0 : k^n,0\to k,0$ be a $\mathcal{R}_V$-finitely determined germ and $h : k^n\times k,0\to k,0$ be a good deformation of h_0. If $\rho(x,t)=\sum_{i=1}^m|dh_t\alpha_i|^2$, then $\mathcal{V}(\rho(x,t))=\{0\}\times k$.*

Proof of the Theorem 4.6:

With the above notations, it follows that

$$|\rho_i|^2\frac{\partial h}{\partial t}=dh_t(\overline{\rho_i}\frac{\partial h}{\partial t}\alpha_i).$$

Since $\rho=\sum_{i=1}^m|\rho_i|^2$, then

$$\rho\frac{\partial h}{\partial t}=dh_t\left(\frac{\partial h}{\partial t}(\overline{\rho_1}\alpha_1+...+\overline{\rho_m}\alpha_m)\right)$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

hence

$$\frac{\partial h}{\partial t} = dh_t\left(\frac{\partial h}{\partial t}\frac{1}{\rho}(\overline{\rho_1}\alpha_1 + ... + \overline{\rho_m}\alpha_m)\right)$$

From Lemma 4.8, $\mathcal{V}(\rho(x,t)) = \{0\} \times k$. We define the vector field X in $k^n \times k, 0$,

$$X(x,t) = \begin{cases} \left(\frac{\partial h}{\partial t}\frac{1}{\rho}(\overline{\rho_1}\alpha_1 + ... + \overline{\rho_m}\alpha_m), 1\right) & \text{if } x \neq 0 \\ (0,1) & \text{if } x = 0 \end{cases}$$

The vector field $X(x,t)$ is real analytic away from $\{0\} \times k$.

¿From the hypothesis, $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta^0_V)}$ and hence by item b) of Theorem 3.3

$$|\frac{\partial h}{\partial t}| \leq csup\{|\rho_i|\}$$

Then

$$\begin{aligned} |X(x,t) - X(0,t)| &= |\tfrac{\partial h}{\partial t}\tfrac{1}{\rho}(\overline{\rho_1}\alpha_1 + ... + \overline{\rho_m}\alpha_m)| \\ &\leq |\tfrac{\partial h}{\partial t}|\tfrac{1}{\rho}(|\overline{\rho_1}||\alpha_1| + ... + |\overline{\rho_m}||\alpha_m|) \\ &\leq csup\{|\rho_i|\}\tfrac{1}{\rho}(|\rho_1||\alpha_1| + ... + |\rho_m||\alpha_m|) \\ &\leq c(|\alpha_1| + ... + |\alpha_m|) \leq C|x| \end{aligned}$$

Thus, X satisfies a Lipschitz condition around the solution $(0,t)$, and it follows from [17] or [4] that $X(x,t)$ is locally integrable in a neighbourhood of $(0,0) \in k^n \times k$. Then, there exists a family of homeomorphisms $\phi(x,t,\tau)$, $\phi : k^n \times k \times \mathbb{R}, 0 \to k^n \times k, 0$ such that $\frac{\partial \phi}{\partial \tau} = -X \circ \phi$ e $\phi(x,t,0) = (x,t)$. The proof will follow now from the following lemma (see Lemma 6.2, in [9]). ♢

Lemma 4.9. *Let $h : k^n \times k, 0 \to k, 0$ be a deformation of h_0. Suppose there is a continuous vector field $(W,1) \in \Theta_{V\times k}$ such that:*
(i) $\rho\frac{\partial h}{\partial t} = dh_t(W)$, where ρ is a control function, that is, $\rho : k^n \times k, 0 \to \mathbb{R}$ with $\rho(x,t) \geq 0$ and $\rho(x,t) = 0$ if and only if $x = 0$.
(ii) $(\frac{W}{\rho}, 1)$ is locally integrable.
Then h is topologically $\mathcal{R}_V$-trivial.

Proof: Let $\phi(x,t,\tau)$ be a family of homeomorphisms such that $\frac{\partial \phi}{\partial \tau} = -(\frac{W}{\rho}, 1) \circ \phi$, $\phi(x,t,0) = (x,t)$. When $k = \mathbb{R}$, we then define

$$\varphi(x,t) = \phi(x,0,t),\ \varphi(x,t) = (\overline{\varphi}(x,t),t).$$

Taking the derivative of $h(\varphi(x,t)) = h(\overline{\varphi}(x,t),t)$ with respect to t, we get

$$\begin{aligned} \frac{\partial}{\partial t}(h(\varphi(x,t))) &= \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(\overline{\varphi}(x,t),t)\frac{\partial \overline{\varphi_i}}{\partial t}(x,t) + \frac{\partial h}{\partial t}(\overline{\varphi}(x,t),t) \\ &= -\sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(\overline{\varphi}(x,t),t)\frac{W_i}{\rho}(\overline{\varphi}(x,t),t) + \frac{\partial h}{\partial t}(\overline{\varphi}(x,t),t) \\ &= (\frac{\partial h}{\partial t} - \sum_{i=1}^{n} \frac{W_i}{\rho}\frac{\partial h}{\partial x_i})(\overline{\varphi}(x,t),t) = 0 \end{aligned}$$

where W_i are the components of W. Hence, fixing x, it follows that $h(\varphi(x,t))$ is constant, that is, $h(\varphi(x,t)) = h(\varphi(x,0)) = h(x,0) = h_0(x)$ for all t and x. Therefore h is topologically $\mathcal{R}_V$-trivial. When $k = \mathbb{C}$, we consider the restriction

$$h^1 = h|\mathbb{C}^n \times \mathbb{R} \times \{0\} \to \mathbb{C}.$$

It is sufficient to show that h is a $\mathcal{R}_V$-topologically trivial deformation of h^1, which in turn is a $\mathcal{R}_V$-topologically trivial deformation of h_0.

Let $\phi(x,t,\tau)$ be such that $\frac{\partial \phi}{\partial \tau} = -(\frac{W}{\rho},1)\circ\phi$ and $\phi(x,t,0) = (x,t)$. We consider $\phi_1(x,u+iv) = \phi(x,u,v)$ and $\phi_2(x,u) = \phi(x,0,u)$. It follows that $h\circ\phi_1$ is constant with respect to v and hence $h(\phi_1(x,u+iv)) = h(\phi_1(x,u)) = h(\phi(x,u,0)) = h(x,u) = h^1(x,u)$. One can also show that $h^1\circ\phi_2$ is constant with respect to u, then $h^1(\phi_2(x,u)) = h^1(\phi_2(x,0)) = h^1(x,0) = h_0$ and the result follows. $\diamondsuit$

5. THE INTEGRAL CLOSURE AND THE NEWTON POLYHEDRON

The applicability of Theorem 4.6 depends on whether we can calculate the integral closure of the tangent space to the $\mathcal{R}_V$-orbit.

The Newton polyhedron of an ideal in $\mathcal{O}_n$ is defined as follows (see [16], [20]). We fix a coordinate system x in k^n, so that $\mathcal{O}_n$ is identified with the ring $k\{x\}$ of convergent power series. For each germ $g(x) = \sum a_k x^k$, we define *supp* $g = \{k \in \mathbb{Z}^n : a_k \neq 0\}$.

DEFINITION 5.10. *(i) Let I be an ideal in $\mathcal{O}_n$, we define*

$$supp\ I = \cup\{supp\ g : g \in I\}$$

(ii) The Newton polyhedron of I, denoted by $\Gamma_+(I)$, is the convex hull in $\mathbb{R}^n_+$ of the set $\cup\{k+v : k \in supp\ I, v \in \mathbb{R}^n_+\}$.
(iii) $\Gamma(I)$ is the union of all compact faces of $\Gamma_+(I)$.
(iv) $I =< g_1,...,g_s >$ is Newton non-degenerate if for each compact face $\triangle \subset \Gamma(I)$, the equations $g_{1\triangle}(x) = g_{2\triangle}(x) = ... = g_{s\triangle}(x) = 0$ have no common solution in $(k-\{0\})^n$, where $g_{i\triangle}$ is the restriction of g_i to the face $\triangle$, that is , if $g_i(x) = \sum a_k x^k$ then $g_{i\triangle}(x) = \sum_{k\in\triangle} a_k x^k$.

We denote by $C(\overline{I})$ the convex hull in $\mathbb{R}^n_+$ of the set $\{m : x^m \in \overline{I}\}$. The main result in [20] is the following theorem:

THEOREM 5.11. *Let $I =< g_1, ..., g_s >$ be an ideal of finite codimension in $\mathcal{O}_n$. Then I é Newton non-degenerate if and only if $\Gamma_+(I) = C(\overline{I})$ (the inclusion $C(\overline{I}) \subseteq \Gamma_+(I)$ always holds).*

We shall take $I = dh_t(\Theta_V^0)$ and $I_0 = dh_0(\Theta_V^0)$. The following results show that $C(\overline{I_0})$ also gives a condition for C^0-$\mathcal{R}_V$-triviality.

LEMMA 5.12. *Let $h(x,t) = h_0(x) + tg(x)$ be a deformation of h_0 with $dg(\alpha_i) \in \overline{dh_0(\Theta_V^0)}$, where α_i are generators of Θ_V^0 then:*
a) h is a good deformation;
b) $\overline{dh_0(\Theta_V^0)} \subseteq \overline{dh_t(\Theta_V^0)}$ for t sufficiently small.
Thus, if $g \in \overline{dh_0(\Theta_V^0)}$ then $g \in \overline{dh_t(\Theta_V^0)}$.

Proof: By hypothesis there exists a neighbourhood U of 0 in k^n such that

$$|t||dg(\alpha_i)| \leq |t|csup_j|dh_0(\alpha_j)|$$

On the other hand,

$$sup_j|dh_t(\alpha_j)| = sup_j|dh_0(\alpha_j) + tdg(\alpha_j)|$$

$$\geq sup_j|dh_0(\alpha_j)| - |t|sup_j|dg(\alpha_j)|$$

$$\geq sup_j|dh_0(\alpha_j)| - |t|csup_j|dh_0(\alpha_j)| \geq (1-\alpha)sup_j|dh_0(\alpha_j)|$$

for some $0 < \alpha < 1$ and $|t| \leq \frac{\alpha}{c}$. Thus, $sup_j|dh_t(\alpha_j)| \geq Csup_j|dh_0(\alpha_j)|$ for t sufficiently small and $C > 0$. From this inequality, it follows that

$$\overline{dh_0(\Theta_V^0)} \subseteq \overline{dh_t(\Theta_V^0)},$$

and $b)$ follows. Now,

$$\rho(x,t) \geq c_1 \sum_j |dh_0(\alpha_j)|^2 \geq c_2|x|^\alpha$$

for some constants c_1, c_2 and α, where $\rho(x,t) = \sum_j |dh_t(\alpha_j)|^2$, which implies $a)$. ♢

The following example shows that the condition $g \in \overline{dh_0(\Theta_V^0)}$ does not imply that $dg(\alpha_i) \in \overline{dh_0(\Theta_V^0)}$.

EXAMPLE 5.5.1. *Let $V, 0 \subseteq k^2, 0$ defined by $\varphi(x,y) = x^3 - y^2 = 0$. The module Θ_V is generated by $\alpha_1 = (2x, 3y), \alpha_2 = (2y, 3x^2)$. Let $h_0(x,y) = x^2 + y^3$. The ideal $I_0 =< 4x^2 + 9y^3, 4xy + 9x^2y^2 >$ is non degenerate, hence $C(\overline{I_0}) = \Gamma_+(I_0)$. We have that $g = xy \in \overline{I_0}$ but $dg(\alpha_2) = 2y^2 + 3x^3 \notin \overline{I_0}$. In fact, let $\phi : k, 0 \to k^2, 0$, $\phi(s) = (0, s)$. Then $dg(\alpha_2) \circ \phi = 2s^2 \notin (\phi^*(I_0))\mathcal{O}_1$, and from Theorem 3.3 part c), $dg(\alpha_2) \notin \overline{I_0}$.*

As an immediate consequence of Lemma5.12 and Theorem5.11 it follows that:

Theorem 5.13. *Let $h(x,t) = h_0(x) + tg(x)$ be a deformation of the germ h_0 with $\Gamma_+(g) \subset C(\overline{I_0})$ and $\Gamma_+(dg(\alpha_i)) \subset C(\overline{I_0})$ then $\Gamma_+(g) \subset C(\overline{I})$ and, thus h_t is C^0-$\mathcal{R}_V$-trivial.*

Proof: The hypothesis imply that $g \in \overline{I_0}$ and $dg(\alpha_i) \in \overline{I_0}$. By Lemma 5.12, $g \in \overline{I}$ then $\Gamma_+(g) \subset C(\overline{I})$. By Theorem 4.6, h_t is C^0-$\mathcal{R}_V$-trivial. ♢

Corollary 5.14. *Let I_0 be non-degenerate and $h(x,t) = h_0(x) + tg(x)$ a deformation of the germ h_0 with $\Gamma_+(g) \subset \Gamma_+(I_0)$ and $\Gamma_+(dg(\alpha_i)) \subset \Gamma_+(I_0)$ then $\Gamma_+(g) \subset C(\overline{I})$ and, thus h_t is C^0-$\mathcal{R}_V$-trivial.*

Proof: Since I_0 is non generate, $\Gamma_+(I_0) = C(\overline{I_0})$ and the result follows from the above result. ♢

Example 5.5.2. *Let $V, 0 \subseteq \mathbb{C}^2, 0$ be defined by $\varphi(x,y) = x^3 - y^2 = 0$. The module Θ_V is generated by $\alpha_1 = (2x, 3y), \alpha_2 = (2y, 3x^2)$. In [3], Theorem 4.9, the $\mathcal{R}_V$ classification of germs $h : \mathbb{C}^2, 0 \to \mathbb{C}, 0$ is given, and we find the following normal form $y^2 + ax^n + tx^{n+1}$, $n \geq 4$, which is finitely determined for $a \neq 0$. Let $h_0(x,y) = y^2 + ax^n$. Then $I_0 =< 2anx^n + 6y^2, 2anx^{n-1}y + 6x^2y >$ is non degenerate, hence $C(\overline{I_0}) = \Gamma_+(I_0))$. From Theorem 5.13, it follows that h_t is C^0-$\mathcal{R}_V$-trivial.*

Example 5.5.3. *Let $V, 0 \subseteq \mathbb{C}^3, 0$ be the swallowtail parametrized by $(x, -4y^3 - 2xy, -3y^4 - xy^2)$. The module Θ_V is generated by $\eta_1 = (2x, 3y, 4z)$, $\eta_2 = (6y, -2x^2 - 8z, xy)$ and $\eta_3 = (-4x^2 - 16z, -8xy, y^2)$. The $\mathcal{R}_V$ classification of germs $h : \mathbb{C}^3, 0 \to \mathbb{C}, 0$ given by Theorem 4.10 in [3], gives the normal form $z + ax^n + tx^{n+1}$, $n \geq 2$ which is finitely determined for $a \neq 0, n \neq 2$, and $a \neq 0, a \neq 1/12, n = 2$. Let $h_0(x,y,z) = z + ax^n$, $I_0 =< 2anx^n + 4z$, $6anx^{n-1}y + xy$, $-4anx^{n+1} - 16anx^{n-1}z + y^2 >$. From Theorem 5.13, h_t is C^0-$\mathcal{R}_V$-trivial.*

6. WEIGHTED HOMOGENEOUS GERMS AND VARIETIES

Definition 6.15. *(a) Given $(w_1, ..., w_n : d_1, ..., d_p)$, w_i, $d_j \in \mathcal{Q}^+$, a map germ $f : k^n, 0 \to k^p, 0$ is weighted homogeneous of type $(w_1, ..., w_n : d_1, ..., d_p)$ if for all $\lambda \in k - \{0\}$:*

$$f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, ..., \lambda^{w_n} x_n) = (\lambda^{d_1} f_1(x), \lambda^{d_2} f_2(x), ..., \lambda^{d_p} f_p(x))$$

In this case, the value w_i is called weight of the variable x_i and the value d_i, is the filtration of f_i with respect to the weights $(w_1, ..., w_n)$. Notation: weight$(x_i) = w(x_i) = w_i$ and filtration$(f) =$ fil$(f) = (d_1, ..., d_p)$.

(b) Given $(w_1, ..., w_n)$, and any monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n}$, we define fil$(x^\alpha) = \sum_{i=1}^n \alpha_i w_i$.

(c) We define a filtration in the ring $\mathcal{O}_n$ via the function defined by fil$(f) = inf_{|\alpha|}\{$fil$(x^\alpha) : \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0) \neq 0\}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Definition 6.16. *A germ of an analytic variety $V,0 \subseteq k^n,0$ is weighted homogeneous if it is defined by a weighted homogeneous map germ $f: k^n,0 \to k^p,0$.*

Definition 6.17. *Let $V,0 \subseteq k^n,0$ be the germ of a weighted homogeneous analytic variety. We say that a set $\{\alpha_1,...,\alpha_r\}$ of generators of Θ_V is weighted homogeneous of type $(w_1,...,w_n : d_1,...,d_r)$ if $\alpha_i = \sum_{j=1}^{n} \alpha_{ij} \frac{\partial}{\partial x_j}$, $w(x_j) = w_j$, $w(\frac{\partial}{\partial x_j}) = -w_j$ and fil$(\alpha_i) = d_i =$ fil$(\alpha_{ij}) - w_j$ for each $i = 1...r$.*

We learned from D. Mond and J. Damon that when V is a weighted homogeneos variety, we always can choose weighted homogeneous generators for Θ_V. A proof can be found in [10].

Following [7], we define:

Definition 6.18. *Let V be defined by weighted homogeneous polynomials . We say that h is weighted homogeneous consistent with V if h is weighted homogeneous with respect to the same set of weights assigned to V.*

Example 6.6.1. *Let $V = \phi^{-1}(0) \subset k^3$ where $\phi(x,y,z) = 2x^2y^2 + y^3 - z^2 + x^4y$. We have that ϕ is weighted homogeneous of type $(1,2,3:6)$. Let $h(x,y,z) = x^3 + xy + z$ and $f(x,y,z) = x^3 + xy + z^2$. Then h is consistent with V, f is weighted homogeneous but not consistent with V.*

The following result follows as a corollary of the proof of Theorem 4.6. It was previously proved by J. Damon in [8], but we include it here for completeness.

Theorem 6.19. *Let V be a weighted homogeneous subvariety of $k^n,0$ and $h_0 : k^n,0 \to k,0$ be weighted homogeneous consistent with V and $\mathcal{R}_V$-finitely determined. Then any deformation h of h_0 by terms of filtration greater than or equal to the filtration of h_0, is C^0-$\mathcal{R}_V$-trivial.*

Proof:

Under the above conditions, any such h is a good deformation of h_0 (see [19]).

We have that $dh_0(\alpha_i)$ is weighted homogeneous, where $\{\alpha_1,...,\alpha_m\}$ is a set of weighted homogeneous generators of Θ_V. Let r_i be the filtration of $dh_0(\alpha_i)$, $i = 1,...,m$ and

$$\omega_0(x) = |dh_0(\alpha_1)(x)|^{2s_1} + ... + |dh_0(\alpha_m)(x)|^{2s_r}$$

with $s_i = k/r_i$, e $k = m.m.c.\{r_i\}$. Let $\rho_i = dh_t(\alpha_i)$ and $\omega = \sum_{i=1}^{m} |\rho_i|^{2s_i}$. Since

$$|\rho_i|^2 \frac{\partial h}{\partial t} = dh_t(\overline{\rho_i} \frac{\partial h}{\partial t} \alpha_i),$$

it follows that

$$\omega\frac{\partial h}{\partial t} = dh_t\left(\frac{\partial h}{\partial t}(\overline{\rho_1}|\rho_1|^{2s_1-2}\alpha_1 + ... + \overline{\rho_m}|\rho_m|^{2s_m-2}\alpha_m)\right).$$

Then

$$\frac{\partial h}{\partial t} = dh_t\left(\frac{\partial h}{\partial t}\frac{1}{\omega}(\overline{\rho_1}|\rho_1|^{2s_1-2}\alpha_1 + ... + \overline{\rho_m}|\rho_m|^{2s_m-2}\alpha_m)\right).$$

The proof now follows analogously to the proof of Theorem 4.6. ◊

EXAMPLE 6.6.2. *Let $V, 0 \subset \mathbb{R}^3, 0$ (or $\mathbb{C}^3, 0$) be defined by $\varphi(x,y,z) = 2x^{k+1}y^2 + y^3 - z^2 + x^{2(k+1)}y = 0$. This is the implicit equation for the S_k-singularities classified by D. Mond [18]. The function-germ φ is weighted homogeneous of weights 2, $2k+2$ and $3k+3$ for x, y and z respectively. We have that $h(x,y,z) = y + a_{k+1}x^{k+1}$ is $\mathcal{R}_V$-finitely determined for $a_{k+1} \neq 0, 1$ and consistent with V. Therefore deformations of h by terms of order higher than or equal to fil(h) are C^0-$\mathcal{R}_V$-trivial. For k odd, $h_1(x,y,z) = z + ax^{3(k+1)/2}$ and $h_2(x,y,z) = z + bx^{(k+1)/2}y$ are consistent with V and $\mathcal{R}_V$-finite for all $a^2 \neq -4/27$ and $b \neq \pm 2$. Thus deformations of h_1 and h_2, respectively by terms of order higher than or equal to fil(h_1) and fil(h_2) are C^0-$\mathcal{R}_V$-trivial.*

The following example shows that the hypothesis in Theorem6.19 can hold even when condition $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ does not hold.

EXAMPLE 6.6.3. *Taking $k = 1$ in the above example, the module Θ_V is generated by $\alpha_1 = (2x, 4y, 6z), \alpha_2 = (0, 2z, x^4 + 4x^2y + 3y^2), \alpha_3 = (x^2 + 3y, -4xy, 0), \alpha_4 = (z, 0, 2x^3y + 2xy^2)$. Any deformation of the germ $h_0(x,y,z) = y + ax^2, a \neq 0, 1$ by terms of filtrations higher than or equal to fil(h_0) $= 2$ are $\mathcal{R}_V$-topologically trivial. In particular $h(x,y,z,t) = y + (a+t)x^2$ is $\mathcal{R}_V$-topologically trivial. However, $\frac{\partial h}{\partial t} = x^2$ is not in the integral closure of the ideal $dh_t(\Theta_V^0)$. In fact, given $\phi : k, 0 \to k^4, 0$, $\phi(s) = (s, -as^2, 0, 0)$, it follows that $\frac{\partial h}{\partial t} \circ \phi$ is not in $(\phi^*(dh_t(\Theta_V^0)))\mathcal{O}_1$, then by Theorem 3.3, $\frac{\partial h}{\partial t} = x^2 \notin \overline{dh_t(\Theta_V^0)}$.*

7. *V*-EQUISINGULARITY

Bernard Teissier developed in [22] an infinitesimal theory and a theory of geometrical invariants to study the equisingularity of families of complex analytic hypersurfaces X_t^d with isolated singularities. The integral closure of an ideal I is the right object to the infinitesimal part of that theory. T. Gaffney in [11] extended Teissier results, using the integral closure of a convenient module to obtain necessary and sufficient conditions for the equisingularity of families of complete intersections with isolated singularities.

DEFINITION 7.20. *Suppose (X, x) is a complex analytic germ, $\mathcal{O}_{X,x}$ its local ring and M a submodule of $\mathcal{O}_{X,x}^p$. Then an element $h \in \mathcal{O}_{X,x}^p$ is in $\overline{M}$ if and only if for all $\phi : \mathbb{C}, 0 \to X, x$, $h \circ \phi$ is in $(\phi^*(M))\mathcal{O}_1$.*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Theorem 7.21. *([11], Theorem 2.5) Let $F : \mathbb{C}^t \times \mathbb{C}^N \to \mathbb{C}^p, 0$, defining $X = \overline{F^{-1}(0)}$ with reduced structure, $Y = \mathbb{C}^t \times 0$ and X_0 the smooth part of X. Then $\frac{\partial F}{\partial s} \in \overline{< z_i \frac{\partial F}{\partial z_j} >}_{\mathcal{O}_X}$ for all tangent vectors $\frac{\partial}{\partial s}$ to $\mathbb{C}^t \times 0$ iff (X_0, Y) are Whitney regular.*

Our purpose in this section is to show that the integral closure of the tangent space to the group $\mathcal{R}_V$ also gives an adequate infinitesimal object to the study of the equisingularity of families of sections of analytic varieties.

Let $V \subset \mathbb{C}^n$ be an analytic variety. The family of sections of V is defined by $h(x,t) = 0$, where $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$, $h(0,t) = 0$, is a good deformation of a $\mathcal{R}_V$-finitely determined map germ $h_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$.

We want to define the concept of equisingularity in this context, that we will call V-equisingularity of the family h. With this purpose, we will construct a stratified diagram of mappings, which satisfies to the Thom's second isotopy lemma.

From now on, we assume that V admits a Whitney stratification $\mathcal{S}_V$ in a neighbourhood U of the origin, for which $\{0\}$ is a stratum. We can also extend this stratification to the neighbourhood U of the origin in a natural way, that is, the strata are the strata of $\mathcal{S}_V$ and the connected components of the complement of V in U. We denote by $\tilde{V}$ the subvariety of $\mathbb{C}^n \times \mathbb{C}, 0$ defined by $\tilde{V} = V \times \mathbb{C}$. The product stratification is clearly Whitney regular. Since the germ $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$ is a good deformation, we can choose a representative, which we also denote by h, given by $h : U \times B_r, 0 \to \mathbb{C}, 0$, where B_r is an open ball in $\mathbb{C}$ centered at the origin with the property that h is transversal to the strata of $\tilde{V}$ away from $0 \times B_r$.

We refine the stratification $\tilde{\mathcal{S}}$ of $U \times B_r$ as follows: given a stratum S of $\mathcal{S}$, we define the new strata $\tilde{S}$ of $\tilde{\mathcal{S}}$ as one of the following types: $(S \times B_r) - h^{-1}(0)$ and $(S \times B_r) \cap h^{-1}(0)$. This refinement defines a new stratification $U \times B_r$, since h is transversal to $\tilde{V}$ away from zero. We denote this new stratification by the same notation $\tilde{\mathcal{S}}$.

Definition 7.22. *With the above notation, h is V-equisingular if there exists $\varepsilon > 0$ such that:*

(1) $(B_\varepsilon \times B_r, \tilde{\mathcal{S}})$ is Whitney regular;

(2) $B_\varepsilon \times B_r \xrightarrow{F} \mathbb{C} \times B_r \xrightarrow{\pi} B_r$ satisfies the second isotopy lemma, where B_ε is the closed ball in $\mathbb{C}^n$ with radius ε, B_r is the closed ball in $\mathbb{C}$ of radius r, and $F : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C} \times \mathbb{C}, 0$ is given by $F(x,t) = (h(x,t), t)$.

As a consequence of the second isotopy lemma, it follows that if h is V-equisingular, then h is topologically $\mathcal{R}_V$-trivial.

In the following theorem we show that $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ is a sufficient condition for V-equisingularity.

Theorem 7.23. *Let $V = \phi^{-1}(0)$, $\phi : \mathbb{C}^n, 0 \to \mathbb{C}, 0$, $h_0 : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ $\mathcal{R}_V$-finitely determined and $h : \mathbb{C}^n \times \mathbb{C}, 0 \to \mathbb{C}, 0$ a good deformation of h_0. Let $h^{-1}(0) \cap \Sigma_\phi = \{0\} \times \mathbb{C}$, where Σ_ϕ is the singular set of ϕ. If $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$, then h is V-equisingular.*

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

J.W. Bruce in [1] considers an analogous question. He describes the topological type of generic families of sections of a semialgebraic stratification $\mathcal{T}$ of a neighbourhood of the origin in $\mathbb{R}^n$, with 0 being a stratum. Such families are *generalised transverse* (G.T) with respect to the stratification, that is, for every pair of strata S_1 and S_2, and a sequence of points $(x_i) \in S_1$ such that $\lim_{i\to\infty} x_i = x \in S_2$ and the limit of the tangent spaces $\lim_{i\to\infty} T_{x_i}S_1 = T$ then $dh(x) : T \to \mathbb{R}$ has maximal rank, that is, $h^{-1}(h(x))$ is transversal to T.

The following theorem is proved in [1]:

THEOREM 7.24. *([1], Proposition 1.4) Let $\mathcal{T}$ a Whitney stratification of an open neighbourhood U of the origin in $\mathbb{R}^n$, with 0 being a stratum. Let $h : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be a family of submersions, with $h(0,t) = 0$ and $h_t(x) = h(x,t)$. If the family h is generalised transverse with respect to $\mathcal{T}$, for all $t \in [0,1]$, then there exists a germ of homeomorphism $G : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ preserving the strata of $\mathcal{T}$ such that $h_0 \circ G = h_1$.*

Good examples of families satisfying the G.T. condition are the families of sections of an analytic variety defined by generic families of hyperplanes in $\mathbb{R}^n$. In this work, we substitute the G.T. condition by the finite determinacy of h_0 and the integral closure condition. Under these hypothesis we are able to obtain the topological triviality of families that do not satisfy the G.T. condition.

EXAMPLE 7.7.1. *We saw in example 5.5.3 that the family $h_t(x,y,z) = z + ax^n + tx^{n+1}$ is topologically $\mathcal{R}_V$-trivial where V is the swallowtail in $\mathbb{C}^3$. However this family h_t is not G.T. at 0, since $dh_t(0,0,0) = (0,0,1)$ and the limit of tangent planes to the smooth part of V is the xy-plane.*

To prove Theorem 7.23, we first prove the following Lemma.

LEMMA 7.25. *Let $\phi : \mathbb{C}^n, 0 \to \mathbb{C}, 0$ and $V = \phi^{-1}(0)$. Given $h : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$, define $G : \mathbb{C}^{n+1}, 0 \to \mathbb{C}^2, 0$, by $G(x,t) = (h(x,t), \phi(x))$. If $g \in \overline{dh_t(\Theta_V^0)}_{\mathcal{O}_{n+1}}$ then $(g,0) \in \overline{< x_i \frac{\partial G}{\partial x_j} >}_{\mathcal{O}_{G^{-1}(0)}}$.*

Proof: By hypothesis, for any analytic curve $\varphi : \mathbb{C}, 0 \to \mathbb{C}^{n+1}, 0$, it follows that $g \circ \varphi \in < dh_t(\alpha_i) \circ \varphi >$ where α_i are generators of Θ_V^0 . Then for all $\varphi : \mathbb{C}, 0 \to V \times \mathbb{C}, 0$, we also have $(g \circ \varphi, 0) \in < dh_t(\alpha_i) \circ \varphi, d\phi(\alpha_i) \circ \varphi >$, since $d\phi(\alpha_i) \in < \phi >$ and $\phi(V) = 0$. Therefore $(g \circ \varphi, 0) \in < (x_i \frac{\partial h}{\partial x_j}, x_i \frac{\partial \phi}{\partial x_j}) \circ \varphi >$. Thus, $(g,0) \in \overline{< (x_i \frac{\partial h}{\partial x_j}, x_i \frac{\partial \phi}{\partial x_j}) >}_{\mathcal{O}_{V\times\mathbb{C}}} = \overline{< x_i \frac{\partial G}{\partial x_j} >}_{\mathcal{O}_{V\times\mathbb{C}}}$. In special, $(g,0) \in \overline{< x_i \frac{\partial G}{\partial x_j} >}_{\mathcal{O}_{G^{-1}(0)}}$. ♢

Remark. The above result remains true under the weaker hypothesis $g \in \overline{dh(\Theta_V^0)}_{\mathcal{O}_{V\times\mathbb{C}}}$.

The example below shows that the condition $g \in \overline{dh(\Theta_V^0)}_{\mathcal{O}_{V\times\mathbb{C}}}$ is stronger than the condition $(g,0) \in \overline{< x_i \frac{\partial G}{\partial x_j} >}_{\mathcal{O}_{G^{-1}(0)}}$.

Example 7.7.2. *Let $V,0 \subset k^3,0$ be defined by $\phi(x,y,z) = 2x^2y^2 + y^3 - z^2 + x^4y = 0$ and $h : \mathbb{C}^4,0 \to \mathbb{C},0$, $h(x,y,z,t) = y + (a+t)x^2$ and $G : \mathbb{C}^4,0 \to \mathbb{C}^2,0$ given by $G(x,y,z,t) = (y + (a+t)x^2, 2x^2y^2 + y^3 - z^2 + x^4y)$. We have that Θ_V is generated by $\eta_1 = (2x, 4y, 6z)$, $\eta_2 = (0, 2z, x^4 + 4x^2y + 3y^2)$, $\eta_3 = (x^2 + 3y, -4xy, 0)$, $\eta_4 = (z, 0, 2x^3y + 2xy^2)$. As we saw in the example 6.6.3, $x^2 \notin \overline{< dh(\eta_i) >_{\mathcal{O}_4}}$ (it also follows that $x^2 \notin \overline{< dh(\eta_i) >_{\mathcal{O}_{V\times\mathbb{C}}}}$). We can verify that $(x^2,0) \in \overline{< x_i \frac{\partial G}{\partial x_j} >_{\mathcal{O}_{G^{-1}(0)}}}$. In fact, we will show that $(x^2,0) \in \overline{< x_i \frac{\partial G}{\partial x_j}, e_iG_j >_{\mathcal{O}_4}}$ and the result will follow from this. We have*

$(a)\ zG_z = (0, -2z^2)$	$(b)\ e_1G_1 = (y + (a+t)x^2, 0)$
$(c)\ x^2G_y = (x^2, 4x^4y + 3x^2y^2 + x^6)$	$(d)\ e_2G_2 + \frac{1}{2}zG_z = (0, 2x^2y^2 + y^3 + x^4y)$

Let $\varphi : \mathbb{C},0 \to \mathbb{C}^4,0$ be given by $\varphi(u) = (\varphi_1(u), \varphi_2(u), \varphi_3(u), \varphi_4(u))$. We shall see that $(\varphi_1^2, 0) \in < (x_i\frac{\partial G}{\partial x_j}, e_iG_j) \circ \varphi >_{\mathcal{O}_1}$. Let $r = ord(\varphi_1)$ and $s = ord(\varphi_2)$, if $s \leq r$ or $2s = r$ then it follows from (b) that $(\varphi_1^2, 0) \in < (x_i\frac{\partial G}{\partial x_j}, e_iG_j) \circ \varphi >_{\mathcal{O}_1}$.

If $s > r$ then it follows from (c) that

$$x^2G_y \circ \varphi = (\varphi_1^2, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6) = (\varphi_1^2, 0) + (0, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6)$$

and from (d) we get that $(0, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6) \in < (x_i\frac{\partial G}{\partial x_j}, e_iG_j) \circ \varphi >_{\mathcal{O}_1}$, hence, $(\varphi_1^2, 0) \in < (x_i\frac{\partial G}{\partial x_j}, e_iG_j) \circ \varphi >_{\mathcal{O}_1}$, or, $(x^2,0) \in \overline{< x_i\frac{\partial G}{\partial x_j}, e_iG_j >_{\mathcal{O}_4}}$.

We now proceed to prove Theorem 7.23; our proof is analogous to the proof of Theorem 7.24 in [1]. As in [1], we divide the proof in steps:

Step 1. The stratification $\tilde{S}$ is Whitney regular.

Proof: The Whitney regularity of a pair of strata (S_1, S_2) follows easily, with exception of the regularity condition of the strata over $\{0\} \times \mathbb{C}$. Clearly the strata of type $(S \times \mathbb{C}) - h^{-1}(0)$ are regular with respect to $\{0\} \times B_r$, since the original stratification satisfies the Whitney conditions. Then we only have to verify that $(S \times B_r) \cap h^{-1}(0)$ is regular over $\{0\} \times B_r$. From hypothesis, $\frac{\partial h}{\partial t} \in \overline{dh_t\Theta_V^0}$ and from Lemma 7.25 it follows that $(\frac{\partial h}{\partial t}, 0) \in \overline{< x_i\frac{\partial G}{\partial x_j} >_{\mathcal{O}_{G^{-1}(0)}}}$. Now, from Theorem 7.21, $(G^{-1}(0) - \Sigma_{G^{-1}(0)}, \{0\} \times B_r) = (h^{-1}(0) \cap \tilde{V} - \{0\} \times B_r, \{0\} \times B_r)$ is W. regular. ♢

Step 2. For some $\varepsilon' > 0$ and all $0 < \varepsilon \leq \varepsilon'$ the product of the boundary of the ε-ball, ∂B_ε, by B_r meets the strata of $\tilde{S}$ transversaly.

Proof: The argument is the same as in Theorem 7.24, in [1]. Let us suppose that the statement is false. Then we can find a sequence of points (x_i, t_i) in some stratum $\tilde{S}$ with $x_i \to 0$ and $T_{(x_i,t_i)}\tilde{S} \subset T_{(x_i,t_i)}(\partial B_{\varepsilon_i} \times B_r)$ where $\varepsilon_i = ||x_i||$. Then $(x_i, 0)$ is perpendicular to $T_{(x_i,t_i)}\tilde{S}$. This contradicts the Whitney condition B. ♢

We then have the first approximation to our stratified diagram, that is,

$$B_\varepsilon \times B_r \xrightarrow{F} \mathbb{C} \times B_r \xrightarrow{\pi} B_r$$

where B_ε is the closed ball in $\mathbb{C}^n$ of radius ε, $\varepsilon \leq \varepsilon'$, $F(x,t) = (h(x,t),t)$ and π is the projection to the second factor. We stratify $\mathbb{C} \times B_r$ by $(\mathbb{C} - \{0\}) \times B_r \cup \{0\} \times B_r$ and we refine the stratification of $B_\varepsilon \times B_r$, taking the intersection of the strata in $\tilde{\mathcal{S}}$ with $\partial B_\varepsilon \times B_r$ and $int B_\varepsilon \times B_r$. We would like to show that this stratification satisfies Thom's condition, but h_t might have critical points on ∂B_ε. To get around this difficulty we need the following.

Step 3. For some $\delta > 0$, $B_\delta - \{0\}$ in $\mathbb{C}$ consists only of regular values of h_t for every $t \in B_r$.

Proof: It follows from the fact that h is a good deformation of h_0. ♢

In the above diagram we change $\mathbb{C}$ by B_δ, where B_δ is the ball with radius δ, with the stratification $\partial B_\delta \cup \{0\} \cup int B_\delta - \{0\}$, and satisfying the conditions in Step 3. We then get a new stratification of $F^{-1}(B_\delta \times B_r)$ pulling back the strata. We consider now

$$F^{-1}(B_\delta \times B_r) \xrightarrow{F} B_\delta \times B_r \xrightarrow{\pi} B_r$$

Step 4. The above diagram is Thom stratified.

Proof: We have to show that the diagram satisfies condition A_{h_t}. Given two strata $\tilde{S_1}$, $\tilde{S_2}$ with $(x_i,t_i) \in \tilde{S_1}$, and $(x_i,t_i) \to (x,t) \in \tilde{S_2}$, the restriction of the kernel of $dF(x_i,t_i)$ to $T_{(x_i,t_i)}\tilde{S_1}$, say K_i, is $T_{(x_i,t_i)}\tilde{S_1} \cap (\ker dh_{t_i}(x_i) \times \{0\})$. The limit of this sequence of spaces is contained $T \cap (\ker dh_t(x) \times \{0\})$ where $T = \lim_{i\to\infty} T_{(x_i,t_i)}\tilde{S_1}$. If $x \neq 0$ then $\ker dh_t(x) \times \{0\}$ is transversal to T, hence $\lim_{i\to\infty} K_i = T \cap (\ker dh_t(x) \times \{0\})$. Since $T \supset T_{(x,t)}\tilde{S_2}$ (Whitney condition A), then $\lim_{i\to\infty} K_i$ contains the restriction of the kernel of $dF(x,t)$ to $T_{(x,t)}\tilde{S_2}$. If $x = 0$ then $\tilde{S_2} = \{0\} \times B_r$, and Thom condition follows trivially. ♢

Another condition associated to equisingularity of families of functions was introduced by J. Henry, M. Merle and C. Sabbah in [14], namely the condition W_f. T. Gaffney and S.Kleiman in recent papers ([12], [13], [15]) have studied the equisingularity of families of complete intersection with isolated singularity and of families of functions defined on analytic varieties. In special, in [13] T.Gaffney and S.Kleiman relate the condition W_f with the integral closure of a convenient module. More precisely, let $X, 0$ be the germ of an equidimensional analytic variety in $\mathbb{C}^n, 0$, and $Y, 0$ the germ of a smooth subvariety of $X, 0$. Let $f : \mathbb{C}^n, 0 \to \mathbb{C}^q, 0$ be a map germ with the property that the restriction $f|Y$ is a submersion onto the germ of an analytic closed set in $\mathbb{C}^q$. Let us assume that there exists an open and dense analytic subset X_0 of X such that $f|(X_0, 0)$ is a submersion onto its image and has equidimensional fibres.

Definition 7.26. *The pair (X_0, Y) satisfies condition W_f at 0 if there exists an euclidian neighbourhood U of 0 in X and a constant $C > 0$ such that, for every $y \in U \cap Y$ and $x \in U \cap X_0$, we have*

$$dist(T_yY(f(y)), T_xX(f(x))) \leq C dist(x, Y)$$

where $T_yY(f(y))$ and $T_xX(f(x))$ are tangent spaces to the fibres of the restrictions $f|Y$ and $f|X$ (are tangent spaces to $f^{-1}(f(y)) \cap Y$ and $f^{-1}(f(x)) \cap X$).

Let $X = F^{-1}(0)$, $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, $\mathbb{C}^n = \mathbb{C}^l \times Y$, $Y = \mathbb{C}^{n-l}$. Denote by $JM(F; f)$ the jacobean module generated by the columns of the jacobean matrix of (F, f), that is, the module generated by $< (x_i \frac{\partial F}{\partial x_j}, x_i \frac{\partial f}{\partial x_j}) >$. Let $JM(F; f)_l$ and $JM(F; f)_{n-l}$ be the submodules of $JM(F; f)$ generated by the partial derivatives with respect to the first l variables and with respect to the last $n - l$ variables of $\mathbb{C}^n$, respectively. Finally, let $\mathcal{M}_Y$ the ideal generated by the first l variables of $\mathbb{C}^n$. With these notations T. Gaffney and S. Kleiman show in [13], that the following conditions are equivalent:
(i) the pair (X_0, Y) satisfies W_f in 0;
(ii) $JM(F; f)_{n-l} \subseteq \overline{\mathcal{M}_Y JM(F; f)_l}$;

Our theorem on V-equisingularity, Theorem 7.23, was obtained independently of the above result of T. Gaffney and S. Kleiman. However, we note that taking X in the above theorem as $\tilde{V} = V \times \mathbb{C}$ and making $f = h$, we get that the condition $\frac{\partial h}{\partial t} \in \overline{dh(\Theta_V^0)}$ implies that $(0, \frac{\partial h}{\partial t}) \in \overline{\mathcal{M}_Y < (\frac{\partial \phi}{\partial x_i}, \frac{\partial h}{\partial x_i}) >} = \overline{\mathcal{M}_Y JM(\phi; f)_l}$, where ϕ is the defining equation for V. Then, the following result follows:

Corollary 7.27. *With the above notation and taking $Y = \mathbb{C}$, if $\frac{\partial h}{\partial t} \in \overline{dh(\Theta_V^0)}$ then the pair $(\tilde{V} - \mathbb{C}, \mathbb{C})$ satisfies W_h in 0.*

Acknowledgement 7.1. *We are very grateful to Terence Gaffney and Marcelo Saia for many helpful conversations. We have also benefited from discussions with Jim Damon and David Mond.*

REFERENCES

1. J. W. Bruce, *Generic functions on semialgebraic sets*, Quart. J. Oxford (2), **37**(1986), 137-165.
2. J. W. Bruce and M. Roberts, *Critical points of functions on analytic varieties*, Topology Vol.**27** No.1 (1988), 57-90.
3. J. W. Bruce, N. P. Kirk and A. A. du Plessis, *Complete transversals and the classification of singularities*, Nonlinearity **10** (1997), 253-275.
4. S. H. Chang and Y. C. Lu, *On C^0-sufficiency of complex jets*, Canad. J. Math. **25**(1973), 874-880.
5. J. Damon, *The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$* , Memoirs Am. Math. Soc., **306**(1984).
6. J. Damon, *Topological triviality and versality for subgroups of $\mathcal{A}$ and $\mathcal{K}$*, Memoirs Am. Math. Soc., **389**(1988).
7. J. Damon, *Deformations of sections of singularities and Gorenstein surface singularities*, Am. Journal of Mathematics, **109**(1987),695-722.
8. J. Damon, *Topological triviality and versality for subgroups of $\mathcal{A}$ and $\mathcal{K}$: II. Sufficient conditions and applications*, Nonlinearity **5**(1992), 373-412.
9. J. Damon, *Finite determinacy and topological triviality I*, Invent. Math. **62**, (1980) 299-324.
10. J. Damon, A note on the freeness of equisingular deformations of plane curve singularitires, preprint.
11. T. Gaffney, *Integral closure of modules*, Invent. Math. **107**, (1992) 301-322.
12. T. Gaffney and S. L. Kleiman, *Specialization of integral dependence for modules*, preprint.

13. T. Gaffney and S. L. Kleiman, *W_f and integral dependence*, preprint.

14. J. P. G. Henry, M. Merle and C. Sabbah, *Sur la condition de Thom stricte pour un morphime analytique complexe*, Ann. Scient. Éc. Norm. Sup. **17**(1984), 227-268.

15. S. L. Kleiman, *Equisingularity, multiplicity, and dependence*, preprint.

16. A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32**(1976), 1-31.

17. T. C. Kuo, *On C^0-sufficiency of jets of potential functions*, Topology, **8**(1969), 167-171.

18. D. Mond, *On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$*, Proc. of London Math. Soc. **3**, **50**(1985), 333-369.

19. M. A. S Ruas, M. J. Saia, *C^l-determinacy of weighted homogeneous germs*, Hokkaido Math. Journal, **26**(1997), 89-99.

20. M. Saia, *The integral closure of ideals and the Newton filtration*, J. Algebraic Geometry, **5**(1996), 1-11.

21. B. Teissier, *Introduction to Equisingularity Problems*, Proceedings of Symposia in Pure Mathematics **29** (1977), 593-631.

22. B. Teissier, *Multiplicities polaires, sections planes, et conditions de Whitney*, Lecture Notes in Mathematics **961** (1982), 314-491.

23. J. N. Tomazella, *Seções de Variedades Analíticas*, Ph.D. Thesis, ICMSC-USP.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC