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Regularity of solutions of Partial Neutral Functional Differential Equations with Unbounded Delay

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We prove the existence of regular solutions for a class partial neutral functional differential equations with unbounded delay that can to be described in the from $\frac{d}{dt}(x(t) + F(t, x_t)) = Ax(t) + G(t, x_t)$, where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on a Banach space X and F, G are appropriated functions. March, 2001 ICMC-USP

1. INTRODUCTION

The purpose of this paper is to prove the existence of regular solutions for a class of quasi-linear neutral functional differential equations with unbounded delay that can be described in the form

$$\begin{aligned} \frac{d}{dt}(x(t) + F(t, x_t)) &= Ax(t) + G(t, x_t), \qquad t \geq \sigma, \\ x_\sigma &= \varphi \in \mathcal{B}, \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on a Banach space X, the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t + \theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically, $\Omega \subset \mathcal{B}$ is open, $0 \leq \sigma < T$ and $F, G : [\sigma, T] \times \Omega \to X$ are appropriate continuous functions.

Equation (1) is called Abstract Neutral Functional Differential Equation (ANFDE) with Unbounded Delay . The results obtained in this paper are the continuation of the papers [11], [12] on the existence of mild, strong and periodical solutions for the Neutral system (1). For the theory of abstract retarded functional differential equations, ARFDE, we refer the reader to [19], [6], [14], [3], [17] for the basic properties in the case with finite delay, and to [7], [9], [10], [2], [11], [12], for the case with unbounded delay.

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Throughout this paper, X will be a Banach space provided with norm $\| \cdot \|$ and $A : D(A) \to X$ will be the infinitesimal generator of an analytic semigroup $T = (T(t))_{t\geq 0}$ of bounded linear operators on X. For the theory of strongly continuous semigroup, refer to Pazy [18] and Krein [15]. We will point out here some notations and properties that will be used in this work. It is well know that there exist constants $\tilde{M}$ and $w \in \mathbb{R}$ such that

$$\| T(t) \| \leq \tilde{M} e^{wt}, \qquad t \geq 0.$$

If T is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$, then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D(-A)^\alpha$. Furthermore, the subspace $D(-A)^\alpha$ is dense in X, and the expression

$$\| x \|_\alpha = \| (-A)^\alpha x \|$$

defines a norm in $D(-A)^\alpha$. If X_α represents the space $D(-A)^\alpha$ endowed with the norm $\| \cdot \|_\alpha$, then the following properties are well known ([18], pp. 74):

Lemma 1.1. *If the previous conditions hold:*

1.Let $0 < \alpha \leq 1$. Then X_α is a Banach space.
2.If $0 < \beta \leq \alpha$ then $X_\alpha \to X_\beta$ is continuous.
3.For every constant $a > 0$, there exists $C_a > 0$ such that

$$\| (-A)^\alpha T(t) \| \leq \frac{C_a}{t^\alpha}, \qquad 0 < t \leq a.$$

4.For every $a > 0$ there exists a positive constant C'_a such that

$$\| (T(t) - I)(-A)^{-\alpha} \| \leq C'_a t^\alpha, \qquad 0 < t \leq a.$$

In this work we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [7]. To establish the axioms of the space $\mathcal{B}$ we follows the terminology used in Hino-Murakami-Naito [13], and thus, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X, endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$. We will assume that $\mathcal{B}$ satisfies the following axioms:

(**A**) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, is continuous in $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

i) x_t is in $\mathcal{B}$.
ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.
iii) $\| x_t \|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t - \sigma) \| x_\sigma \|_{\mathcal{B}}$.
Where $H > 0$ is a constant; $K, M : [0, \infty) \to [0, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

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$(\mathbf{A-1})$ For the function $x(\cdot)$ in $(\mathbf{A})$, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma + a)$.

$(\mathbf{B})$ The space $\mathcal{B}$ is complete.

For the literature on phase space, we refer the reader to [13]. while noting here that from the axiom $(\mathbf{A-1})$, it follows that the operator functions $S(\cdot)$, $W(\cdot)$, given by

$$[S(t)\varphi](\theta) := \begin{cases} \varphi(0) & \text{for } -t \leq \theta \leq 0, \\ \varphi(t+\theta) & \text{for } -\infty < \theta < -t, \end{cases} \tag{2}$$

and

$$[W(t)\varphi](\theta) := \begin{cases} T(t+\theta)\varphi(0) & \text{for } -t \leq \theta \leq 0, \\ \varphi(t+\theta) & \text{for } -\infty < \theta < -t, \end{cases} \tag{3}$$

are strongly continuous semigroups of bounded linear operators on $\mathcal{B}$. We will denote by A_S and A_W the infinitesimal generators of $S(\cdot)$ and $W(\cdot)$ and by $D(A_S)$ and $D(A_W)$ their respective domains.

To obtain some of our results we will require additional properties for the phase space $\mathcal{B}$, in particular we consider the following axiom (see [9], pp. 526 for details);

$(\mathbf{C_3})$ Let $a > 0$. Let $x : (-\infty, \sigma + a] \to X$ be a continuous function such that $x_\sigma \equiv 0$ and the right derivative, denoted $\dot{x}(0^+)$, exists. If the function ψ defined by $\psi(\theta) = 0$ for $\theta < 0$ and $\psi(0) := \dot{x}(0)$ belongs to $\mathcal{B}$, then $\| (\frac{1}{h})x_h - \psi \|_{\mathcal{B}} \to 0$ as $h \to 0^+$.

On the other hand, for a linear map $P : D(P) \subset X \to X$ and $\varphi \in \mathcal{B}$, we denote by $P\varphi$ the function defined by $P\varphi(\theta) = P(\varphi(\theta))$ for all $-\infty < \theta \leq 0$ and for any $0 < \alpha \leq 1$ we use the notation $\mathcal{B}_\alpha$ for the vector space

$$\mathcal{B}_\alpha = \{(-A)^{-\alpha}\varphi : \varphi \in \mathcal{B}\}.$$

It is easily to prove that $\mathcal{B}_\alpha$, endowed with the seminorm defined by

$$\| \psi \|_{\mathcal{B}_\alpha} := \| (-A)^\alpha \varphi \|_{\mathcal{B}},$$

is a phase space of functions with values in X_α.

The paper is organized as follows. In section 2, we define the different concepts used in this work and establish the existence of N-Classical and Classical solutions for the initial value problem (1) Our results are based on the properties of analytic semigroups and the ideas contained in [18] chapter VI and Henríquez [9]. Also in section 2, we cite reference for an application of our results to an abstract Volterra integrodifferential equation.

Throughout this work we assume that X is an abstract Banach space. The terminology and notations are those generally used in operator theory. In particular, if X and Y are

Banach spaces, we indicate by $\mathcal{L}\,(X:Y)$ the Banach space of the bounded linear operators of X in Y and we abbreviate to $\mathcal{L}\,(X)$ whenever $X=Y$. In addition $B_r(x:X)$ will denote the closed ball in the space X with center at x and radius r.

For some bounded function $\xi:[\sigma,T]\to X$ and $\sigma\le s<t\le T$ we employ the notation

$$\| \xi(\cdot) \|_{[s,t]}=\sup\{\| \xi(\theta) \|: \theta\in[s,t]\} \tag{4}$$

and we will write simply ξ_t for $\| \xi(\cdot) \|_{[\sigma,t]}$ when no confusion arises.

If $x\in X$ we will use the notation χ_x for the function $\chi_x:(-\infty,0]\to X$ where $\chi_x(\theta)=0$ for $\theta<0$ and $\chi_x(0)=x$.

Finally, a function $f:I\subset\mathbb{R}\to X$ is α-Hölder continuous, $0<\alpha<1$, if there exists a positive constant $L>0$ such that

$$\| f(s)-f(t) \|\le L \mid t-s \mid^{\alpha}, \quad s,t\in I.$$

2. REGULAR SOLUTIONS

In this section we will study the regularity of the mild solutions of the abstract Cauchy problem (1). Our results are based on those of regularity of mild solutions to the abstract Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t)+f(t),\\ x(0) &= x_0. \end{aligned} \tag{5}$$

By analogy with the abstract Cauchy problem (5) we adopt the following definitions:

DEFINITION 2.1. We will say that a function $x:(-\infty,\sigma+a)\to X$, $a>0$, is a mild solution of the abstract Cauchy problem (1) if: $x_\sigma=\varphi$; the restriction of $x(\cdot)$ to the interval $[\sigma,\sigma+a)$ is continuous; for each $\sigma\le t<\sigma+a$ the function $AT(t-s)F(s,x_s)$, $s\in[\sigma,t)$, is integrable and

$$\begin{aligned} x(t) &= T(t-\sigma)(\varphi(0)+F(\sigma,\varphi))-F(t,x_t)-\int_\sigma^t AT(t-s)F(s,x_s)ds\\ &\quad+\int_\sigma^t T(t-s)G(s,x_s)ds, \end{aligned}$$

for every $t\in[\sigma,\sigma+a)$.

DEFINITION 2.2. We will say that a function $x:(-\infty,\sigma+a)\to X$, $a>0$, is a classical solution of the abstract Cauchy problem (1) if: $x_\sigma=\varphi$, the restriction of $x(\cdot)$ to the interval $[\sigma,\sigma+a)$ is continuous, $x(t)\in D(A)$ for all $t\in(\sigma,\sigma+a)$, $\dot{x}$ is continuous in $(\sigma,\sigma+a)$ and $x(\cdot)$ satisfies equation (1) in $(\sigma,\sigma+a)$.

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DEFINITION 2.3. We will say that a function $x : (-\infty, \sigma + a) \to X$, $a > 0$, is an N-classical solution of the abstract Cauchy problem (1) if: $x_\sigma = \varphi$, the restriction of $x(\cdot)$ to the interval $[\sigma, \sigma + a)$ is continuous, $x(t) \in D(A)$ for all $t \in (\sigma, \sigma + a)$, $\frac{d}{dt}(x(t) + F(t, x_t))$ is continuous in $(\sigma, \sigma + a)$ and $x(\cdot)$ satisfies equation (1) in $(\sigma, \sigma + a)$.

The existence and uniqueness of mild solution of system (1), under application of the contraction principle, was proved in ([12]). More precisely:

THEOREM 2.1. *Let $\varphi \in \Omega$ and assume that the following conditions hold:*

a)There exist $\beta \in (0, 1)$ and $L \geq 0$ such that the function F is X_β-valued and satisfies the Lipschitz condition

$$\| (-A)^\beta F(t, \psi_1) - (-A)^\beta F(s, \psi_2) \| \leq L(| t - s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}) \tag{6}$$

for every $\sigma \leq s, t \leq T$, $\psi_1, \psi_2 \in \Omega$ and

$$K(0)L \| (-A)^{-\beta} \| < 1. \tag{7}$$

b)The function G is continuous and there exist $N > 0$ such that

$$\| G(t, \psi_1) - G(s, \psi_2) \| \leq N(| t - s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}) \tag{8}$$

for every $\sigma \leq s, t \leq T$ and $\psi_1, \psi_2 \in \Omega$.

Then there exists a unique mild solution $x(\cdot, \varphi)$ of the abstract Cauchy problem (1) defined on $(-\infty, \sigma + r)$, for some $0 < r < T - \sigma$. Furthermore, if $\Omega = X$ then r can be chosen as $r := min\{T - \sigma, r_0\}$ where r_0 is a constant, independent of σ and φ.

Remark 2. 1. Clearly the assertions in Theorem (2.1) remains true if the assumptions on F and G are substituted by locally Lipschitz conditions, in particular if $(-A)^\beta F$ and G are continuously differentiable in $[\sigma, T] \times \Omega$.

Remark 2. 2. In the rest of this work, to simplify notations, we only consider the case $\sigma = \mathbf{0}$.

Now we establish a first result about the existence of regular solutions; the existence of N-classical solution.

THEOREM 2.2. *Assume that there exist constants $0 < \alpha < \beta < 1$, $0 < \gamma < 1$ and an open subset $\Omega_\alpha \subset \mathcal{B}_\alpha$ such that $F : [0, T] \times \Omega_\alpha \to D(A)$ and $G : [0, T] \times \Omega_\alpha \to X$ are continuous functions and that the following conditions holds:*

a)The function F is X_β-valued and there exists $L > 0$ such that

$$\| (-A)^\beta F(t,\psi_1) - (-A)^\beta F(s,\psi_2) \| \leq L(| t-s |^\gamma + \| \psi_1 - \psi_2 \|_{\mathcal{B}_\alpha})$$

$$\| G(t,\psi_1) - G(s,\psi_2) \| \leq L(| t-s |^\gamma + \| \psi_1 - \psi_2 \|_{\mathcal{B}_\alpha})$$

for every $0 \leq s,t \leq T$,$\psi_1,\psi_2 \in \Omega_\alpha$.
b)$K(0)L \| (-A)^{\alpha-\beta} \| < 1$.

Let $\varphi \in \Omega_\alpha$. If the function $W(\cdot)(-A)^\alpha\varphi$ is ξ-Hölder continuous in the interval $[0,T]$ and $\beta + Min\{\beta-\alpha, 1-\alpha, \xi, \gamma\} > 1$ then there exists a unique N-Classical solution $x(\cdot,\varphi)$ of the abstract Cauchy problem (1) defined on $(-\infty, b)$, for some $0 < b < T$.

Proof: From our assumptions on the operator A, see lemma (1.1), we fix positive constants $C_\alpha > 0$ and $C_{\alpha+1-\beta}$ such that for all $t \in (0,T]$

$$\| (-A)^\alpha T(t) \| \leq C_\alpha t^{-\alpha}$$

and

$$\| (-A)^{1-\beta+\alpha} T(t) \| \leq C_{\alpha+1-\beta} t.^{-(1-\beta+\alpha)}$$

Let $b_1 > 0$ and $\delta > 0$ such that $V = \{(s,\psi) : 0 \leq s \leq b_1, \quad \| \psi - \varphi \|_{\mathcal{B}_\alpha} < \delta\} \subset [0,T] \times \Omega_\alpha$ and $\mu = K_{b_1} \| (-A)^{\alpha-\beta} \| L < 1$. Now we choose $0 < b < b_1$ such that

$$\sup_{\theta\in[0,b]} \| W(\theta)(-A)^\alpha(\varphi) - (-A)^\alpha(\varphi) \|_{\mathcal{B}} < \frac{(1-\mu)}{6}\delta, \tag{9}$$

$$K_b \sup_{\theta\in[0,b]} \| T(\theta)(-A)^\alpha\varphi(0) - (-A)^\alpha\varphi(0) \| < \frac{(1-\mu)}{6}\delta, \tag{10}$$

$$K_b \sup_{\theta\in[0,b]} \| T(\theta)(-A)^\alpha F(0,\varphi) - (-A)^\alpha F(0,\varphi) \| < \frac{(1-\mu)}{6}\delta, \tag{11}$$

$$K_b L \| (-A)^{\alpha-\beta} \| b < \frac{(1-\mu)}{6}\delta, \tag{12}$$

$$\frac{b^{\beta-\alpha}}{\beta-\alpha} K_b C_{\alpha+1-\beta}\{L(b^\gamma+\delta) + \| (-A)^\beta F(0,\varphi) \|\} < \frac{(1-\mu)}{6}\delta, \tag{13}$$

$$K_b \frac{b^{1-\alpha}}{1-\alpha} C_\alpha \{L(b^\gamma+\gamma) + \| G(0,\varphi) \|\} < \frac{(1-\mu)}{6}\delta, \tag{14}$$

$$K_b(C_{\alpha+1-\beta} L \frac{b^{\beta-\alpha}}{\beta-\alpha} + C_\alpha L \frac{b^{1-\alpha}}{1-\alpha}) < \frac{(1-\mu)}{2}. \tag{15}$$

In the space $Y = C([0,b] : X)$ provided with the topology of uniform convergence, we define;

$$A(\varphi,\alpha,b) = \{u \in Y : u(0) = (-A)^{\alpha}\varphi(0),\ \| (-A)^{\alpha}\varphi - \tilde{u}_t \|_{\mathcal{B}} \leq \delta, \forall\ t \in [0,b]\ \},$$

where $\tilde{u}$ is the extension of u to $(-\infty, b]$ with $\tilde{u}_0 = (-A)^{\alpha}\varphi$. Clearly $A(\varphi,\alpha,b)$ is a nonempty, convex and closed subset of Y. On $A(\varphi,\alpha,b)$ we define the operator Φ by the expression

$$\begin{aligned} \Phi(u)(t) &= T(t)((-A)^{\alpha}(\varphi(0) + F(0,\varphi)) - (-A)^{\alpha}F(t,(-A)^{-\alpha}\tilde{u}_t) \\ &\quad + \int_0^t (-A)^{\alpha+1-\beta}T(t-s)(-A)^{\beta}F(s,(-A)^{-\alpha}\tilde{u}_s)ds \\ &\quad + \int_0^t (-A)^{\alpha}T(t-s)G(s,(-A)^{-\alpha}\tilde{u}_s)ds. \end{aligned}$$

In order to use the contraction mapping principle, we now show that the range of Φ is included in $A(\varphi,\alpha,b)$. To this end we introduce the functions $y_{\alpha}, z_i : (-\infty, b] \to X$ where $y_{\alpha}(t) = T(t)(-A)^{\alpha}\varphi(0)$ for $t \geq 0$, $(y_{\alpha})_0 = (-A)^{\alpha}\varphi$ and

$$\begin{aligned} z^1(t) &= T(t)(-A)^{\alpha}F(0,\varphi) - (-A)^{\alpha}F(t,(-A)^{-\alpha}\tilde{u}_t), \\ z^2(t) &= \int_0^t (-A)^{\alpha+1-\beta}T(t-s)(-A)^{\beta}F(s,(-A)^{-\alpha}\tilde{u}_s)ds \\ z^3(t) &= \int_0^t (-A)^{\alpha}T(t-s)G(s,(-A)^{-\alpha}\tilde{u}_s)ds \end{aligned}$$

for $t > 0$, $z_0^i = 0$ for $i \in \{1,2,3\}$. Clearly

$$\Phi(u)(t) = y_{\alpha}(t) + z^1(t) + z^2(t) + z^3(t)$$

in $[0,b]$ and $\widetilde{\Phi(u)}_t = (y_{\alpha})_t + z_t^1 + z_t^2 + z_t^3$. With the previous notation for $t \in [0,b]$ we have,

$$\| \widetilde{\Phi(u)}_t - (-A)^{\alpha}\varphi \|_{\mathcal{B}} \ \leq \ \| (y_{\alpha})_t - (-A)^{\alpha}\varphi \|_{\mathcal{B}} + \| z_t^1 \|_{\mathcal{B}} + \| z_t^2 \|_{\mathcal{B}} + \| z_t^3 \|_{\mathcal{B}} . \qquad (17)$$

Using axiom (**A**) concerning the phase space, we estimate each term on the right side of (17) separately. Directly from the choice of b we get the estimate

$$\| (y_{\alpha})_t - (-A)^{\alpha}\varphi \|_{\mathcal{B}} \leq \| W(t)(-A)^{\alpha}\varphi - (-A)^{\alpha}\varphi \|_{\mathcal{B}} \leq \frac{(1-\mu)}{6}\delta.$$

On the other hand, for $t \in [0,b]$

$$\| z_t^1 \|_{\mathcal{B}} \leq K_b \sup_{s\in[0,t]} \| z^1(s) \| \qquad (19)$$

and for $s \in [0,t]$

$$\begin{aligned} \| z^1(s) \| &= \| T(s)(-A)^\alpha F(0,\varphi) - (-A)^\alpha F(s,(-A)^{-\alpha}\tilde{u}_s) \| \\ &\leq \| T(s)(-A)^\alpha F(0,\varphi) - (-A)^\alpha F(0,\varphi) \| \\ &\quad + \| (-A)^\alpha F(0,\varphi) - (-A)^\alpha F(s,(-A)^{-\alpha}\tilde{u}_s) \| \\ &\leq \frac{(1-\mu)}{K_b 6}\delta + \| (-A)^{\alpha-\beta} \| \| (-A)^\beta F(0,\varphi) - (-A)^\beta F(s,(-A)^{-\alpha}\tilde{u}_s) \| \\ &\leq \frac{(1-\mu)}{K_b 6}\delta + L \| (-A)^{\alpha-\beta} \| \{b + \| (-A)^\alpha \varphi - \tilde{u}_s \|_{\mathcal{B}}\} \end{aligned}$$

hence

$$\| z^1(s) \| \leq \frac{2(1-\mu)}{K_b 6}\delta + L \| (-A)^{\alpha-\beta} \| \delta. \tag{20}$$

Using (20) in the inequality (19)

$$\| z_t^1 \|_{\mathcal{B}} \leq \frac{2(1-\mu)}{6}\delta + \mu\delta. \tag{21}$$

Now, for the function $z^2(\cdot)$ we have that

$$\| z_t^2 \|_{\mathcal{B}} \leq K_b \sup_{s\in[0,t]} \| z^2(s) \|$$

and for $s \in [0,t]$

$$\begin{aligned} \| z^2(s) \| &\leq \int_0^s \| (-A)^{\alpha+1-\beta} T(s-\theta)(-A)^\beta F(\theta,(-A)^{-\alpha}\tilde{u}_\theta) \| d\theta \\ &\leq \int_0^s \| (-A)^{\alpha+1-\beta} T(s-\theta)((-A)^\beta F(\theta,(-A)^{-\alpha}\tilde{u}_\theta) - (-A)^\beta F(0,\varphi)) \| d\theta \\ &\quad + \int_0^s \| (-A)^{\alpha+1-\beta} T(s-\theta)(-A)^\beta F(0,\varphi)) \| d\theta \\ &\leq \int_0^s \frac{C_{\alpha+1-\beta}}{(s-\theta)^{\alpha+1-\beta}} L(\theta^\gamma + \| \tilde{u}_\theta - (-A)^\alpha \varphi \|_{\mathcal{B}}) d\theta \\ &\quad + \frac{b^{\beta-\alpha}}{\beta-\alpha} C_{\alpha+1-\beta} \| (-A)^\beta F(0,\varphi) \| \\ &\leq L(b^\gamma + \delta)\frac{b^{\beta-\alpha}}{\beta-\alpha} C_{\alpha+1-\beta} + \frac{b^{\beta-\alpha}}{\beta-\alpha} C_{\alpha+1-\beta} \| (-A)^\beta F(0,\varphi) \| . \end{aligned}$$

Employing the last inequality in (22) we obtain that;

$$\| z_t^2 \| \leq \frac{b^{\beta-\alpha}}{\beta-\alpha} K_b C_{\alpha+1-\beta} \{L(b^\gamma + \delta) + \| (-A)^\beta F(0,\varphi) \|\} \leq \frac{(1-\mu)}{6}\delta. \tag{23}$$

Similarly for z^3,

$$\| z_t^3 \|_{\mathcal{B}} \leq K_b \sup_{s\in[0,t]} \| z^3(s) \|$$

and for $s \in [0,t]$

$$\begin{aligned}
\| z^3(s) \| &\leq \int_0^s \| (-A)^\alpha T(s-\theta)(G(\theta,(-A)^{-\alpha}\tilde{u}_\theta) - G(0,\varphi)) \| \, d\theta \\
&\quad + \int_0^s \| (-A)^\alpha T(s-\theta) G(0,\varphi) \| \, d\theta \\
&\leq \int_0^s \frac{C_\alpha}{(s-\theta)^\alpha} L(b^\gamma + \delta) d\theta + C_\alpha \frac{b^{1-\alpha}}{1-\alpha} \| G(0,\varphi) \| \\
&\leq L(b^\gamma + \delta) C_\alpha \frac{b^{1-\alpha}}{1-\alpha} + C_\alpha \frac{b^{1-\alpha}}{1-\alpha} \| G(0,\varphi) \| .
\end{aligned}$$

Therefore from (24)

$$\| z_t^3 \|_{\mathcal{B}} \leq K_b \frac{b^{1-\alpha}}{1-\alpha} C_\alpha \{ L(b^\gamma + \delta) + \| G(0,\varphi) \| \} \leq \frac{(1-\mu)}{6} \delta. \tag{25}$$

Combining (17), (18), (21), (23) and (25), we conclude that $\Phi(u) \in A(\varphi,\alpha,b)$.

Now we prove that Φ is a contraction. Let $u, v \in A(\varphi,\alpha,b)$, so that

$$\| \Phi(u)(t) - \Phi(v)(t) \|$$

$$\begin{aligned}
&\leq \| (-A)^\alpha F(t,(-A)^{-\alpha}\tilde{u}_t) - (-A)^\alpha F(t,(-A)^{-\alpha}\tilde{v}_t) \| \\
&\quad + \int_0^t \| (-A)^{\alpha+1-\beta} T(t-s)((-A)^\beta F(s,(-A)^{-\alpha}\tilde{u}_s) - (-A)^\beta F(s,(-A)^{-\alpha}\tilde{v}_s)) \| \, ds \\
&\quad + \int_0^t \| (-A)^\alpha T(t-s)(G(s,(-A)^{-\alpha}\tilde{u}_s) - G(s,(-A)^{-\alpha}\tilde{v}_s)) \| \, ds \\
&\leq \| (-A)^{\alpha-\beta} \| L \| \tilde{u}_t - \tilde{v}_t \| + \int_0^t \frac{C_{\alpha+1-\beta} L}{(t-s)^{\alpha+1-\beta}} \| \tilde{u}_s - \tilde{v}_s \|_{\mathcal{B}} \, ds \\
&\quad + \int_0^t \frac{C_\alpha L}{(t-s)^\alpha} \| \tilde{u}_s - \tilde{v}_s \|_{\mathcal{B}} \, ds.
\end{aligned}$$

Therefore

$$\| \Phi u - \Phi v \|_b \leq K_b (\| (-A)^{\beta-\alpha} \| L + C_{\alpha+1-\beta} L \frac{b^{\beta-\alpha}}{\beta-\alpha} + C_\alpha L \frac{b^{1-\alpha}}{1-\alpha}) \| u - v \|_b . \tag{26}$$

From the choice of b, (26) and the contraction mapping principle, we can conclude that Φ has a unique fixed point $x(\cdot)$ in $A(\varphi,\alpha,b)$.

From the assumptions on F and G, it follows that the functions $t \to F(t, (-A)^{-\alpha}\tilde{x}_t)$ and $t \to (-A)^{\beta}G(t, (-A)^{-\alpha}\tilde{x}_t)$ are continuous and bounded in $[0, b]$. In the following N will be a positive constant such that

$$\begin{aligned} \| (-A)^{\beta}F(t, (-A)^{-\alpha}\tilde{x}_t) \| \ &\leq \ N, \\ \| G(t, (-A)^{-\alpha}\tilde{x}_t) \| \ &\leq \ N \end{aligned} \tag{27}$$

for all $t \in \ [0, b]$. Next we will prove that the functions $t \to (-A)^{\beta}F(t, (-A)^{-\alpha}\tilde{x}_t)$ and $t \to G(t, (-A)^{-\alpha}\tilde{x}_t)$ are Hölder continuous in $[0, b]$. From Lemma (1.1) and the condition $\beta + Min\{\beta - \alpha, 1 - \alpha, \xi, \gamma\} > 1$, we fix $0 < \vartheta < \min\{1 - \alpha, \beta - \alpha, \xi, \gamma\}$ with $\vartheta + \beta > 1$ and $\tilde{C} > 0$ such that for all $0 < s < t < b_1$ and $0 < h < 1$

$$\| (T(h) - I)(-A)^{\alpha}T(t - s) \| \ \leq \ \tilde{C}h^{\vartheta}(t - s),^{-(\vartheta + \alpha)} \tag{28}$$

$$\| (T(h) - I)(-A)^{\alpha+1-\beta}T(t - s) \| \ \leq \ \tilde{C}h^{\vartheta}(t - s).^{-(\alpha+1-\beta+\vartheta)} \tag{29}$$

For $t \in [0, b)$ and $0 < h < 1$ with $t + h < b$ we get

$$\| x(t + h) - x(t) \|$$

$$\begin{aligned} \leq \ & \| T(t + h)(-A)^{\alpha}\varphi(0) - T(t)(-A)^{\alpha}\varphi(0) \| \\ & + \| T(t + h)(-A)^{\alpha}F(0, \varphi) - T(t)(-A)^{\alpha}F(0, \varphi) \| \\ & + \| (-A)^{\alpha}F(t + h, (-A)^{-\alpha}\tilde{x}_{t+h}) - (-A)^{\alpha}F(t, (-A)^{-\alpha}\tilde{x}_t) \| \\ & + \int_0^t \| (-A)^{\alpha+1-\beta}(T(h) - I)T(t - s)(-A)^{\beta}F(s, (-A)^{-\alpha}\tilde{x}_s) \| \, ds \\ & + \int_t^{t+h} \| (-A)^{\alpha+1-\beta}T(t + h - s)(-A)^{\beta}F(s, (-A)^{-\alpha}\tilde{x}_s) \| \, ds \\ & + \int_0^t \| (-A)^{\alpha}(T(h) - I)T(t - s)G(s, (-A)^{-\alpha}\tilde{x}_s) \| \, ds \\ & + \int_t^{t+h} \| (-A)^{\alpha}T(t + h - s)G(s, (-A)^{-\alpha}\tilde{x}_s) \| \, ds. \end{aligned}$$

We estimate each term on the right side of the last inequality separately. Clearly there exists positive constants C_1, C_2, independent of $h > 0$, such that

$$I_1(t, h) \ \leq \ C_1 h^{\xi} \tag{30}$$

and

$$I_2(t, h) \ \leq \ C_2 h^{1-\alpha}. \tag{31}$$

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For the third term

$$\begin{aligned} I_3 &\leq \parallel (-A)^{\alpha} F(t+h, (-A)^{-\alpha}\tilde{x}_{t+h}) - (-A)^{\alpha} F(t, (-A)^{-\alpha}\tilde{x}_t) \parallel \\ &\leq \parallel (-A)^{\alpha-\beta} \parallel \parallel (-A)^{\beta} F(t+h, (-A)^{-\alpha}\tilde{x}_{t+h}) - (-A)^{\beta} F(t, (-A)^{-\alpha}\tilde{x}_t) \parallel \\ &\leq \parallel (-A)^{\alpha-\beta} \parallel L\{h^{\gamma} + \parallel \tilde{x}_{t+h} - \tilde{x}_t \parallel_{\mathcal{B}}\} \\ &\leq \parallel (-A)^{\alpha-\beta} \parallel Lh^{\gamma} + \parallel (-A)^{\alpha-\beta} \parallel LM_b \parallel \parallel \tilde{x}_h - (-A)^{\alpha}\varphi \parallel_{\mathcal{B}} + \\ &\quad + \parallel (-A)^{\alpha-\beta} \parallel LK_b \sup_{\theta\in[0,t]} \parallel x(\theta+h) - x(\theta) \parallel \end{aligned}$$

equivalently,

$$\begin{aligned} I_3 &\leq C_3 h^{\gamma} + \parallel (-A)^{\alpha-\beta} \parallel LM_b \parallel \parallel \tilde{x}_h - (-A)^{\alpha}\varphi \parallel_{\mathcal{B}} \\ &\quad + \parallel (-A)^{\alpha-\beta} \parallel LK_b \sup_{\theta\in[0,t]} \parallel x(\theta+h) - x(\theta) \parallel . \end{aligned} \tag{32}$$

With respect to the fourth term, we get

$$\begin{aligned} I_4 &= \int_0^t \parallel (T(h)-I)(-A)^{\alpha+1-\beta} T(t-s)(-A)^{\beta} F(s, (-A)^{-\alpha} x_s) \parallel ds \\ &\leq \int_0^t \frac{h^{\vartheta} N\tilde{C}}{(t-s)^{\alpha+1-\beta+\vartheta}} ds \\ &\leq \frac{t^{\beta-\alpha-\vartheta}}{\beta-\alpha-\vartheta} \tilde{C} N h^{\vartheta} \end{aligned}$$

and therefore

$$I_4 \leq C_4 h^{\vartheta} \tag{33}$$

where C_4 is independent of t and h.

For I_5 we have that

$$\begin{aligned} I_5 &\leq \int_t^{t+h} \parallel (-A)^{\alpha+1-\beta} T(t+h-s)(-A)^{\beta} F(s, (-A)^{-\alpha} x_s) \parallel ds \\ &\leq C_{\alpha+1-\beta} N \frac{h^{\beta-\alpha}}{\beta-\alpha} \leq C_5 h^{\vartheta} \end{aligned}$$

then

$$I_5 \leq C_5 h^{\vartheta}$$

where C_5 is independent of $t \in [0, b)$ and $0 < h < 1$.

In a similar manner we can prove that

$$I_6 \leq C_6 h^{\vartheta}, \tag{35}$$

$$I_7 \leq C_7 h^{\vartheta} \tag{36}$$

where C_6 and C_7 are independent of $t \in [0,b)$ and $0 < h < 1$.

Using the estimates (30)-(36), it follows that there exists $\tilde{C}_1 > 0$, independent of $t \in [0,b)$ such that for $0 < h < 1$ with $0 < t + h < b$,

$$\begin{aligned} \| x(t+h) - x(t) \| \quad &\leq \quad \tilde{C}_1 h^{\vartheta} + M_b L \| (-A)^{\alpha-\beta} \| \| \tilde{x}_h - (-A^{-\alpha})\varphi \|_{\mathcal{B}} \\ &\quad + \| (-A)^{\alpha-\beta} \| L K_b \sup_{\theta \in [0,t]} \| x(\theta + h) - x(\theta) \| \end{aligned}$$

consequently

$$\| x(\theta + h) - x(\theta) \|_{[0,t]} \leq \frac{\tilde{C}_1 h^{\vartheta}}{1-\mu} + \frac{M_b L}{1-\mu} \| (-A)^{\alpha-\beta} \| \| \tilde{x}_h - (-A^{-\alpha})\varphi \|_{\mathcal{B}} \tag{37}$$

since $0 < \mu = K_{b_1} \| (-A)^{\alpha-\beta} \| L < 1$. Moreover, using the definition of y_α and reiterating the previous argument it is possible to show that

$$\begin{aligned} \| \tilde{x}_h - (-A)^{\alpha}\varphi \|_{\mathcal{B}} \ &\leq \| W(h)(-A)^{\alpha}\varphi - (-A)^{\alpha}\varphi \|_{\mathcal{B}} + \textstyle\sum_{i=1}^{3} \| z_h^i \|_{\mathcal{B}} \\ &\leq C_8 h^{\xi} + \| z_h^1 \|_{\mathcal{B}} + C_9 h^{\vartheta}. \end{aligned}$$

Now from axiom (**A**)

$$\| z_h^1 \|_{\mathcal{B}} \leq K_h \sup_{s \in [0,h]} \| T(s)(-A)^{\alpha} F(0,\varphi) - (-A)^{\alpha} F(s, (-A)^{-\alpha}\tilde{x}_s) \| \tag{39}$$

and for $0 \leq s \leq h$

$$\| T(s)(-A)^{\alpha} F(0,\varphi) - (-A)^{\alpha} F(s, (-A)^{-\alpha}\tilde{x}_s) \|$$

$$\begin{aligned} &\leq \ \| T(s)(-A)^{\alpha} F(0,\varphi) - (-A)^{\alpha} F(0,\varphi) \| + L \| (-A)^{\alpha-\beta} \| \| (-A)^{\alpha}\varphi - \tilde{x}_s \|_{\mathcal{B}} \\ &\leq \ C_{10} h^{1-\alpha} + L \| (-A)^{\alpha-\beta} \| \| (-A)^{\alpha}\varphi - \tilde{x}_s \|_{\mathcal{B}} . \end{aligned}$$

Employing this last inequality in (39), it follows that:

$$\| z_h^1 \|_{\mathcal{B}} \leq K_b C_{10} h^{1-\alpha} + K_b L \| (-A)^{\alpha-\beta} \| \| (-A)^{\alpha}\varphi - \tilde{x}_\tau \|_{\tau \in [0,h]} . \tag{40}$$

From (40), (38), the choice of b and the fact that $\vartheta < Min\{\beta - \alpha, 1 - \alpha, \xi, \gamma\}$; for $0 < h < 1$ we have that

$$\| x_\tau - (-A)^{\alpha}\varphi \|_{\tau \in [0,h]} \leq C_{11} h^{\vartheta} + K_b L \| (-A)^{\alpha-\beta} \| \| (-A)^{\alpha}\varphi - \tilde{x}_\tau \|_{\tau \in [0,h]} \tag{41}$$

and thus

$$\| x_h - (-A)^{\alpha}\varphi \|_{\mathcal{B}} \leq \frac{C_{11}}{1-\mu} h^{\vartheta}. \tag{42}$$

The inequalities (42) and (37) show that there exists $C_{12} > 0$, independent of $\theta \in [0, b)$ and $0 < h < 1$, such that

$$\| x(\theta + h) - x(\theta) \| \leq C_{12} h^{\vartheta} \tag{43}$$

where $\vartheta < Min\{\beta - \alpha, 1 - \alpha, \xi, \gamma\}$ and $\vartheta + \beta > 1$.

From (42), (43) and axiom (**A**) it easily follows that functions $(-A)^{\beta} F(t, (-A)^{-\alpha}\tilde{x}_t)$ and $G(t, (-A)^{-\alpha}\tilde{x}_t)$ are ϑ-Hölder with $\vartheta + \beta > 1$. Let the function $y(\cdot)$ be defined by

$$\begin{aligned} y(t) &= T(t)(\varphi(0) + F(0, \varphi)) + F(t, (-A)^{-\alpha}\tilde{x}_t) \\ &+ \int_0^t (-A)^{1-\beta} T(t-s)(-A)^{\beta} F(s, (-A)^{-\alpha}\tilde{x}_s) ds \\ &+ \int_{\sigma}^t T(t-s) G(s, (-A)^{-\alpha}\tilde{x}_s) ds. \end{aligned} \tag{44}$$

The Lemma (2.1) below implies that $y(t) - F(t, (-A)^{-\alpha}\tilde{x}_t) \in C^1((0, b) : X)$ and that $y(t) - F(t, (-A)^{-\alpha}\tilde{x}_t) \in D(A)$ for all $t \in (0, b)$. Operating on (43) with $(-A)^{\alpha}$ we infer that $(-A)^{\alpha} y = x$ and then $y(t) - F(t, y_t) \in C^1((0, b) : X)$. The above remarks prove that $y(\cdot)$ is a unique N-classical solution for system (1). This completes the proof.

The proof of the next Lemma is analogous to the proof of Theorem (2.4.1) in [5]. However there are some differences that require special attention and we include the principal ideas of the proof for completeness.

LEMMA 2.1. *Let $0 < \beta < 1$, $g \in C([0, T] : X_{1-\beta})$ and $f \in C([0, T] : X)$. Assume that $g, f : [0, T] \to X$ are θ-Hölder continuous with $\beta + \theta > 1$. If $y_i : [0, T] \to X$, $i = 1, 2$, are defined by*

$$\begin{aligned} y_1(t) &= \int_0^t (-A)^{1-\beta} T(t-s) g(s) ds \\ y_2(t) &= \int_0^t T(t-s) f(s) ds \end{aligned}$$

then $y_i(t) \in D(A)$ for every $t \in [0, T)$ and $\dot{y}_i \in C([0, T) : X)$.

Proof. We only proof the assertions for the function y_1. For $t \in [0, T)$ we rewrite $y_1(t)$ in the form

$$\int_0^t (-A)^{1-\beta} T(t-s)(g(s) - g(t)) ds + \int_0^t (-A)^{1-\beta} T(t-s) g(t) ds = v(t) + w(t). \tag{45}$$

Clearly, $Aw(t) = T(t)(-A)^{1-\beta}g(t) - (-A)^{1-\beta}g(t) \in C([0,T] : X)$. Now for $\epsilon > 0$ sufficiently small we define the function

$$v_\epsilon(t) := \begin{cases} \int_0^{t-\epsilon}(-A)^{1-\beta}T(t-s)(g(s)-g(t))ds\,, & \text{for } t \in [\epsilon, T), \\ \\ 0 & \text{for } t \in [0,\epsilon]. \end{cases} \tag{46}$$

It is clear that $v_\epsilon(t) \in D(A)$. Moreover for $0 < \delta_1 < \delta_2$

$$\begin{aligned} \| Av_{\delta_2}(t) - Av_{\delta_1}(t) \| &\leq \int_{t-\delta_2}^{t-\delta_1} \| (-A)^{2-\beta}T(t-s)(g(s)-g(t)) \| \, ds \\ &\leq \int_{t-\delta_2}^{t-\delta_1} \frac{C_{2-\beta}}{(t-s)^{2-\beta}} |s-t|^\theta ds \\ &\leq C_{2-\beta}(\delta_2^{\beta+\theta-1} - \delta_1^{\beta+\theta-1}). \end{aligned} \tag{47}$$

The last inequality proves that $A(v_\delta)(t)$ is convergent, $\beta + \theta > 1$, and therefore

$$A(v(t)) = \int_0^t A^{2-\beta}T(t-s)(g(s)-g(t))ds \tag{48}$$

since A is a closed operator.

The continuity of $\partial_t y_1$ is proved in usual manner, we omit details. This completes the proof.

In the rest of this paper, for a function $j : [0,T] \times \mathcal{B} \to X$ and $h \in \mathbb{R}$ we use the notation $\partial_h j$ for the function

$$\partial_h j(t) := \frac{j(t+h) - j(t)}{h}.$$

Moreover, if j is differentiable we will employ the following decomposition

$$j(t+s,\psi) - j(t,\psi) = D_1 j(t,\psi)s + W_1(j,t,s,\psi) \tag{49}$$

and

$$j(t,\psi+\psi_1) - j(t,\psi) = D_2 j(t,\psi)\psi_1 + W_2(j,t,\psi,\psi_1) \tag{50}$$

where

$$\begin{aligned} \frac{W_1(j,t,s,\psi)}{| t-s |} &\to 0, \qquad as \quad s \to t, \\ \frac{W_1(j,t,\psi,\psi_1)}{\| \psi - \psi_1 \|_{\mathcal{B}}} &\to 0, \qquad as \quad \psi_1 \to \psi. \end{aligned}$$

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COROLLARY 2.1. *Assume that $\varphi \in D(A_W)$ and that F is continuously differentiable in $[0,T] \times \Omega$ with $DF_{(0,\varphi)} \equiv 0$. If $\chi_{G(0,\varphi)} \in \mathcal{B}$ and $\mathcal{B}$ satisfies axiom $(\mathbf{C_3})$ or if $G(0,\varphi)=0$, then there exists a unique classical solution of system (1) defined in $[0,b]$ for some $0 < b < T$.*

Proof. Let $y(\cdot)$ be the mild solution of (1) and $0 < 2b < T$ such that $y(\cdot)$ is defined in $(-\infty, 2b]$ and

$$\mu = K_{2b} \sup_{t \in [0,2b]} \| D_2F(t, y_t) \|_{\mathcal{L}(\mathcal{B}:X)} < 1. \tag{51}$$

It is clear that for $h > 0$ sufficiently small, $\partial_h y \in C([0,b] : X)$. Moreover $\partial_h y$ is convergent on the space $C([0,b] : X)$. In fact, for positive numbers $h, \tilde{h}$ and $t \in [0,b]$ we have that

$$\begin{aligned} \| \partial_h y(t) - \partial_{\tilde{h}} y(t) \| \quad \leq \quad & \| \partial_h p(t) - \partial_{\tilde{h}} p(t) \| \\ & + \| \frac{F(t+h, y_{t+h}) - F(t, y_t)}{h} - \frac{F(t+\tilde{h}, y_{t+\tilde{h}}) - F(t, y_t)}{\tilde{h}} \| . \end{aligned}$$

From the proof of Theorem (2.2) and the fact that $\varphi(0) \in D(A)$, we infer that $p(\cdot) \in C^1([0,b] : X)$. Now, using the differentiability of F and the previous notations, see (49)-(50), we get

$$\begin{aligned} \| \partial_h y(t) - \partial_{\tilde{h}} y(t) \| \ \leq \ & \| \partial_h p(t) - \partial_{\tilde{h}} p(t) \| + \| D_1F(t, y_{t+h}) - D_1F(t, y_{t+\tilde{h}}) \| \\ & + \| D_2F(t, y_t)\left(\frac{y_{t+h} - y_t}{h} - \frac{y_{t+\tilde{h}} - y_t}{\tilde{h}}\right) \| \\ & + \| \frac{W_1(F, t, t+h, y_{t+h})}{h} - \frac{W_1(F, t, t+\tilde{h}, y_{t+\tilde{h}})}{\tilde{h}} \| \\ & + \| \frac{W_2(F, t, y_t, y_{t+h})}{h} - \frac{W_2(F, t, y_t, y_{t+\tilde{h}})}{\tilde{h}} \| . \end{aligned}$$

Employing axiom $(\mathbf{A})$ in the last inequality, we have

$$\begin{aligned} \| \partial_h y - \partial_{\tilde{h}} y \|_{[0,b]} \ \leq \ & \frac{1}{1-\mu} \| \partial_h p - \partial_{\tilde{h}} p \|_b + \frac{1}{1-\mu} \| D_1F(s, y_{s+h}) - D_1F(s, y_{s+\tilde{h}}) \|_b \\ & + \frac{M_b}{1-\mu} \| D_2F(s, y_s) \|_b \| \frac{y_h - \varphi}{h} - \frac{y_{\tilde{h}} - \varphi}{\tilde{h}} \|_{\mathcal{B}} \\ & + \frac{1}{1-\mu} \| \frac{W_1(F, s, s+h, y_{s+h})}{h} - \frac{W_1(F, s, s+\tilde{h}, y_{s+\tilde{h}})}{\tilde{h}} \|_b \\ & + \frac{1}{1-\mu} \| \frac{W_2(F, s, y_s, y_{s+h})}{h} - \frac{W_2(F, s, y_s, y_{s+\tilde{h}})}{\tilde{h}} \|_b . \end{aligned}$$

It is clear that $\| \partial_h p(t) - \partial_{\tilde{h}} p(t) \|_{[0,b]} \to 0$ as $h \to 0$ and $\tilde{h} \to 0$. On the other hand,

$$\frac{W_1(F, s, s+h, y_{s+h})}{h} \to 0 \qquad as \quad h \to 0$$

uniformly for $s \in [0,b]$. Moreover, from Proposition (3.1) in [12], we know that $s \to y_s$ is Lipschitz and thus

$$\frac{W_2(F,s,y_s,y_{s+h})}{h} = \frac{W_2(F,s,y_s,y_{s+h})}{\| y_s - y_{s+h} \|_{\mathcal{B}}} \cdot \frac{\| y_s - y_{s+h} \|_{\mathcal{B}}}{h} \to 0, \qquad as \qquad h \to 0,$$

uniformly for $s \in [0,b]$. The above remarks permit us to rewrite the last inequality in the form

$$\| \partial_h y - \partial_{\tilde{h}} y \|_{[0,b]} \leq \chi(h,\tilde{h}) + \frac{M_b}{1-\mu} \| D_2F(s,y_s) \|_b \| \frac{y_h - \varphi}{h} - \frac{y_{\tilde{h}} - \varphi}{\tilde{h}} \|_{\mathcal{B}} \tag{52}$$

where $\chi(h,\tilde{h}) \to 0$ as $h,\tilde{h} \to 0$. Clearly, $\partial_h y$ will be convergent in $C([0,b]:X)$ if $h^{-1}(y_h - \varphi)$ is convergent in $\mathcal{B}$. Next we will prove this convergence. For $h > 0$ we consider the decomposition

$$\begin{aligned} \| \tfrac{y_h - \varphi}{h} - A_W(\varphi) - \chi_{G(0,\varphi)} \|_{\mathcal{B}} &= \| \tfrac{W(h)\varphi - \varphi}{h} - A_W(\varphi) + \tfrac{z_h^1 + z_h^2 + z_h^3}{h} - \chi_{G(0,\varphi)} \|_{\mathcal{B}} \\ &\leq \| \tfrac{W(h)\varphi - \varphi}{h} - A_W(\varphi) \|_{\mathcal{B}} + \| \tfrac{z_h^1 + z_h^2}{h} \|_{\mathcal{B}} \\ &\quad + \| \tfrac{z_h^3}{h} - \chi_{G(0,\varphi)} \|_{\mathcal{B}} \end{aligned}$$

where $z^i = 0$ in $(-\infty, 0]$ and

$$\begin{aligned} z^1(\theta) &= T(\theta)F(0,\varphi) - F(\theta, y_\theta) \\ z^2(\theta) &= \int_0^\theta (-A)^{1-\beta} T(\theta - s)(-A)^\beta F(s,y_s)ds \\ z^3(\theta) &= \int_0^\theta T(\theta - s)G(s,y_s)ds. \end{aligned}$$

for $\theta \in [0,b]$. We estimate each term of (53) separately. Clearly

$$I_1(h) \to 0 \qquad \text{as} \qquad h \to 0, \tag{54}$$

since $\varphi \in D(A_W)$. On the other hand, directly from the condition $\chi_{G(0,\varphi)} \equiv 0$ or from the axiom $(\mathbf{C_3})$ in the case $\chi_{G(0,\varphi)} \in \mathcal{B}$ and $G(0,\varphi) \neq 0$, it follows that

$$I_3(h) \to 0 \qquad \text{as} \qquad h \to 0. \tag{55}$$

For the second term we get

$$\begin{aligned} I_2(h) \;\leq\; & K_b \frac{1}{h} \| (T(\theta) - I)F(0,\varphi) + \int_0^\theta (-A)^{1-\beta} T(\theta - s)(-A)^\beta F(s,y_s)ds \|_{\theta \in [0,h]} \\ & + K_b \frac{1}{h} \| F(0,\varphi) - F(\theta, y_\theta) \|_{\theta \in [0,h]} . \end{aligned} \tag{56}$$

Moreover for $\theta \in [0,h]$,

$$\begin{aligned}
&\frac{1}{h} \ \parallel \ (T(\theta)-I)F(0,\varphi)+\int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^\beta F(s,y_s)ds \parallel \\
&\leq \ \frac{1}{h}\int_0^\theta \parallel (-A)^{1-\beta}T(\theta-s)\{(-A)^\beta F(s,y_s)-(-A)^\beta F(0,\varphi)\} \parallel ds \\
&\leq \ \frac{1}{h}\int_0^\theta \frac{C_{1-\beta}L}{(\theta-s)^{1-\beta}}(s^\gamma+\parallel \varphi-y_s \parallel_{\mathcal{B}})ds \\
&\leq \ \frac{1}{h}\int_0^\theta \frac{C_{1-\beta}L}{(\theta-s)^{1-\beta}}(h^\gamma+Ch)ds \\
&\leq \ \frac{1}{h}C_{1-\beta}L(h^\gamma+Ch)\frac{h^\beta}{\beta}
\end{aligned}$$

where we used the Lipschitz continuity of $s \to y_s$, see Proposition (3.1) in [12], consequently

$$\frac{1}{h} \parallel (T(\theta)-I)F(0,\varphi)+\int_0^\theta (-A)^{1-\beta}T(\theta-s)(-A)^\beta F(s,y_s)ds \parallel \to 0 \tag{57}$$

if $h \to 0$ uniformly for $\theta \in [0,h]$ since $\beta + Min\{\beta-\alpha, 1-\alpha, \gamma, \xi\} > 1$. On the other hand

$$\frac{F(0,\varphi)-F(\theta,y_\theta)}{\theta}\cdot\frac{\theta}{h} \to 0 \tag{58}$$

as $h \to 0$, since $DF(0,\varphi) \equiv 0$. Using (57) and (58) in (55), we conclude that

$$I_2(h) \to 0 \qquad\qquad h \to 0. \tag{59}$$

Finally the convergence of $h^{-1}(y_h-\varphi)$, consequently the convergence of $\partial_h y$, follows from (54), (55), (59) and (53). This complete the proof of the corollary.

The next result establishes the existence of classical solutions for the neutral system (1), making use of usual regularity assumptions for the functions $(-A)^\beta F$ and G.

Theorem 2.3. *Let $\varphi \in D(A_W)\bigcap\Omega$ and assume that $(-A)^\beta F$ and G are continuously differentiable in $[0,T]\times\Omega$, that F is $D(A)$-valued continuous with $D((-A)^\beta F)(0,\varphi) \equiv 0$ and that $K(0)L \parallel (-A)^{-\beta} \parallel < 1$. If $\chi_{G(0,\varphi)} \equiv 0$ or $\chi_{G(0,\varphi)} \in \mathcal{B}$ and $\mathcal{B}$ satisfies axiom $(\mathbf{C_3})$, then there exists a unique classical solution of the system (1) defined in $[0,b]$ for some $0<b<T$.*

Proof. Let $u := u(\cdot,\varphi)$ the mild solution of (1). In the following we assume that $u(\cdot)$ is defined in $(-\infty, 2b]$ where $0<b<T$ and

$$\mu = K_{2b} \parallel D_2F(s,u_s) \parallel_{[0,2b]} < 1. \tag{60}$$

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Let $z(\cdot)$ be the solution of the integral equation

$$\begin{aligned} z(t) &= T(t)A\varphi(0) + h(t) - D_2F(t,u_t)\cdot z_t \\ & \int_0^t (-A)^{1-\beta}T(t-s)D_2(-A)^{\beta}F(s,u_s)\cdot z_s ds \\ & \int_0^t T(t-s)D_2G(s,u_s)\cdot z_s ds, \qquad\qquad t\geq 0 \end{aligned} \tag{61}$$

where

$$z_0 = A_W(\varphi) + \chi_{G(0,\varphi)}, \tag{62}$$

and

$$\begin{aligned} h(t) &= -D_1F(t,u_t) + \int_0^t (-A)^{1-\beta}T(t-s)D_1(-A)^{\beta}F(s,u_s)ds \\ &+ \int_0^t T(t-s)D_1G(s,u_s)ds + T(t)G(0,\varphi). \end{aligned} \tag{63}$$

The existence and uniqueness of local solution to the integral equation (60)-(62) is clear and we omit more details. In what follows we assume that $z \in C([0,b]:X)$. We affirm that $\dot{u}(\cdot) = z(\cdot)$ in $[0,b]$. In order to prove the assertion, for $t\in[0,b]$ and $0<h<1$ sufficiently small,

$$\parallel \partial_h u(t) - z(t) \parallel$$

$$\begin{aligned} \leq\ & \parallel T(t)[\frac{T(h)-I}{h}\varphi(0) - A\varphi(0)] \parallel \\ +\ & \parallel T(t)(\frac{T(h)-I}{h})F(0,\varphi) + \frac{1}{h}\int_0^h (-A)^{1-\beta}T(t+h-s)(-A)^{\beta}F(s,u_s)ds \parallel \\ +\ & \parallel \frac{-F(t+h,u_{t+h}) + F(t,u_t)}{h} + D_1F(t,u_t) + D_2F(t,u_t)\cdot z_t \parallel \\ +\ & \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \parallel \partial_h(-A)^{\beta}F(s,u_s) - D_1(-A)^{\beta}F(s,u_s) - D_2(-A)^{\beta}F(s,u_s)\cdot z_s \parallel ds \\ +\ & \parallel \frac{1}{h}\int_0^h T(t+h-s)G(s,u_s)ds - T(t)G(0,\varphi) \parallel \\ +\ & \int_0^t \tilde{M} \parallel \partial_h G(s,u_s) - D_1G(s,u_s) - D_2G(s,u_s)\cdot z_s \parallel ds \end{aligned}$$

With the notations introduced in (49), (50) we estimate each term in the last inequality separately.

It is clear that

$$I_i(t,h) \to 0 \qquad as \;\; h \to 0, \qquad i \in \{1,2,5\} \tag{64}$$

uniformly for $t \in [0,b]$. On the other hand for the third term

$$\begin{aligned}\| I_3 \| &= \| -\partial_h F(t,u_t) + D_1F(t,u_t) + D_2F(t,u_t)\cdot z_t \| \\ &\leq \| -D_1F(t,u_{t+h}) + D_1F(t,u_t) \| + \| D_2F(t,u_t)\cdot(\frac{u_{t+h}-u_t}{h} - z_t) \| \\ &\quad + \| \frac{1}{h}W_1(F,t,t+h,u_{t+h}) \| + \| \frac{1}{h}W_2(F,t,u_t,u_{t+h}) \| .\end{aligned}$$

It's clear that

$$\frac{1}{h}W_1(F,t,t+h,u_{t+h}) \to 0 \qquad \text{as} \qquad h \to 0,$$

uniformly fot $t \in [0,b]$. Moreover from Proposition (3.1) in [12], the function $s \to u_s$ is Lipschitz continuous in $[0,b]$, then

$$\frac{1}{h} \| W_2(F,t,u_t,u_{t+h}) \| = \frac{\| W_2(F,t,u_t,u_{t+h}) \|}{\| u_{t+h}-u_t \|_{\mathcal{B}}} \cdot \frac{\| u_{t+h}-u_t \|_{\mathcal{B}}}{h} \to 0 \tag{65}$$

uniformly for $t \in [0,b]$ as $h \to 0$, so that we can rewrite the last inequality in the form

$$\| I_3 \| \leq \xi_3(t,h) + \| D_2F(t,u_t) \| \| \frac{u_{t+h}-u_t}{h} - z_t \|_{\mathcal{B}} \tag{66}$$

where $\xi_3(t,h) \to 0$ if $h \to 0$ uniformly for $t \in [0,b]$.

Using similar arguments, we have for I_4 that

$$\begin{aligned}\| I_4 \| \leq & \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_1(-A)^\beta F(s,u_{s+h}) - D_1(-A)^\beta F(s,u_s) \| \, ds \\ & + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^\beta F(s,u_s) \| \| (\frac{u_{s+h}-u_s}{h} - z_s) \|_{\mathcal{B}} \, ds \\ & + \int_0^t (\| \frac{1}{h}W_1((-A)^\beta F,s,s+h,u_{s+h}) \| + \| \frac{1}{h}W_2((-A)^\beta F,s,u_s,u_{s+h}) \|) ds\end{aligned}$$

and so, from (65)

$$\| I_4 \| \leq \xi_4(t,h) + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^\beta F(s,u_s) \| \| (\frac{u_{s+h}-u_s}{h} - z_s) \|_{\mathcal{B}} \, ds$$

where $\xi_4(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$.

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Analogously for I_6

$$\| I_6 \| \leq \xi_6(t,h) + \int_0^t \tilde{M} \| D_2G(s,u_s) \| \| (\frac{u_{s+h}-u_s}{h} - z_s) \|_{\mathcal{B}}\, ds$$

where $\xi_6(t,h) \to 0$ as $h \to 0$ uniformly for $t \in [0,b]$.

Combining (64), (66), (67), (68) and the first inequality we get

$$\begin{aligned} \| \partial_h u(t) - z(t) \| \ &\leq\ \xi_7(t,h) + \| D_2F(t,u_t) \| \| \frac{u_{t+h}-u_t}{h} - z_t \|_{\mathcal{B}} \\ &+ \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^{\beta} F(s,u_s) \| \| (\frac{u_{s+h}-u_s}{h} - z_s) \|_{\mathcal{B}}\, ds \\ &+ \int_0^t \tilde{M} \| D_2G(s,u_s) \| \| (\frac{u_{s+h}-u_s}{h} - z_s) \|_{\mathcal{B}}\, ds, \end{aligned}$$

where $\xi_7(t,h) \to 0$ if $h \to 0$ uniformly for $t \in [0,b]$. Using axiom (**A**) and (60) we infer that,

$$\begin{aligned} \| \partial_h u(\cdot) - z(\cdot) \|_{[0,t]} \ &\leq\ \frac{\xi_7(t,h)}{1-\mu} + M_b \frac{1}{1-\mu} \| D_2F(s,u_s) \|_{[0,t]} \| \frac{u_h - \varphi}{h} - z_0 \|_{\mathcal{B}} \\ &+ \frac{K_b}{1-\mu} \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}} \| D_2(-A)^{\beta} F(s,u_s) \| \| \partial_h u(\theta) - z(\theta) \|_s\, ds \\ &+ C_{1-\beta} M_b \frac{b^{\beta}}{\beta} \frac{1}{1-\mu} \| D_2(-A)^{\beta} F(s,u_s) \|_b \| \frac{u_h - \varphi}{h} - z_0 \|_{\mathcal{B}} \\ &+ \tilde{M} K_b \frac{1}{1-\mu} \int_0^t \| D_2G(\theta,u_\theta) \|_b \| \partial_h u(\theta) - z(\theta) \|_{\theta \in [0,s]}\, ds \\ &+ b\tilde{M} M_b \frac{1}{1-\mu} \| D_2G(\theta,u_\theta) \|_b \| (\frac{u_h - \varphi}{h} - z_0) \|_{[0,b]} \ . \end{aligned}$$

Clearly $\partial_t u(\cdot) = z(\cdot)$ if $h^{-1}(u_h - \varphi) - z_0 \to 0$ as $h \to 0$ and this convergence is consequence of the estimates in the proof of Corollary (2.1), see (53)-(59); we will omit this part of the proof. Finally from lemma 2.1, we conclude that $x(t) \in D(A)$ for $t \in [0,T)$. The proof of the Theorem is complete.

Remark 2. 3. We consider important to remark that, in general, the different technical assumptions used in this work are introduced in the papers [9] and [11]. For this reason we chose omit details.

Example 2.1. We refer to the Example (4.1) in [12]. The functions $F = \Lambda_1$ and Λ_2 are linear bounded operators. If

$$\int_0^{\pi} \int_{-\infty}^0 \int_0^{\pi} \frac{1}{g(\theta)} (\frac{\partial^2}{\partial \xi^2} b(\theta,\eta,\xi))^2 d\eta d\theta d\xi < \infty$$

then $\Lambda_1 : \mathcal{B} \to D(A)$ is continuous, moreover a simple calculation shows that for

$$\int_0^\pi \int_{-\infty}^0 \int_0^\pi \frac{1}{g(\theta)} (\frac{\partial}{\partial t} b(\theta, \eta, \xi))^2 d\eta d\theta d\xi$$

sufficiently small, $\|(-A)^{-\gamma}\| \|(-A)^\gamma \Lambda_1\| K(0) < 1$ for every $0 < \gamma \leq 1$. The existence of N-classical solutions follows from theorem (2.2) for appropriate φ and α, β, in particular for $\varphi \in D(A_W)$ and $1 - \beta < \beta - \alpha$ (or $1 + \alpha < 2\beta$).

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