CADERNOS DE MATEMÁTICA **02**, 165–175 March (2001)
ARTIGO NÚMERO SMA# 105

On the stability in terms of two measures of perturbed neutral functional differential equations

M.A. Bená*

Departamento de Física e Matemática da Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto- USP
Universidade de São Paulo - 14040-901 - Ribeirão Preto, SP, Brasil
E-mail: mabena@ffclrp.usp.br

S.M.S. Godoy

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo-Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: smsgodoy@icmc.sc.usp.br

The aim of this paper is to investigate the stability in terms of two measures in a perturbed neutral linear system. To address this issue we will discuss under what conditions, stability in measure of an unperturbed neutral linear system implies similar stability of the perturbed system. March, 2001 ICMC-USP

AMS-Subject Classification: 34K20, 34K40

Key words and phrases: neutral functional differential equation, stability in measure, perturbed linear systems, Lyapunov functional.

1. INTRODUCTION

Different concepts of stability are presented in the literature. To recall a few, we name the usual stability, partial stability and the conditional stability. The overall concepts of stability can be put forward in the notion of stability in terms of two different measures. Following Movchan [12], many authors have successfully developed the theory of stability in terms of two measures, which became important in the investigation of converse theorems. See [1], [3], [8], [9], [10], [13].

* Supported by FAPESP, proc. 2000/04365-3.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

2. PRELIMINARIES

Let $\mathbb{R}^n$ be a real or complex n-dimensional linear vector space with norm $|\cdot|$. For $r \geq 0$, let $C = C([-r, 0], \mathbb{R}^n)$ be the space of continuous functions taking $[-r, 0]$ into $\mathbb{R}^n$ with $\|\phi\|, \phi \in C$, defined by $||\phi|| = \sup\{\,|\phi(\theta)|,\ \theta \in [-r, 0]\}$. If Ω is an open subset of $\mathbb{R} \times C$, $g, f : \Omega \to \mathbb{R}^n$ are continuous functions, we define the operator $D : \Omega \to \mathbb{R}^n$ by

$$D(t, \phi) = \phi(0) - g(t, \phi).$$

A functional differential equation is a system of the form

$$(d/dt)D(t, x_t) = f(t, x_t)\,, \tag{1}$$

where $x_t \in C$ is defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$.

For any $\phi \in C$, $t_0 \in \mathbb{R}$, $(t_0, \phi) \in \Omega$, a function $x = x(t_0, \phi)$ defined on $[t_0 - r, t_0 + A], A > 0$, is said to be a solution of (1) on $(t_0, t_0 + A)$ with initial value ϕ at t_0 if x is continuous on $[t_0 - r, t_0 + A)$, $x_{t_0} = \phi$, $D(t, x_t)$ is continuously differentiable on $(t_0, t_0 + A)$ and relation (1) is satisfied on $(t_0, t_0 + A)$.

The initial value problem

$$\begin{aligned} &(d/dt)D(t, x_t) = f(t, x_t) \\ &x_{t_0} = \phi \end{aligned} \tag{2}$$

is equivalent to the integral equation $D(t, x_t) = D(t_0, \phi) + \int_{t_0}^{t} f(s, x_s)\, ds,\ \ t \geq t_0$.

For $g = 0, D(t, \phi) = \phi(0)$, equation (1) is the standard functional differential equation of retarded type.

System (1) is said to be a functional differential equation of neutral type if $g(t, \phi)$ is non-atomic at zero.

In this paper, it is always assumed that $g(t, \phi)$ is linear in ϕ and there are an $n \times n$ matrix $\mu(t, \theta)$, $t \geq t_0$, $\theta \in [-r, 0]$, of bounded variation on θ and a scalar function $l(s)$ continuous, nondecreasing for $s \in [0, r]$, $l(0) = 0$, such that

$$\begin{aligned} &g(t, \phi) = \int_{-r}^{0} [d_\theta \mu(t, \theta)]\, \phi(\theta), \\ &|\int_{-s}^{0} [d_\theta \mu(t, \theta)]\, \phi(\theta)| \leq l(s) \sup_{-s \leq \theta \leq 0} |\phi(\theta)|. \end{aligned} \tag{3}$$

Furthermore, f takes bounded sets of $\mathbb{R} \times C$ into bounded sets, $f(t, 0) = 0, D(t, 0) = 0,\ t \in \mathbb{R}_+$ and f, g and D satisfy enough additional properties to ensure the existence, uniqueness, continuation and continuous dependence of the solution $x(t_0, \phi)$ of (2) on the initial data.

More information about neutral functional differential equations may be found in [2], [5], [6], [7], [11].

Definition 2.1. The operator D is said to be uniformly stable if there are positive constants K and M such that for any $(t_0, \phi) \in \mathbb{R}_+ \times C$, $H \in C(\mathbb{R}_+, \mathbb{R}^n)$, the solution $x(t_0, \phi, H)$ of $D(t, x_t) = D(t_0, \phi) + [H(t) - H(t_0)]$, $t \geq t_0$, $x_{t_0} = \phi$ satisfies the condition: $\|x_t(t_0, \phi, H)\| \leq K\|\phi\| + M \sup_{t_0 \leq u \leq t} |H(u) - H(t_0)|,\ \ t \geq t_0$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

3. STABILITY CONCEPTS IN TERMS OF TWO MEASURES

DEFINITION 3.1. A continuous function $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is called a measure if $h(t, 0) = 0$.

Consider the following classes of functions:
$\mathfrak{K} = \{a \in C(\mathbb{R}_+, \mathbb{R}_+) : a \text{ is strictly increasing and } a(0) = 0\}$,
$\mathfrak{CK} = \{a \in C(\mathbb{R}_+^2, \mathbb{R}_+) : a(t, .) \in \mathfrak{K} \text{ for each } t\}$.

DEFINITION 3.2. Let h_0 and h be two measures and $x(t_0, \phi)(t)$ a solution of (1). We say that
(i) h_0 is finer than h if there exist a $\rho > 0$ and a function $b \in \mathfrak{CK}$ such that if $h_0(t, x(t_0, \phi)(t)) < \rho$, then $h(t, x(t_0, \phi)(t)) \le b(t, h_0(t, x(t_0, \phi)(t)))$;
(ii) h_0 is uniformly finer than h if in (i) b is independent of t.

Let us now establish the stability concepts for equation (1) in terms of two measures h_0 and h.

DEFINITION 3.3. Equation (1) is said to be

(I) (h_0, h) - equistable, if for each $\varepsilon > 0$ and $t_0 \ge 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that $h_0(t_0 + \theta, \phi(\theta)) < \delta$, for $\theta \in [-r, 0]$, implies $h(t, x(t_0, \phi)(t)) < \varepsilon$, $t \ge t_0$;

(II) (h_0, h) - uniformly stable, if (I) holds with δ being independent of t_0;

(III) (h_0, h) - equiasymptotically stable, if given $t_0 \ge 0$, there exists a $\delta_0 = \delta_0(t_0)$, and for each $\varepsilon > 0$, there exists $T = T(\varepsilon, t_0)$ such that $h_0(t_0 + \theta, \phi(\theta)) < \delta_0$, for $\theta \in [-r, 0]$, implies $h(t, x(t_0, \phi)(t)) < \varepsilon$, for $t \ge t_0 + T(\varepsilon, t_0)$;

(IV) (h_0, h) - uniform equiasymptotically stable, if δ_0 and T in (III) are independent of t_0;

(V) (h_0, h) - uniform asymptotically stable, if (II) and (IV) hold;

(VI) (h_0, h) - exponentially stable if there exist positive constants α, β, ρ such that $h_0(t_0 + \theta, \phi(\theta)) < \rho$, for $\theta \in [-r, 0]$, implies $h(t, x(t_0, \phi)(t)) \le \beta e^{-\alpha(t - t_0)} h_0(t_0 + \theta, \phi(\theta))$.

We shall give a few choices of the two measures (h_0, h) to demonstrate the generality of the above definition. Definition (4) reduces to

(1) the usual stability of the null solution of equation (1) if $h(t, x(t_0, \phi)(t)) = h_0(t, x(t_0, \phi)(t)) = |\, x(t_0, \phi)(t) \,|$.

(2) partial stability of the null solution of equation (1) if $h(t, x(t_0, \phi)(t)) = |\, x(t_0, \phi)(t) \,|_s = (x_1^2(t_0, \phi)(t) + x_2^2(t_0, \phi)(t) + ... + x_s^2(t_0, \phi)(t))^{1/2}, 1 \le s < n$ and $h_0(t, x(t_0, \phi)(t)) = |\, x(t_0, \phi)(t) \,| = (x_1^2(t_0, \phi)(t) + x_2^2(t_0, \phi)(t) + ... + x_n^2(t_0, \phi)(t))^{1/2}$.

(3) the stability of an invariant set $A \subset \mathbb{R}^n$ if $h(t, x(t_0, \phi)(t)) = h_0(t, x(t_0, \phi)(t)) = d(x, A)$, d being the distance function.

4. MAIN RESULTS - STABILITY IN TERMS OF TWO MEASURES OF PERTURBED NEUTRAL LINEAR SYSTEMS

Consider the system

$$(d/dt)D(t,y_t)=f(t,y_t), \tag{4}$$

where $D(t,\phi)$ and $f(t,\phi)$ are bounded, linear in ϕ, continuous and D is uniformly stable. Consider now the perturbed system

$$(d/dt)\bar{D}(t,x_t)=\bar{f}(t,x_t). \tag{5}$$

System (5) can be transformed into the system

$$(d/dt)D(t,x_t)=f(t,x_t)+G(t,x_t)\,, \tag{6}$$

where only the function f is perturbed, $G:\mathbb{R}_+\times C\to\mathbb{R}^n$ is continuous, $G(t,0)=0$ and $G(t,x_t)=\bar{f}(t,x_t)-f(t,x_t)+\dfrac{d}{dt}[D(t,x_t)-\bar{D}(t,x_t)]$.

To study the properties of stability in terms of two measures of system (6), we will need the following results:

LEMMA 4.1. *Let $x(t_0,\phi)(t)$ be a solution of (6) and $y(t_0,\phi)(t)$ a solution of (4). Suppose $f(t,\phi)$ locally Lipschitzian in ϕ with constant $L>0$, uniformly with respect to t and h is a measure satisfying $|\,h(t,x)-h(t,y)\,|\le M|\,x-y\,|$; $M>0$, $(t,x),(t,y)\in\mathbb{R}_+\times\mathbb{R}^n$. Then the following inequality is valid,*

$$h(t,x(t_0,\phi)(t)-y(t_0,\phi)(t))\le\ [e^{L(t-t_0)}-1]\,|\,G(t,x_t)\,|,\ \ t\ge t_0. \tag{7}$$

Proof: The inequality (7) is verified for $t=t_0$. Let t_1 be the supremum of the $t's$ for which inequality (7) is satisfied. From the continuity of both sides of (7) there exists a sequence $t_n\,,t_n\to 0$ for $n\to\infty$, t_n small enough such that

$h(t_1+t_n,x(t_0,\phi)(t_1+t_n)-y(t_0,\phi)(t_1+t_n))>[e^{L(t_1+t_n-t_0)}-1]\,|G(t_1+t_n,x_{t_1+t_n})\,|$.

Since h is Lipschitz continuous and $h(t,0)=0$, it follows that

$h(t_1+t_n,x(t_0,\phi)(t_1+t_n)-y(t_0,\phi)(t_1+t_n))\le M|\,x(t_0,\phi)(t_1+t_n)-y(t_0,\phi)(t_1+t_n)\,|\le M\|x_{t_1+t_n}(t_0,\phi)-y_{t_1+t_n}(t_0,\phi)\|$.

From [2], there exist a k_0 small enough and a function $l(s)$ as in (3) such that if $0\le t_n\le k_0$,

$\|x_{t_1+t_n}(t_0,\phi)-y_{t_1+t_n}(t_0,\phi)\|\le\dfrac{1}{1-l(k_0)}\displaystyle\int_{t_1}^{t_1+t_n}|\,G(s,x_s)\,|\,ds$, with $l(k_0)<1$.

Then we can write

$h(t_1+t_n,x(t_0,\phi)(t_1+t_n)-y(t_0,\phi)(t_1+t_n))\le\dfrac{M}{1-l(k_0)}\displaystyle\int_{t_1}^{t_1+t_n}|\,G(s,x_s)\,|\,ds$.

Thus, $\dfrac{M}{1-l(k_0)}\displaystyle\int_{t_1}^{t_1+t_n}|\,G(s,x_s)\,|\,ds>[e^{L(t_1+t_n-t_0)}-1]\,|\,G(t_1+t_n,x_{t_1+t_n})\,|$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Hence $\dfrac{M}{1-l(k_0)}\, t_n \sup_{t_1 \le s \le t_1+t_n} |\,G(s,x_s)\,| \;\ge\; [e^{L(t_1+t_n-t_0)}-1]\,|\,G(t_1+t_n, x_{t_1+t_n})\,|$ and $\limsup_{t_n \to 0^+} \dfrac{M}{1-l(k_0)} \sup_{t_1 \le s \le t_1+t_n} |\,G(s,x_s)\,| \ge \limsup_{t_n \to 0^+} \dfrac{[e^{L(t_1+t_n-t_0)}-1]}{t_n} |G(t_1 + t_n, x_{t_1+t_n})|$.

Then $\dfrac{M}{1-l(k_0)} |\,G(t_1,x_{t_1})\,| \;\ge\; L\,e^{(t_1-t_0)}\,|\,G(t_1,x_{t_1})\,|$ and this is a contradiction if we take $L > \dfrac{M}{1-l(k_o)}$. Therefore inequality (7) is valid for all $t \ge t_0$.

Definition 4.1. If $V : \mathbb{R}_+ \times C \to \mathbb{R}_+$ is continuous and $x(t,\phi)$ is the solution of (1) through (t,ϕ) we define

$$\dot{V}(t,\phi) = \overline{\lim_{k \to 0^+}} \frac{1}{k}\,[V(t+k, x_{t+k}(t,\phi)) - V(t,\phi)].$$

Theorem 4.1. *Let us suppose that for $\phi \in C([-r,0],\mathbb{R}^n)$, $h_0(t_0+\theta,\phi(\theta)) \le H$, there exists a functional $V(t,\phi)$ defined for $t_0 \ge 0$ with the following properties:*

(i) There exist functions $a(\tau)$, $b(\tau)$, $c(\tau)$, continuous, positive and monotone increasing for $\tau > 0$, $a(0) = b(0) = c(0) = 0$, such that $a(h(t,x(t_0,\phi)(t))) \le V(t,\phi) \le b(h_0(t_0 + \theta,\phi(\theta)))$.

(ii) $\overline{\lim}_{k\to 0^+} \dfrac{V(t+k,x_{t+k}(t,\phi)) - V(t,\phi)}{k} \le -c\,(h_0(t_0+\theta,\phi(\theta)))$.

Then equation (1) is (h_0,h) - uniform asymptotically stable.

Proof: We must show that (1) is (h_0,h) - uniformly stable and (h_0,h) - uniform equiasymptotically stable.
Let $\delta(\epsilon) < b^{-1}(a(\epsilon))$, $h_0(t_0+\theta,\phi(\theta)) < \delta(\epsilon)$, $V^*(t) = V(t,x_t(t_0,\phi))$. We have

$$\begin{aligned}\overline{\lim}_{k\to 0^+} \frac{V^*(t+k)-V^*(t)}{k} &= \overline{\lim}_{k\to 0^+} \frac{V(t+k,x_{t+k}(t_0,\phi)) - V(t,x_t(t_0,\phi))}{k}\\ &\le -c(h_0(t_0+\theta,x_t(t_0,\phi)(\theta))) < 0\end{aligned}$$

if $h_0(t_0+\theta,x_t(t_0,\phi)(\theta)) \ne 0$. It follows that $V^*(t)$ is monotone decreasing. Thus,

$$V^*(t) < V^*(t_0) = V(t_0,x_{t_0}(t_0,\phi)) = V(t_0,\phi) \le b(h_0(t_0+\theta,\phi(\theta))) < b(\delta(\epsilon)) < a(\epsilon).$$

By hypothesis (i), $a(h(t,x(t_0,\phi)(t))) < a(\epsilon)$. From the monotonicity of the function a, we obtain $h(t,x(t_0,\phi)(t)) < \epsilon$, for $t \ge t_0$. Hence we proved equation (1) is (h_0,h) - uniformly stable. Since $V^*(t)$ is monotone decreasing, it is differentiable almost everywhere and thus (ii) is verified almost everywhere by the derivative of V^*.
Let $\delta_0 = \delta(H)$, $T(\epsilon) = b(\delta_0)/c(\delta(\epsilon))$, $h_0(t_0+\theta,\phi(\theta)) < \delta_0$.
If for $t \in [t_0, t_0+T]$ we would have $h_0(t_0+\theta,x_t(t_0,\phi)(\theta)) \ne \delta(\epsilon)$, it would follow that $c(h_0(t_0+\theta,x_t(t_0,\phi)(\theta))) \ge c(\delta(\epsilon))$. Hence

$\overline{\lim}_{k\to 0^+}\ \dfrac{V^*(t+k)-V^*(t)}{k}\ \leq\ -c(h_0(t_0+\theta, x_t(t_0,\phi)(\theta)))\ \geq -c(\delta(\epsilon))$, that implies $V^*(t)-V^*(t_0)\ \leq\ -c(\delta(\epsilon))\,(t-t_0)$. Therefore $V^*(t)\ \leq\ V^*(t_0)\ -\ c(\delta(\epsilon))\,(t-t_0)\ \leq b\,(h_0(t_0+\theta,\phi(\theta)))\ -\ c(\delta(\epsilon))\,(t-t_0)$ $\leq\ b(\delta_0)-c(\delta(\epsilon))\,(t-t_0)$. From this follows that $V^*(t_0+T)\ <\ b(\delta_0)-c(\delta(\epsilon))\,T\ =0$, which is a contradiction. So, there exists $t'\in[t_0,t_0+T]$ such that $h_0(t_0+\theta,x_{t'}(t_0,\phi)(\theta))\ <\ \delta(\epsilon)$. Hence, $h(t,x(t',x_{t'}\,(t_0,\phi)(t)))\ <\epsilon$, for $t\geq t'$. This means that for $t\geq t_0+T$ we have $h(t,x_t(t_0,\phi)(t))\ <\ \epsilon$ and the theorem is proved.

LEMMA 4.2. *Let h_0 and h be measures satisfying $h(t,x+y)\ \leq\ h(t,x)+h(t,y)$, $h(t,p\,x)=|\,p\,|\,h(t,x), p\in\mathbb{R}$. If equation (4) is (h_0,h) - uniform asymptotically stable, then it is (h_0,h) - exponentially stable, that is, there exist positive constants α, β, ρ such that if $h_0(t_0+\theta,\,\phi(\theta))<\rho$, for $\theta\in[-r,0]$, then $h(t,x(t_0,\phi)(t))\ \leq\ \beta e^{-\alpha(t-t_0)}\,h_0(t_0+\theta,\,\phi(\theta))$.*

Proof: We define the operator U_{t,t_0} by the relation $U_{t,t_0}\phi=y(t_0,\phi)(s),\ t-r\leq s\leq t$, where $y(t_0,\phi)(t)$ is the solution of (4). The operator maps the space of continuous functions defined in $[t_0-r,t_0]$ in the space of the continuous functions in $[t-r,t]$. The linearity of equation (4) implies the operator is linear and it is also continuous. See [4]. Consider the quantity $\|\,U_{t,t_0}\,\|$ with the property

$$\|\,U_{t,t_0}\,\|=\sup_{h_0(t_0+\theta,\phi(\theta))\,\leq 1} h(t,U_{t,t_0}\phi),\ t\geq t_0.$$

By (h_0,h) - uniform stability, for $h_0(t_0+\theta,\phi(\theta))\leq\delta_0,\ t\geq t_0+T,$ we have $h(t,y(t_0,\phi)(t))<\epsilon$; hence for $t\geq t_0+T+r$ it follows that $h(t,U_{t,t_0}\phi)<\epsilon$. We fix ϵ with $0<\epsilon<1$. Let ϕ_0 arbitrary with $h_0(t_0+\theta,\phi_0(\theta))\leq 1$. By hypothesis on h we can write $h_0(t_0+\theta,\,\delta_0\,\phi_0(\theta))\,=\,\delta_0\,h_0(t_0+\theta,\phi_0(\theta))\,\leq\,\delta_0$. Hence $h(t,U_{t.t_0}\delta_0\,\phi_0)\,<\,\epsilon$, for $t\,\geq\,t_0+T+r$ and $h(t,U_{t,t_0}\phi_0)\,<\,\epsilon/\delta_0$. Since ϕ_0 is arbitrary with $h_0(t_0+\theta,\phi_0(\theta))\leq 1$, we will have $\|\,U_{t,t_0}\,\|\ \leq\ \epsilon/\delta_0$ for $t\geq t_0+T+r$, or, $\|\,U_{t,t_0}\,\|\leq\epsilon$ for $t\geq t_0+T_1+r$, where we set $T_1(\epsilon)=T(\delta_0\,\epsilon)$. We have the relation $U_{t,t_0}\phi=U_{t,t_0+T_1+r}\,(U_{t_0+T_1+r,t_0}\,\phi)$ which amounts to $y(t_0,\phi)(s)=y(t_0+T_1+r,y(t_0,\phi)(u))(s),\ u\in[t_0+T_1,t_0+T_1+r]$. From here follows, $\|\,U_{t,t_0}\,\|\ \leq\ \|\,U_{t,t_0+T_1+r}\,\|\,.\,\|\,U_{t_0+T_1+r,t_0}\,\|\ \leq\epsilon^2$ for $t\,\geq\,t_0+2(T_1+r)$. By induction it follows that $\|\,U_{t,t_0}\,\|\ \leq\ \epsilon^m$, for $t\,\geq\,t_0+m(T_1+r)$. Now let $t\,\geq\,t_0$, arbitrary. There exists an $m\geq 1$ such that $t_0+(m-1)(T_1+r)\leq t<t_0+m(T_1+r)$. Furthermore, $m(T_1+r)>t-t_0,\ \ m>(1/(T_1+r))\,(t-t_0)$, $\epsilon^m<\epsilon^{(1/(T_1+r))(t-t_0)}$, since $\epsilon<1$.

On the other hand, there exists a $\delta(\epsilon)$ such that $h(t,y(t_0,\phi)(t))<\epsilon$ if $h_0(t_0+\theta,\phi(\theta))\leq\delta(\epsilon)$ and $t\geq t_0+T(\epsilon)$. As above, we will have $h(t,\delta(\epsilon)\,U_{t,t_0}\phi)<\epsilon,\ t\geq t_0+T(\epsilon)$. Hence $h(t,U_{t,t_0}\,\phi)\ <\ \dfrac{\epsilon}{\delta(\epsilon)},t\geq t_0+T(\epsilon)$. Since $t\geq t_0+(m-1)(T_1+r)$, we will have $\|\,U_{t,t_0}\|<[\epsilon/\delta(\epsilon)]\,\epsilon^{m-1}$. Indeed, if $m\geq 2$ this inequality follows from the fact that $\delta(\epsilon)\leq\epsilon$, and if $m=1$ from the fact that $\|\,U_{t,t_0}\,\|<\epsilon/\delta(\epsilon)$, for $t\geq t_0+T(\epsilon)$. Finally, we have shown that for every $t\geq t_0+T(\epsilon)$, $\|\,U_{t,t_0}\,\|\ \leq[1/\delta(\epsilon)]\epsilon^m$ and thus $\|\,U_{t,t_0}\,\|<[1/\delta(\epsilon)]\epsilon^{(1/(T_1+r))(t-t_0)}$.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Let us $1/\delta(\epsilon) = \beta, \alpha = -(1/(T_1 + r))\ln\epsilon$, and the previous estimate becomes $||U_{t,t_0}|| < \beta e^{-\alpha(t-t_0)}$.Let ψ be arbitrary, $\psi \neq 0$. We observe that

$$h_0(t_0 + \theta, \frac{\psi(\theta)}{h_0(t_0 + \theta, \psi(\theta))}) = [1/h_0(t_0 + \theta, \psi(\theta))].h_0(t_0 + \theta, \psi(\theta)) = 1.$$

Then we have $h(t, U_{t,t_0}\psi) = h(t, U_{t,t_0}[\psi h_0(t_0+\theta, \psi(\theta))/h_0(t_0+\theta, \psi(\theta))]) = h_0(t_0+\theta, \psi(\theta))$ $h(t, U_{t,t_0}\ [\psi/h_0(t_0+\theta, \psi(\theta))]) < h_0(t_0+\theta, \psi(\theta))||U_{t,t_0}|| < \beta e^{-\alpha(t-t_0)}h_0(t_0+\theta, \psi(\theta))$. Thus, equation (4) is (h_0, h)-exponentially stable.

In what follows, $y(t, \phi)$ is understood as any solution of equation (4) with the initial condition ϕ at time t.

Theorem 4.2. *Assume h and h_0 two measures satisfying the hypotheses of Lemma (2) and $\mid h_0(t,x) - h_0(t,y) \mid \leq h_0(t, x-y)$; g and l as in (3).*
Then, if equation (4) is (h_0, h) - uniform asymptotically stable, there is a continuous scalar functional $V(t,\phi)$ for $(t,\phi) \in [t_0,\infty) \times C$, such that
(a) $(h_0(t+\theta, \phi(\theta)))^2 \leq V(t,\phi) \leq k_1\ (h_0(t+\theta, \phi(\theta)))^2$
(b) $\mid V(t,\phi_1) - V(t,\phi_2) \mid \leq k_2\ [h_0(t+\theta, \phi_1(\theta)) + h_0(t+\theta, \phi_2(\theta))].[h_0(t+\theta, \phi_1(\theta) - \phi_2(\theta))]$
(c) $\dot{V}(t,\phi) \leq -(h(t+\theta,\ y_t(t_0,\phi)(\theta)))^2$.

Proof: (a) Define

$$V(t,\phi) = \int_0^\infty [h(t+s+\theta,\ y_{t+s}(t,\phi)(\theta))]^2\, ds\ + \sup_{s\geq 0} [h(t+s+\theta,\ y_{t+s}(t,\phi)(\theta))]^2.$$

Then $V(t,\phi) \geq \sup_{s\geq 0} [h(t+s+\theta,\ y_{t+s}(t,\phi)(\theta))]^2$.
As $h(t+\theta,\ y_t(t,\phi)(\theta)) = h_0(t+\theta,\ \phi(\theta))$, it follows that $V(t,\phi) \geq (h_0(t+\theta, \phi(\theta)))^2$.
Since equation (4) is (h_0, h) - uniform asymptotically stable, by Lemma(2) it is (h_0, h) - exponentially stable, that ensures the convergence of the integral in the definition of $V(t,\phi)$.
This implies the existence of positive constants α, β, ρ such that $h_0(t+\theta, \phi(\theta)) \leq \rho$, then $h(t+s+\theta, y(t,\phi)(t+s+\theta)) \leq \beta\, e^{-\alpha(t+s+\theta-t)}\, h_0(t+\theta, \phi(\theta))$
$= \beta e^{-\alpha\, s}\, e^{-\alpha\theta}\, h_0(t+\theta, \phi(\theta)) \leq \beta\, e^{-\alpha\, s}\, e^{\alpha\, r}\, h_0(t+\theta, \phi(\theta))$.
Thus, $(h(t+s+\theta, y(t,\phi)(t+s+\theta)))^2 \leq \beta^2\, e^{-2\alpha\, s}\, e^{2\alpha\, r}\, (h_0(t+\theta, \phi(\theta)))^2$.
It follows that
$\sup_{s\geq 0} (h(t+s+\theta, y(t,\phi)(t+s+\theta)))^2 \leq \beta^2\, e^{-2\alpha\, s}\, e^{2\alpha\, r}\, (h_0(t+\theta, \phi(\theta)))^2$. So

$$\begin{aligned} V(t,\phi) &\leq \textstyle\int_0^\infty \beta^2 e^{-2\alpha s} e^{2\alpha\, r} (h_0(t+\theta, \phi(\theta)))^2\, ds + \beta^2 e^{2\alpha r}\, (h_0(t+\theta, \phi(\theta)))^2 \\ &= \beta^2 e^{2\alpha r} (h_0(t+\theta, \phi(\theta)))^2 \textstyle\int_0^\infty e^{-2\alpha s} ds + \beta^2 e^{2\alpha r} (h_0(t+\theta, \phi(\theta)))^2 \\ &= \beta^2 e^{2\alpha r} (h_0(t+\theta, \phi(\theta)))^2 \frac{1}{2\,\alpha} + \beta^2 e^{2\alpha r} (h_0(t+\theta, \phi(\theta)))^2 \\ &= \beta^2 e^{2\alpha r} (\frac{1}{2\alpha} + 1)(h_0(t+\theta, \phi(\theta)))^2 = k_1 (h_0(t+\theta, \phi(\theta)))^2. \end{aligned}$$

Hence, $V(t,\phi) \leq k_1 (h_0(t+\theta, \phi(\theta)))^2$. As $V(t,\phi) \geq (h_0(t+\theta, \phi(\theta)))^2$, it follows that $(h_0(t+\theta, \phi(\theta)))^2 \leq V(t,\phi) \leq k_1 (h_0(t+\theta, \phi(\theta)))^2$ and (a) is proved.

(b) $|V(t,\phi_1) - V(t,\phi_2)|$
$\leq \mid \int_0^\infty (h(t+s+\theta, y(t,\phi_1)(t+s+\theta)))^2\, ds - \int_0^\infty (h(t+s+\theta, y(t,\phi_2)(t+s+\theta)))^2\, ds| +$

$|\sup_{s\geq 0}(h(t+s+\theta,y(t,\phi_1)(t+s+\theta)))^2 - \sup_{s\geq 0}(h(t+s+\theta,y(t,\phi_2)(t+s+\theta)))^2|$
$\leq \int_0^\infty |(h(t+s+\theta,y(t,\phi_1)(t+s+\theta)))^2\,ds - (h(t+s+\theta,y(t,\phi_2)(t+s+\theta)))^2|\,ds +$
$\sup_{s\geq 0}|(h(t+s+\theta,y(t,\phi_1)(t+s+\theta)))^2 - (h(t+s+\theta,y(t,\phi_2)(t+s+\theta)))^2|$
$\leq \int_0^\infty |h(t+s+\theta,y(t,\phi_1)(t+s+\theta)) + h(t+s+\theta,y(t,\phi_2)(t+s+\theta))|\times$
$|h(t+s+\theta,y(t,\phi_1)(t+s+\theta)) - h(t+s+\theta,y(t,\phi_2)(t+s+\theta))|\,ds +$
$\sup_{s\geq 0}|h(t+s+\theta,y(t,\phi_1)(t+s+\theta)) + h(t+s+\theta,y(t,\phi_2)(t+s+\theta))|\times$
$\sup_{s\geq 0}|h(t+s+\theta,y(t,\phi_1)(t+s+\theta)) - h(t+s+\theta,y(t,\phi_2)(t+s+\theta))|$
$\leq \int_0^\infty[\beta\,e^{-\alpha s}\,e^{\alpha r}\,h_0(t+\theta,\phi_1(\theta)) + \beta\,e^{-\alpha s}\,e^{\alpha r}\,h_0(t+\theta,\phi_2(\theta))]\times$
$[\beta\,e^{-\alpha s}\,e^{\alpha r}\,h_0(t+\theta,\phi_1(\theta)) - \beta\,e^{-\alpha s}\,e^{\alpha r}\,h_0(t+\theta,\phi_2(\theta))]\,ds +$
$\beta\,e^{\alpha r}\,[h_0(t+\theta,\phi_1(\theta)) + h_0(t+\theta,\phi_2(\theta))]\,.\,\beta\,e^{\alpha r}\,[h_0(t+\theta,\phi_1(\theta)) - h_0(t+\theta,\phi_2(\theta))].$

The hypothesis of the measure h_0 allows us write
$|V(t,\phi_1) - V(t,\phi_2)|$
$\leq \int_0^\infty \beta^2\,e^{-2\alpha\,s}e^{2\alpha r}\,[h_0(t+\theta,\phi_1(\theta)) + h_0(t+\theta,\phi_2(\theta))].[h_0(t+\theta,\phi_1(\theta)-\phi_2(\theta))]\,ds]$
$+\beta\,e^{\alpha\,r}\,[h_0(t+\theta,\phi_1(\theta)) + h_0(t+\theta,\phi_2(\theta))]\,.\,\beta\,e^{\alpha\,r}\,[h_0(t+\theta,\phi_1(\theta)-\phi_2(\theta))]$
$=\beta^2\,e^{2\alpha\,r}\,[h_0(t+\theta,\phi_1(\theta)) + h_0(t+\theta,\phi_2(\theta))].[h_0(t+\theta,\phi_1(\theta)-\phi_2(\theta))].\int_0^\infty e^{-2\alpha\,s}ds$
$+\beta^2\,e^{2\alpha\,r}\,[h_0(t+\theta,\phi_1(\theta)) + h_0(t+\theta,\phi_2(\theta))].[h_0(t+\theta,\phi_1(\theta)-\phi_2(\theta))]$
$=\beta^2\,e^{2\alpha\,r}\,(1+\dfrac{1}{2\alpha})\,[h_0(t+\theta,\phi_1(\theta)) + h_0(t+\theta,\phi_2(\theta))].[h_0(t+\theta,\phi_1(\theta)-\phi_2(\theta))]$
Thus,
$|V(t,\phi_1) - V(t,\phi_2)|$
$\leq \beta^2\,e^{2\alpha r}(1+\dfrac{1}{2\alpha})[h_0(t+\theta,\phi_1(\theta)) + h_0(t+\theta,\phi_2(\theta))].[h_0(t+\theta,\phi_1(\theta)-\phi_2(\theta))]$
$=k_2[h_0(t+\theta,\phi_1(\theta)) + h_0(t+\theta,\phi_2(\theta))].[h_0(t+\theta,\phi_1(\theta)-\phi_2(\theta))]$
and (b) is proved.

(c) $V(t,y_t(t_0,\phi)) = \int_0^\infty (h(t+s+\theta,y_{t+s}\,(t,y_t(t_0,\phi))(\theta)))^2\,ds +$
$\sup_{s\geq 0}(h(t+s+\theta,y_{t+s}\,(t,y_t(t_0,\phi))(\theta)))^2$
$= \int_0^\infty (h(t+s+\theta,y_{t+s}\,(t_0,\phi)(\theta)))^2\,ds + \sup_{s\geq 0}(h(t+s+\theta,y_{t+s}\,(t_0,\phi)(\theta)))^2$
$= \int_t^\infty (h(u+\theta,\,y_u(t_0,\phi)(\theta))^2\,du + \sup_{s\geq 0}(h(t+s+\theta,y_{t+s}(t_0,\phi)(\theta)))^2$
$\dot{V}(t,y_t(t_0,\phi)) = \overline{\lim}_{k\to 0^+}\,\dfrac{1}{k}\,[\int_{t+k}^\infty\,(h(u+\theta,y_u(t_0,\phi)(\theta))^2\,du +$
$\sup_{s\geq 0}(h(t+k+s+\theta,y_{t+k+s}\,(t_0,\phi))(\theta))^2 - \int_t^\infty (h(u+\theta,y_u(t_0,\phi)(\theta)))^2\,du -$
$\sup_{s\geq 0}(h(t+s+\theta,y_{t+s}(t_0,\phi)(\theta)))^2].$

Since $\sup_{s\geq 0}(h(t+s+\theta,y_{t+s}(t_0,\phi)(\theta)))^2$ is a decreasing function, it follows that
$\dot{V}(t,y_t(t_0,\phi)) \leq \overline{\lim}_{k\to 0^+}\,\dfrac{1}{k}\,[\int_{t+k}^\infty\,(h(u+\theta,y_u(t_0,\phi)(\theta)))^2\,du$
$-\int_t^\infty (h(u+\theta,y_u(t_0,\phi)(\theta)))^2\,du] = \dfrac{d}{dt}\int_t^\infty (h(u+\theta,y_u(t_0,\phi)(\theta)))^2\,du$
$= -(h(t+\theta,y_t(t_0,\phi)(\theta)))^2$ and the theorem is proved.

THEOREM 4.3. *Let h_0 and h be two measures with h_0 satisfying $|h_0(t,x) - h_0(t,y)| \leq h_0(t,x-y)$, f and h are as in Lemma (1), h is uniformly finer than h_0. Suppose D and f satisfying the hypotheses of equation (4) and assume that*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

$$| \frac{d}{dt} D(t,\phi) - \frac{d}{dt} \bar{D}(t,\phi) \,| + | \,\bar{f}(t,\phi) - f(t,\phi) \,| \leq \gamma \, h(t, x(t_0,\phi)(t)), \tag{8}$$

where γ is chosen small enough and $(t,x) \in S(h,\rho)$ for some $\rho > 0$;
$S(h,\rho) = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : h(t,x) < \rho\}$. If equation (4) is (h_0,h) - uniform asymptotically stable then equation (6) is (h_0,h) - uniform asymptotically stable.

Proof: From (8), $|\,G(t,x_t)\,| \leq \gamma\, h(t,x(t_0,\phi)(t))$. Let $V(t,\phi)$ be as in Theorem (2). Define $V^*(t) = V(t, x_t(t_0,\phi))$. We have

$\overline{\lim}_{k\to 0^+} \dfrac{V^*(t+k) - V^*(t)}{k} = \overline{\lim}_{k\to 0^+} \dfrac{V(t+k, x_{t+k}(t_0,\phi) - V(t, x_t(t_0,\phi))}{k}$

$\leq \overline{\lim}_{k\to 0^+} \dfrac{V(t+k, y_{t+k}(t, x_t(t_0,\phi)) - V(t, x_t(t_0,\phi))}{k} +$

$\overline{\lim}_{k\to 0^+} \dfrac{V(t+k, x_{t+k}(t, x_t(t_0,\phi))) - V(t+k, y_{t+k}(t, x_t(t_0,\phi)))}{k}$,

where $x(t_0,\phi)(t)$ is a solution of (6) and $y(t_0,\phi)(t)$ is a solution of (4). From (c) of Theorem (2) , we have:

$\overline{\lim}_{k\to 0^+} \dfrac{V^*(t+k) - V^*(t)}{k} \leq -(h(t+\theta, x_t(t_0,\phi)(\theta)))^2 +$

$\overline{\lim}_{k\to 0^+} \dfrac{V(t+k, x_{t+k}(t, x_t(t_0,\phi))) - V(t+k, y_{t+k}(t, x_t(t_0,\phi)))}{k}$.

From part (b) of Theorem (2) we obtain:

$|\,V(t+k, x_{t+k}(t, x_t(t_0,\phi))) - V(t+k, y_{t+k}(t, x_t(t_0,\phi)))\,|$

$\leq k_2\,[h_0(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) + h_0(t+k+\theta, y_{t+k}(t, x_t(t_0,\phi)(\theta))] \times$

$[h_0(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta)) - y_{t+k}(t, x_t(t_0,\phi)(\theta)))]$.

From Lemma (1) and condition (8) we have

$$\begin{aligned} h_0\,(t+k+\theta, & x_{t+k}(t, x_t(t_0,\phi)(\theta)) - y_{t+k}(t, x_t(t_0,\phi)(\theta))) \\ & \leq [e^{L(t+k+\theta-t)} - 1].|G(t+k+\theta, x_{t+k})| \\ & \leq [e^{L(k+\theta)} - 1)]\,\gamma\, h(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) \\ & \leq [e^{Lk} - 1]\,\gamma h(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))). \end{aligned} \tag{9}$$

Now, by means of Lemma (1) and hypothesis on h_0, we can write

$h_0(t+k+\theta, y_{t+k}(t, x_t(t_0,\phi)(\theta)))$

$= |\,h_0(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) - h_0(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta)))$

$+h_0(t+k+\theta, y_{t+k}(t, x_t(t_0,\phi)(\theta)))|$

$\leq [e^{Lk} - 1]\,\gamma h(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) + h_0(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta)))$.

Thus,

$$\begin{aligned} h(t+k+\theta, y_{t+k}(t, x_t(t_0,\phi)(\theta))) \; & \leq [e^{Lk} - 1]\gamma h(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) \\ & +h_0(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))). \end{aligned} \tag{10}$$

Hence using (9)and (10) , we obtain:

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

$$\begin{aligned}
&|V(t+k, x_{t+k}(t, x_t(t_0,\phi))) - V(t+k, y_{t+k}(t, x_t(t_0,\phi)))| \\
&\leq [2k_2 h_0(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) + k_2\,[e^{Lk}-1]\gamma h(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta)))] \\
&\times [e^{Lk}-1]\gamma h(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) \\
&= 2k_2[e^{Lk}-1]\gamma h_0(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) \times h(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))) \\
&+ k_2\gamma^2(e^{Lk}-1)\,(e^{Lk}-1)(h(t+k+\theta, x_{t+k}(t, x_t(t_0,\phi)(\theta))))^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{\lim}_{k\to 0^+}\; & k^{-1}|V(t+k, x_{t+k}(t, x_t(t_0,\phi))) - V(t+k, y_{t+k}(t, x_t(t_0,\phi)))| \\
& \leq (h(t+\theta, x_t(t_0,\phi)(\theta)))^2\, 2k_2\,\gamma\, \overline{\lim}_{k\to 0^+} k^{-1}[e^{Lk}-1] \\
& + (h(t+\theta, x_t(t_0,\phi)(\theta)))^2 k_2\gamma^2\, \overline{\lim}_{k\to 0^+}\; k^{-1}[e^{Lk}-1][e^{Lk}-1] \\
& = 2k_2\gamma L(h(t+\theta, x_t(t_0,\phi)(\theta)))^2.
\end{aligned}$$

Thus if we take $\gamma < 1/(2\,k_2\,L)$, it follows that,

$$\begin{aligned}
\overline{\lim}_{k\to 0^+}\; \frac{V^*(t+k)-V^*(t)}{k} & \leq -(h(t+\theta, x_t(t_0,\phi)(\theta)))^2 + 2k_2\gamma L(h(t+\theta, x_t(t_0,\phi)(\theta)))^2 \\
& = (h(t+\theta, x_t(t_0,\phi)(\theta)))^2(2k_2\gamma L-1) = -\gamma_1(h(t+\theta, x_t(t_0,\phi)(\theta)))^2, \gamma_1 > 0.
\end{aligned}$$

Since h is uniformly finer than h_0 for $(t,x) \in S(h,\rho)$, we have
$h_0(t+\theta, x_t(t_0,\phi(\theta)) \leq b(h(t+\theta, x_t(t_0,\phi)(\theta))), b \in \mathfrak{CK}$.
This implies that $(h(t+\theta, x_t(t_0,\phi)(\theta)))^2 \geq c(h_0(t+\theta, x_t(t_0,\phi)(\theta))),\; c \in \mathfrak{CK}$.
Hence, using this inequality, we can write,

$$\overline{\lim_{k\to 0^+}}\; \frac{V^*(t+k)-V^*(t)}{k} \leq -\gamma_1 c(h_0(t+\theta, x_t(t_0,\phi)(\theta))),$$

which shows, on basis of Theorem (1) that equation (6) is (h_0, h)-uniform asymptotically stable.

REFERENCES

1. Bená, M.A., Dos Reis, J.G., Santos, J.S., Stability of neutral functional differential equations in terms of two measures, in *Dynamics of Continuous, Discrete and Impulsive Systems,* 6, 279-287 (1999).
2. Cruz, M.A., Hale, J.K., Stability of functional differential equations of neutral type, in *J.Differential Equations,* 7, 334-355 (1970).
3. Dos Reis, J.G., Bená, M.A., A stability theorem in terms of two measures, in *Nonlinear Analysis, T.M.A.,* 29(7), 737-740 (1997).
4. Halanay, A., Differential Equations: stability, oscillations, time lags, Academic Press Inc., New York (1966).
5. Hale, J.K., Forward and backward continuation for neutral functional differential equations, in *J. Differential Equations,* 9, 168-181 (1971).
6. Hale, J.K., Lunel, S.M.V., Introduction to Functional Differential Equations, Applied Math. Sciences 99, Springer-Verlag New York, Inc., New York (1993).

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

7. Izé, A.F., Dos Reis, J.G., Stability of perturbed neutral functional differential equations, in *Nonlinear Analysis, T.M.A.,* 2(5), 563-571 (1978).

8. Lakshmikantham, V., Uniform asymptotic stability criteria for functional differential equations in terms of two measures, in *Nonlinear Analysis, T.M.A.,* 34, 1-6 (1998).

9. Lakshmikantham, V., Leela, S., Martynyuk, A. A., Stability Analysis of Nonlinear Systems, Marcel Dekker, New York, (1989).

10. Lakshmikantham, V., Salvadori, L., On Massera type converse theorem in terms of two different measures, in *Bull. U.M.I.,* 13, 293-301 (1976).

11. Lopes, O., Periodic solutions of perturbed neutral differential equations, in *J. Differential Equations,* 15, 70-76 (1974).

12. Movchan, A.A., Stability of process with respect to two measures , in *Prikl.Mat.Mekh.,* 24, 988-1001 (1960).

13. Xu, D.Y., Uniform asymptotic stability in terms of two measures for functional differential equations, in *Nonlinear Analysis, T.M.A.,* 27, 413-427 (1996).

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC