

## Newton filtrations, graded algebras and codimension of non-degenerate ideals

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We investigate a generalization of the method introduced by Kouchnirenko to compute the codimension (colength) of an ideal under a certain non-degeneracy condition on a given system of generators of  $I$ . We also discuss Newton non-degenerate ideals and give characterizations using the notion of reductions and Newton polyhedra of ideals. March, 2001 ICMC-USP

### 1. INTRODUCTION

The computation of  $\dim_{\mathbb{C}} \mathcal{O}_n/I$ , the codimension of an ideal  $I$  in the ring  $\mathcal{O}_n$  of holomorphic map germs at the origin in  $\mathbb{C}^n$ , is one of the main tools used to calculate geometric invariants of singularities. In general, this calculus is done using the theory of standard basis of the ring  $\mathcal{O}_n/I$ . For weighted homogeneous map germs  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  it is known that some geometric invariants are given in terms of its weights and degrees. For instance, Milnor and Orlik gave a formula in [13] for the Milnor number of a weighted homogeneous map germ  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  with an isolated singularity at the origin. There are other invariants that are also computed for weighted homogeneous map germs

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in terms of its weights and degrees, Gaffney and Mond in [8] show how to compute the number of cusps in a generic deformation of a map germ  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  and Mond [14] shows how to compute the number of cross caps in a generic deformation of a map germ  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ .

On the other side, Kouchnirenko considered in [11] the ring  $\mathcal{F}_n$  of formal power series in the variables  $x_1, \dots, x_n$  around the origin and gave a formula, in terms of the Newton polyhedron  $\Gamma_+(f)$ , for the Milnor number of a Newton non-degenerate germ of function  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  with an isolated singularity at the origin. The key object needed to obtain his result is the Newton filtration of  $\mathcal{F}_n$  constructed from  $\Gamma_+(f)$ . Therefore, we observe that also in this case, the Milnor number of a Newton non-degenerate germ of function only depends on such a basic object like its Newton polyhedron.

In this article, we show a generalization of the method introduced by Kouchnirenko in order to formalize the above idea. In Section 3 we use the notion of Newton filtration to introduce a new non-degeneracy condition. This gives rise to a class of ideals that contains the Newton non-degenerate ideals in the sense of [17], that we call NND ideals for short, and the ideals of finite codimension generated by weighted homogeneous germs. When an ideal of this class has finite codimension and is a complete intersection, we use the Koszul complex to obtain a formula for its codimension that only depends on a suitable fixed Newton polyhedron  $\Gamma_+$  and the levels (or degrees) of a system of generators with respect to the filtration induced by  $\Gamma_+$ . This formula also characterizes complete intersection ideals which satisfy this non-degeneracy condition. Before showing this we make a study of the multiplicity of NND ideals in  $\mathcal{F}_n$ .

The definition of NND ideal, which first appeared in the work [17] of Saia, is motivated by the notion of Newton non-degeneracy of germs of functions  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ , which is used by several authors. For instance Kouchnirenko computes in [11] the Milnor number of Newton non-degenerate functions with an isolated singularity at the origin and Yoshinaga [24] gave conditions for the topological triviality of families defined by Newton non-degenerate functions. Varchenko [21] also dealt with the Newton polyhedron of this kind of functions to compute the zeta function of its monodromy. More recently Wall gave in [23] a weaker condition of non-degeneracy for germs of functions in order to characterize the Milnor number of any isolated singularity germ which satisfies this condition.

As is proved in [17], the integral closure of a NND ideal  $I$  is generated by all the monomials whose exponents belong to the Newton polyhedron of  $I$  (hence, this is a generalization of the result of Yoshinaga in [24] characterizing Newton non-degenerate functions). We shall see in Section 2 that this fact determines the expression for the multiplicity  $e(I)$  of a given NND ideal  $I \subseteq \mathcal{F}_n$  of finite codimension.

In Section 4, we reproduce Kouchnirenko's argument for modules  $M$  with free resolution:

$$0 \longrightarrow \mathcal{F}_n^{e_\ell} \xrightarrow{\varphi_\ell} \dots \xrightarrow{\varphi_2} \mathcal{F}_n^{e_1} \xrightarrow{\varphi_1} \mathcal{F}_n^{e_0} \longrightarrow M,$$

coming from a free resolution of a perfect module. From this resolution we consider induced complexes of the graded ring and the Rees ring associated to the filtration and compare the acyclicity of these complexes. In this setup, we give a definition of non-degeneracy of the matrix  $(g_{ij})$  defining the morphism  $\varphi_1 : \mathcal{F}_n^{e_1} \rightarrow \mathcal{F}_n^{e_0}$  and obtain that acyclicity of the complexes there is equivalent to non-degeneracy of the matrix  $(g_{ij})$ . We also show that if

$(g_{ij})$  is non-degenerate and if  $\text{Im } \varphi_1$  has finite codimension in  $\mathcal{F}_n^{e_0}$ , we can compute this codimension in terms of the volume of a suitable Newton polyhedron.

In this article, we shall consider the formal power series ring  $\mathcal{F}_n$ , but we assume that all map germs  $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  and all germs of functions  $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  are holomorphic. We remember that if  $I$  denotes an ideal generated by holomorphic germs of functions  $g_1, \dots, g_s$  in  $\mathcal{O}_n$  and  $\hat{I}$  denotes the ideal generated by the same  $g_1, \dots, g_s$  in  $\mathcal{F}_n$  it follows from the Artin Theorem (see [20], Theorem 4.2), that  $\dim_{\mathbb{C}} \mathcal{O}_n/I = \dim_{\mathbb{C}} \mathcal{F}_n/\hat{I}$ , if they are finite.

In the language of commutative algebra, it is proper to say that the number  $\dim_{\mathbb{C}} \mathcal{O}_n/I$  is the colength of the ideal  $I$ , to avoid confusion with Krull codimension (i.e., height). But we apologize that we sometimes prefer to say this is just codimension, since this is actually the codimension of an appropriate orbit in some geometric setup. However, we sometimes use the word ‘codimension’ as Krull codimension (height  $\text{ht}(I)$ ). In such cases, we mention this like ‘codimension (or height)’ to avoid the reader’s confusion.

## 2. THE MULTIPLICITY OF NEWTON NON-DEGENERATE IDEALS

We shall denote by  $\mathcal{A}$  the ring  $\mathcal{F}_n$  of formal power series in the variables  $x_1, \dots, x_n$  around the origin. Let  $g = \sum_k a_k x^k$  be a series in  $\mathcal{A}$ , the *support of  $g$* , denoted by  $\text{supp } g$ , is the set of points  $k \in \mathbb{Z}^n$  such that  $a_k \neq 0$ . If  $I$  is an ideal of  $\mathcal{A}$ , we define the *support of  $I$*  as  $\text{supp } I = \cup_{g \in I} \text{supp } g$ .

**DEFINITION 2.1.** The *Newton polyhedron* of an ideal  $I \subseteq \mathcal{A}$ , denoted by  $\Gamma_+(I)$ , is the convex hull in  $\mathbb{R}_+^n$  of  $\{k + v : v \in \mathbb{R}_+^n, k \in \text{supp}(I)\}$ .

We say that  $\Gamma_+ \subseteq \mathbb{R}_+^n$  is a *Newton polyhedron* if there exist some  $k_1, \dots, k_r \in \mathbb{Q}_+^n$  such that  $\Gamma_+$  is the convex hull in  $\mathbb{R}_+^n$  of the set  $\{k_i + v : v \in \mathbb{R}_+^n, i = 1, \dots, r\}$  and  $\Gamma_+$  intersects all the coordinate axis. We denote by  $\Gamma$  the union of the compact faces of  $\Gamma_+$  and by  $\Gamma_-$  the set  $\mathbb{R}^n \setminus \Gamma_+$ . If  $\Gamma_+ = \Gamma_+(I)$ , the sets  $\Gamma$  and  $\Gamma_-$  are denoted by  $\Gamma(I)$  and  $\Gamma_-(I)$ , respectively.

Moreover, we denote the  $n$ -dimensional volume of a compact subset  $K \subseteq \mathbb{R}_+^n$  by  $V_n(K)$ . When an ideal  $J \subseteq \mathcal{A}$  has finite codimension, the  $n$ -dimensional volume of  $\Gamma_-(J)$  is finite, in this case we denote the number  $V_n(\Gamma_-(J))$  by  $v(J)$ .

Given a face  $\Delta \subseteq \Gamma$ , where  $\Gamma_+$  is some Newton polyhedron, we denote the union of half-rays emanating from the origin and passing through  $\Delta$  by  $C(\Delta)$ . The set  $C(\Delta) \cap \mathbb{Z}^k$  is a subsemigroup of  $\mathbb{Z}^k$ , hence the subset of  $\mathcal{A}$  given by  $\mathcal{A}_\Delta = \{g \in \mathcal{A} : \text{supp } g \subseteq C(\Delta) \cap \mathbb{Z}^k\}$  is a subring with unity of  $\mathcal{A}$ . Moreover, if  $D$  is a fixed subset of  $\Gamma_+$  and  $g = \sum_k a_k x^k \in \mathcal{A}$ , we set  $g_D = \sum_{k \in D} a_k x^k$ .

**DEFINITION 2.2.** Let  $I$  be an ideal of finite codimension in  $\mathcal{A}$ . We say that  $I$  is *Newton non-degenerate* if there exists a system of generators  $g_1, \dots, g_s$  of  $I$  such that, for each compact face  $\Delta \subseteq \Gamma(I)$ , the ideal  $I_\Delta$  generated by  $g_{1_\Delta}, \dots, g_{s_\Delta}$  has finite codimension in  $\mathcal{A}_\Delta$ .

In the sequel, we shall abbreviate the words ‘Newton non-degenerate’ by NND, hence we will talk about NND ideals. By Theorem 6.2 of [11], the ideal  $I_\Delta$  generated by  $g_{1\Delta}, \dots, g_{s\Delta}$  has finite codimension in  $\mathcal{A}_\Delta$  if and only if, for each compact face  $\Delta_1$  of  $\Delta$ , the equations  $g_{1\Delta_1} = \dots = g_{s\Delta_1} = 0$  have no common solution in  $(\mathbb{C} \setminus \{0\})^n$ .

A series  $g = \sum_k a_k x^k$  in  $\mathcal{A}$  is said *Newton non-degenerate*, or NND for short, if the ideal generated by the system  $x_1 \partial g / \partial x_1, \dots, x_n \partial g / \partial x_n$  is NND.

As we shall see, NND ideals are related with the notion of integral closure. The integral closure of an ideal  $I$  in the ring  $\mathcal{O}_n$  is a key ingredient to study the problem of Whitney equisingularity for families of complex analytic hypersurfaces with isolated singularities. For ideals with finite colength, the invariant which controls the integral closure is the multiplicity of  $I$ . In this section we show how to calculate the multiplicity of any NND ideal  $I$  with finite colength in terms of  $\Gamma_+(I)$ .

Given an ideal  $I$  in a ring  $R$ , an element  $h \in R$  is *integral over  $I$*  if it satisfies a relation

$$h^k + a_1 h^{k-1} + \dots + a_{k-1} h + a_k = 0, \quad \text{with } a_i \in I^i.$$

The set of such elements is an ideal  $\bar{I}$  in  $R$ , called the *integral closure* of  $I$  in  $R$ .

If  $I$  is an ideal in  $\mathcal{A}$ , then we denote by  $I_0$  the ideal generated by those monomials  $x^k$  such that  $k$  belongs to  $\Gamma_+(I)$ .

LEMMA 2.1. *Let  $I$  be an ideal of  $\mathcal{A}$ , then  $\Gamma_+(I) = \Gamma_+(\bar{I})$ .*

*Proof.* Since  $I_0$  is a monomial ideal, we have the equality  $\Gamma_+(I_0) = \Gamma_+(\bar{I}_0)$  (see [19, p. 129]). Then,  $\Gamma_+(I) \subseteq \Gamma_+(\bar{I}) \subseteq \Gamma_+(\bar{I}_0) = \Gamma_+(I_0) = \Gamma_+(I)$ .

Here we state the characterization of the integral closure of ideals in  $R = \mathcal{O}_n$  given by Teissier in [18, p. 288].

PROPOSITION 2.1. *Let  $I$  be an ideal in  $\mathcal{O}_n$ . The following statements are equivalent:*

1.  $h \in \bar{I}$ ,
2. For each system of generators  $g_1, \dots, g_s$  of  $I$  there exists a neighbourhood  $U$  of 0 and a constant  $C > 0$  such that  $|h(x)| \leq C \sup\{|g_1(x)|, \dots, |g_s(x)|\}$ , for all  $x \in U$ .
3. For each analytic curve  $\varphi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$ , the germ  $h \circ \varphi$  lies in  $(\varphi^*(I)) \mathcal{O}_1$ .

If  $I$  is an ideal in  $\mathcal{A}$ , we define the set  $C(\bar{I})$  as the convex hull in  $\mathbb{R}_+^n$  of  $\{k \in \mathbb{Z}^k : x^k \in \bar{I}\}$ . That is,  $C(\bar{I})$  is the Newton polyhedron of the ideal  $K_I$  generated by all the monomials belonging to the integral closure of  $I$ . If  $I$  has finite codimension, we also set  $w(I) = v(K_I)$ .

THEOREM 2.1 ([17, 3.4]). *Let  $I$  be an ideal of  $\mathcal{O}_n$  of finite codimension, then  $C(\bar{I}) \subseteq \Gamma_+(I)$ . The equality  $\Gamma_+(I) = C(\bar{I})$  holds if and only if  $I$  is NND.*

In the proof of the previous theorem, Proposition 2.1 plays an essential role. Therefore, we find it’s worth to show that Theorem 2.1 implies a similar result for ideals in the ring  $\mathcal{A}$  of formal power series around the origin.

COROLLARY 2.1. *Let  $I \subseteq \mathcal{A}$  an ideal of finite codimension, then  $I$  is NND if and only if  $\Gamma_+(I) = C(\bar{I})$*

*Proof.* Let  $\{g_1, \dots, g_s\}$  be a system of generators of  $I$ . Then, given an  $i = 1, \dots, s$ , if  $g_i = \sum_k a_k x^k$  we denote by  $p(g)$  the sum of those  $a_k x^k$  such that  $k$  belongs to  $\Gamma(I)$ . Let  $J$  denote the ideal of  $\mathcal{A}_n$  generated by  $p(g_1), \dots, p(g_s)$ , then we observe that  $J$  is generated by polynomials (and hence can be considered as an ideal in  $\mathcal{O}_n$ ) and that  $I$  is NND if and only if  $J$  is NND.

Since  $I$  has finite codimension, the ring  $\mathcal{A}/I$  is Artinian, so we can consider the minimum integer  $\ell$  such that  $m_n^\ell \subseteq I$ , and we denote this by  $\ell_0$ . Therefore, if we express  $g_i$  as above, for a given  $i = 1, \dots, s$ , we denote by  $g'_i$  the sum of those  $a_k x^k$  such that  $k_1 + \dots + k_n \leq \ell_0 + 1$  and  $k$  does not belong to  $\Gamma(I)$ . Let  $J'$  be the ideal generated by  $p(g_i) + g'_i$ ,  $i = 1, \dots, s$ . By the definition of  $\ell_0$  it is obvious that  $J' \subseteq I$  and that  $I = J' + m_n I$ . Then  $J' = I$ , by Nakayama's lemma and we observe that the result follows from applying Theorem 2.1 to  $J'$ , since  $J'$  is generated by polynomials.

DEFINITION 2.3. [[12, p. 109]] The multiplicity of an ideal  $I$  of finite codimension in  $\mathcal{A}$  is defined as  $e(I) = \lim_{k \rightarrow \infty} \frac{n!}{k^n} \dim_{\mathbb{C}} \mathcal{A}/I^k$ .

Now, we quote the following result of Teissier about the multiplicity of monomial ideals.

LEMMA 2.2 ([19, p. 131]). *If  $I$  is an ideal of  $\mathcal{A}$  of finite codimension that is generated by monomials, then  $e(I) = n!v(I)$ .*

From Lemma 2.2, we obtain the following inequalities

$$n!w(I) = e(K_I) \geq e(\bar{I}) = e(I) \geq e(I_0) = n!V_n(\Gamma_-(I)) = n!v(I), \quad (1)$$

where we have used the fact that  $\Gamma_+(I) = \Gamma_+(I_0)$  and that  $e(I_1) \geq e(I_2)$  whenever  $I_1, I_2$  is any pair of ideals in  $\mathcal{A}$  such that  $I_1 \subseteq I_2$ .

As we shall see, the Newton non-degeneracy condition characterizes all ideals  $I$  satisfying the equality  $e(I) = n!v(I)$ .

Given two ideals  $J \subseteq I$  in an arbitrary ring  $R$ , the ideal  $J$  is said to be a *reduction* of  $I$  when there is an integer  $r > 0$  such that  $I^{r+1} = JI^r$ . From [3] Lemma 4.6.5 and the work of Rees [16] we have the following characterization of reductions in  $\mathcal{A}$ .

THEOREM 2.2. *Let  $J \subseteq I$  be a pair of ideals of  $\mathcal{A}$  with  $I$  of finite codimension. Then, the following conditions are equivalent:*

1.  $J$  is a reduction of  $I$ ;
2.  $e(I) = e(J)$ ;
3.  $J$  and  $I$  have the same integral closure.

The following result is immediate from Lemma 2.1, Corollary 2.1 and 2.2.

COROLLARY 2.2. *If  $I$  is a NND ideal of  $\mathcal{A}$ , then any reduction of  $I$  is also NND.*

THEOREM 2.3. *Let  $I \subseteq \mathcal{A}$  be an ideal of finite codimension. Then, the following conditions are equivalent:*

1.  $I$  is a NND ideal;
2.  $\Gamma_+(I) = \mathbb{C}(\bar{I})$ ;
3.  $e(I) = n!v(I)$ ;
4.  $\bar{I}$  is generated by monomials;
5.  $I_0 \subseteq \bar{I}$ ;
6.  $\bar{I} = \{f \in \mathcal{A} : \Gamma_+(f) \subseteq \Gamma_+(I)\}$ .

*Proof.* The equivalence between (i) and (ii) comes from Corollary 2.1. Let us see (ii)  $\iff$  (iii). If  $\Gamma_+(I) = \mathbb{C}(\bar{I})$ , then  $v(I) = w(I)$  and it follows that  $e(I) = n!v(I)$ , by (1). Suppose that  $e(I) = n!v(I)$ . Since the multiplicity of  $I_0$  is also equal to  $n!v(I_0) = n!v(I)$  and  $I \subseteq I_0$ , we find that  $I$  and  $I_0$  have the same integral closure, by Theorem 2.2. In particular, item (ii) is true. The equivalence between (ii), (iv), (v) and (vi) is obvious.

In order to compute the Milnor number of a germ of a function  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  with isolated singularity at the origin, Kouchnirenko considered in [11] the ideal  $I(f)$  generated by  $\{x_1 \partial f / \partial x_1, \dots, x_n \partial f / \partial x_n\}$ . He showed in Theorem AI that, if  $I(f)$  has finite codimension and  $f$  is NND then  $\dim_{\mathbb{C}} \mathcal{A}/I(f) = n!v(I(f))$ . As a corollary of Theorem 2.3 we have that the equality  $\dim_{\mathbb{C}} \mathcal{A}/I = n!v(I)$  also holds for any NND ideal  $I$  generated by  $n$  elements, since the multiplicity  $e(I)$  is equal to the codimension of  $I$  in this case.

EXAMPLE 2.1. We consider the ideal  $I = \langle xy + z^3, xz + y^3, yz + x^3 \rangle$  in  $\mathbb{C}[[x, y, z]]$ . Since  $I$  is NND and  $v(I) = 11/6$ , we obtain  $\dim_{\mathbb{C}} \mathbb{C}[[x, y, z]]/I = 11$ .

As a consequence of Theorem 2.3, if  $I \subseteq \mathcal{A}$  is not NND, then  $I$  satisfies the strict inequality  $e(I) > n!v(I)$ . Moreover, the inclusion  $\mathbb{C}(\bar{I}) \subseteq \Gamma_+(I)$  must be strict. Hence, there arises the question of determining the cases in which the multiplicity of a Newton degenerate ideal can be calculated as  $n!w(I)$ . As we shall see in the following lemma this expression for the multiplicity only works with NND ideals.

COROLLARY 2.3. *Let  $I \subseteq \mathcal{A}$  be an ideal of finite codimension. Then, the relation  $e(I) = n!w(I)$  holds if and only if  $\Gamma_+(I) = \mathbb{C}(\bar{I})$ .*

*Proof.* The *if* part is an immediate consequence of (1). If  $e(I) = n!w(I)$ , then  $K_I$  is a reduction of  $\bar{I}$ , since  $e(K_I) = n!V_n(\Gamma_-(K_I)) = n!w(I)$ . Then, the integral closure of  $K_I$  is equal to  $\bar{I}$ , which implies that  $\bar{I}$  is a monomial ideal. As a consequence, the ideal  $I$  is NND, by Corollary 2.2.

In [19, p. 132] Teissier states that if  $I$  and  $J$  are monomial ideals in  $\mathcal{A}$  of finite codimension, then the number  $v(I^r J^s) = V_n(\mathbb{R}_+^n \setminus \Gamma_+(I^r J^s))$  admits the following expression:

$$v(I^r J^s) = \sum_{i=0}^n \binom{n}{i} v_i r^i s^{n-i},$$

for certain positive rational numbers  $v_i$ . These numbers are called the *mixed covolumes* of  $\Gamma_+(I)$  and  $\Gamma_+(J)$  and are also denoted by  $v_i(I, J)$ . It can be proved that  $v_0(I, J) = v(J)$  and  $v_n(I, J) = v(I)$ . Since the function  $v(I^r J^s)$  only depends on the polyhedra  $\Gamma_+(I)$  and  $\Gamma_+(J)$ , we can also consider the mixed covolumes  $v_i(I, J)$  for arbitrary ideals  $I$  and  $J$  of finite codimension in  $\mathcal{A}$ .

Moreover, as can be seen in [19, p. 132], given two ideals  $I$  and  $J$  of finite codimension in  $\mathcal{A}$ , the multiplicity  $e(I^r J^s)$  is expressed as

$$e(I^r J^s) = \sum_{i=0}^n \binom{n}{i} e_i r^i s^{n-i},$$

for certain positive integers  $e_i$ . These are also denoted by  $e_i(I, J)$ , for all  $i = 0, \dots, n$  and are called the *mixed multiplicities* of  $I$  and  $J$ . It is also proved in [18, p. 307] that  $e_0(I, J) = e(J)$  and  $e_n(I, J) = e(I)$ .

**COROLLARY 2.4.** *Suppose that  $I$  and  $J$  are NND ideals of  $\mathcal{A}$ , then  $IJ$  is also NND.*

*Proof.* Since  $I$  and  $J$  are NND ideals, we have that  $\bar{I}$  and  $\bar{J}$  are generated by monomials. The integral closure of  $IJ$  can be always expressed as  $\overline{IJ} = \bar{I}\bar{J}$ . Then,  $\overline{IJ}$  is also generated by monomials and  $IJ$  is NND, by Theorem 2.3.

**COROLLARY 2.5.** *Suppose that  $I$  and  $J$  are NND ideals of  $\mathcal{A}$ , then  $e_i(I, J) = n!v_i(I, J)$ , for all  $i = 0, 1, \dots, n$ .*

*Proof.* By Corollary 2.4 we know that  $I^r J^s$  is NND, for all  $r, s$ . Then  $e(I^r J^s) = n!v(I^r J^s)$ , for all  $r, s$ .

This result can be generalized to the case of more than two ideals. Let  $I_1, \dots, I_m$  be ideals of finite codimension in  $\mathcal{A}$ . We define the numbers  $v_{i_1, \dots, i_m}(I_1, \dots, I_m)$  and  $e_{i_1, \dots, i_m}(I_1, \dots, I_m)$  by

$$\begin{aligned} v(I_1^{r_1} \dots I_m^{r_m}) &= \sum_{i_1 + \dots + i_m = n} \binom{n}{i_1 \dots i_m} v_{i_1, \dots, i_m}(I_1, \dots, I_m) r_1^{i_1} \dots r_m^{i_m}, \quad \text{and} \\ e(I_1^{r_1} \dots I_m^{r_m}) &= \sum_{i_1 + \dots + i_m = n} \binom{n}{i_1 \dots i_m} e_{i_1, \dots, i_m}(I_1, \dots, I_m) r_1^{i_1} \dots r_m^{i_m}, \end{aligned}$$

where  $\binom{n}{i_1 \dots i_m} = n!/(i_1! \dots i_m!)$ .

COROLLARY 2.6. *If  $I_1, \dots, I_m$  are NND, then we have*

$$e_{i_1, \dots, i_m}(I_1, \dots, I_m) = n!v_{i_1, \dots, i_m}(I_1, \dots, I_m).$$

### 3. NON-DEGENERATE SYSTEMS ON A NEWTON POLYHEDRON

Next, we show some preliminary concepts which allow us to give a new definition of non-degeneracy. The key point is that we can consider any Newton polyhedron and not only the Newton polyhedron of the ideal that we want to compute the codimension.

From the boundary of a Newton polyhedron  $\Gamma_+ \subseteq \mathbb{R}^n$  we can construct a piecewise-linear function  $\phi_\Gamma : \mathbb{R}^n \rightarrow \mathbb{R}$  with the following properties:

1.  $\phi_\Gamma$  is linear on each cone  $C(\Delta)$ , where  $\Delta$  is a compact face of  $\Gamma$ ;
2.  $\phi_\Gamma$  takes positive integer values on the lattice points of  $\mathbb{R}_+^n \setminus \{0\}$ ;
3. there exists a positive integer  $M$  such that  $\phi_\Gamma(k) = M$ , for all  $k \in \Gamma$ .

When the map  $\phi_\Gamma$  is constructed we define the ideals

$$\mathcal{A}_q = \{g \in \mathcal{A} : \text{supp } g \subseteq \phi_\Gamma^{-1}(q + \mathbb{N})\}, \quad \text{for all } q \geq 0,$$

then

$$\mathcal{A} = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \dots$$

is a filtration of ideals of  $\mathcal{A}$ , that is  $\mathcal{A}_p \mathcal{A}_q \subseteq \mathcal{A}_{p+q}$ . This is called the *Newton filtration of  $\mathcal{A}$  associated to  $\Gamma_+$*  (see also [11, 2.1]). For any compact face  $\Delta$  of  $\Gamma$ , this filtration induces a filtration on  $\mathcal{A}_\Delta$  in a natural way.

Let  $g = \sum_k a_k x^k$  be a series in  $\mathcal{A}$ , the *level of  $g$*  with respect to the filtration given above is defined as

$$\nu_\Gamma(g) = \min\{\phi_\Gamma(k) : k \in \text{supp } g\} = \max\{q : g \in \mathcal{A}_q\}.$$

The *principal part of  $g$* , denoted by  $\text{in}(g)$ , is the sum of terms  $a_k x^k$  such that  $\phi_\Gamma(k) = \nu_\Gamma(g)$ . Given a face  $\Delta$  of  $\Gamma$ , the *principal part of  $g$  over  $\Delta$* , denoted by  $\text{in}_\Delta(g)$ , is the polynomial

$$\text{in}_\Delta(g) = \sum \{a_k x^k : k \in \text{supp } g \cap C(\Delta) \text{ and } \phi_\Gamma(k) = \nu_\Gamma(g)\}.$$

DEFINITION 3.1. Let  $I$  be an ideal of finite codimension of  $\mathcal{A}$ . We say that a system of generators  $g_1, \dots, g_s$  of  $I$  is *non-degenerate on  $\Gamma_+$*  if, for each compact face  $\Delta \subseteq \Gamma$ , the ideal of  $\mathcal{A}_\Delta$  generated by  $\text{in}_\Delta(g_1), \dots, \text{in}_\Delta(g_s)$  has finite codimension in  $\mathcal{A}_\Delta$ . When the system  $g_1, \dots, g_s$  does not satisfy the above definition, we say that this is *degenerate on  $\Gamma_+$* .

Analogously to Definition 2.2, the ideal of  $\mathcal{A}_\Delta$  generated by  $\text{in}_\Delta(g_1), \dots, \text{in}_\Delta(g_s)$  has finite codimension in  $\mathcal{A}_\Delta$  if and only if, for each compact face  $\Delta_1 \subseteq \Delta$ , the equations  $\text{in}_{\Delta_1}(g_1)(x) = \dots = \text{in}_{\Delta_1}(g_s)(x) = 0$  have no common solution in  $(\mathbb{C} \setminus \{0\})^n$ .

*Remark 3.* 1. (i) This definition is quite different from Definition 2.2. Given an ideal  $I \subseteq \mathcal{A}$  we could have two different systems of generators  $S$  and  $S'$  of  $I$  and a Newton polyhedron  $\Gamma_+ \subseteq \mathbb{R}^n$  such that  $S$  is non-degenerate on  $\Gamma_+$  and  $S'$  is degenerate on  $\Gamma_+$ . As is easy to check, this fact does not happen when dealing with NND ideals.

(ii) The ideal  $I$  generated by a non-degenerate system on  $\Gamma_+(I)$  could be Newton degenerate. For example, this occurs with the system  $\{x^3 + xy + y^3, x^2y^2\} \subset \mathbb{C}[[x, y]]$ .

(iii) Observe that if  $I$  is an ideal of finite codimension and  $\Gamma_+$  is any Newton polyhedron, then  $I$  always admits a generating system which is non-degenerate on  $\Gamma_+$ . We know from the finite codimension of the ideal  $I$  that there exists an integer  $\ell$  such that  $m_n^\ell \subseteq I$ . But this implies that we can add all the monomials of degree  $\ell$  to a given generating system  $S$  of  $I$  to obtain a non-degenerate system on  $\Gamma_+$ . The problem is that if we do this, the number of generators of the system  $S$  increases. For example, the ideal  $I = \langle x^5 + x^2y, xy^2 + y^5 \rangle$  is not NND and the given generating system of  $I$  is degenerate on any Newton polyhedron. This fact is a consequence of Theorem 3.2. On the other hand, we have that the system  $\{x^5 + x^2y, xy^2 + y^5, x^4y^2, x^2y^4\}$  also generates  $I$  and is non-degenerate on  $\Gamma_+(I)$ .

Here we give the main result of this section.

**THEOREM 3.1.** *Let  $g_1, \dots, g_n$  be a system of generators of an ideal  $I$  with finite codimension in  $\mathcal{A}$  and let  $\Gamma_+ \subseteq \mathbb{R}^n$  be a Newton polyhedron. If  $M$  is the value on  $\Gamma$  of the filtration induced by  $\Gamma_+$  and  $d_1 = \nu_\Gamma(g_1), \dots, d_n = \nu_\Gamma(g_n)$  are the levels of the given set of generators of  $I$  with respect to this filtration, then*

$$1. \text{ we have the inequality } \dim_{\mathbb{C}} \mathcal{A}/I \geq \frac{d_1 \dots d_n}{M^n} n! V_n(\Gamma_-),$$

2. the equality holds if and only if the system  $g_1, \dots, g_n$  is non-degenerate on  $\Gamma_+$ .

*Proof of part (i).* We first observe that each set  $\mathcal{A}_q$  with  $q \geq 0$ , of the Newton filtration induced by  $\Gamma_+$  is a monomial ideal, then for  $d = d_1 \dots d_n$ , we have

$$\frac{d^n}{M^n} n! V_n(\Gamma_-) = n! V_n(\phi^{-1}([0, d])) \leq n! v(\mathcal{A}_d).$$

Now consider the ideal  $J$  generated by the system  $\{g_1^{d/d_1}, \dots, g_n^{d/d_n}\}$ , since  $J \subseteq \mathcal{A}_d$ , we conclude that

$$n! v(\mathcal{A}_d) \leq n! v(J) \leq \dim_{\mathbb{C}} \mathcal{A}/J = \frac{d^n}{d_1 \dots d_n} \dim_{\mathbb{C}} \mathcal{A}/\langle g_1, \dots, g_n \rangle, \quad (2)$$

where the last equality is given in [1, 12.3], and the second inequality comes from (1).

The part ( $\implies$ ) of (ii) in the above theorem will be given in the Theorem 3.6, and the part ( $\impliedby$ ) in the Section 4.

A map germ  $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  is said to be *weighted homogeneous of weights*  $w_1, \dots, w_n$  and *degrees*  $d_1, \dots, d_p$  if

$$g(\lambda^{w_1}x_1, \dots, \lambda^{w_n}x_n) = (\lambda^{d_1}g_1(x_1, \dots, x_n), \dots, \lambda^{d_p}g_p(x_1, \dots, x_n)), \quad \text{for all } \lambda \in \mathbb{C}.$$

We say that a map germ  $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  is *semi weighted homogeneous* of weights  $w_1, \dots, w_n$  and degrees  $d_1, \dots, d_p$  if  $f$  can be expressed as a sum  $f = g + h$ , where  $g$  is a finitely determined weighted homogeneous map germ of weights  $w_1, \dots, w_n$  and degrees  $d_1, \dots, d_p$ ,  $h = (h_1, \dots, h_p)$  and each  $h_i$  is a germ of function such that  $w_1k_1 + \dots + w_nk_n > d_i$  for all  $k = (k_1, \dots, k_n) \in \text{supp } h_i$ .

Observe that the component functions  $f_1, \dots, f_p$  of any semi weighted homogeneous map germ  $f$  constitute a non-degenerate system on the polyhedron determined by the hyperplane in  $\mathbb{R}^n$  with equation  $w_1k_1 + \dots + w_nk_n = 1$ , where  $w_1, \dots, w_n$  denote the weights of  $f$ .

The *multiplicity of a finite map germ*  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$  is defined as the codimension (i.e., colength) in  $\mathcal{O}_n$  of the ideal generated by the component functions of  $f$ . For a semi weighted homogeneous map germ  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$  we see in the following corollary of Theorem 3.1 (see also [1, 12.3]) that the multiplicity of  $f$  only depends on its weighted initial part.

**COROLLARY 3.1** ([1, 12.3]). *Let  $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$ ,  $g = (g_1, \dots, g_n)$  be a semi weighted homogeneous map germ of weights  $w_1, \dots, w_n$  and degrees  $d_1, \dots, d_n$ . Suppose that  $g_1, \dots, g_n$  is a system of generators of an ideal  $I$  of finite codimension, then*

$$\dim_{\mathbb{C}} \mathcal{O}_n/I = \frac{d_1 \dots d_n}{w_1 \dots w_n}.$$

More generally, as a particular case of Theorem 3.1, when the component functions of a finite map germ  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$  give a non-degenerate system with respect to some Newton polyhedron  $\Gamma_+$ , then the multiplicity of  $f$  only depends on the ideal generated by the principal parts of its component functions with respect to the filtration induced by  $\Gamma_+$ .

**EXAMPLE 3.1.** Consider the ideal in  $\mathcal{A} = \mathbb{C}[[x, y]]$  given by  $I = \langle x^\alpha + x^r y^s + y^\beta, x^{\lambda r} y^{\lambda s} \rangle$ , where  $\lambda > 0$ . The Newton polyhedron of  $I$  has two faces if and only if  $\alpha s < \beta(\alpha - r)$ . Suppose that this is the case. The associated Newton filtration of  $\mathcal{A}$  is given via the map  $\phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  defined by

$$\phi(a, b) = \begin{cases} r\beta(sa + (\alpha - r)b), & \text{if } b \leq \frac{s}{r}a \\ \alpha s((\beta - s)a + rb), & \text{if } b \geq \frac{s}{r}a. \end{cases}$$

The value of  $\phi$  along  $\Gamma(I)$  is  $M = \alpha\beta rs$ . We observe that this ideal is Newton degenerate but the system of generators of  $I$  given above is non-degenerate on  $\Gamma_+(I)$ . Then, by Theorem 3.1, we have that  $\dim_{\mathbb{C}} \mathcal{A}/I = \lambda(\alpha s + \beta r)$ .

Next, we give a characterization of non-degenerate systems on a given Newton polyhedron that is related with NND ideals. For a fixed Newton polyhedron  $\Gamma_+$  and a system of generators  $\{g_1, \dots, g_n\}$  of an ideal  $I$  with  $\nu_\Gamma(g_i) = d_i$ , we set  $d = d_1 \dots d_n$ .

**THEOREM 3.2.** *Under the same hypotheses as in Theorem 3.1, the following conditions are equivalent:*

1. *the system  $\{g_1, \dots, g_n\}$  is non-degenerate on  $\Gamma_+$ ;*
2. *All inequalities in (1) become inequalities;*
3. *the ideal  $J = \langle g_1^{d/d_1}, \dots, g_n^{d/d_n} \rangle$  is NND and  $\Gamma_+(J) = \Gamma_+(\mathcal{A}_d)$ .*

*Proof.* The proof that (i)  $\implies$  (ii) will be given in section 4 (proof of the Theorem 3.3.).

Suppose now that (ii) holds. In particular, we have  $v(J) = v(\mathcal{A}_d)$  and this implies that  $\Gamma_+(J) = \Gamma_+(\mathcal{A}_d)$ . Therefore,  $J$  is a NND ideal, by Theorem 2.3.

Now, suppose condition (iii). From the condition  $\Gamma_+(J) = \Gamma_+(\mathcal{A}_d)$ , the Newton polyhedron of  $J$  is face-wise parallel to  $\Gamma_+$ . Then, if  $\Delta$  is a face of  $\Gamma_+$  and we denote the elements  $g^{d/d_i}$  by  $h_i$ , for  $i = 1, \dots, n$ , we have that the ideal in  $\mathcal{A}_\Delta$  generated by  $\text{in}_\Delta(h_1), \dots, \text{in}_\Delta(h_n)$  has finite codimension. Moreover,

$$\langle \text{in}_\Delta(h_1), \dots, \text{in}_\Delta(h_n) \rangle = \langle \text{in}_\Delta(g_1)^{d/d_1}, \dots, \text{in}_\Delta(g_n)^{d/d_n} \rangle \subseteq \langle \text{in}_\Delta(g_1), \dots, \text{in}_\Delta(g_n) \rangle.$$

Then  $\langle \text{in}_\Delta(g_1), \dots, \text{in}_\Delta(g_n) \rangle$  has also finite codimension in  $\mathcal{A}_\Delta$  and  $\{g_1, \dots, g_n\}$  verifies the condition of non-degeneracy on the polyhedron  $\Gamma_+$ .

**EXAMPLE 3.2.** Consider the ideal in  $\mathcal{O}_2$  given by  $I = \langle x^\alpha + x^r y^s, x^s y^r + y^\beta \rangle$ . Suppose that  $r > s$  and the parameters  $r, s$  are such that the polyhedron  $\Gamma_+(I)$  has three compact faces. The vertexes of  $\Gamma_+(I)$  are  $(\alpha, 0), (r, s), (s, r)$  and  $(0, \beta)$ . By Theorem 3.2 we see that the generating system given above is always degenerate with respect to any Newton polyhedron. Anyway, we can use the filtration induced by  $\Gamma_+(I)$  and part (i) of Theorem 3.1 to give the inequality

$$\dim_{\mathbb{C}} \mathcal{A}/I > r^2 + s^2 + s(\alpha + \beta - 2s).$$

Next we show some applications of our results to Singularity Theory.

### 3.1. Multiplicities of Implicit Differential Equations

Bruce and Tari considered in [2] the problem of the computation of the local multiplicity of a germ of a singularity of an implicit differential equation  $F(x, y, dy/dx) = 0$ . They showed that when the discriminants are simple curve singularities, these multiplicities are equal to the codimension of ideals in  $\mathcal{O}_2$  generated by two elements. We shall show here how to calculate the codimension of a family of ideals that appears in this work. This is given by  $I_{k,\ell,m} = \langle y^2 + x^k, y^2 + x^\ell y + x^m \rangle$ , we compute the codimension of  $I_{k,\ell,m}$  for some integer values of  $k, \ell, m$ , with  $k$  and  $m > 2$ .

(i) Suppose first that  $\ell \geq r/2$ , where  $r = \min\{k, m\}$ . The Newton polyhedron  $\Gamma_+(I_{k,\ell,m})$  has only one face  $\Delta$  of maximal dimension with vertices  $(0, 2)$  and  $(r, 0)$ . From Corollary 3.1 we have  $\dim_{\mathbb{C}} \mathcal{O}_2/I_{k,\ell,m} = 2r$ , if  $k \neq m$  and  $\dim_{\mathbb{C}} \mathcal{O}_2/I = 2\ell + k$ , if  $k = m$ .

(ii) Suppose now that  $\ell < r/2$ . Then,  $\Gamma_+(I)$  has two 1-dimensional compact faces  $\Delta_1$  and  $\Delta_2$  with vertex  $(0, 2)$ ,  $(\ell, 1)$  and  $(r, 0)$ .

If  $k \leq m$ , we have  $I_{\Delta_1} = \langle y^2, y^2 + x^\ell y \rangle$  and  $I_{\Delta_2} = \langle x^k, x^\ell y \rangle$ , hence  $I$  is Newton non-degenerate and by Theorem 2.3,  $\dim_{\mathbb{C}} \mathcal{O}_2/I = n!v(I) = 2\ell + k$ .

If  $m < k$ , the ideal  $I$  is Newton degenerate and the system of generators  $\{y^2 + x^k, y^2 + x^\ell y + x^m\}$  is also degenerate with respect to  $\Gamma_+(I)$ . If we prolong the face  $\Delta_2$ , we obtain another polyhedron  $\Delta$  that intersects the  $Y$  axis at  $(0, \frac{m}{m-\ell})$ . The filtration induced by  $\Delta$  is given by the map  $\phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  such that  $\phi(a, b) = a + (m - \ell)b$ . Observe that the above system is non-degenerate on this polyhedron if and only if  $\phi(0, 2) \leq \phi(k, 0)$ , which is equivalent to say that  $2m - k \leq 2\ell$ . In this case,  $\dim_{\mathbb{C}} \mathcal{O}_2/I = 2m$ , by Theorem 3.1. We observe that we cannot apply Theorem 3.1 in the remaining cases.

### 3.2. Thom-Boardman singularities

The Thom-Boardman singularities which appear in a generic deformation of a finitely determined map germ  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  give some of the main invariants used in Singularity Theory. In some cases, see [7, p. 141], these invariants can be defined algebraically by means of the iterated jacobian extension of the map germ.

In [8] the authors considered finitely determined weighted homogeneous map germs  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  and gave an algebraic formula, in terms of the weights and degrees of  $f$ , for the number of *cusps* (or  $\Sigma^{1,1,0}$  singularities) that appear in a generic deformation of  $f$ .

Next we shall consider the case of corank one map germs  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$  to illustrate how to compute these invariants in terms of a suitable Newton polyhedron, for map germs that are not weighted homogeneous. We denote by  $c_{\mathbf{i}}(f)$  the number of singularities of type  $\Sigma^{\mathbf{i}}$  that appear in a one parameter generic deformation of  $f$ , where  $\mathbf{i} = (i_1, \dots, i_k)$  is a given Boardman symbol.

EXAMPLE 3.3. Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0)$  be a corank one map germ of the form

$$f(x_1, \dots, x_{n-1}, z) = (x_1, \dots, x_{n-1}, z^{n+t} + g_1(x)z^{n-1} + g_2(x)z^{n-2} + \dots + g_{n-1}(x)z),$$

for some  $t > 0$ . Let us consider the ideal  $I = \langle z^t, g_1, g_2, \dots, g_{n-1} \rangle$  and let  $\mathbf{i}$  be the Boardman symbol  $(1, \dots, 1)$ , where 1 is repeated  $n$  times. Then the number  $c_{\mathbf{i}}(f)$  is equal to  $\dim_{\mathbb{C}} \mathcal{A}/I$ , by Proposition 2.5 of [7]. Then, if  $I$  is NND, the number  $c_{\mathbf{i}}(f)$  is equal to  $n!v(I)$ , by Theorem 2.3. More generally, if the system of generators  $\{z^t, g_1, \dots, g_{n-1}\}$  is non-degenerate on a given Newton polyhedron, we can apply Theorem 3.1 in order to compute  $c_{\mathbf{i}}(f)$ .

EXAMPLE 3.4. Let  $f(x, y, z) = (x, y, z^{t+3} + (x^{r+2} + x^2 y^2)z^2 + (y^{s+2} + x^2 y^2)z)$ , for some positive integer numbers  $r, s, t$ . The number of swallowtails, or  $\Sigma^{1,1,1,0}$  singularities, that appear in a 1-parameter generic deformation of  $f$  is the codimension of the ideal

$I = \langle z^t, x^{r+2} + x^2y^2, y^{s+2} + x^2y^2 \rangle$ . Then, we obtain that  $\#(\Sigma^{1,1,1,0}(f)) = 2t(4 + r + s)$ , since  $I$  is NND.

Let  $f(x, y, z) = (x, y, z^{t+3} + (x^r + xy + y^s)z^2 + x^2y^2z)$ , for some positive integer numbers  $r, s, t \geq 2$ . Then, we find that the system of generators  $S = \{z^{t+3}, x^r + xy + y^s, x^2y^2\}$  is non-degenerate on the Newton polyhedron determined by  $S$ . Therefore  $\#(\Sigma^{1,1,1,0}(f)) = 2t(r + s)$ , by Theorem 3.1.

#### 4. GRADED ALGEBRAS INDUCED BY NEWTON FILTRATIONS

In [11], Kouchnirenko uses the Koszul complex for computing the codimension of an ideal of the form  $I = \langle x_1 \partial f / \partial x_1, \dots, x_n \partial f / \partial x_n \rangle \subseteq \mathcal{F}_n$ . The problem we treat here is how to apply the method of Kouchnirenko to other complexes. It is natural to expect that similar computations work for some other complexes, and we will show that this is actually true for complexes coming from a free resolution of a perfect module. Therefore, we set our work in a more general context, that is, we consider finite  $\mathcal{A}$ -modules of projective dimension  $\leq n$ . As a particular cases, we give the proof of part (ii) of Theorem 3.1 and we compute the codimension of some determinantal ideals which are non-degenerate on a given Newton polyhedron.

##### 4.1. Preliminaries

Let  $R$  be a Noetherian ring,  $I$  an ideal of  $R$ , and  $M$  a finite  $R$ -module such that  $IM \neq M$ . We recall that a sequence of elements  $x_1, \dots, x_n \in I$  is said to be an  $M$ -sequence in  $I$  when  $x_1$  is not a zero divisor in  $M$ , and  $x_i$  is not a zero divisor in  $M/\langle x_1, \dots, x_{i-1} \rangle M$ , for all  $i = 2, \dots, n$ . If  $x_1, \dots, x_n$  is an  $M$ -sequence in  $I$ , then we say that this is a *maximal  $M$ -sequence* in  $I$  of length  $n$  when  $x_1, \dots, x_n, x$  is not an  $M$ -regular sequence in  $I$ , for all  $x \in I$ . By a theorem of Rees (see [3, p. 10]) all maximal  $M$ -regular sequences in  $I$  have the same length. Then, the *depth* of  $I$  with respect to  $M$  (or *grade* of  $I$  with respect to  $M$ ), denoted  $\text{depth}(I, M)$ , is defined as the length of any maximal  $M$ -sequence in  $I$ . We often use the fact that  $\text{depth}(I, M)$  is determined by the radical ideal  $\sqrt{I}$  and  $M$ .

Suppose that  $R$  is a local ring and let  $\mathfrak{m}$  be its maximal ideal. Then, we denote the number  $\text{depth}(\mathfrak{m}, M)$  simply by  $\text{depth}(M)$  and we refer to this as the *depth* (or *grade*) of  $M$ . In general we have that  $\text{depth}(M) \leq \text{pd } M$ , where  $\text{pd } M$  denotes the projective dimension of  $M$ . When the equality holds, then  $M$  is said to be a *perfect module* (see [4, 16C], [3, p. 25]).

Given any  $p \times q$  matrix  $U$  with entries in  $R$  and an integer  $s < \min\{p, q\}$ , we denote by  $I_s(U)$  the ideal of  $R$  generated by all minors of  $U$  of order  $s$ . Then, if  $\varphi : R^n \rightarrow R^m$  is a morphism of free modules, we define the *rank* of  $\varphi$  as  $\text{rank } \varphi = \max\{r : I_r(\varphi) \neq 0\}$ , where we identify  $\varphi$  with some of its representation matrices.

Let  $\mathcal{F}$  denote a complex of finite free  $R$ -module:

$$\mathcal{F} : 0 \longrightarrow F_\ell \xrightarrow{\varphi_\ell} F_{\ell-1} \xrightarrow{\varphi_{\ell-1}} \dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0.$$

Let  $r_k = \sum_{i=k}^{\ell} (-1)^{i-k} \text{rank } F_i$ . We remark that  $\text{rank } \varphi_k = r_k$ ,  $k = 1, 2, \dots, \ell$ , if  $\mathcal{F}$  is exact.

Now, we recall two theorems in commutative algebra which are important in this section.

**THEOREM 4.1** (Buchsbaum-Eisenbud acyclicity criterion). *The complex  $\mathcal{F}$  is acyclic if and only if  $\text{depth}(I_{r_i}(\varphi_i), R) \geq i$  for  $i = 1, \dots, \ell$ .*

For a proof of the above result, we refer to [4, (16.15)] or [3, Theorem 1.4.12].

**THEOREM 4.2** (Transfer of perfection). *Let  $M$  be a perfect  $R$ -module with  $\text{pd } M = \ell$ . Let  $S$  be a Noetherian  $R$ -algebra such that  $\text{depth}(\text{Ann}(M \otimes S), S) \geq \ell$  and  $M \otimes S \neq 0$ . Then  $M \otimes S$  is perfect with  $\text{pd } M = \ell$ . Furthermore  $\mathcal{F} \otimes S$  is a free resolution of  $M \otimes S$  for every free resolution  $\mathcal{F}$  of  $M$  of length  $\ell$ .*

We refer to [4, (3.5)] for the proof of the above theorem.

## 4.2. Graded algebras

Let  $\Gamma_+ \subseteq \mathbb{R}^n$  be a Newton polyhedron and  $\{\mathcal{A}_q\}_{q \geq 0}$  the filtration of  $\mathcal{A}$  induced by this polyhedron. Consider the graded ring  $A = \text{gr } \mathcal{A} = \bigoplus_q A_q$ , where  $A_q = \mathcal{A}_q / \mathcal{A}_{q+1}$ , for all  $q \geq 0$ . As a consequence of Theorem 4.5.6 of [3], we have that the dimension of  $A$  is equal to  $n$ . We also consider the following graded rings associated to  $\{\mathcal{A}_q\}_{q \geq 0}$ :

1. the *Rees ring*  $R(\mathcal{A}) = \bigoplus_{q \geq 0} \mathcal{A}_q T^q \subseteq \mathcal{A}[T]$ ,  $T$  an indeterminate
2. the *extended Rees ring*  $\widehat{R}(\mathcal{A}) = R(\mathcal{A}) \oplus \bigoplus_{q < 0} \mathcal{A} T^q \subseteq \mathcal{A}[T, T^{-1}]$ .

We understand these rings as graded rings graded by the degree of the indeterminate  $T$ . Observe that one has the representation  $A = \widehat{R}(\mathcal{A}) / T^{-1} \widehat{R}(\mathcal{A})$ . The natural map  $\text{Proj}(R(\mathcal{A})) \rightarrow \text{Spec}(\mathcal{A})$  is the ‘formal completion’ of the projective toric modification defined by  $\Gamma_+$ , and  $\text{Proj}(A)$  is its exceptional set.

We recall that a morphism  $\alpha$  between filtered rings  $\mathcal{A}$  and  $\mathcal{B}$ , with filtrations  $\{\mathcal{A}_q\}_{q \in \mathbb{Z}}$  and  $\{\mathcal{B}_q\}_{q \in \mathbb{Z}}$  respectively, is said to be *strict* if  $\alpha(\mathcal{A}_q) = \alpha(\mathcal{A}) \cap \mathcal{B}_q$ , for each  $q \in \mathbb{Z}$ . A complex of filtered modules is *strict* when each of its morphisms is strict.

**PROPOSITION 4.1.** *The rings  $\widehat{R}(\mathcal{A})$  and  $A$  are Cohen-Macaulay.*

*Proof.* Since  $A$  is a quotient of  $\widehat{R}(\mathcal{A})$  by a nonzero divisor  $T^{-1}$ , the ring  $\widehat{R}(\mathcal{A})$  is Cohen-Macaulay if and only if  $A$  is so.

By Proposition 4.10 of [10], it is enough to show that the localization  $A_{A_+}$  is Cohen-Macaulay, where  $A_+ = \bigoplus_{q > 0} A_q$ . Therefore, by [5] Lemma 18.1, it is enough to show that there is an  $A$ -sequence in  $A_+$  of length  $n$ . Using [11] we show that this is true. Let  $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be a Newton non-degenerate function germ such that  $\Gamma_+(f) = \Gamma_+$  and  $M$  be the degree of  $f$  with respect to the filtration induced by  $\Gamma_+$ . Then, the Koszul complex generated by  $x_1 \partial f / \partial x_1, \dots, x_n \partial f / \partial x_n$  is strict and  $x_1 \partial f / \partial x_1 T^M, \dots, x_n \partial f / \partial x_n T^M$  form a regular sequence in  $\widehat{R}(\mathcal{A})$ . This also represents a regular sequence in  $A$ .

Next, we give some preliminary definitions and results from [11] that we shall use in this section.

Let  $F_q(\Gamma)$  be the family of all  $q$ -dimensional faces of  $\Gamma$  not contained in the union of the coordinate planes, for any  $q \geq 0$ . Now, we define the rings

$$C_q = \bigoplus_{\Delta \in F_q(\Gamma)} A_\Delta, \quad \text{for any } q \geq 0.$$

These rings are very important, since they take part on an exact sequence that provides a method for the calculation of the Hilbert series of  $A$ .

PROPOSITION 4.2 ([11, 2.4]). *For each face  $\Delta \subseteq \Gamma$ , there exists a grade preserving epimorphism  $\pi_\Delta : A \rightarrow A_\Delta$ . Moreover, for each pair  $\Delta \subset \Delta_1$  there exists a grade preserving epimorphism  $\pi_{\Delta, \Delta_1} : A_{\Delta_1} \rightarrow A_\Delta$  such that  $\pi_{\Delta, \Delta_1} \circ \pi_{\Delta_1} = \pi_\Delta$ .*

The construction of the morphisms of the above proposition is quite simple. The image of an element  $f + \mathcal{A}_{q+1} \in A_q$  through  $\pi_\Delta$  is given by the image in  $A_\Delta$  of  $f_{C(\Delta)}$ . The morphisms  $\pi_{\Delta, \Delta_1}$ , where  $\Delta \subset \Delta_1$ , are defined in an analogous way.

THEOREM 4.3 ([11, 2.6]). *There exists an exact sequence of graded  $A$ -modules with grade preserving morphisms of the form*

$$0 \longrightarrow A \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0. \quad (3)$$

THEOREM 4.4 ([11, 2.9]). *Let  $M$  be the value of the filtration over  $\Gamma$ , then*

*1. given a  $q$ -dimensional face of  $\Gamma$ , the Hilbert series of  $A_\Delta$  is written as*

$$H_{A_\Delta}(t) = \frac{p(t)}{(1-t^M)^{q+1}},$$

*where  $p(t)$  is a polynomial with rational coefficients.*

*2. Moreover, if  $\Delta$  is  $(n-1)$ -dimensional, we have that*

$$\lim_{t \rightarrow 1} (1-t^M)^n H_{A_\Delta}(t) = n! V_n(P(\Delta)),$$

*where  $P(\Delta)$  is the pyramid of base  $\Delta$  and vertex at 0.*

COROLLARY 4.1. *Under the same hypotheses as in the previous theorem, we have the following equivalent equalities:*

- 1.  $\lim_{t \rightarrow 1} (1-t)^n H_A(t) = n! V_n(\phi_\Gamma^{-1}([0, 1]))$ ,*
- 2.  $\lim_{t \rightarrow 1} (1-t^M)^n H_A(t) = n! V_n(\Gamma_-)$ .*

LEMMA 4.1 ([11, 4.3]). *Let  $\mathcal{K} \rightarrow \mathcal{A} \rightarrow \mathcal{B}$  be a complex of filtered  $\mathbb{C}$ -modules which preserve filtrations. If the induced morphism  $\text{gr } \mathcal{K} \rightarrow \text{gr } \mathcal{A} \rightarrow \text{gr } \mathcal{B}$  is exact, then the morphism  $\mathcal{A} \rightarrow \mathcal{B}$  is strict.*

LEMMA 4.2 ([11, 4.4]). *Let  $\alpha : \mathcal{A} \rightarrow \mathcal{B}$  be a strict morphism between filtered  $\mathbb{C}$ -modules. Then, there is a  $\mathbb{C}$ -module isomorphism  $\text{gr}(\mathcal{B}/\alpha(\mathcal{A})) \cong \text{gr } \mathcal{B}/\text{gr}(\alpha)(\text{gr } \mathcal{A})$ .*

Now we introduce some notation. If  $R = \bigoplus_{q \geq 0} R_q$  is a graded ring, we denote by  $R(-a)$  the graded  $R$  module given by  $R(-a)_q = R_{q-a}$ , for all  $q \geq 0$ . Consider the graded complex of free  $R$ -modules:

$$\mathcal{F} : 0 \longrightarrow \bigoplus_{i=1}^{e_\ell} R(-a_i^{(\ell)}) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{e_1} R(-a_i^{(1)}) \longrightarrow \bigoplus_{i=1}^{e_0} R(-a_i^{(0)}) \longrightarrow 0$$

so that all differentials are of degree 0. Then, we define the polynomial

$$q_{\mathcal{F}}(t) = \sum_{i=0}^{\ell} (-1)^i (t^{a_1^{(i)}} + \cdots + t^{a_{e_i}^{(i)}}).$$

Remember that for two graded complexes  $\mathcal{F}$  and  $\mathcal{G}$ ,  $q_{\mathcal{F} \otimes \mathcal{G}}(t) = q_{\mathcal{F}}(t)q_{\mathcal{G}}(t)$ .

### 4.3. Comparing exactness of complexes

We first consider the following complex of free  $\mathcal{A}$ -modules

$$0 \longrightarrow \mathcal{A}^{e_\ell} \xrightarrow{\varphi_\ell} \cdots \xrightarrow{\varphi_2} \mathcal{A}^{e_1} \xrightarrow{\varphi_1} \mathcal{A}^{e_0}. \quad (4)$$

Let  $\Gamma_+ \subseteq \mathbb{R}^n$  be a Newton polyhedron, as usual, we denote by  $\nu_\Gamma(g)$  the level of a series  $g \in \mathcal{A}$  with respect to the filtration induced by  $\Gamma_+$ . Let  $(g_{ij}^{(s)})$  be a representation matrix of the map  $\varphi_s : \mathcal{A}^{e_s} \rightarrow \mathcal{A}^{e_{s-1}}$ ,  $s = 0, 1, \dots, \ell$ . We also fix non-negative integers  $a_i^{(s)}$  for  $s = 0, 1, \dots, \ell$ ,  $i = 1, \dots, e_s$  such that  $\nu_\Gamma(g_{ij}^{(s)}) \geq a_j^{(s)} - a_i^{(s-1)}$ , for any  $s, i, j$ . Therefore the modules  $\mathcal{A}^{e_s}$  are considered as filtered modules by saying that

$$(h_1, \dots, h_{e_s}) \in (\mathcal{A}^{e_s})_q \quad \text{if and only if} \quad h_j \in \mathcal{A}_{q-a_j^{(s)}} \text{ for } j = 1, \dots, e_s.$$

Then, complex (4) becomes a filtered complex. Next, we construct the following graded free complex

$$0 \longrightarrow \widehat{F}_\ell \xrightarrow{\widehat{\varphi}_\ell} \widehat{F}_{\ell-1} \xrightarrow{\widehat{\varphi}_{\ell-1}} \cdots \xrightarrow{\widehat{\varphi}_2} \widehat{F}_1 \xrightarrow{\widehat{\varphi}_1} \widehat{F}_0, \quad (5)$$

where we set the modules

$$\widehat{F}_s = \bigoplus_{i=1}^{e_s} \widehat{R}(\mathcal{A})(-a_i^{(s)}), \quad s = 0, 1, \dots, \ell,$$

and we assume that the maps  $\widehat{\varphi}_s : \widehat{F}_s \longrightarrow \widehat{F}_{s-1}$  are defined by the matrix

$$\left( g_{ij}^{(s)} T^{a_j^{(s)}} - a_i^{(s-1)} \right)_{i=1, \dots, e_s; j=1, \dots, e_{s-1}}.$$

Therefore, the morphism  $\widehat{\varphi}_s : \widehat{F}_s \longrightarrow \widehat{F}_{s-1}$  is graded of degree 0, for all  $s = 1, \dots, \ell$ .

Now, we set  $F_s = \bigoplus_{i=1}^{e_s} A(-a_i^{(s)})$ , for all  $s = 0, 1, \dots, \ell$ , and consider the complex

$$0 \longrightarrow F_\ell \xrightarrow{\overline{\varphi}_\ell} F_{\ell-1} \xrightarrow{\overline{\varphi}_{\ell-1}} \dots \xrightarrow{\overline{\varphi}_2} F_1 \xrightarrow{\overline{\varphi}_1} F_0, \quad (6)$$

where  $\overline{\varphi}_s : F_s \rightarrow F_{s-1}$  is the map induced by the map  $\widehat{\varphi}_s : \widehat{F}_s \rightarrow \widehat{F}_{s-1}$ .

Let  $\Delta \in F_q(\Gamma)$ ,  $q = 0, 1, \dots, n-1$ . We set

$$F_{s\Delta} = \bigoplus_{i=1}^{e_s} A_\Delta(-a_i^{(s)}), \quad s = 0, 1, \dots, \ell,$$

and consider the complex induced from the previous one

$$0 \longrightarrow F_{\ell\Delta} \xrightarrow{\overline{\varphi}_\ell|_\Delta} F_{\ell-1\Delta} \xrightarrow{\overline{\varphi}_{\ell-1}|_\Delta} \dots \xrightarrow{\overline{\varphi}_2|_\Delta} F_{1\Delta} \xrightarrow{\overline{\varphi}_1|_\Delta} F_{0\Delta}. \quad (6_\Delta)$$

LEMMA 4.3. *If the complex (4) is exact and each  $\varphi_s$  is strict then the complexes (5) and (6) are exact.*

*Proof.* This is proved in a straightforward way, and we omit the details.

We set the number  $r_k = \sum_{i=k}^{\ell} (-1)^{i-k} e_i$ , for all  $k = 0, \dots, \ell$ .

PROPOSITION 4.3. *The following conditions are equivalent:*

1. the complex (4) is exact and each  $\varphi_s$  is strict;
2. the complex (6) is exact;
3. the complex (6<sub>Δ</sub>) is exact in dimensions  $\geq n-q$ , for any  $\Delta \in F_q(\Gamma)$ ,  $q = 0, 1, \dots, n-1$ ;
4. the complex (5) is exact.

*Proof.* The previous lemma implies (i)  $\implies$  (ii) and (i)  $\implies$  (iv). We first show (ii)  $\implies$  (i). If we suppose that the complex (6) is exact, by Lemma 4.1 we have that (4) is a complex and each  $\varphi_s$  is strict. We remark that, by the exactness of (6), for any  $a_q \in (\mathcal{A}^{e_i})_q$  with  $\varphi_i(a_q) = 0$ , there exists  $b_q \in (\mathcal{A}^{e_{i+1}})_{q+1}$  such that  $a_q - \varphi_{i+1}(b_q) \in (\mathcal{A}^{e_i})_{q+1}$ . Let  $a \in \mathcal{A}^{e_i}$  with  $\varphi_i(a) = 0$ . We choose  $q_0$  so that  $a \in (\mathcal{A}^{e_i})_{q_0} \setminus (\mathcal{A}^{e_i})_{q_0+1}$ . Setting  $a_{q+1} = a_q - \varphi_{i+1}(b_q)$ , for  $q \geq q_0$ , we have  $\varphi_{i+1}(\sum_{q \geq q_0} b_q) = a$ , and we are done by completeness of  $\mathcal{A}$ .

For the proof of (iii)  $\implies$  (ii), we consider the two (horizontal and vertical) spectral sequences of the double complex which is the tensor product of the complex (6) and the complex in Theorem 4.3. The  $(p, q)$ -terms in  ${}^0E$  terms are isomorphic to  $F_{-q} \otimes C_{n-p}$ .

Since the horizontal sequences in the double complex are exact, we obtain  ${}_{\text{hor}}^1 E^{p,q} = 0$ , and we thus obtain all the cohomologies of the total complex vanish. Next we observe that the condition (iii) implies  ${}_{\text{vert}}^1 E^{p,q} = 0$  if  $p + q \leq 0$  and  $p \geq 1$ . So we obtain

$${}_{\text{vert}}^1 E^{0,q} = {}_{\text{vert}}^2 E^{0,q} = \dots = {}_{\text{vert}}^\infty E^{0,q} = 0, \quad \text{if } q < 0,$$

and therefore the complex (6) is exact.

We show here that (ii) $\implies$ (iii). Suppose that (6) is exact and let  $\Delta \in F_q(\Gamma)$ . We first remark that

$$\begin{aligned} \dim A_\Delta / I_{r_i}(\overline{\varphi}_i|_\Delta) &\leq \dim A / I_{r_i}(\overline{\varphi}_i) \\ &\leq n - \text{ht } I_{r_i}(\overline{\varphi}_i) \\ &\leq n - \text{depth}(I_{r_i}(\overline{\varphi}_i), A), \quad (\text{by [3] Proposition 1.2.14}) \\ &\leq n - i \quad (\text{by acyclicity of the complex (6)}). \end{aligned}$$

Since  $A_\Delta$  is Cohen-Macaulay [9], we obtain

$$\text{depth}(I_{r_i}(\overline{\varphi}_i|_\Delta), A_\Delta) = \dim A_\Delta - \dim A_\Delta / I_{r_i}(\overline{\varphi}_i|_\Delta) \geq q + 1 - n + i.$$

Then, item (iii) follows from Buchsbaum-Eisenbud acyclicity criterion.

We assume (iv). Since  $\widehat{R}(\mathcal{A})$  is Cohen-Macaulay, we have that

$$\begin{aligned} \dim A / I_{r_i}(\overline{\varphi}_i) + 1 &= \dim \widehat{R}(\mathcal{A}) / I_{r_i}(\widehat{\varphi}_i) \\ &= n + 1 - \text{depth}(I_{r_i}(\widehat{\varphi}_i), \widehat{R}(\mathcal{A})) \\ &\leq n + 1 - i \quad (\text{by acyclicity of (5)}). \end{aligned}$$

Since  $A$  is Cohen-Macaulay, we obtain

$$\text{depth}(I_{r_i}(\overline{\varphi}_i), A) = \dim A - \dim A / I_{r_i}(\overline{\varphi}_i) \geq n - n + i = i.$$

Then, by Buchsbaum-Eisenbud acyclicity criterion, we have (iv) $\implies$ (ii).

#### 4.4. Non-degeneracy of the matrix $(g_{ij})$

In this section we assume that  $(g_{ij})$  is the matrix defining the morphism  $\varphi_1 : \mathcal{A}^{e_1} \rightarrow \mathcal{A}^{e_0}$  of complex (4). We also denote the map  $\varphi_1$  by  $\varphi$ . Suppose that  $\text{rank } \varphi = e_0$  and  $\text{depth}(I_{e_0}(\varphi)) = \ell$ . Assuming  $g_{ij} = \sum_k a_k x^k$  and  $\Delta \in I_q$ , we set

$$G_{ij,\Delta} = \text{the class of } \sum \{a_k x^k : k \in C(\Delta), \nu_\Gamma(x^k) = a_j^{(1)} - a_i^{(0)}\} \in A_\Delta.$$

DEFINITION 4.1. We say that the matrix  $(g_{ij})$  is *non-degenerate on  $\Gamma_+$*  if  $I_{e_0}(G_{ij,\Delta})$  is an ideal of depth  $\ell - n + q + 1$  in  $A_\Delta$ , for each face  $\Delta \in F_q(\Gamma)$ .

We remember that  $\text{depth } I_{e_0}(G_{ij,\Delta})$  is just the codimension (or height) of  $I_{e_0}(G_{ij,\Delta})$  in  $A_\Delta$  because  $A_\Delta$  is Cohen-Macaulay.

PROPOSITION 4.4. *Suppose that the complex  $(6_\Delta)$  is exact in dimensions  $\geq n - q$ , for all  $\Delta \in F_q(\Gamma)$ ,  $q = 0, 1, \dots, n - 1$ . Then, the matrix  $(g_{ij})$  is non-degenerate on  $\Gamma_+$ .*

*Proof.* Let  $\Delta \in F_q(\Gamma)$ . We choose  $\Delta_1 \in F_{n-1}(\Gamma)$  such that  $\Delta$  is a face of  $\Delta_1$ . Since (6) is exact and  $\sum_{i=0}^{\ell} (-1)^i e_i = 0$ , we have  $e_0 = r_1$ , and we obtain  $\sqrt{I_{r_\ell}(\overline{\varphi}_\ell|_{\Delta_1})} = \sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}$ , by [5] Corollary 20.12 (observe that we can use this result, since the annihilator of  $M$  is not zero, by Theorem 19.8 of [12]). Therefore

$$\begin{aligned} \text{depth}(I_{r_1}(\overline{\varphi}_1|_\Delta), A_\Delta) &= \text{depth}(I_{r_1}(\overline{\varphi}_1|_{\Delta_1})A_\Delta, A_\Delta) && \text{(by [5] Corollary 20.5)} \\ &= \text{depth}(\sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}A_\Delta, A_\Delta) \\ &= \text{depth}(\sqrt{I_{r_\ell}(\overline{\varphi}_\ell|_{\Delta_1})}A_\Delta, A_\Delta) && \text{(by the above equality)} \\ &= \text{depth}(I_{r_\ell}(\overline{\varphi}_1|_{\Delta_1})A_\Delta, A_\Delta) \\ &= \text{depth}(I_{r_\ell}(\overline{\varphi}_1|_\Delta), A_\Delta) && \text{(by [5] Corollary 20.5)} \\ &\geq \ell - n + q + 1, \end{aligned}$$

where the last inequality is a consequence of the hypothesis and the Buchsbaum-Eisenbud acyclicity criterion applied to complex  $(6_\Delta)$ .

#### 4.5. The case for complexes coming from a free resolution of perfect modules

Set  $R = \mathbb{C}[X_1, \dots, X_m]$  then we define a grading on  $R$  by setting the degree of  $X_i$  equal to  $d_i$ , for all  $i = 1, \dots, m$ .

Let  $\mathfrak{M}$  be a graded perfect  $R$ -module with projective dimension  $\ell \geq 1$ . We consider a graded free resolution of  $\mathfrak{M}$ :

$$0 \longrightarrow \tilde{F}_\ell \xrightarrow{\tilde{\varphi}_\ell} \dots \xrightarrow{\tilde{\varphi}_2} \tilde{F}_1 \xrightarrow{\tilde{\varphi}_1} \tilde{F}_0 \longrightarrow \mathfrak{M} \longrightarrow 0, \quad (7)$$

where  $\tilde{F}_s = \bigoplus_{i=1}^{e_s} R(-\tilde{a}_i^{(s)})$  and the map  $\tilde{\varphi}_s : \tilde{F}_s \rightarrow \tilde{F}_{s-1}$  are of degree 0.

The perfect  $R$ -module  $\mathfrak{M}$  has nonzero annihilator, since

$$\text{depth}(\text{Ann}(\mathfrak{M}), R) = \ell.$$

By Theorem 19.8 of [12], we have that  $\sum_{i=0}^{\ell} (-1)^i e_i = 0$ .

We consider homogeneous elements  $f_1, \dots, f_m \in \widehat{R}(\mathcal{A})$ , then we define an  $R$ -module structure on  $\widehat{R}(\mathcal{A})$  via the substitution  $X_i \mapsto f_i$ . Then we can consider the tensor product  $\mathfrak{M} \otimes_R \widehat{R}(\mathcal{A})$ .

PROPOSITION 4.5. *Suppose that the complex (5) is obtained from (7) by tensoring  $\widehat{R}(\mathcal{A})$  over  $R$ , that is,  $(5) = (7) \otimes_R \widehat{R}(\mathcal{A})$ . We suppose that  $\text{ht}(I_{r_1}(\varphi_1)) = \ell$ . Then, the conditions in Proposition 4.3 are equivalent to the non-degeneracy of  $(g_{ij})$  on  $\Gamma_+$ .*

*Proof.* It is enough to prove that condition (iii) of Proposition 4.3 is obtained from the hypothesis of non-degeneracy. By Lemma 4.3 we have that  $\text{pd}(\text{coker } \widehat{\varphi}) \leq \ell$ . Since

$\mathcal{A} = \widehat{R}(\mathcal{A}) / (T^{-1} - 1) \widehat{R}(\mathcal{A})$ , we have

$$\text{depth}(I_{e_0}(\widehat{\varphi}), \widehat{R}(\mathcal{A})) \geq \text{depth}(I_{e_0}(\varphi), \mathcal{A}).$$

Since  $\text{ht}(I_{r_1}(\varphi)) = \ell$ ,  $\text{depth}(I_{e_0}(\varphi), \mathcal{A}) = \ell$ . Then, we have that  $\text{coker } \widehat{\varphi}$  is a perfect  $\widehat{R}(\mathcal{A})$ -module.

Let  $\Delta \in F_{n-1}$ , consider  $A_\Delta$  as an  $\widehat{R}(\mathcal{A})$ -module through the natural maps  $\widehat{R}(\mathcal{A}) \rightarrow A \rightarrow A_\Delta$ . Since  $\text{coker } \widehat{\varphi}$  is a perfect  $\widehat{R}(\mathcal{A})$ -module and  $\text{depth}(I_{e_0}(\overline{\varphi}|_\Delta), A_\Delta) \geq \ell$ , Theorem 4.2 implies that the complex  $(6_\Delta)$  is exact.

Let  $\Delta \in F_q(\Gamma)$ ,  $q = 0, 1, \dots, n-2$ . By Buchsbaum-Eisenbud acyclicity criterion, the complex  $(6_\Delta)$  is acyclic in dimensions  $\geq n - q$  if and only if the following conditions hold.

$$\text{depth}(I_{r_i}(\overline{\varphi}_i|_\Delta), A_\Delta) \geq i - n + q + 1, \quad \text{for } i = n - q, \dots, \ell. \quad (E_\Delta)$$

We choose  $\Delta_1 \in F_{n-1}$  such that  $\Delta$  is a face of  $\Delta_1$ . Because  $(6_{\Delta_1})$  is exact, we have  $\sqrt{I_{r_i}(\overline{\varphi}_i|_{\Delta_1})} \supseteq \sqrt{I_{r_i}(\overline{\varphi}_i|_\Delta)}$  for  $i = 1, \dots, \ell$ . Then, we obtain the following inequalities:

$$\begin{aligned} \text{ht}(I_{r_i}(\overline{\varphi}_i|_\Delta)) &= \text{depth}(I_{r_i}(\overline{\varphi}_i|_\Delta), A_\Delta) && \text{(since } A_\Delta \text{ is Cohen-Macaulay)} \\ &= \text{depth}(\sqrt{I_{r_i}(\overline{\varphi}_i|_\Delta)}, A_\Delta) \\ &\geq \text{depth}(\sqrt{I_{r_i}(\overline{\varphi}_i|_{\Delta_1})} A_\Delta, A_\Delta) && \text{(since } \sqrt{I_{r_i}(\overline{\varphi}_i|_\Delta)} \supseteq \sqrt{I_{r_i}(\overline{\varphi}_i|_{\Delta_1})} A_\Delta) \\ &\geq \text{depth}(\sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})} A_\Delta, A_\Delta) && \text{(since } \sqrt{I_{r_i}(\overline{\varphi}_i|_{\Delta_1})} \supseteq \sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})}) \\ &= \text{ht}(\sqrt{I_{r_1}(\overline{\varphi}_1|_{\Delta_1})} A_\Delta) && \text{(since } A_\Delta \text{ is Cohen-Macaulay)} \\ &= \text{ht}(I_{r_1}(\overline{\varphi}_\Delta)) = \ell - n + q + 1. \end{aligned}$$

In the last equality we have applied the hypothesis. Then, we see that the conditions in  $(E_\Delta)$  are satisfied and we obtain condition (iii) of Proposition 4.3.

We summarize the result we obtained as follows.

**THEOREM 4.5.** *Suppose that there exist a perfect  $R$ -module  $\mathfrak{M}$ , and a resolution (7) of  $\mathfrak{M}$  so that (5) = (7)  $\otimes_R \widehat{R}(\mathcal{A})$ . Then the following conditions are equivalent:*

1. the complex (4) is exact and each  $\varphi_s$  is strict;
2. the complex (6) is exact;
3. the complex  $(6_\Delta)$  is acyclic in dimensions  $\geq n - q$  for any  $\Delta \in F_q(\Gamma)$ ,  $q = 0, 1, \dots, n-1$ ;
4. the complex (5) is exact;
5. the matrix  $(g_{ij})$  is non-degenerate.

**THEOREM 4.6.** *If we suppose that  $\text{coker } \varphi$  has finite codimension, then it is automatically perfect with projective dimension  $n$ . Suppose that there exist a perfect  $R$ -module  $\mathfrak{M}$ , and*

a resolution (7) of  $\mathfrak{M}$  so that (5) = (7)  $\otimes_R \widehat{R}(\mathcal{A})$ . Suppose that the matrix  $(g_{ij})$  is non-degenerate on  $\Gamma_+$  and that  $\text{coker } \varphi$  has finite codimension, then

$$\dim_{\mathbb{C}} \text{coker } \varphi = (-1)^n \frac{q^{(n)}(1)}{M^n} V_n(\Gamma_-), \quad \text{where} \quad q(t) = q_{(6)}(t) = \sum_{s=0}^n \sum_{i=1}^{e_s} (-1)^s t^{a_i^{(s)}},$$

$M$  denotes the common value of the filtration induced by  $\Gamma_+$  along its boundary and  $q^{(n)}(1)$  is the  $n$ -th derivative of  $q_{(6)}$  at 1.

*Proof.* By Proposition 4.5 the complex (6) is exact, and the Hilbert series of  $\text{coker } \bar{\varphi}$  is given by  $H_{\text{coker } \bar{\varphi}}(t) = H_A(t)q(t)$ , where  $H_A(t)$  is the Hilbert series of  $A$  and  $q(t)$  is the polynomial described above. Then, we conclude

$$\begin{aligned} \dim_{\mathbb{C}} \text{coker } \varphi &= \dim_{\mathbb{C}} \text{coker } \bar{\varphi} && \text{(by Lemmas 4.1 and 4.2)} \\ &= \lim_{t \rightarrow 1} H_A(t)q(t) \\ &= n! V_n(\Gamma_-) \lim_{t \rightarrow 1} \frac{q(t)}{(1-t^M)^n} && \text{(by Corollary 4.1)} \\ &= (-1)^n \frac{q^{(n)}(1)}{M^n} V_n(\Gamma_-). \end{aligned}$$

*Proof of Theorem 3.1, part  $\implies$  of (ii).* We apply here the above theorem to the Koszul complex associated to the system  $\{g_1, \dots, g_n\}$ . Since they generate an ideal of finite codimension, this complex is acyclic. It is given by the tensor product of

$$0 \longrightarrow A(-d_i) \xrightarrow{g_i} A \longrightarrow 0.$$

Therefore, the polynomial  $q(t)$  of Theorem 4.6 associated to this complex is given by  $q(t) = \prod_{i=1}^n (1-t^{d_i})$ . Moreover,

$$\lim_{t \rightarrow 1} \frac{q(t)}{(1-t)^n} = \lim_{t \rightarrow 1} \prod_{i=1}^n (1+t+\dots+t^{d_i-1}) = d_1 \dots d_n,$$

and we obtain that

$$\dim_{\mathbb{C}} \mathcal{A}/\langle g_1, \dots, g_n \rangle = \frac{d_1 \dots d_n}{M^n} n! V_n(\Gamma_-).$$

### 5. COMPLEXES

To deduce formulas for  $\dim_{\mathbb{C}} \mathcal{A}/\langle g_1, \dots, g_{e_1} \rangle$ , we consider free resolutions of perfect modules which are explicitly described. Then we can determine the grade shifting (i.e.,  $(-a_i^{(s)})$ ) at each term so that each differential is of degree 0. As we have seen, it can be done easily for

the Koszul complex. Other complexes that we also consider here are the Eagon-Northcott complex, which gives the resolution for determinantal ideals of maximal order and the Buchsbaum-Rim complex, which provides a resolution for  $\text{coker } \varphi$  where  $\varphi : \mathcal{A}^p \rightarrow \mathcal{A}^s$  is a morphism of free modules. We also remark that the Lascoux complex [15], a resolution for determinantal ideals which involve minors of lower order, can also be used to obtain formulas for the codimension of these ideals.

### 5.1. Eagon-Northcott complex

Let  $R$  be a Noetherian ring and  $U = (g_{ij})$  a  $p$  by  $s$  matrix,  $p \geq s$ , with  $g_{ij} \in R$ ,  $I_s(U)$  denotes the ideal generated by the maximal minors of  $U$ . We set

$$F_k = \wedge^s(R^s)^* \otimes S_{k-1}(R^s)^* \otimes \wedge^{s+k-1}R^p, \quad k = 1, \dots, p-s+1,$$

where  $S_k(R^s)$  denotes the  $k$ -th symmetric algebra of  $R^s$ , and  $F_0 = R$ . Then, the Eagon-Northcott complex is given by

$$EN : 0 \longrightarrow F_{p-s+1} \longrightarrow F_{p-s} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0.$$

More details on the construction of this complex are given in [6]. This is a free resolution of  $R/I_s(U)$  if  $\text{depth}(I_s(U), R) = p-s+1$ . Let  $w_1, \dots, w_p, d_1, \dots, d_s$  be non-negative integers. If  $R$  is a graded ring and  $g_{ij}$  are homogeneous elements of degree  $d_j - w_i$ , this is a graded complex and

$$q_{EN}(t) = 1 + \sum_{k=1}^{p-s+1} (-1)^k \sum_{I_k, J_k} t^{d+d_{J_k}-w_{I_k}},$$

where  $d = d_1 + \cdots + d_s$ ,  $d_{J_k} = d_{j_1} + \cdots + d_{j_{k-1}}$  for  $J_k = (j_1 \leq \cdots \leq j_{k-1})$ , and  $w_{I_k} = w_{i_1} + \cdots + w_{i_{s+k-1}}$ , for  $I_k = (i_1 < \cdots < i_{s+k-1})$ .

Thus we obtain that

$$q_{EN}^{(p-s+1)}(1) = \sum_{k=1}^{p-s+1} (-1)^k \sum_{I_k, J_k} (d + d_{J_k} - w_{I_k})_{p-s+1},$$

where  $(n)_k = n(n-1)\cdots(n-k+1)$ .

EXAMPLE 5.1. Consider the matrix with entries in  $\mathcal{A} = \mathbb{C}[[x, y]]$  given by

$$U = \begin{pmatrix} x^\alpha + x^r y^s + 2y^\beta & -x^r y^s + y^\beta \\ -x^\alpha & x^\alpha + x^r y^s - y^\beta \\ 2x^\alpha - x^r y^s + y^\beta & x^\alpha + x^r y^s + 2y^\beta \end{pmatrix}$$

Let us consider the polyhedron  $\Gamma_+$  of the ideal generated by  $\{x^\alpha, x^r y^s, y^\beta\}$ . The Newton filtration associated to  $\Gamma_+$  is given in Example 3.1, then we assume that  $\alpha s < \beta(\alpha - r)$ .

We observe that this matrix is non-degenerate on  $\Gamma_+$  and that  $I_2(U)$  has finite codimension, then the associated Eagon-Northcott complex is acyclic. In this case, the Eagon-

Northcott complex has the form

$$0 \longrightarrow \mathcal{A}^2 \xrightarrow{\varphi_2} \mathcal{A}^3 \xrightarrow{\varphi_1} \mathcal{A} \longrightarrow \mathcal{A}/I_2(U) \longrightarrow 0.$$

Here the morphism  $\varphi_2 : \mathcal{A}^2 \rightarrow \mathcal{A}^3$  is defined by the matrix  $U$  and  $\varphi_1 : \mathcal{A}^3 \rightarrow \mathcal{A}$  is given by  $(J_{2,3} \ J_{3,1} \ J_{1,2})$ , where  $J_{i,j}$  is the minor of the matrix  $U$  constructed taking rows  $i$  and  $j$ . Let us denote the graded ring associated to the mentioned filtration on  $\mathcal{A}$  by  $A$ . Then, the associated graded complex is given by

$$0 \longrightarrow F_2 \xrightarrow{\bar{\varphi}_2} F_1 \xrightarrow{\bar{\varphi}_1} F_0 \longrightarrow F_0/I_2(U') \longrightarrow 0,$$

where  $F_2 = A(-\alpha\beta rs) \oplus A(-\alpha\beta rs)$ ,  $F_1 = A(-2\alpha\beta rs) \oplus A(-2\alpha\beta rs) \oplus A(-2\alpha\beta rs)$ ,  $F_0 = A$  and  $U'$  is the matrix whose entries are the initial forms of the entries of  $U$ . Then, by Theorem 4.6 we obtain that  $\dim_{\mathbb{C}} \mathcal{A}/I_2(U) = \frac{2q^{(2)}(1)}{M^2} V_2(\Gamma_-) = 3(\alpha s + \beta r)$ .

## 5.2. Buchsbaum-Rim complex

Let  $R$  be a Noetherian ring, and  $U = (g_{ij})$  a  $p$  by  $s$  matrix,  $p \geq s$ , with  $g_{ij} \in R$ . The matrix  $U$  defines a map  $\varphi : R^p \rightarrow R^s$ . We set

$$F_k = \wedge^s(R^s)^* \otimes S_{k-2}(R^s)^* \otimes \wedge^{s+k-1} R^p, \quad k = 2, \dots, p-s+1,$$

$F_1 = R^p$ ,  $F_0 = R^s$ , and consider the Buchsbaum-Rim complex

$$BR : 0 \longrightarrow F_{p-s+1} \longrightarrow F_{p-s} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0.$$

More details on the construction of this complex are given in [5, pp. 594–595].

This is a free resolution of  $\text{coker } \varphi$  if  $\text{depth}(I_s(\varphi), R) = p-s+1$ . Let  $w_1, \dots, w_p, d_1, \dots, d_s$  be non-negative integers. If  $R$  is a graded ring and  $g_{ij}$  are homogeneous elements of degree  $w_j - d_i$ , this is a graded complex and

$$q_{BR}(t) = \sum_{i=1}^s t^{d_i} - \sum_{j=1}^p t^{w_j} + \sum_{k=2}^{p-s+1} (-1)^k \sum_{I_k, J_k} t^{d + d_{J_k} - w_{I_k}},$$

where  $d = d_1 + \dots + d_s$ ,  $d_{J_k} = d_{j_1} + \dots + d_{j_{k-2}}$  for  $J_k = (j_1 \leq \dots \leq j_{k-2})$ , and  $w_{I_k} = w_{i_1} + \dots + w_{i_{s+k-1}}$  for  $I_k = (i_1 < \dots < i_{s+k-1})$ . Thus we obtain that

$$q_{BR}^{(p-s+1)}(1) = \sum_{i=1}^s (d_i)_{p-s+1} - \sum_{j=1}^p (w_j)_{p-s+1} + \sum_{k=2}^{p-s+1} (-1)^k \sum_{I_k, J_k} (d + d_{J_k} - w_{I_k})_{p-s+1}.$$

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