CADERNOS DE MATEMÁTICA **02**, 159–164 March (2001)
ARTIGO NÚMERO SMA# 106

Foliations by planes in the complement of a compact set

J. L. Arraut and C. Biasi

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: arraut@icmc.sc.usp.br and biasi@icmc.sc.usp.br

Let N be a closed orientable n-manifold, $n \geq 3$, and K be a compact non-empty subset such that $\pi_1(N \setminus K)$ is finitely generated. We prove that the existence of a transversally orientable cod.1 foliation on $N \setminus K$ with leaves homeomorphic to $\mathbb{R}^{n-1}$, in the relative topology, implies that K must be connected. If in addition one imposes some restrictions on the homology of K, then N must be a homotopy sphere. An application to Lie group actions is also given. March, 2001 ICMC-USP

AMS Subject Classification Primary 57S28; Secondary 57R25, 57R30

Keywords foliation by planes, homotopy spheres, actions.

1. INTRODUCTION

A codimension one C^2 foliation defined on a n-manifold such that all leaves are diffeomorphic to $\mathbb{R}^{n-1}$ is called a *foliation by planes.* Two foliated manifolds $(V, \mathcal{F})$ and $(V', \mathcal{F}')$ are said to be *conjugated* if there exists a homeomorphism $h : V \to V'$ that takes leaves of $\mathcal{F}$ onto leaves of $\mathcal{F}'$. Several authors have studied foliation by planes. The common idea behind these studies is that very few manifolds admit such foliations. In [6], Rosenberg and Sondow proved that the torus T^3 is the only closed 3-manifold which admits a foliation by planes. Let M be a compact 3-manifold whose boundary is a union of tori. A foliation of M tangent to the boundary and such that the interior leaves are planes is called a Reeb foliation. In [7], Rosenberg and Roussarie proved that if M admits a C^2 Reeb foliation, then M is diffeomorphic either to $S^1 \times D^2$ or to $[0,1] \times T^2$. In [5], Palmeira studied foliations by planes on open manifolds. Among others, he proved:

THEOREM 1.1. *([3] page 125) If V is an orientable open n-manifold, $n \geq 3$, which has finitely generated fundamental group and with a transversally orientable C^2 foliation by closed planes $\mathcal{F}$, then there exists an orientable surface Σ and an orientable one dimensional foliation $\mathcal{F}_0$ of Σ such that $(V, \mathcal{F})$ is conjugated by a diffeomorphism to $(\Sigma \times \mathbb{R}^{n-2}, \mathcal{F}_0 \times \mathbb{R}^{n-2})$. The leaves of the foliation $\mathcal{F}_0 \times \mathbb{R}^{n-2}$ are of the form $L_0 \times \mathbb{R}^{n-2}$, where $L_0 \in \mathcal{F}_0$.*

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

When V is simply connected it is not necessary to assume that the leaves are closed and besides the surface Σ is precisely $\mathbb{R}^2$.

In this paper we shall make extensive use of Palmeira's theorem to study foliations by planes on a closed manifold N minus a compact subset K. The results obtained apply to the case of a singular foliation on N defined by a C^2 integrable one form in which all regular leaves are planes that cluster in K, where K is the union of all singular leaves. The conclusions that we obtain point in the same direction that the former results i.e., that very few closed manifolds admit singular foliations by planes. Here, we prove the following propositions:

Proposition 1.1. *Let V be an open orientable manifold of dimension $n \geq 3$ and K a compact subset such that $\pi_1(V \setminus K)$ is finitely generated. If there exists a transversally orientable foliation by planes of $V \setminus K$ such that each leaf is closed, then $K = \emptyset$.*

Proposition 1.2. *Let N be a closed connected and orientable n-manifold, with $n \geq 3$, and K a compact non-empty subset such that $\pi_1(N \setminus K)$ is finitely generated. If there exists a transversally orientable foliation by planes of $N \setminus K$ such that each leaf is closed, then K is connected.*

Example 1.1. Consider the singular foliation of S^2 whose regular leaves are the meridians and the singular ones are the poles P_1 and P_2 and form the product $S^2 \times [0,1]$. Next, identify $(x,1)$ with $(\psi(x),0)$, where $\psi : S^2 \to S^2$ is a rotation, that fix the poles, of an angle α such that the numbers $\{\alpha, 2\pi\}$ are linearly independent as elements of the vector field $\mathbb{R}$ over the rationals. In this way one obtains a foliation by planes of $N = S^2 \times S^1$ with a singular set $K = \{P_1\} \times S^1 \cup \{P_2\} \times S^1$ which is not connected. Notice that here the regular leaves are not closed in $N \setminus K$ instead they are dense in N.

Due to Proposition 1.2 there is no lost of generality if one assumes, in the next two theorems, that K is connected.

Theorem 1.2. *Let N be a closed connected and orientable n-manifold, with $n \geq 3$, and $K \subset N$ a non-empty compact and connected ANR such that $\pi_1(N \setminus K)$ is finitely generated. Assume that $H_p(K;\mathbb{Z}) = 0$, for each $0 < p \leq [\frac{n}{2}]$. If there exists a transversally orientable foliation by planes on $N \setminus K$ such that each leaf is closed, then N is a homology sphere.*

Theorem 1.3. *Let N and K be as in Theorem 1.2 and assume, besides, that $H^{n-2}(K) = 0$ and that $dim_{top} K \leq n-2$. Then N is a homotopy sphere for $n = 3$ and homeomorphic to S^n if $n \geq 4$.*

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

COROLLARY 1.1. *Let N be a closed connected and orientable n-manifold, with $n \geq 3$, and K a compact non-empty subset with $dim_{top}K = 0$ and such that $\pi_1(N \setminus K)$ is finitely generated. If there exists a transversally orientable foliation by planes on $N \setminus K$ such that each leaf is closed, then*
i) K contains only one point,
ii) N is homeomorphic to S^n.

EXAMPLE 1.2. If $(x_1, x_2, ..., x_n)$ are the coordinates of a point in $\mathbb{R}^n$, then the form dx_n defines a foliation by closed planes of $\mathbb{R}^n = S^n \backslash \{\infty\}$. The form $e^{-r^2} dx_n$, where $r^2 = x_1^2 + x_2^2 + ... + x_n^2$, defines a foliation by planes of S^n with $\{\infty\}$ as the only singular leave.

Now, let G denote a Lie group diffeomorphic to $\mathbb{R}^{n-1}$. For $n - 1 = 2$ there are two such Lie groups: $\mathbb{R}^2$ and A^2 the connected component of the identity of the group of affine transformations of $\mathbb{R}$. Given an action of G on N, a point p is said to be a *singular* point of the action if the orbit of p has topological dimension strictly less than $(n - 1)$.

THEOREM 1.4. *Let N be a closed connected orientable n-manifold, $n \geq 3$, with $\pi_1(N)$ finite and G a Lie group diffeomorphic to $\mathbb{R}^{n-1}$ acting in class C^2 on N. Assume that the set K of singular points of the action is a non-empty finite subset. Then*
i) K contains only one point,
ii) N is homeomorphic to S^n.

EXAMPLE 1.3. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid x_1^2 + ... + x_{n+1}^2 = 1\}$, $F = (0, ..., 0, 1)$, $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$ and $P : S^n \backslash F \to \mathbb{R}^n$ the projection using F as focus. The vector fields $P_*^{-1} \partial/\partial x_j$, $1 \leq j \leq n - 1$, defined on $S^n \backslash F$ extend to C^∞ vector fields X_j on S^n and clearly any two of them commute. They define an action of $\mathbb{R}^{n-1}$ on S^n where all orbits are planes that cluster in the stationary point F.

Notice that in Theorem 1.4 we did not assume neither that the leaves were planes nor that they were closed in $N \backslash K$. On the other hand we imposed that $\pi_1(N)$ be finite. It would be interesting to decide if this assumption is really necessary.

THEOREM 1.5. *Let N be a closed connected an orientable 3-manifold and K a circle. Suppose that there exists a transversally orientable foliation by planes of $N \setminus K$ such that each leaf is closed, then N admits a Heegaard diagram of genus one and thus $\pi_1(N)$ is a cyclic group. Thus*
i) If $\pi_1(N) = 0$, then N is homeomorphic to S^3,
ii) if $\pi_1(N) = \mathbb{Z}$, then N is homeomorphic to $S^1 \times S^2$.

EXAMPLE 1.4. Consider the following three foliations on $S^1 \times D^2$, the compact solid torus. Using $(\phi, (x, y))$ as coordinates put $\omega_1 = d\phi$ and $\omega_2 = q^*(-y\,dx + x\,dy)$, where $q : S^1 \times D^2 \to D^2$ is the projection. The leaves of the foliation $\mathcal{F}_1$, defined by ω_1, are the disks $\{\phi\} \times D^2$. The regular leaves of $\mathcal{F}_2$, defined by ω_2, are of the form $S^1 \times \{ray\}$, and the singular leaf is the central circle $K = S^1 \times \{0\}$. $\mathcal{F}_3$ is obtained from $\mathcal{F}_1$ by turbulaizing the disks along the central circle. Now consider a copy of the solid torus with the foliation $\mathcal{F}_1$ and another copy with $\mathcal{F}_2$ and identify their boundaries through the map that sends meridians onto parallels. One obtains a foliation by closed planes of $S^3 \setminus K$. If one uses $\mathcal{F}_1$ on one copy and $\mathcal{F}_3$ on the other and identify them with the identity map of the boundary, then one obtains a foliation by closed planes of $S^1 \times S^2 \setminus K$.

It should be remarked that we only considered foliations by planes on $N \setminus K$ such that each plane was closed. The analogous questions without this assumption remain open.

2. PROOF OF THE RESULTS

In this section we give the proof of the statements that appear in the introduction with the exception of Proposition 1.1 whose proof is very similar to that of Proposition 1.2. We start with a lemma that translates Theorem 1.1 into homological information on $N \setminus K$.

LEMMA 2.1. *Let N be a closed connected and orientable n-manifold, $n \geq 3$, and K a compact non-empty subset such that $N \setminus K$ is connected and $\pi_1(N \setminus K)$ is finitely generated. If there exists a transversally orientable foliation by planes of $N \setminus K$ such that each leaf is closed, then $H_p(N \setminus K) = 0$ for each $p \geq 2$.*

Proof: $N \setminus K$ is a connected orientable n-manifold with a finitely generated fundamental group and with a transversally orientable foliation $\mathcal{F}$ by closed planes. By Theorem 1.1 there exists an orientable connected surface Σ and an orientable one dimensional foliation $\mathcal{F}_0$ of S such that $(N \setminus K, \mathcal{F})$ is conjugated by a diffeomorphism to $(\Sigma \times \mathbb{R}^{n-2}, \mathcal{F}_0 \times \mathbb{R}^{n-2})$. In particular, $N \setminus K$ and Σ have the same homotopy type and thus $H_p(N \setminus K) \cong H_p(\Sigma)$ for each p. Σ can only be an open surface or the torus $S^1 \times S^1$. Since the leaves of $\mathcal{F}_0$ must be homeomorphic to $\mathbb{R}$, it follows that Σ can not be the torus. Thus Σ is an open surface and $H_p(N \setminus K) \cong H_p(\Sigma) = 0$, for each $p \geq 2$.

Proof of Proposition 1.2. Consider the exact sequence of singular homology groups with coefficients in $\mathbb{Z}$

$$\to H_{p+1}(N, N \setminus K) \to H_p(N \setminus K) \to H_p(N) \to H_p(N, N \setminus K) \to \qquad (1)$$

and of the isomorphisms

$$H_p(N, N \setminus K) \cong H^{n-p}(K) \qquad (2)$$

for each $0 \leq p \leq n$, given by the Alexander-Poincaré duality. The cohomology we use for K is the Čech cohomology. Replacing $H_n(N, N \setminus K)$ by $H^0(K)$ in the exact sequence 1

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

and using Lemma 2.1 one obtains the short exact sequence

$$0 \to \mathbb{Z} \to H^0(K) \to 0$$

Thus, $H^0(K) = \mathbb{Z}$ and consequently, K is connected.

Proof of Theorem 1.2. Consider the singular homology sequence of the pair (N, K) with coefficients in $\mathbb{Z}$

$$\to H_p(K) \to H_p(N) \to H_p(N, K) \to H_{p-1}(K) \to \tag{3}$$

and also the singular homology and cohomology groups of $N \backslash K$. Since K is an ARN (absolute neighborhood retract) we have, by duality, the isomorphisms $H_p(N, K) \cong H^{n-p}(N \backslash K)$ for each $p \geq 0$. Notice that By Lemma 2.1 $H_p(N \setminus K) = 0$, for each $p \geq 2$, and therefore $H^p(N \setminus K) = 0$, too. Write

$$H_p(N) = F_p \oplus T_p \ \ and \ \ H^p(N) = F_p \oplus T_{p-1} \tag{4}$$

where F denotes the free part and T the torsion part. From 3 and duality we have

$$\to H_p(K) \to H_p(N) \to H^{n-p}(N \setminus K) \to$$

By assumption $H_p(K) = 0$ for each $1 \leq p \leq [\frac{n}{2}]$, and since $n - p \geq 2$, we know that $H^{n-p}(N \setminus K) = 0$, too. The sequence above looks $0 \to H_p(N) \to 0$ and therefore $H_p(N) = 0$, for each $1 \leq p \leq [\frac{n}{2}]$. By Poincaré duality $H^{n-p}(N) = 0$. Using this information in 4 one obtains that $H_p(N) = 0$ for each $1 \leq p \leq n - 1$.

Proof of Theorem 1.3. We already know, from Theorem 1.2, that N is a homology sphere. By a theorem of Hurewicz, it is enough to prove that $\pi_1(N) = 0$. The exact sequence 1 for $p = 1$, after using that $H^{n-2}(K) = 0$ and that $H^1(N) = 0$ gives $H^1(N \backslash K) = H^1(\Sigma) = 0$. Thus Σ is diffeomorphic to $\mathbb{R}^2$ and by Theorem 1.1 that $N \backslash K$ is diffeomorphic to $\mathbb{R}^n$ and therefore that $\pi_1(N \setminus K) = 0$. Finally, since $\dim_{top}(K) \leq n - 2$ and N is a Cantor manifold, it follows from ([4], page 93), that thee map $\pi_1(N \setminus K) \to \pi_1(N)$ induced by the inclusion is surjective and consequently $\pi_1(N) = 0$, as we wanted. That N is homeomorphic to S^n follows from celebrated theorems of Freedman [1], $n = 4$, and Smale [8], $n \geq 5$.

Proof of Corollary 1.1. By Proposition 1.2 K is connected and since $\dim_{top} K = 0$, it follows that K reduces to a point. By Theorem 1.3 N is a homotopy sphere and therefore homeomorphic to S^n, for $n \geq 4$. For $n = 3$ the fact that K is a point and $N \setminus K$ is homeomophic to $\mathbb{R}^3$ implies that N is homeomorphic to S^3.

Proof of Theorem 1.4. We know that $\pi_1(N \backslash K)$ is isomorphic to $\pi_1(N)$ and therefore finite. Denote by $\mathcal{G}$ the regular codimension one C^2-foliation on $N \backslash K$ defined by the orbits of the action and let L be any leaf of $\mathcal{G}$. Fix a point $p \in L$ and consider the map

$j_* : \pi_1(L, p) \to \pi_1(N \backslash K, p)$. We are going to show that j_* is injective. Proposition 3.10 in [1] says that a codimension one C^1-foliation of a manifold defined by a locally free action of a Lie group has no vanishing cycle. But Theorem 3.3, in the same book, guarantees that if j_* were not injective, then $\mathcal{G}$ would have a vanishing cycle. Thus, it is injective, and consequently $\pi_1(L, p)$ is finite. Since L is the injective image of G/G_p and $\dim(L) = n-1$, it follows that G_p is a discrete subgroup of G. Therefore $A_p : G \to L$ is a covering map and $\pi_1(L, p)$ is isomorphic to G_p. From $\chi(G_p) > 0$ and $1 = \chi(G) = \chi(G_p) \times \chi(G/G_p)$, it follows that $\chi(G_p) = 1$, i.e., $G_p = \{e\}$. Thus, L is obtained from an injective immersion of $\mathbb{R}^{n-1}$. It follows from a now classical argument by Haefliger, see [3], that every leaf of $\mathcal{G}$ is closed in $N \backslash K$. Finally, we apply theorem 1.1 and the proof is complete.

Proof of Theorem 1.5 Let $T(K)$ be a tubular neighborhood of K diffeomorphic to $S^1 \times D^2$ and put $V = N \setminus \overline{T(K)}$. Since V is diffeomorphic to $N \setminus K$ and $\pi_1(V)$ is finitely generated, then it satisfies the assumptions of Theorem 1.1 and therefore V is diffeomorphic to $\Sigma \times \mathbb{R}$, where Σ is an open connected orientable surface. By Lemma 2.1 $H_p(V) = H_p(\Sigma) = 0$ for each $p \geq 2$. From the formula $\chi(N) = \chi(V) - \chi(\overline{T(K)})$, that relates the Euler characteristics, we obtain

$$0 = \beta_0(\Sigma) - \beta_1(\Sigma) = 1 - \beta_1(\Sigma)$$

i.e., $\beta_1(\Sigma) = 0$. Since β_1 is a complete invariant for orientable surfaces, it follows that Σ is homeomorphic to $S^1 \times (0, 1)$ and in consequence V is homeomorphic to $S^1 \times D^2$. Thus, N is obtained by pasting two copies of $S^1 \times D^2$ which means that N admits a Heegaard Splitting of genus 1.

REFERENCES

1. M. H. Freedman, *The topology of four-dimensional manifolds,* J. Differential Geom., 17 (1982), no. 3, 357-453.
2. C. Godbillon, *Feuilletages, Études géométriques II,* Institut de Recherche Mathématique Avancée, Université Louis Pasteur, 1986.
3. A. Haefliger, *Sur les feuilletages des variétés de dimension n* par des feuilles fermées de dimension $n-1$, Colloque de Topologie de Strasbourg, 1955.
4. W. Hurewicz and H. Wallman, *Dimension Theory,* Princeton University Press, Princeton 1948.
5. C. F. B. Palmeira, *Open manifolds foliated by planes,* Annals of Math., 107 (1970), 109-131.
6. H. Rosenberg, *Foliations by Planes,* Topology, vol 7 (1968), 131-138
7. H. Rosenberg and R. Roussarie, *Reeb Foliations,* Annals of Math., 91 (1970), 1-24.
8. S. Smale, *Generalized Poincaré's conjecture in dimensions greater than 4,* Annals of Mathematics, vol. 64 (1956), 399-405.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC