

Attractors for Strongly Damped Wave Equations with Critical Nonlinearities

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In this paper we obtain global well posedness results for the strongly damped wave equation $u_{tt} + (-\Delta)^\theta u_t = \Delta u + f(u)$, for $\theta \in [\frac{1}{2}, 1]$, in $H_0^1(\Omega) \times L^2(\Omega)$ when Ω is a bounded smooth domain and the map f grows like $|u|^{\frac{n+2}{n-2}}$. The local well posedness is considered in [7]. If $f = 0$, then this equation generates an analytic semigroup with generator $-\mathcal{A}_{(\theta)}$. Special attention is devoted to the case when $\theta = 1$ since in this case the generator $-\mathcal{A}_{(1)}$ does not have compact resolvent, contrary to the case $\theta \in [\frac{1}{2}, 1)$. Under the dissipativeness condition, $\limsup_{|s| \rightarrow \infty} \frac{f(s)}{s} \leq 0$ we prove the existence of compact global attractors for this problem. In the critical growth case we use Alekseev's nonlinear variation of constants formula (see [3]) to obtain that the semigroup is asymptotically smooth. March, 2001 ICMC-USP

1. INTRODUCTION

For $\theta \in [\frac{1}{2}, 1]$, $\eta > 0$, we consider the global well posedness and existence of global attractors for a family of problems of the form

$$\begin{cases} u_{tt} + \eta(-\Delta)^\theta u_t + (-\Delta)u = f(u), & t > 0, x \in \Omega, \\ u(0, x) = u_0(x), u_t(0, x) = v_0(x), & x \in \Omega, \\ u(t, x) = 0, & t \geq 0, x \in \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded C^2 -smooth domain in \mathbb{R}^n , $n \geq 3$, and $A = (-\Delta)$ with Dirichlet boundary conditions. It is well known that A is a positive, self-adjoint operator with

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domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ and therefore $-A$ generates an analytic semigroup on $X = X^0 = L^2(\Omega)$. We denote by X^α the fractional power spaces associated to the operator A ; that is, $X^\alpha = D(A^\alpha)$ endowed with the graph norm.

The problems (1) will be viewed as ordinary differential equations in a product space $Y = Y^0 = X^{\frac{1}{2}} \times X^0$:

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{A}_{(\theta)} \begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{F} \begin{bmatrix} u \\ v \end{bmatrix}, \quad t > 0, \quad \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \quad (2)$$

with $\mathcal{A}_{(\theta)} : D(\mathcal{A}_{(\theta)}) \subset Y^0 \rightarrow Y^0$ and \mathcal{F} given by

$$\mathcal{A}_{(\theta)} = \begin{bmatrix} 0 & -I \\ A & \eta A^\theta \end{bmatrix} = \begin{bmatrix} -\psi \\ A^\theta(A^{1-\theta}\varphi + \eta\psi) \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in D(\mathcal{A}_{(\theta)}), \quad \mathcal{F} = \begin{bmatrix} 0 \\ F \end{bmatrix}, \quad (3)$$

where F is the Nemitskii map associated to $f(u)$ and

$$D(\mathcal{A}_{(\theta)}) = Y_{(\theta)}^1 = \left\{ \begin{bmatrix} \varphi \\ \psi \end{bmatrix}; \varphi \in X^{\frac{3}{2}-\theta}, \psi \in X^{\frac{1}{2}}, A^{1-\theta}\varphi + \eta\psi \in X^\theta \right\}, \quad \theta \in \left[\frac{1}{2}, 1 \right]. \quad (4)$$

Of course,

$$\mathcal{A}_{(\theta)} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} -\psi \\ A\varphi + \eta A^\theta \psi \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X^1 \times X^\theta,$$

$X^1 \times X^\theta$ being a dense subset of $D(\mathcal{A}_{(\theta)})$.

The linear problem associated to (2) in Y^0 ,

$$\ddot{u} + \eta A^\theta \dot{u} + Au = 0, \quad t > 0, \quad u(0) = u_0, \quad \dot{u}(0) = v_0, \quad (5)$$

is studied in [8, 9, 10], where the sectoriality of $\mathcal{A}_{(\theta)}$ is established and a description of the fractional power spaces $Y_{(\theta)}^\alpha$, $\alpha \in [0, 1]$ is given.

We choose as a base space for (1) the product space $Y^0 = X^{\frac{1}{2}} \times X^0$. This choice of space seems to be the best possible to study the asymptotic behavior of (1) since in it we may exhibit an energy functional to (1).

In the cases $\alpha = \frac{1}{2}$ and $\theta = 1$ will deserve special attention. For the case $\theta = \frac{1}{2}$, because of the form of the damping term $A^{\frac{1}{2}}u_t$, a more complete description of the fractional power spaces associated to $\mathcal{A}_{(\frac{1}{2})}$ is available. Using this, we have been able to: (i) completely describe the extrapolated fractional power scale generated by $(Y^0, \mathcal{A}_{(\frac{1}{2})})$ [7]; (iii) to obtain the convergence of bounded sets from Y^0 to the attractor in the strong topology of $H^{1+\alpha}(\Omega) \times H^\alpha(\Omega)$ -norm, $\alpha \in [\frac{n-2}{n+2}, 1)$. The fact that $-\mathcal{A}_{(\frac{1}{2})}$ generates a compact analytic semigroup is essential to the the analysis here. The cases $\theta \in (\frac{1}{2}, 1)$ can be treated similarly. For the case $\theta = 1$ we have that: (i) The nonlinearity becomes subcritical; (ii) we loose compactness of the semigroup and of the nonlinearity (so subcritical is of no help) but we are still able to ensure the existence of a global attractor with the aid of a nonlinear variation of constants formula.

The main result of [7] that we will use is that

THEOREM 1.1. *If f satisfies*

$$|f(u) - f(u')| \leq c|u - u'|(1 + |u|^{\rho-1} + |u'|^{\rho-1}) \quad (6)$$

with $\rho \leq \frac{n+2}{n-2}$ then (1) is locally well posed in $H_0^1(\Omega) \times L^2(\Omega)$.

Our main result, concerning the asymptotic behavior of (1), can be stated in the following form

THEOREM 1.2. *If in addition to (6) we have that f satisfies the dissipativeness condition*

$$\limsup_{|u| \rightarrow \infty} \frac{f(u)}{u} \leq 0. \quad (7)$$

Then, the problem (1) with $\theta \in [\frac{1}{2}, 1]$ has a compact global attractor \mathbf{A}_0 .

Some regularity results for the local solutions and for the attractors are also obtained. Among other things we prove that \mathbf{A}_0 is a bounded subset of $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$.

This paper is organized as follows. In Section 2 we prove state the solvability results for (1) proved in [7] for f satisfying (6). Section 3 is devoted to obtaining some additional regularity for solutions of (1). In Section 4 we treat the global solvability and the existence of global attractors for (1). This section is divided into three subsections. In Subsection 4.1 we prove the existence of a compact global attractor (2) for the case $\theta = [\frac{1}{2}, 1]$, $\rho < \frac{n+2}{n-2}$ and f satisfying (7) and (6). In Section 4.2 we treat the subcritical case $\rho < \frac{n+2}{n-2}$ for $\theta = 1$. In Subsection 4.3 we treat the critical case $\rho = \frac{n+2}{n-2}$ for $\theta = 1$. We remark that for $\theta = 1$ the resolvent of $\mathcal{A}_{(\theta)}$ is not compact. However, we show that the semigroup $\{T(t)\}$ corresponding to (1) is asymptotically smooth. This is proved decomposing of $\{T(t)\}$ on a sum of the exponentially decaying semigroup and a family of compact maps (cf. [12]). In the subcritical case this is accomplished using the fact that the nonlinearity is compact and in the critical case we employ the nonlinear variation of constants formula as in [3].

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2. LOCAL SOLVABILITY IN Y^0

In this section we state the results of [7] on local well posedness and regularity for (2) with initial conditions in Y^0 and nonlinearities f growing critically. We first recall that

PROPOSITION 2.1. *$\mathcal{A}_{(\theta)}$, $\theta \in [\frac{1}{2}, 1]$, is a sectorial, positive operator in Y^0 . The semigroup of contractions $\{e^{-\mathcal{A}_{(\theta)}t}\}$ is analytic in $Y_{(\theta)}^\alpha$, $\alpha \in [0, 1)$. It is also compact for $t > 0$ except*

the case $\theta = 1$. Furthermore,

$$\begin{aligned} Y_{(\theta)}^\alpha &= [Y^0, Y_{(\theta)}^1]_\alpha \\ &= \begin{cases} X^{\frac{1}{2} + \alpha(1-\theta)} \times X^{\theta\alpha}, & \alpha \in [0, \frac{1}{2}], \\ \left\{ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \in X^{\frac{1}{2} + \alpha(1-\theta)} \times X^{\theta - \frac{1}{2} + \alpha(1-\theta)}; A^{1-\theta}\varphi + \eta\psi \in X^{\theta\alpha} \right\}, & \alpha \in [\frac{1}{2}, 1]. \end{cases} \end{aligned} \quad (8)$$

Also recall that the extrapolation space $Y_{(\theta)-1}$ of Y^0 generated by $\mathcal{A}_{(\theta)}$ is the completion of the normed space $(Y^0, \|\mathcal{A}_{(\theta)}^{-1} \cdot\|_{Y^0})$. Similarly as in [7, Lemma 2] one may infer that

- $\mathcal{A}_{(\theta)-1}$ ($\mathcal{A}_{(\theta)-1}$ being the closure of $\mathcal{A}_{(\theta)}$ in $Y_{(\theta)-1}$) is sectorial and positive operator in $Y_{(\theta)-1}$ with $D(\mathcal{A}_{(\theta)-1}) = Y_{(\theta)-1}^1 = Y^0$.
- Imaginary powers of $\mathcal{A}_{(\theta)-1}$ are bounded.
- If $\theta \neq 1$, the $\mathcal{A}_{(\theta)-1}$ has compact resolvent.

We shall thus study (1) as a sectorial problem (2) in $Y_{(\theta)-1}$. Our concern will be the solutions to (2) originating at the elements of Y^0 .

The embeddings below relate the spaces in the extrapolated fractional power scale and known spaces. They will be needed to obtain regularity results and asymptotic compactness of the semigroup generated by (2).

LEMMA 2.1. *Let $[(X^\alpha, A_\alpha), \alpha \in \mathbb{R}]$ (A_α being the realization of A in X^α) be generated by $(L^2(\Omega), (-\Delta_D))$. Then: i) for $\alpha(1-\theta) < \frac{1}{4}$ and $n = 3$ or $n > 3$,*

$$Y_{(\theta)-1}^{1+\alpha} \subset H^{1+2\alpha(1-\theta)}(\Omega) \times H^{2\theta\alpha}(\Omega) \subset L^{q_1}(\Omega) \times L^{q_2}(\Omega), \quad (9)$$

$1 \leq q_1 \leq \frac{2n}{n-2-4\alpha(1-\theta)}$, $1 \leq q_2 \leq \frac{2n}{n-4\alpha\theta}$, $\alpha \in [0, \frac{1}{2}]$, $\theta \in [\frac{1}{2}, 1]$, ii) for $n = 3$ and $\alpha = \theta = \frac{1}{2}$ the embedding (9) holds for $1 \leq q_1 \leq \infty$, $1 \leq q_2 \leq 3$, iii) for $n \geq 3$ $Y_{(\frac{1}{2})-1}^{1+\alpha} \subset H^{1+\alpha}(\Omega) \times H^\alpha(\Omega)$ for any $\alpha \in [0, 1]$. Furthermore,

$$Y_{(\theta)-1}^\alpha \supset X^{\frac{1}{2} - (1-\alpha)(1-\theta)} \times X^{-\frac{1}{2} + \alpha(1-\theta)} \supset X^{\frac{1}{2} - (1-\alpha)(1-\theta)} \times L^q(\Omega), \quad (10)$$

$q \geq \frac{2n}{n+2-4\alpha(1-\theta)}$, $\alpha \in [0, \frac{1}{2}]$, $\theta \in [\frac{1}{2}, 1]$, $n \geq 3$ and also $Y_{(\frac{1}{2})-1}^\alpha \supset X^{\frac{\alpha}{2}}(\Omega) \times H^{-1+\alpha}(\Omega)$ for any $\alpha \in [0, 1]$, $n \geq 3$.

For a proof, see [7].

We will now recall the concept of ε -regularity of maps and solutions. If P is a sectorial, positive operator acting in a Banach space $Z = Z^0$ and ε is a nonnegative number, then

DEFINITION 2.1. $G : D(G) \rightarrow Z$ is ε -regular relatively to (Z^1, Z^0) (equivalently, G is of class $\mathcal{F}(\varepsilon, \rho, \gamma(\varepsilon), C)$) if and only if there are constants $\rho > 1$, $\gamma(\varepsilon) \geq 0$, $C > 0$ such that $\rho\varepsilon \leq \gamma(\varepsilon) < 1$, G takes $Z^{1+\varepsilon}$ into $Z^{\gamma(\varepsilon)}$, and the following estimate holds:

$$\|G(z_1) - G(z_2)\|_{Z^{\gamma(\varepsilon)}} \leq C \|z_1 - z_2\|_{Z^{1+\varepsilon}} (\|z_1\|_{Z^{1+\varepsilon}}^{\rho-1} + \|z_2\|_{Z^{1+\varepsilon}}^{\rho-1} + 1), \quad z_1, z_2 \in Z^{1+\varepsilon}. \quad (11)$$

For the maps in (1) we have the following result

THEOREM 2.1. *Assume that f satisfies (6) with $1 < \rho \leq \frac{n+2}{n-2}$. Let \mathcal{F} be the map defined by*

$$\mathcal{F}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ F(u) \end{pmatrix} \tag{12}$$

where $F(u)$ is the Nemitskiĭ map associated to f . Then, \mathcal{F} is an ε -regular map relatively to $(Y_{(\theta)-1}^1, Y_{(\theta)-1})$ for each $\varepsilon \in [0, \frac{1}{2\rho}]$ ($\varepsilon \in [0, 1)$ if $\theta = \frac{1}{2}$) and $\gamma(\varepsilon) = \rho\varepsilon$. That is,

$$\begin{aligned} & \|\mathcal{F}\begin{pmatrix} u \\ v \end{pmatrix} - \mathcal{F}\begin{pmatrix} u' \\ v' \end{pmatrix}\|_{Y_{(\theta)-1}^{\gamma(\varepsilon)}} \\ & \leq c \left\| \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u' \\ v' \end{pmatrix} \right\|_{Y_{(\theta)-1}^{1+\varepsilon}} \left(1 + \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{Y_{(\theta)-1}^{1+\varepsilon}}^{\rho-1} + \left\| \begin{pmatrix} u' \\ v' \end{pmatrix} \right\|_{Y_{(\theta)-1}^{1+\varepsilon}}^{\rho-1} \right), \quad \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \in Y_{(\theta)-1}^{1+\varepsilon}, \end{aligned} \tag{13}$$

for $\varepsilon \in [0, \frac{1}{2\rho}]$ ($\varepsilon \in [0, 1)$ if $\theta = \frac{1}{2}$).

For a proof see [7]. The above result plays an important role in the regularity of the solutions of (1) and we will refer to it later in the paper.

Consider an abstract problem:

$$\dot{z} + Pz = G(z), \quad t > 0, \quad z(0) = z_0 \tag{14}$$

with P being a sectorial, positive operator in a Banach space Z^0 . Let $\varepsilon \geq 0, \tau > 0, z_0 \in Z^1$, and $z = z(\cdot, z_0) : [0, \tau] \rightarrow Z^1$. Recall that

DEFINITION 2.2. z is an ε -regular solution to the problem (14), if and only if $z \in C([0, \tau], Z^1) \cap C((0, \tau], Z^{1+\varepsilon})$, and

$$z(t) = e^{-Pt}z_0 + \int_0^t e^{-P(t-s)}G(z(s))ds \quad \text{for } z \in [0, \tau].$$

Concerning the local existence and regularity of ε -regular solutions we quote the following result from [7].

THEOREM 2.2. *Let (6) be satisfied, $\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix} \in Y^0$ and let $B_{Y^0}(\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix}, r)$ denote a ball in Y^0 with radius $r > 0$ centered at $\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix}$.*

Then, there are $r > 0$ and $\tau_0 > 0$ such that for each $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B_{Y^0}(\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix}, r)$ there exists a unique $\tilde{\varepsilon}$ -regular solution $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot, u_0, v_0)$ to (2). In addition,

- (i) $t^\zeta \left\| \begin{bmatrix} u \\ v \end{bmatrix} (t, u_0, v_0) \right\|_{Y_{(\theta)_-1}^{1+\zeta}} \rightarrow 0$ as $t \rightarrow 0^+$, ($0 < \zeta < \frac{1}{2}$, $0 \leq \zeta \leq \zeta_0 < 1$ if $\theta = \frac{1}{2}$)
- (ii) $t^\zeta \left\| \begin{bmatrix} u \\ v \end{bmatrix} (t, u_1, v_1) - \begin{bmatrix} u \\ v \end{bmatrix} (t, u_2, v_2) \right\|_{Y_{(\theta)_-1}^{1+\zeta}} \leq C' \left\| \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} - \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \right\|_{Y^0}$ whenever $t \in [0, \tau_0]$,
 $0 \leq \zeta \leq \zeta_0 < \frac{1}{2}$ ($0 \leq \zeta \leq \zeta_0 < 1$ if $\theta = \frac{1}{2}$), $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} \in B_{Y^0}(\begin{bmatrix} \bar{u}_0 \\ \bar{v}_0 \end{bmatrix}, r)$,
- (iii) $\begin{bmatrix} u \\ v \end{bmatrix} (\cdot, u_0, v_0) \in C((0, \tau_0], Y_{(\zeta)_-1}^{1+\frac{1}{2}}) \cap C^1((0, \tau_0], Y_{(\zeta)_-1}^{1+\zeta})$ for $0 \leq \zeta < \frac{1}{2}$ ($0 \leq \zeta \leq \zeta_0 < 1$ if $\theta = \frac{1}{2}$); in particular, $\begin{bmatrix} u \\ v \end{bmatrix} (\cdot, u_0, v_0)$ satisfies both relations in (2).

The existence of the ε -regular solution to (2) under the assumptions (6) has been already discussed in [7]. We remark that such considerations are the extension of the original results reported in [4], [5].

3. SMOOTHING ACTION OF ε -REGULAR SOLUTIONS

Assume that f satisfies (6). Then we have the following results

LEMMA 3.1. For $\alpha \in [\frac{n-2}{n+2}, 1)$, the map \mathcal{F} corresponding to (2) takes $Y_{(\theta)}^\alpha$ into Y^0 and is Lipschitz continuous on bounded subsets of $Y_{(\theta)}^\alpha$.

Proof: Description of $Y_{(\theta)}^\alpha$ spaces has been given in [7]. The proof follows by standard calculations based on the Hölder inequality and Sobolev embedding. ■

The above lemma and the general results of [13] imply:

LEMMA 3.2. For $\alpha \in [\frac{n-2}{n+2}, 1)$ and $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y_{(\theta)}^\alpha$ there exists a unique $Y_{(\theta)}^\alpha$ -solution to (1) defined on a maximal interval of existence $[0, \tau_{u_0, v_0})$. That is, there exists a unique function $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix} \in C([0, \tau_{u_0, v_0}), Y_{(\theta)}^\alpha)$ such that:

- (i) $\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix} \in C((0, \tau_{u_0, v_0}), Y_{(\theta)}^1)$,
- (ii) $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix} \in C^1((0, \tau_{u_0, v_0}), Y_{(\theta)}^\beta)$, $\beta \in [0, 1)$,
- (iii) both relations in (2) are satisfied.

THEOREM 3.1. If, in addition to (6), we assume that either $3 \leq n$ and $\theta = \frac{1}{2}$ or $3 \leq n \leq 5$ and $\theta \in (\frac{1}{2}, 1]$ are satisfied. Then, the ε -regular solutions from Theorem 2.2 fulfill the conditions (i)-(iii) of Lemma 3.2.

Proof: Take $\begin{bmatrix} u \\ v \end{bmatrix}$ and choose $\varepsilon > 0$ such that $\gamma(\varepsilon) \geq \frac{n-2}{n+2}$. Let $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix}$ be ε -regular solution obtained in Theorem 2.2.

Since $Y_{(\theta)-1}^{1+\gamma(\varepsilon)} = Y_{(\theta)}^{\gamma(\varepsilon)} \subset Y_{(\theta)}^{\frac{n-2}{n+2}}$ we find from Theorem 2.2 (iii) that

$$\begin{bmatrix} u(s, u_0, v_0) \\ v(s, u_0, v_0) \end{bmatrix} \in Y_{(\theta)}^{\frac{n-2}{n+2}} \text{ for each } s \in (0, \tau_0).$$

According to Lemma 3.2 there exists $Y_{(\theta)}^{\frac{n-2}{n+2}}$ -solution $\begin{bmatrix} \tilde{u}(\cdot, u(s, u_0, v_0)) \\ \tilde{v}(\cdot, v(s, u_0, v_0)) \end{bmatrix}$ to (1). This proves that

$$\begin{bmatrix} u(t+s, u_0, v_0) \\ v(t+s, u_0, v_0) \end{bmatrix} = \begin{bmatrix} \tilde{u}(t, u(s, u_0, v_0)) \\ \tilde{v}(t, v(s, u_0, v_0)) \end{bmatrix}, \quad t \in [0, \tau_{u_0, v_0}),$$

and consequently,

$$\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix} \in Y_{(\theta)}^1, \quad t \in (s, \tau_{u_0, v_0}), \quad \begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix} \in C^1((s, \tau_{u_0, v_0}), Y_{(\theta)}^\beta), \quad \beta \in [0, 1).$$

Since $s > 0$ could be arbitrarily small, the proof is complete. ■

4. GLOBAL SOLVABILITY AND GLOBAL ATTRACTOR

4.1. Subcritical Case: $\theta \in [\frac{1}{2}, 1)$

In this section we consider the existence of a global compact attractor for (1) when f is subcritical; that is, it satisfies (6) with $\rho < \frac{n+2}{n-2}$. We restrict our attention to the cases when either $\theta = \frac{1}{2}$ and $n \geq 3$ or $\theta \in (\frac{1}{2}, 1)$ and $3 \leq n \leq 5$.

LEMMA 4.1. *If f satisfies (6) with $\rho < \frac{n+2}{n-2}$; then, for any bounded set $B \subset Y^0$ there is a time $\tau_B > 0$ such that the ε -regular solutions $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix}$, given by Theorem 2.2, originating at $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B$ exists and are bounded in $Y_{(\theta)}^{\theta_0}$ for arbitrary $\theta_0 < \gamma(\varepsilon)$, where $\gamma(\varepsilon) > \max\{\frac{n-2}{n+2}, \rho\varepsilon\}$. In particular, the set*

$$\left\{ \begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix}, \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B \right\}$$

is precompact in $Y_{(\theta)}^{\frac{n-2}{n+2}}$ for each $t \in (0, \tau_B)$.

Proof: The proof is a direct consequence of Theorem 2.2 (ii). We remark that, for $\gamma(\varepsilon) > \rho\varepsilon$ a number r in Theorem 2.2 can be chosen arbitrarily large so that the time of the existence of ε -regular solutions is uniform on bounded sets of Y^0 (cf. [4, Corollary 1]). ■

Remark 4. 1. We mention for completeness that one may choose in Lemma 4.1 e.g. $\gamma(\varepsilon) = \varepsilon \frac{n+2}{n-2}$ with certain $\varepsilon > \left(\frac{n-2}{n+2}\right)^2$.

In the considerations below, devoted to the existence of the global attractor to (1) in a subcritical case, we shall follow the general abstract scheme developed in [11, 6]. For convenience we recall this scheme in the closing Section 5.

THEOREM 4.1. *Let the assumption of Lemma 4.1 hold and, in addition, f satisfies the dissipative condition (7). Then,*

(i) *For any $\alpha \in [\frac{n-2}{n+2}, 1)$ there exists corresponding to (1) a compact C^0 -semigroup $\{T(t)\}$ of global $Y_{(\theta)}^\alpha$ -solutions to (1) which possesses a compact global attractor \mathbf{A}_α in $Y_{(\theta)}^\alpha$,*

(ii) $\mathbf{A}_\alpha = \mathbf{A}_{\frac{n-2}{n+2}} =: \mathbf{A}$, $\alpha \in [\frac{n-2}{n+2}, 1)$,

(iii) $T(t) : Y^0 \rightarrow Y_{(\theta)}^\alpha$, $t > 0$, where $T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix}$, $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix}$ being an ε -regular solution from Theorem 2.2, are well defined maps which are the extensions of $T(t)$ ($t > 0$) to Y^0 ,

(iv) \mathbf{A}_α attracts bounded subset of Y^0 under $\{T(t)\}$ in $Y_{(\theta)}^\alpha$ -norm.

Proof: The proof of (i) occurs in four steps.

Step 1. (Y^0 -estimate and the Lyapunov function) Take $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y_{(\theta)}^\alpha$ and consider the cor-

responding $Y_{(\theta)}^\alpha$ -solution $\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix}$. Lemma 3.2 ensures that we have enough regularity to work with the starting equation (1). Multiplying thus (1) by $v = u_t$ in $L^2(\Omega)$ and using the properties of the negative Laplacian with Dirichlet boundary conditions we obtain that

$$\frac{d}{dt} \left(\frac{1}{2} \|u_t\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega)}^2 - \int_{\Omega} \int_0^u f(s) ds \right) = -\eta \|A^{\frac{\theta}{2}} u_t\|_{L^2(\Omega)}^2.$$

This ensures in particular that

$$\left\| \begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix} \right\|_{Y^0} \leq c + c' \mathcal{L} \left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right) \leq C \left(\left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right\|_{Y^0} \right), \quad (15)$$

where c, c' do not depend on η, β ,

$$\mathcal{L} \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \frac{1}{2} \|w_2\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla w_1\|_{L^2(\Omega)}^2 - \int_{\Omega} \int_0^{w_1} f(s) ds, \quad w_1, w_2 \in Y^0, \quad (16)$$

and $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a locally bounded function independent of η, β .

Step 2. (*subordination of the nonlinearity to a power of \mathcal{A}*) Since $1 < \rho < \frac{n+2}{n-2}$, then based on the Nirenberg-Gagliardo type inequality we obtain that

$$\|f(u(t, u_0, v_0))\|_{L^2(\Omega)} \leq g(\|u(t, u_0, v_0)\|_{H^1(\Omega)}) (1 + \|u(t, u_0, v_0)\|_{H^{1+\alpha_1}(\Omega)}^{\theta_1}), \quad (17)$$

$t \in (0, \tau_{u_0, v_0})$, with certain $\theta_1 \in [0, 1)$, $\alpha_1 \in [0, 1)$ and some nondecreasing function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ (cf. [1]). Next, based on (17), we get the relation:

$$\begin{aligned} \|\mathcal{F}\left(\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix}\right)\|_{Y^0} &= \|f(u(t, u_0, v_0))\|_{L^2(\Omega)} \\ &\leq g\left(\left\|\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix}\right\|_{Y^0}\right) \left(1 + \left\|\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix}\right\|_{Y_{(\theta)}^{\alpha_1}}\right). \end{aligned} \quad (18)$$

Step 3. (*global solvability and compactness*) Conditions (15) and (18) plus the compactness of the resolvent of \mathcal{A} ensures that to (1) corresponds a compact C^0 -semigroup $\{T(t)\}$ of global $Y_{(\theta)}^\alpha$ -solutions having bounded orbits of bounded sets. For the proof of the existence of the global attractor for $\{T(t)\}$ in $Y_{(\theta)}^\alpha$ it now suffices to show that the estimate (15) is asymptotically independent of $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y_{(\theta)}^\alpha$.

Step 4. (*point dissipativeness of $\{T(t)\}$ - the role of the Lyapunov function*). Functional \mathcal{L} defined in (16) is a Lyapunov function for $\{T(t)\}$ in $Y_{(\theta)}^\alpha$. Therefore, ω -limit sets of points from $Y_{(\theta)}^\alpha$ lie within the set \mathcal{E} of all stationary solutions to (1). Our concern now is to prove that \mathcal{E} is bounded in Y^0 .

Let $\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathcal{E}$. Then $\tilde{v} = 0$, whereas \tilde{u} is an $H^2(\Omega)$ -solution of the elliptic problem

$$\begin{cases} -\Delta \tilde{u} = f(\tilde{u}), & x \in \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases} \quad (19)$$

With the use of (7) it is easy to show that if \tilde{u} solves (19), then $\|\tilde{u}\|_{H^1(\Omega)} \leq c''$ where $c'' = c''(\Omega, f) > 0$ is independent of \tilde{u} . Consequently, we have

$$\left\|\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}\right\|_{Y^0} \leq c''', \quad \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathcal{E}. \quad (20)$$

Since each ω -limit set $\omega\left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right)$, lies in \mathcal{E} , is compact and attracts $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y_{(\theta)}^\alpha$ under $\{T(t)\}$ in $Y_{(\theta)}^\alpha$ -norm, condition (20) ensures in particular that

$$\limsup_{t \rightarrow +\infty} \left\|\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix}\right\|_{Y^0} \leq c''', \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y_{(\theta)}^\alpha. \quad (21)$$

Therefore, the estimate (15) is asymptotically independent of initial data from $Y_{(\theta)}^\alpha$ which completes the proof of the assertion (i).

Part (ii) is a consequence of the smoothing action of $\{T(t)\}$. Part (iii) follows from Theorem 3.1. Finally, part (iv) results from Lemma 4.1. Theorem 4.1 is thus proved. ■

4.2. Subcritical Case: $\theta = 1$

In this section we restrict our attention to the case $\theta = 1$ studied previously by many authors (cf. [16], [14], [12], [17]).

Remark 4. 2. In the recent paper [17]) the dimension of the global attractor was estimated. One can find however in this paper rather very strange errors. First, the author takes $X^1 \times X^1$ as the domain of $\mathcal{A}_{(1)}$. However, if the base space is Y^0 , this operator is not closed with such a domain. This is the case, when one needs to choose $Y_{(1)}^1$ as the domain of $\mathcal{A}_{(1)}$ following the description of [10]. In this case it is thus rather unknown if the solution possesses the regularity stated in [10, Lemma 1 (ii)] for initial data from Y^0 . Next in the proof of [10, Theorem 2] the author says that the semigroup $\{e^{-\mathcal{A}_{(1)}t}\}$ is compact. But this cannot be true because the resolvent of $\mathcal{A}_{(1)}$ is not compact. The latter may be easily seen if we look at the embeddings of $Y_{(1)}^\alpha$ spaces. Of course it is impossible for $Y_{(1)}^{\frac{1}{2}} = X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ to be compactly embedded in $Y^0 = X^{\frac{1}{2}} \times X^0$.

Throughout this section we shall consider functions f satisfying subcritical growth; that is, (6) with $\rho < \rho(n)$. In this particular case, \mathcal{F} takes $Y_{(1)-1}^1$ into $Y_{(1)-1}^{\frac{1}{2}}$ and is Lipschitz continuous in bounded sets. This says that the map \mathcal{F} is subcritical and the Theorem 2.2 can be rewritten in the following form.

THEOREM 4.2. *For any initial data $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ lying in a bounded subset B of $Y_{(1)-1}^1$ there exists a $\tau = \tau(B)$ and a unique 0-regular solution $[0, \tau] \ni t \mapsto \begin{bmatrix} u \\ v \end{bmatrix}(t, u_0, v_0) \in Y_{(1)-1}^1$ to (2) which depends continuously on the initial data. Furthermore,*

$$\begin{bmatrix} u \\ v \end{bmatrix}(\cdot, u_0, v_0) \in C((0, \tau], Y_{(1)-1}^{\frac{3}{2}}) \cap C^1((0, \tau], Y_{(1)-1}^{\frac{3}{2}-})$$

and $\begin{bmatrix} u \\ v \end{bmatrix}(t, u_0, v_0)$ satisfies both relations in (2).

The theorem above is proved as in [13].

If we show that local solutions from Theorem 4.2 are bounded in the norm of $X^{\frac{1}{2}} \times X$ uniformly on bounded sets, then we shall obtain the existence of a C^0 -semigroup $\{T(t)\}$ corresponding to (2) in Y^0 having bounded orbits of bounded sets (cf. [6], [11]).

Our concern is thus to prove the following:

LEMMA 4.2. *Let $\begin{bmatrix} u \\ v \end{bmatrix}(\cdot, u_0, v_0)$ denote a solution obtained in Theorem 4.2 and f satisfies the dissipativeness condition (7). Then,*

$$\left\| \begin{bmatrix} u \\ v \end{bmatrix}(t, u_0, v_0) \right\|_{Y^0} \leq c \left(\left\| \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \right\|_{Y^0} \right),$$

where $c : R^+ \rightarrow R^+$ is a locally bounded function.

The proof of this result is similar to the proof of Step 1 in Theorem 6 in [7].

LEMMA 4.3. *Under the assumptions of Lemma 4.2 0-regular solutions from Theorem 4.2 exist globally in time. Therefore, the equation (2) defines a C^0 -semigroup $\{T(t), t \geq 0\}$ on Y^0 such that*

- (i) $\{T(t), t \geq 0\}$ has bounded orbits of bounded sets;
(ii) $\{T(t), t \geq 0\}$ is asymptotically smooth.

Proof: The existence of a C^0 -semigroup with bounded orbits of bounded sets follows from Lemma 4.2. To prove condition (ii) we start writing the variation of constants formula

$$T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = e^{-\mathcal{A}_{(1)-1}t} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + \int_0^t e^{-\mathcal{A}_{(1)-1}(t-s)} \mathcal{F}(T(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) ds.$$

Note that $e^{-\mathcal{A}_{(1)-1}t}$ decays exponentially and that f takes bounded subsets of $X^{\frac{1}{2}}$ into bounded subsets of $X^{-\frac{1}{2}+\delta}$, for some $\delta > 0$. From this we have that \mathcal{F} is a compact map from $Y_{(1)-1}^1 = Y^0$ into $Y_{(1)-1}^{\frac{1}{2}} = Y^{-\frac{1}{2}}$. Since $e^{-\mathcal{A}_{(1)-1}t}$ is a bounded linear operator from $Y_{(1)-1}^{\frac{1}{2}}$ to $Y_{(1)-1}^1$ we have that the operator

$$U(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \int_0^t e^{-\mathcal{A}_{(1)-1}(t-s)} \mathcal{F}(T(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) ds$$

as a map from Y^0 into Y^0 is compact. It follows from the results in [12] that $T(t)$ is an asymptotically smooth as a sum of an exponentially decaying semigroup with a compact family of maps. This completes the proof. \blacksquare

As an immediate consequence of these lemmas and of Step 4 in Theorem 6 of [7] we have the following result

THEOREM 4.3. *Under the assumptions of Lemma 4.2, $\{T(t), t \geq 0\}$ has a compact global attractor \mathbf{A}_0 in Y^0 .*

THEOREM 4.4. *Under the assumptions of Lemma 4.2, the problem (2) defines a C^0 semigroup $\{T_\alpha(t), t \geq 0\}$ on $Y_{(1)-1}^{1+\alpha}$ for each $\alpha \in [0, \frac{1}{2})$ which possesses a compact global attractor \mathbf{A}_α . Furthermore, \mathbf{A}_0 is bounded in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ and $\mathbf{A}_\alpha = \mathbf{A}_0$ for $\alpha \in [0, \frac{1}{2})$.*

Proof: Noting that $\{T_0(t), t \geq 0\}$ is a dissipative C^0 -semigroup in $Y_{(1)-1}^1$ having bounded orbits of bounded sets, with simple computations based on the variation of constants formula one can easily see that $\{T_\alpha(t), t \geq 0\}$ is a point dissipative C^0 -semigroup in $Y_{(1)-1}^{1+\alpha}$ with bounded orbits of bounded sets, for each $\alpha \in [0, \frac{1}{2})$ (cf. [11, Corollary 4.3.2]). The proof that the semigroups $\{T_\alpha(t), t \geq 0\}$ are asymptotically smooth follows as in Lemma 4.3. For the proof that \mathbf{A}_0 is bounded in $X^{\frac{1}{2}} \times X^{\frac{1}{2}} = Y_{(1)-1}^{\frac{3}{2}}$ we refer to Lemma 3.2.1 in [11].

4.3. Attractors in the Critical Growth Case: $\theta = 1$

In this section we shall consider the case when f satisfies (6) with the critical exponent $\rho = \rho(n)$.

4.3.1. The case of strong dissipation

We begin from the simpler case when the semigroup $\{T(t)\}$ corresponding to (2) is exponentially decaying and the attractor is a one point set $\{(0, 0)\}$.

Throughout this section we assume the following stronger dissipativeness condition

$$sf(s) \leq 0, \quad s \in \mathbb{R}. \quad (22)$$

We remark that we may replace the above condition by $sf(s) \leq \lambda_1 s$, $s \in \mathbb{R}$, where λ_1 is the first eigenvalue of A . This will be clear from the proof of the results.

PROPOSITION 4.1. *Under (22) the equation (2) defines a C^0 -semigroup on Y^0 which has a compact global attractor $\mathbf{A}_0 = \{(0, 0)\}$.*

Proof: Note that both Theorem 4.2 and Lemma 4.2 remain true under the assumptions of the present section. Therefore, there exists corresponding to (2) semigroup $\{T(t), t \geq 0\}$ of global 0-regular solutions with bounded orbits of bounded sets.

Based on (22) we shall next prove that

$$\|T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\|_{Y^0} \leq h(r)e^{-M_\delta(r)t}, \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B_r = B_{Y^0} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, r \right), \quad t \geq 0, \quad r > 0, \quad (23)$$

where $h(r)$ and $M_\delta(r)$ are given in (31) and (27),(29) respectively. In particular, $\{(0, 0)\}$ is a unique equilibrium which attracts bounded subsets of Y^0 .

Following [3] we introduce a functional

$$\mathcal{L}_\delta \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \mathcal{L}_0 \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) + \delta \int_{\Omega} w_1 w_2 dx, \quad \delta \geq 0 \quad (24)$$

where \mathcal{L}_0 is a standard Lyapunov functional to (1);

$$\mathcal{L}_0 \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) = \frac{1}{2} \|w_2\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla w_1\|_{L^2(\Omega)}^2 - \int_{\Omega} \int_0^{w_1} f(s) ds dx. \quad (25)$$

We remark that as a consequence of (22) the integral $\int_{\Omega} \int_0^{w_1} f(s) ds dx$ is nonpositive. Therefore, for δ sufficiently small $\mathcal{L}_\delta \left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right)$ majorizes the norm $\left\| \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\|_{L^2(\Omega)}$ (see (30)).

We then have

$$\begin{aligned}
\frac{d}{dt} \mathcal{L}_\delta(T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) &= -\eta \|\nabla v(t, u_0, v_0)\|_{L^2(\Omega)}^2 + \delta \|v(t, u_0, v_0)\|_{L^2(\Omega)}^2 \\
&+ \delta \int_{\Omega} u(t, u_0, v_0) (\Delta u(t, u_0, v_0) + \eta \Delta v(t, u_0, v_0) + f(u(t, u_0, v_0))) dx \\
&\leq -\frac{\eta}{2} \|\nabla v(t, u_0, v_0)\|_{L^2(\Omega)}^2 + \delta \|v(t, u_0, v_0)\|_{L^2(\Omega)}^2 - \delta (1 - \frac{\delta \eta}{2}) \|\nabla u(t, u_0, v_0)\|_{L^2(\Omega)}^2 \\
&\leq -\frac{\delta}{2} \|\nabla u(t, u_0, v_0)\|_{L^2(\Omega)}^2 - \frac{\eta \lambda_1}{4} \|v\|_{L^2(\Omega)}^2, \quad 0 < \delta < \min\{\frac{\eta \lambda_1}{4}, \frac{1}{\eta}\},
\end{aligned} \tag{26}$$

where λ_1 is the first eigenvalue of A . Since (22) implies that $f(0) = 0$, therefore (6) ensures that

$$\exists \tilde{c} \geq 1 \mid \int_{\Omega} \int_0^w f(s) ds dx \leq \tilde{c} (1 + \|\nabla w\|_{L^2(\Omega)}^{\frac{4}{n-2}}) \|\nabla w\|_{L^2(\Omega)}^2, \quad w \in H_0^1(\Omega).$$

Defining

$$\begin{aligned}
M_r &= \sup\{\|\nabla u(t, u_0, v_0)\|_{L^2(\Omega)}^{\frac{4}{n-2}}; \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B_r, t \geq 0\}, \\
M_\delta(r) &= \frac{\delta}{4\tilde{c}(1 + M_r^{\frac{4}{n-2}})}, \quad \text{where } \tilde{c} \geq \max\{1, \lambda_1\} \text{ and } 0 < \delta < \{\frac{\eta \lambda_1}{4}, \frac{1}{\eta}\}
\end{aligned} \tag{27}$$

we may increase the right hand side of (26) to get

$$\begin{aligned}
\frac{d}{dt} \mathcal{L}_\delta(T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) &\leq -M_\delta(r) \mathcal{L}_\delta(T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) - \frac{\delta}{4} \|\nabla u(t, u_0, v_0)\|_{L^2(\Omega)}^2 - \frac{\eta \lambda_1}{8} \|v\|_{L^2(\Omega)}^2 \\
&+ M_\delta(r) \int_{\Omega} u(t, u_0, v_0) v(t, u_0, v_0) dx, \quad 0 < \delta < \min\{\frac{\eta \lambda_1}{4}, \frac{1}{\eta}\}.
\end{aligned} \tag{28}$$

Based on the Poincaré and Young inequalities we have that

$$\begin{aligned}
&-\frac{\delta}{4} \|\nabla u(t, u_0, v_0)\|_{L^2(\Omega)}^2 - \frac{\eta \lambda_1}{8} \|v(t, u_0, v_0)\|_{L^2(\Omega)}^2 \\
&+ M_\delta(r) \int_{\Omega} u(t, u_0, v_0) v(t, u_0, v_0) dx \leq 0,
\end{aligned} \tag{29}$$

and

$$\frac{1}{4} \|T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\|_{Y^0} \leq \mathcal{L}_\delta(T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) \tag{30}$$

are satisfied. For such value of δ inequality (28) reads:

$$\frac{d}{dt} \mathcal{L}_\delta(T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) \leq -M_\delta(r) \mathcal{L}_\delta(T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}), \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B_r$$

and we obtain the estimate

$$\|T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\|_{Y^0} \leq 4\mathcal{L}_\delta(T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) \leq h(r)e^{-M_\delta(r)t}, \quad t \geq 0, \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B_r.$$

where

$$h(r) = 4 \sup\{\mathcal{L}_\delta(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}); \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B_r\} \quad (31)$$

The proof is complete. ■

4.3.2. Nonlinear variation of constants formula

Our next concern is to prove for a pair of problems (32) and (33) the Alekseev's nonlinear variation of constants formula (35) (cf. [3, Theorem 2.2]). In these considerations we shall need the following assumptions:

(H_0) Let P be a sectorial, positive operator in a Banach space $Z = Z^0$ with the domain Z^1 . Fix $\alpha \in [0, 1)$ and consider functions $G_1 : Z^\alpha \rightarrow Z^0$, $G_2 : Z^\alpha \rightarrow Z^\alpha$ such that G_1 has continuous Fréchet derivative and G_2 is Lipschitz continuous on bounded sets.

(H_α) There exists a Banach space $\mathcal{Y} = \mathcal{Y}^0$ densely embedded in Z^α such that $P|_{\mathcal{Y}^0}$ ($P|_{\mathcal{Y}^0}$ being a realization of P in Y^0) is sectorial and positive in \mathcal{Y}^0 with the domain \mathcal{Y}^1 and G_1 , G_2 are Lipschitz continuous as the maps from \mathcal{Y}^0 into \mathcal{Y}^0 .

We remark that in the special case when $\alpha = 0$ conditions of (H_α) are consequence of the assumptions in (H_0). Thus, for $\alpha = 0$, the requirements of (H_α) are inessential.

For $v \in Z^\alpha$ let $z = z(t, v)$ be a solution to

$$\dot{z} + Pz = G_1(z), \quad t > 0, \quad z(0) = v. \quad (32)$$

Similarly, let $\tilde{z} = \tilde{z}(t, v)$ be a solution to

$$\dot{\tilde{z}} + P\tilde{z} = G_1(\tilde{z}) + G_2(\tilde{z}), \quad t > 0, \quad \tilde{z}(0) = v. \quad (33)$$

LEMMA 4.4. *Suppose that the requirements of (H_0) and (H_α) are satisfied. Then, the following conditions hold:*

function $(0, +\infty) \times Z^\alpha \ni (t, \omega) \rightarrow z(t, \omega) \in Z^\alpha$ has continuous Fréchet derivative (34)

$$\tilde{z}(t, v) = z(t, v) + \int_0^t \frac{\partial z}{\partial \omega}(t-s, \tilde{z}(s, v)) G_2(\tilde{z}(s, v)) ds, \quad t > 0. \quad (35)$$

Proof: Condition (34) is a consequence of [13, Corollary 3.4.6]. Next, since $z(t, \omega)$ in (34) is a C^1 function, using the chain rule we obtain:

$$\frac{d}{ds} [z(t-s, \tilde{z}(s, v))] = -\dot{z}(t-s, \tilde{z}(s, v)) + \frac{\partial z}{\partial \omega}(t-s, \tilde{z}(s, v)) \dot{\tilde{z}}(s, v). \quad (36)$$

For $v \in \mathcal{Y}^0$, assumptions of (H_α) guarantee that $\tilde{z}(s, v) \in \mathcal{Y}^1$ and $\dot{z}(0, \tilde{z}(s, v))$ exists in \mathcal{Y}^0 -norm. Since $\mathcal{Y}^0 \subset Z^\alpha$, the derivative $\dot{z}(0, \tilde{z}(s, v))$ exists in Z^α -norm and we have:

$$\begin{aligned} \dot{z}(t-s, \tilde{z}(s, v)) &= \lim_{h \rightarrow 0^+} \frac{z(t-s+h, \tilde{z}(s, v)) - z(t-s, \tilde{z}(s, v))}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{z(t-s, z(h, \tilde{z}(s, v))) - z(t-s, z(0, \tilde{z}(s, v)))}{h} \\ &= \frac{\partial z}{\partial \omega}(t-s, \tilde{z}(s, v)) \dot{z}(0, \tilde{z}(s, v)). \end{aligned} \quad (37)$$

Connecting (36), (37), and (33) we get

$$\begin{aligned} \frac{d}{ds} [z(t-s, \tilde{z}(s, v))] &= \frac{\partial z}{\partial \omega}(t-s, \tilde{z}(s, v)) (Pz(0, \tilde{z}(s, v)) - G_1(z(0, \tilde{z}(s, v))) \\ &\quad - P\tilde{z}(s, v) + G_1(\tilde{z}(s, v)) + G_2(\tilde{z}(s, v))) = \frac{\partial z}{\partial \omega}(t-s, \tilde{z}(s, v)) G_2(\tilde{z}(s, v)). \end{aligned} \quad (38)$$

Integrating both sides of (38) we show that (35) holds for $v \in \mathcal{Y}^0$.

Now choose $v_0 \in Z^\alpha$ and consider a sequence $\{v_n\} \in \mathcal{Y}^0$ convergent to v_0 in Z^α . We know that

$$\tilde{z}(t, v_n) = z(t, v_n) + \int_0^t \frac{\partial z}{\partial \omega}(t-s, \tilde{z}(s, v_n)) G_2(\tilde{z}(s, v_n)) ds, \quad t > 0, \quad n \in \mathbb{N} \quad (39)$$

where $z(\cdot, v_n)$ and $\tilde{z}(\cdot, v_n)$ tend in Z^α to $z(\cdot, v_0)$ and $\tilde{z}(\cdot, v_0)$ respectively. Since convergence of $z(\cdot, v_n)$ and $\tilde{z}(\cdot, v_n)$ is uniform with respect to t varying in compact subintervals of $[0, +\infty)$ (cf. [13, Theorem 3.4.1]), passing to the limit in (39) we obtain (35) for $v \in Z^\alpha$. The proof is complete. \blacksquare

Remark 4.3. Lemma 4.4 remains true if instead of (H_0) and (H_α) we assume that (H'_0) and (H'_α) hold;

(H'_0) P is a sectorial, positive operator in a Banach space $Z = Z^0$ with the domain Z^1 , $\alpha \geq \beta \geq 0$ satisfy $\alpha - \beta \in [0, 1)$ and functions $G_1 : Z^\alpha \rightarrow Z^\beta$, $G_2 : Z^\alpha \rightarrow Z^\alpha$ are such that G_1 has continuous Frechét derivative and G_2 is Lipschitz continuous on bounded sets.

Proof: Indeed, since $P|_{Z^\beta}$ ($P|_{Z^\beta}$ being the realization of P in $W^0 := Z^\beta$) is a sectorial, positive operator with $D(P|_{Z^\beta}) = Z^{\beta+1} =: W^1$ (cf. [11, Proposition 1.3.8]) and, for $\alpha' = \alpha - \beta$, $W^{\alpha'} = (Z^\beta)^{\alpha-\beta} = Z^\alpha$, (cf. [2, p. 260]) we repeat the arguments of Lemma 4.4 with $P|_{Z^\beta}$, W^0 and W^α instead of P , Z^0 , Z^α . \blacksquare

The next lemma shows validity of the Alekseev's formula for a pair of sectorial problems connected to the strongly damped wave equation (1).

LEMMA 4.5. *Let $n = 3, 4, 5, 6$. Suppose that $f(u, v) = f(u) - \beta v$, where*

- $f = f_{11} + f_{12}$, $f_{1i} : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2$,
- f_{11} has second order derivative, $|f_{11}''(s)| \leq c(1+|s|^{\rho(n)-2})$, f_{11} satisfies (6) with $\rho = \rho(n)$ and in addition $sf_{11}(s) \leq 0$ for $s \in \mathbb{R}$,
- f_{12} satisfies (6) with $\rho \leq \frac{n}{n-2}$ and, moreover, $\limsup_{|s| \rightarrow \infty} \frac{f_{12}(s)}{s} \leq 0$.

Then,

(i) Assumptions of (H'_0) hold with $P = \mathcal{A}_{(1)_{-1}}$, $Z^0 = Y_{(1)_{-1}}$, $\alpha = 1$, $\beta = \frac{1}{2}$, and

$$G_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ F_{11}(u) \end{pmatrix}, \quad G_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ F_{12}(u) \end{pmatrix},$$

where F_{11} , F_{12} are Nemitskiĭ maps corresponding to f_{11} and f_{12} respectively.

(ii) Assumptions of (H_α) hold with $\alpha = 1$, $\mathcal{Y}^0 = X^1 \times X^0$, $\mathcal{Y}^1 = X^1 \times X^1$, and $P|_{\mathcal{Y}^0} = \mathcal{A}_{(1)|_{X^1 \times X^0}}$ (cf. [16, Proposition 2.2]).

(iii) Alekseev's formula (35) holds with $v := \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$, $\tilde{z} := T(\cdot) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ denoting the solution to

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{A}_{(1)_{-1}} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ F_{11}(u) \end{bmatrix} + \begin{bmatrix} 0 \\ F_{12}(u) \end{bmatrix}, \quad t > 0, \quad \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \quad (40)$$

and $z := S(\cdot) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ denoting the solution to

$$\frac{d}{dt} \begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{A}_{(1)_{-1}} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ F_{11}(u) - \beta v \end{bmatrix}, \quad t > 0, \quad \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \quad (41)$$

4.3.3. A regularity result

Here we state a regularity result taken from [3] that will play a crucial role on the proof that the semigroup $T(t)$ of global solutions to (1) can be decomposed as a sum of an exponentially decaying semigroup (in bounded subsets) $S(t)$ and a compact operator $U(t)$. In fact it will be used to prove the compactness of $U(t)$. This result has been used in [3] with the exact same purpose.

If $A : D(A) \subset Z \rightarrow Z$ is a sectorial operator, let Z_A denote the Banach space $D(A)$ endowed with the graph norm.

PROPOSITION 4.2. *Assume Z is a Banach space and $A : D(A) \subset Z \rightarrow Z$ is sectorial on Z . Let $T \in (t_0, \infty]$ and $f : [t_0, T) \times Z \rightarrow Z$ be a function which satisfies enough conditions to have uniqueness of mild solutions of the following problem,*

$$\dot{z} = Az + f(t, z) \quad t_0 < t < T, \quad z \in Z; \quad (42)$$

$$z(t_0) = z_0 \in Z. \quad (43)$$

Define the function

$$\begin{aligned} g : [t_0, T) \times Z^1 \times Z &\rightarrow Z \\ (t, z, y) &\rightarrow f_t(t, z) + f_z(t, z)y, \end{aligned} \tag{44}$$

where the subscripts denote the usual partial derivatives, and assume that

(H) g is continuous in $t \in [t_0, T)$ and locally Lipschitz continuous in $(z, y) \in Z^1 \times Z$ uniformly on compact intervals of $[t_0, T)$.

If $z_0 \in D(A)$, then there exists a unique solution $z(t)$ of (42) through (t_0, z_0) on a maximal interval $[t_0, t_1)$ which is a strict solution on $[t_0, \tau)$ for some $t_0 < \tau \leq t_1$; that is,

$$z \in C^1([t_0, \tau), Z) \cap C^0([t_0, \tau), Z^1) \cap C^0([t_0, t_1), Z).$$

In fact we will use the following consequence of 4.2 (see [3])

COROLLARY 4.1. *Assume that:*

- i) f satisfies a global Lipschitz condition in Z uniformly on compact intervals of $t \in [t_0, T)$
- ii) g satisfies a global Lipschitz condition in $Z^1 \times Z$ uniformly on compact intervals of $t \in [t_0, T)$

then $\tau = t_1 = T$ and there exists a continuous function $C : R^+ \times [t_0, T) \rightarrow R^+$ such that

$$\|z(t, z_0)\|_{Z^1} \leq C(\|z_0\|_{Z^1}, t) \tag{45}$$

4.3.4. Existence theorem

With the use of Alekseev's formula we may finally obtain the existence of a compact global attractor for the semigroup $\{T(t)\}$ corresponding to (1) in the critical growth case.

THEOREM 4.5. *Under the assumptions of Lemma 4.5 we may assume that, f_{12} is continuously differentiable function satisfying growth restriction $|f'_{12}| \leq c$, the problem (40) defines in Y^0 a C^0 -semigroup $\{T(t)\}$ of 0-regular solutions which possesses a compact global attractor in Y^0 .*

Proof: The assertions of Theorem 4.2 and Lemma 4.2 remain valid under the assumptions of the present theorem. The existence of a C^0 -semigroup $\{T(t)\}$ in Y^0 with bounded orbits of bounded sets is thus straightforward.

If we proved that $\{T(t)\}$ is asymptotically smooth, then the existence of a Lyapunov functional \mathcal{L}_0 (cf. (25)) and the boundedness of the set of stationary solutions would guarantee that $\{T(t)\}$ is point dissipative (cf. [7, Theorem 6]). Consequently, $\{T(t)\}$ would possess a compact global attractor in Y^0 .

To prove that $\{T(t)\}$ is asymptotically smooth we apply Lemma 4.5 (iii) decomposing $\{T(t)\}$ so that

$$T(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = S(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} + U(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \quad \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^0,$$

where $S(\cdot) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$ is a solution to (41) and

$$U(t) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \int_0^t \frac{\partial S}{\partial \omega}(t-s, T(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) G_2(T(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}) ds.$$

By Proposition 4.1, (23), $\{S(t)\}$ is a C^0 -semigroup in Y^0 , asymptotically decaying uniformly on bounded subsets of Y^0 . Therefore, to justify that $\{T(t)\}$ is asymptotically smooth we only need to prove that $U(t) : Y^0 \rightarrow Y^0$ is a compact map for each $t > 0$ (cf. [12, Lemma 3.2.3]).

As a consequence of the growth restriction for f_{12} , for any $\delta_0 \in (0, \frac{1}{2}]$, F_{12} takes bounded subsets of $X^{\frac{1}{2}}$ into bounded subsets of X^δ (cf. [3, Lemma 5.2]).

Let $E^0 = E = X^{\frac{1}{2}+\delta_0} \times X^{\delta_0}$. Then $\mathcal{A}_{(1)}$ is sectorial on E^0 (see [14, Theorem 1.1]).

We also remark that $\mathcal{A}_{(1)}$ considered on a base space E^0 with the domain $E^1 = \left\{ \begin{bmatrix} \phi \\ \psi \end{bmatrix} \in X^{\frac{1}{2}+\delta_0} \times X^{\delta_0}; \phi + \eta\psi \in X^{1+\delta_0} \right\}$ is maximal accretive with zero in the resolvent set.

We then check that

- $X^{\frac{1}{2}+\delta_0} \times X^{\delta_0} = E^0 = E_{-1}^1 \subset E_{-1}^\alpha \subset E_{-1} \supset X^{\frac{1}{2}+\delta_0} \times X^{-\frac{1}{2}+\delta_0}$, $\alpha \in [0, 1]$,
- $E^0 = E_{-1}^1$,
- for each $\delta \in (0, \delta_0]$,

$$H \left(t, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right) = \begin{pmatrix} 0 \\ f'(w(t))z_1 \end{pmatrix}$$

is strongly continuous in t and, for t fixed, is Lipschitz continuous on bounded sets of E_0 with values in E_{-1} .

Next we apply Corollary 4.1 to obtain that the operator $U(t)$ is compact. For that we note that if we let

$$U_0 = T(s) \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

and

$$U(t) = \begin{bmatrix} v(t) \\ v_t(t) \end{bmatrix} = \frac{\partial S}{\partial w}(t, U_0) G_2(U_0),$$

$$S(t) = \begin{bmatrix} w(t) \\ w_t(t) \end{bmatrix},$$

then we have that U satisfies the equation

$$\begin{aligned} U_t &= \mathcal{A}U + h(t, U) \\ U(0) &= G_2(U_0), \quad h(t, U) = \begin{bmatrix} 0 \\ f'_{11}(w(t))v - \beta w_t(t) \end{bmatrix}. \end{aligned} \quad (46)$$

The following lemma implies that we may apply Corollary 4.1 to obtain compactness of $U(t)$

LEMMA 4.6. *Let*

$$g : [t_0, \infty) \times E_{-1}^1 \times E_{-1} \rightarrow E_{-1}$$

be the function defined by

$$g \left(t, \begin{pmatrix} \mu \\ \eta \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} 0 \\ f''_{11}(w(t))w_t(t)\mu \end{pmatrix} + \begin{pmatrix} 0 \\ f'_{11}(w(t))u \end{pmatrix}$$

Then, g is continuous in t and globally Lipschitz in $\begin{pmatrix} \mu \\ \eta \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}$ uniformly for $t > 0$ and $U_0 \in B_r$.

Now we are ready to complete the proof of existence of global attractors. Since $G(U_0) \in E_0$ and since $\|G(U_0)\|_{E_0} \leq C(\|U_0\|_{Y_0})$ for some continuous function $C(\cdot)$ and since orbits of bounded subsets of Y_0 are bounded subsets of Y_0 we have from Corollary 4.1 that $U(t, U_0)$ is a bounded subset of E_0 for any bounded subset B of Y_0 . Since the embedding of E_0 into Y_0 is compact we have that $U(t)$ is a compact map.

5. APPENDIX: ABSTRACT RESULT FOR THE EXISTENCE OF A COMPACT GLOBAL ATTRACTOR

Consider the Cauchy problem (14) assuming that $Z = Z^0$ is a Banach space, $P : D(P) \rightarrow Z$ sectorial and positive operator in Z and, for some $\alpha \in [0, 1)$, $G : Z^\alpha \rightarrow Z$ is Lipschitz continuous on bounded subsets of Z^α .

Under these assumptions to any $z_0 \in Z^\alpha$ corresponds a unique Z^α -solution $z(\cdot, z_0)$ of (14) defined on a maximal interval of existence $[0, \tau_{z_0})$. We then have (cf. [11, 6]):

PROPOSITION 5.1. *Suppose that the above assumptions are satisfied and, in addition, P has compact resolvent. Then the following two conditions are equivalent:*

(i) *Relation $T(t)z_0 = z(t, z_0)$, $t \geq 0$, defines on Z^α a compact C^0 -semigroup $\{T(t)\}$ of global Z^α -solutions to (14) which has a compact global attractor in Z^α .*

(ii) There are:

- a Banach space \mathcal{Y} , with $D(P) \subset \mathcal{Y}$,
- a locally bounded function $C : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,
- a nondecreasing function $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$,
- certain number $\theta_1 \in [0, 1)$,

such that, for each $z_0 \in Z^\alpha$, both conditions:

$$\|z(t, z_0)\|_{\mathcal{Y}} \leq C(\|z_0\|_{Z^\alpha}), \quad t \in (0, \tau_{z_0}), \quad (47)$$

and

$$\|G(z(t, z_0))\|_Z \leq g(\|z(t, z_0)\|_{\mathcal{Y}})(1 + \|z(t, z_0)\|_{Z^\alpha}^\theta), \quad t \in (0, \tau_{z_0}), \quad (48)$$

hold, where the estimate (47) is asymptotically independent of $z_0 \in Z^\alpha$.

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