CADERNOS DE MATEMÁTICA **02**, 1–13 March (2001)
ARTIGO NÚMERO SMA#88

# A necessary condition for formation of internal transition layer in a reaction-difusion system

Arnaldo Simal do Nascimento

*Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, 13560-905, São Carlos, SP, Brasil*
E-mail: arnaldon@dm.ufscar.br

Janete Crema

*Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 15560-970, São Carlos SP, Brazil*
E-mail: janete@icmc.sc.usp.br

We prove that an equal-area like condition is necessary for the formation of internal transition layer in a singularly perturbed elliptic problem in a varying or fixed domain. As an application it is formally derived that such condition is also necessary for the existence of a family of stable stationary solutions to a reaction-diffusion system with Neumann boundary conditions on a fixed domain. March, 2001 ICMC-USP

## 1. INTRODUCTION

The setting in which the issue of spatial pattern formation in reaction-diffusion systems is considered typically involves the choice of a specific parameter region (the rates of diffusion and/or reaction) and the geometry of the nullcline of the reaction terms. By spatial pattern we mean spatial inhomogeneity with some regularity.

In an activator-inhibitor model, for instance, when the activator can stabilizes at only two different concentrations and its diffusibility (considered as a parameter) tends to zero, there are stationary solutions which oscillates (in space) between these two concentrations.

Loosely speaking, when this parameter goes to zero, this component of the solution may settle down in many different types of interfacial configurations (asymptotic spatial concentration distributions). Among these configurations some exhibit relatively simple geometrical configuration and draw more attention.



Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

In one of them, the interfacial configuration is a well-defined limiting set of discontinuity which corresponds to a nontrivial partition of the domain, on each of which connected component the activator assume different constant concentration values. The boundaries of these connected components which lie inside the domain, are called internal transition layers.

In another one the concentration tends to cluster at certain shrinking regions, showing a spiky geometric profile, while presenting almost uniform concentrations out of these regions. The asymptotic interfacial configuration corresponds to a concentration which is constant almost everywhere in the domain.

These two types will be somehow contemplated in our discussions.

On the other hand, for a large class of two-component systems of semilinear reaction-diffusion equation, whenever the issue is the formation of internal transition layer the so-called equal-area condition is always assumed. However it is never clear in the vast avaluable literature on the issue the reason for assuming this hypothesis. It might look like at first as an unessential hypothesis which only simplifies the computations. We prove herein that this is not the case. Rather it is a necessary condition.

More specifically, we consider a system in two different scaling settings and study the role played by the equal-area condition in the formation of internal transition layers in one of them, and in the existence of stable patterns in the other one.

Let us start with the reaction-diffusion system

$$\begin{cases} u_t = \varepsilon^2 \triangle u + f(u,v), \ \ x \in \Omega \\ v_t = \mu(\varepsilon) \triangle v + g(u,v), \ \ x \in \Omega \\ \dfrac{\partial v}{\partial \hat{n}} = 0 \text{ (or } v = 0) \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\varepsilon$ is a small positive parameter, $\mu(\varepsilon) \to \infty$, as $\varepsilon \to 0$ and $\Omega \subset I\!R^N, \ N \geq 2$.

Note that, for small $\varepsilon$, the inhibitor diffuses faster than the activator and then according to Turing [10], spatially inhomogeneous patterns can be created through diffusion-driven instability.

In particular, our result imply that if an uniformly bounded (in $\varepsilon$) family $\{(u_\varepsilon, v_\varepsilon)\}_{0<\varepsilon\leq\varepsilon_0}$ of stationary solutions to (1.1) develops internal transition layers, as $\varepsilon \to 0$, with $u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} u_0(x) \in \{\alpha, \beta\}$, $v_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} v_0$, where $\alpha$, $\beta$ and $v_0$ are constants, then

$$\int_\alpha^\beta f(\xi, v_0) d\xi = 0, \tag{2}$$

In other words, the equal-area condition (1.2) is a necessary hypothesis for the existence of such solutions. Note that this imposes restrictions on the geometry of the nullcline of $f$.

We also conclude that $|\Omega_\alpha| g(\alpha, v_0) = -|\Omega_\beta| g(\beta, v_0)$ where $\Omega_\alpha = \{u_0 = \alpha\}$ and $\Omega_\beta = \{u_0 = \beta\}$. This relation provides some clues about the location of the interface. For instance, if $g(\alpha, v_0) = -g(\beta, v_0)$ then the interface must partition the domain into two subsets each having half the volume (or area) of the domain $\Omega$.

Actually our results are proved for varying domains, i.e., $\Omega_\varepsilon$ and for variable diffusibility. The reason we do so is to study the role played by (1.2) in the existence of **stable** spatial

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

patterns to the following system

$$\begin{cases} u_t = \varepsilon^2 \triangle u + f(u,v), \\ v_t = D\triangle v + g(u,v), \quad (x,t) \in \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \hat{n}} = \dfrac{\partial v}{\partial \hat{n}} = 0, \ (x,t) \in \partial\Omega \times (0,\infty) \end{cases} \tag{3}$$

More specifically, when $N = 1$ and a set of hypotheses (which comprises (1.2)) is satisfied, it is shown in [6] that (1.3) has a stable family of stationary solutions. However when $N \geq 2$ this is no longer true and any family of stationary solutions to (1.3) becomes unstable for $\varepsilon$ sufficiently small [8].

On the other hand, according to some results announced by Nishiura, stable solutions having interfaces with width $O(\varepsilon^{1/3})$ do exist when $\Omega$ is a retangular domain. This suggests a final configuration of the latter type mentioned at the beginning.

Then our results along with a formal argument provided in [8] strongly suggests that, for $N \geq 2$, a condition like (1.2) is necessary for the existence of stable spatial patterns.

We should mention that location of internal layers is not our concern herein, although some conclusions on the issue are drawn. Instead the focus is on the importance of the equal-area condition in two different situations.

## 2. NECESSITY FOR FORMATION OF INTERNAL LAYERS

As said before we state our results in a slightly more general framework than we need for the application we have in mind.

We start by considering the following system in varying domains:

$$\begin{cases} \varepsilon\, div\,(h_1(x)\nabla u_\varepsilon) + f(u_\varepsilon, v_\varepsilon) = 0, \ \ x \in \Omega_\varepsilon \\ div\,(h_2(x)\nabla v_\varepsilon) + \delta(\varepsilon) g(u_\varepsilon, v_\varepsilon) = 0, \ \ x \in \Omega_\varepsilon \\ \dfrac{\partial\, v_\varepsilon}{\partial\, \hat{n}} = 0 \text{ (or } v_\varepsilon = 0) \text{ on } \partial\Omega_\varepsilon, \end{cases} \tag{1}$$

where $\varepsilon$ is a small positive parameter, $\delta(\varepsilon)$ a real function continuous at $\varepsilon = 0$, $f$ and $g$ are functions in $C^1(I\!R^2)$ and $h_i \in C^{1,\nu}(\Omega_\varepsilon)$, $0 < \nu < 1$, satisfies $0 < m < h_i < M$ $(i = 1,2)$, for some constants $m$ and $M$.

As for $\Omega_\varepsilon$ it is supposed to be an $\varepsilon$-family of uniformly bounded domains in $I\!R^N$, i.e., $\mid \Omega_\varepsilon \mid \leq M$, where $M$ is a constant independent of $\varepsilon$ and $\mid \cdot \mid$ denotes the $N$-dimensional Lebesgue measure. The boundary of $\Omega_\varepsilon$, denoted by $\partial\Omega_\varepsilon$, is at least piecewise smooth.

Note that when only stationary solutions are considered, (1.1) is a particular case of (1) with $\delta(\varepsilon) = (1/\mu(\varepsilon))$, constant diffusibility functions and a fixed domain.

Next we define what we mean by formation of internal transition layer. The definition comprises the minimal requirements that any family of solutions to (1.1) which develops a sharp transition from one state to another inside $\Omega$, is expected to fulfill.

DEFINITION 2.1. Let $\mathcal{O}$ be an open connected set in $R^N$ and $\Gamma \subset \mathcal{O}$ an $(N-1)$-dimensional smooth compact manifold without boundary. We will say that a family of functions $\{(u_\varepsilon, v_\varepsilon)\}_{0<\varepsilon\leq\varepsilon_0}$, with $u_\varepsilon$ and $v_\varepsilon$ in $C^1(\overline{\mathcal{O}}) \cap C^2(\mathcal{O})$ develops internal transition layer, as $\varepsilon \to 0$, in $\mathcal{O}$ with interface $\Gamma \subset \mathcal{O}$ and limit $(u_0(x), v_0(x))$ if:

- $u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} u_0$, uniformly on any compact set $K \subset \{\mathcal{O}\backslash\Gamma\}$, where $u_0$ is given by

$$u_0(x) = \alpha(x)\chi_{\mathcal{O}_\alpha}(x) + \beta(x)\chi_{\mathcal{O}_\beta}(x) \tag{2}$$

for some functions $\alpha$ and $\beta$ in $C(\mathcal{O})$ with $\alpha(x) < \beta(x)$ for $x \in \Gamma$, $\mathcal{O}_\alpha$ is the open region in $\mathcal{O}$ enclosed by $\Gamma$ and $\mathcal{O}_\beta = \mathcal{O}\backslash\{\Gamma \cup \mathcal{O}_\alpha\}$

- $v_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} v_0$ uniformly on any compact set $K \subset \mathcal{O}$.

Note that $\mathcal{O} = \mathcal{O}_\alpha \cup \Gamma \cup \mathcal{O}_\beta$. Also $\mathcal{O}$ can be either a narrow tubular neighborhood of $\Gamma$ or a simply connected domain. In the latter case $\partial\mathcal{O}_\alpha = \Gamma$.

The above definition, which is local in space, suffices for our purposes and it allows for the existence of more than one transition-layer surface.

*Remark 2. 1.*

Note that the case in which $\Gamma_0 \cap \partial\Omega \neq \emptyset$ has been ruled out although when $h_1$ and $h_2$ are constant functions this is more likely to happen. This was done just for simplicity in notation. The analysis poses no additional difficulties and at the end we will indicate how that can be accomplished.

With the above definitions and notations we now state our main result.

THEOREM 2.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family of solutions to problem (1) which develops internal transition layer, as $\varepsilon \to 0$, in $\mathcal{U}$ with interface $\Gamma$ and limit $(u_0(x), v_0(x))$ satisfying $f(u_0, v_0) = 0$, a.e. in $\mathcal{U}$, where $\mathcal{U}$ is an open tubular neighborhood of $\Gamma$ such that $\mathcal{U} \subset \Omega_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, for $\varepsilon_0$ small enough.*

*Moreover suppose that the following hypotheses are satisfied.*

$(H.1)\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C$, $\|v_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq C$, *for* $0 < \varepsilon \leq \varepsilon_0$ *and some constant* $C > 0$,

$(H.2)\lim\limits_{\varepsilon\to 0} \varepsilon \displaystyle\int_{\partial\mathcal{U}} |\nabla u_\varepsilon(x)|^2 \; dS = 0.$

*Then the $N$-vector equation*

$$\int_\Gamma \left\{ \int_{\alpha(x)}^{\beta(x)} f(s, v_0(x))ds \right\} \hat{\nu}(x) \; dS = 0 \tag{3}$$

*holds. Here $\hat{\nu}$ stands for the outward unit normal vector on $\Gamma$.*

*Remark 2. 2.* At the end of this section we will see that in some important cases which appear in the applications, $(H.2)$ is automatically satisfied.

**Proof:** By taking a narrower tubular neighborhood $\widetilde{\mathcal{U}}$ of $\Gamma$, i.e., $\widetilde{\mathcal{U}} \subset \mathcal{U}$, it follows from Definition 1.1 that $u_\varepsilon$ and $v_\varepsilon$ converge uniformly to $u_0$ and $v_0$, respectively, on $\partial\widetilde{\mathcal{U}}$. For simplicity we drop $\tilde{}$ and write just $\mathcal{U}$. As in the Pohozaev procedure, multiplying the first equation in (1) by $x.\nabla u_\varepsilon$ and integrating over $\mathcal{U}$ we obtain

$$\int_{\mathcal{U}} \{\varepsilon div(h_1(x)\nabla u_\varepsilon)(x \cdot \nabla u_\varepsilon) + f(u_\varepsilon, v_\varepsilon)x \cdot \nabla u_\varepsilon\}\ dx = 0 \tag{4}$$

Working with the first term of this equality and using that

$$\text{div}[x \cdot \nabla u\ h_1 \nabla u] = x \cdot \nabla u\ \text{div}(h_1 \nabla u) + h_1 \left[|\nabla u|^2 + x \cdot \nabla(|\nabla u|^2)/2\right]$$

along with the Divergence theorem, it follows that

$$\begin{aligned} &- \varepsilon \int_{\partial\mathcal{U}} h_1(x)\ x \cdot \nabla u_\varepsilon\ \frac{\partial u_\varepsilon}{\partial \hat{n}}\ dS - \frac{\varepsilon}{2} \int_{\partial\mathcal{U}} h_1(x)|\nabla u_\varepsilon|^2\, x \cdot \hat{n} dS \\ &- \frac{\varepsilon}{2} \int_{\mathcal{U}} |\nabla u_\varepsilon|^2 x \cdot \nabla h_1\, dx + \varepsilon(2-N)/2 \int_{\mathcal{U}} h_1(x)|\nabla u_\varepsilon|^2\, dx \\ &= \int_{\mathcal{U}} f(u_\varepsilon, v_\varepsilon)x \cdot \nabla u_\varepsilon\ dx. \end{aligned} \tag{5}$$

We claim that the left hand side of this equality goes to 0, as $\varepsilon \to 0$. In fact, this holds for the first and second terms by virtue of $(H.2)$. We claim that the third and fourth terms approach zero, as $\varepsilon \to 0$, too. It suffices to show that

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\mathcal{U}} |\nabla u_\varepsilon(x)|^2\, dx\ =\ 0. \tag{6}$$

But multiplying the first equation in (1) by $u_\varepsilon$ and integrating over $\mathcal{U}$ we obtain

$$\varepsilon \int_{\mathcal{U}} h_1 |\nabla u_\varepsilon|^2\ dx = \varepsilon \int_{\partial\mathcal{U}} h_1 \frac{\partial u_\varepsilon}{\partial \nu}\, dS + \int_{\mathcal{U}} f(u_\varepsilon, v_\varepsilon)u_\varepsilon\ dx \overset{\varepsilon \to 0}{\longrightarrow} 0, \tag{7}$$

by $(H.1)$, $(H.2)$ and the fact that $f(u_\varepsilon, v_\varepsilon) \to f(u_0, v_0) = 0$, a.e. in $\mathcal{U}$. Hence we have (6) and going back to (5) we conclude

$$\lim_{\varepsilon \to 0} \int_{\mathcal{U}} f(u_\varepsilon, v_\varepsilon)\, x \cdot \nabla u_\varepsilon\, dx =\ 0. \tag{8}$$

Note that

$$\begin{aligned} \text{div}\ [\,x \int_\theta^u f(s,v)\, ds] \ &=\ (\nabla \int_\theta^u f(s,v) ds) \cdot x +\ div\, x\ \cdot \int_\theta^u f(s,v)\, ds \\ &=\ f(u,v)\nabla u \cdot x + \int_\theta^u x \cdot \nabla_x f(s,v) ds + N \int_\theta^u f(s,v) ds \end{aligned}$$

Therefore

$$\int_{\mathcal{U}} f(u_\varepsilon, v_\varepsilon)\, x \cdot \nabla u_\varepsilon \; dx \; = \tag{9}$$

$$\int_{\mathcal{U}} div\,[x \int_\theta^{u_\varepsilon} f(s, v_\varepsilon)\, ds]dx - \int_{\mathcal{U}} \{ \int_\theta^{u_\varepsilon} x \cdot \nabla_x f(s, v_\varepsilon) ds + N \int_\theta^{u_\varepsilon} f(s, v_\varepsilon) ds\} dx =$$

$$\int_{\partial\mathcal{U}} x \cdot \hat{n} \int_\theta^{u_\varepsilon} f(s, v_\varepsilon)\, ds\, dS - \int_{\mathcal{U}} \{ \int_\theta^{u_\varepsilon} x \cdot \nabla_x f(s, v_\varepsilon) ds + N \int_\theta^{u_\varepsilon} f(s, v_\varepsilon) ds\} dx$$

Recall that by hypothesis, $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ converge uniformly on $\partial\mathcal{U}$, so that the first integral of the last term in (2.13) satisfies

$$\int_{\partial\mathcal{U}} x \cdot \hat{n} \int_\theta^{u_\varepsilon} f(s, v_\varepsilon)\, ds\, dS \stackrel{\varepsilon \to 0}{\longrightarrow} \int_{\partial\mathcal{U}} x \cdot \hat{n} \int_\theta^{u_0} f(s, v_0)\, ds\, dS \tag{10}$$

Also it is easy to see that

$$\begin{aligned} \int_{\mathcal{U}_\alpha} \int_\theta^{u_\varepsilon} f(s, v_\varepsilon) ds \; dx &\stackrel{\varepsilon \to 0}{\longrightarrow} \int_{\mathcal{U}_\alpha} \int_\theta^{\alpha} f(s, v_0) ds \; dx \\ \int_{\mathcal{U}_\beta} \int_\theta^{u_\varepsilon} f(s, v_\varepsilon) ds \; dx &\stackrel{\varepsilon \to 0}{\longrightarrow} \int_{\mathcal{U}_\beta} \int_\theta^{\beta} f(s, v_0) ds \; dx \end{aligned} \tag{11}$$

where $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ are as in Definition 2.1.

Finally note that $\nabla_x f(s, v) = \partial_2 f(s, v)\, \nabla v(x)$ (here $\partial_2$ denotes the partial derivative with respect to the second argument) and $\partial_2 f$ is uniformly continuous in compact sets. Hence by (H.1),

$$x \cdot \int_\theta^{u_\varepsilon} \partial_2 f(s, v_\varepsilon(x))\, ds \to x \cdot \int_\theta^{u_0} \partial_2 f(s, v_0(x))\, ds \tag{12}$$

strongly in $L^2(\mathcal{U})$. Then if $\nabla v_\varepsilon$ converged weakly in $L^2(\mathcal{U})$ we would have

$$\int_{\mathcal{U}} \int_\theta^{u_\varepsilon} x \cdot \nabla_x f(s, v_\varepsilon) ds\, dx = \tag{13}$$

$$\int_{\mathcal{U}} x \cdot \nabla v_\varepsilon(x) \int_\theta^{u_\varepsilon} \partial_2 f(s, v_\varepsilon(x))\, ds\, dx \stackrel{\varepsilon \to 0}{\longrightarrow} \int_{\mathcal{U}} x \cdot \nabla v_0(x) \int_\theta^{u_0} \partial_2 f(s, v_0(x))\, ds\, dx =$$

$$\int_{\mathcal{U}} \int_\theta^{u_0} x \cdot \nabla_x f(s, v_0) ds\, dx.$$

In order to establish the weak convergence of $\{v_\varepsilon\}$ in $H^1(\mathcal{U})$ note that multiplying the second equation in (1) by $v_\varepsilon$ and integrating on $\Omega_\varepsilon$ we conclude that

$$\int_{\Omega_\varepsilon} h_2(x) |\nabla v_\varepsilon|^2 dx \; = \; \int_{\Omega_\varepsilon} \delta(\varepsilon) g(u_\varepsilon, v_\varepsilon) v_\varepsilon \, dx$$

By our hypotheses and since $\delta(\varepsilon)$ is continuous at $\varepsilon = 0$, the righthand term in the above equality is uniformly bounded in $\varepsilon$, and therefore so is $\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}$. Moreover $v_\varepsilon \to v_0$ uniformly in compact sets of $\mathcal{U}$ hence $v_\varepsilon$ is bounded in $H^1(\mathcal{U})$. It follows that $v_\varepsilon \rightharpoonup v_0$ in $H^1(\mathcal{U})$ and so $\nabla v_\varepsilon \rightharpoonup \nabla v_0$ in $L^2(\mathcal{U})$, as we wanted.

Passing to the limit in (9), as $\varepsilon \to 0$, and using (8), (10), (11) and (13) we obtain

$$\begin{aligned} 0 \;=\; & \int_{\partial\mathcal{U}} x\cdot\hat{n}\int_\theta^{u_0(x)} f(s,v_0(x))ds\,dS \\ -\; & \int_{\mathcal{U}_\alpha}\{\int_\theta^{\alpha(x)} x\cdot\nabla_x f(s,v_0) \;+\; Nf(s,v_0)\,ds\}\,dx \\ -\; & \int_{\mathcal{U}_\beta}\{\int_\theta^{\beta(x)} x\cdot\nabla_x f(s,v_0) \;+\; Nf(s,v_0)\,ds\}\,dx\,. \end{aligned}$$

We will rewrite this expression remarking that if $F(\beta(x)) = \int_\theta^{\beta(x)} f(s,v_0(x))ds$ then

$$div\,(xF(\beta(x)) = N\,F(\beta(x)) + x\cdot\,\nabla_x F(\beta(x)) =$$

$$N\,F(\beta(x)) + x\cdot[f(\beta(x),v_0(x))\cdot\nabla\beta(x) + \int_\theta^{\beta(x)}\nabla_x f(s,v_0(x)ds)],$$

with an analogous formulation for $\alpha$. So we obtain

$$0 = \int_{\partial\mathcal{U}} x\cdot\hat{n}\;\;F(u_0(x))\;dS - \int_{\mathcal{U}_\alpha}\{div(xF(\alpha(x)) - f(\alpha,v_0)\nabla\alpha\cdot x\}\;dx -$$

$$\int_{\mathcal{U}_\beta}\{div(xF(\beta(x)) - f(\beta,v_0)\nabla\beta\cdot x\}\;dx.$$

Since $f(\alpha,v_0) = f(\beta,v_0) = 0$ the Divergence theorem implies

$$\int_\Gamma\{\int_{\alpha(x)}^{\beta(x)} f(s,v_0(x))ds\,\}\,x\cdot\hat{\nu}\;dS = 0. \tag{14}$$

In order to obtain (3) it suffices to follow above procedure with $x\cdot\nabla v$ replaced with $(x+p)\cdot\nabla v$, with $p$ being an arbitrary $N$-vector. We end up with the following equation

$$\int_\Gamma\{\int_{\alpha(x)}^{\beta(x)} f(s,v_0(x))ds\,\}\,(x+p)\cdot\hat{\nu}\;dS = 0.$$

Since $p$ is an arbitrary vector, (3) is proved. ■

*Remark 2. 3.* Condition $(H.2)$ is satisfied in $\partial\mathcal{U}$ if $\{\|\nabla u_\varepsilon\|\}_{0<\varepsilon\leq\varepsilon_0}$ can be controlled there. For example, if $\|\nabla u_\varepsilon(x)\| = o(\varepsilon^{-1/2})$, uniformly in $0<\varepsilon\leq\varepsilon_0$, in some neighborhood of $\partial\mathcal{U}$, then $(H.2)$ holds.

Corollary 2.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family of solutions to problem (1) which develops internal transition layer, as $\varepsilon \to 0$, in $\mathcal{U}$ with interface $\Gamma$ and limit $(u_0(x), v_0(x))$ satisfying $f(u_0, v_0) = 0$, a.e. in $\mathcal{U}$, where $\mathcal{U}$ is an smooth tubular neighborhood of $\Gamma$ such that $\mathcal{U} \subset \Omega_\varepsilon$, $0 < \varepsilon \le \varepsilon_0$, for $\varepsilon_0$ small enough.*

*Moreover suppose that (H.1) is satisfied and that $\alpha$ and $\beta$ are constant functions. Then*

$$\int_\Gamma \int_\alpha^\beta f(s, v_0(x))ds \ \hat{\nu}(x) \ dS = 0. \tag{15}$$

**Proof:** All that remains to prove is that (H.2) holds and that will be the case if we show that

$$\lim_{\varepsilon\to 0} \left|\varepsilon^{1/2}\, \nabla u_\varepsilon(x)\right| = 0\,, \text{ uniformly on } \partial\mathcal{U}\ .$$

A standard procedure will be used and therefore we only sketch the proof. It is based on first flattening locally the boundary, using a blow-up technique and then some Schauder estimates.

It suffices to prove that any sequence $\left(\varepsilon_k, x^k\right) \in \mathbb{R}^+ \times \partial\mathcal{U}$, with $\varepsilon_k \to 0$, has a subsequence $\left(\varepsilon_{k_i}, x^{k_i}\right)$, for which the following equality holds

$$\lim_{i\to\infty} \left|\varepsilon_{k_i}^{1/2}\, \nabla u_{\varepsilon_{k_i}}(x^{k_i})\right| = 0\ .$$

If $\left(\varepsilon_k, x^k\right) \in \mathbb{R}^+ \times \partial\mathcal{U}$ is such a sequence, since $\partial\mathcal{U}$ is compact there exists a subsequence, still labeled $\left(\varepsilon_k, x^k\right)$ such that $x^k \overset{k\to\infty}{\longrightarrow} \overline{x}$, for some $\bar{x} \in \partial\mathcal{U}$.

Define a $C^2$ change of variables $\Sigma(x) = y$ which straightens $\partial\mathcal{U}$, near $\overline{x}$ and then set

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(\Sigma^{-1}(y)), \quad \text{for} \quad y \in \overline{B_\rho^+},$$

where $\overline{B_\rho^+}$ stands for the positive hemisphere of the ball of radius $\rho$ and center at the origin. Now define scaled functions $\omega_k(z)$ and $\theta_k(z)$ by

$$\omega_k(z) = \tilde{u}_{\varepsilon_k}\left(y^k + \varepsilon_k^{1/2} z\right)$$

$$\theta_k(z) = \tilde{u}_{\varepsilon_k}\left(y^k + \varepsilon_k^{1/2} z\right)$$

for $z \in \overline{B^+_{\rho/\varepsilon_k^{1/2}}}$.

All the coefficients in the new equation for $\omega_k$ are $C^\nu$ bounded, uniformly in $k$. For a fixed $\rho$, we set

$$\rho_k \overset{\text{def}}{=} \left(\rho/\varepsilon_k^{1/2}\right) \overset{k\to\infty}{\longrightarrow} \infty\ .$$

Let $R_m$ be a monotone increasing sequence of positive numbers such that $R_m \to +\infty$, as $m \to \infty$. For each $m$, there is $k_m$ such that $2R_m < \rho_k$, for $k \geq k_m$.

Since $\{u_\varepsilon\}$ and $\{v_\varepsilon\}$ are bounded on $\overline{\mathcal{U}}$, uniformly in $\varepsilon$, it follows that

$$\|\theta_k\|_{C\left(\overline{B^+_{2R_m}}\right)}, \ \ \|\omega_k\|_{C\left(\overline{B^+_{2R_m}}\right)} \leq K_1 \ ,$$

for some constant $K_1$, independent of $k$.

By Theorem 8.24, [1], we conclude that $\theta_k$ and $\omega_k$ are locally $C^\nu$ bounded in $\overline{B^+_{2R_m}}$, uniformly in $k$.

Using interior Schauder estimate in $B_{R_m}$ (see Theorem 6.13 and Exercise 6.1, [1]) we conclude that $\omega_k$ is $C^{2,\nu}$ bounded in $\overline{B^+_{2R_m}}$, uniformly for $k \geq k_m$. Then by a diagonal process we can extract a subsequence, still labeled $\{\omega_k\}$, such that $\omega_k \to \omega_o$ in $C^2_{\text{loc}}\left(\mathbb{R}^N_+\right)$ where $\mathbb{R}^N_+ = \left\{z \in \mathbb{R}^N : z_N \geq 0\right\}$.

Clearly from our hypotheses $\omega_0 \equiv \beta$ or $\omega_0 \equiv \alpha$. It follows that $\lim_{k\to\infty} |\nabla\omega_k(0)| = 0$ and therefore

$$\lim_{k\to\infty} \left|\varepsilon_k^{1/2}\, \nabla u_{\varepsilon_k}(x^k)\right| = 0 \ .$$

Thus our claim is proved. ∎

Next we provide sufficient conditions so that $\alpha(x)$, $\beta(x)$ and $v_0(x)$ are constant functions thus obtaining a simpler equal-area condition.

Corollary 2.2. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family of solutions as in Corollary 2.1 which satisfies (H.1). Hence*

- *if $\delta(\varepsilon) \xrightarrow{\varepsilon\to 0} 0$ then $v_0$ is constant.*
- *and if the nullcline of $f$ ($\stackrel{\text{def}}{=} \{(u,v) : f(u,v) = 0\}$) intersects $\{(s, v_0) : \alpha(x) \leq s \leq \beta(x),\, x \in \mathcal{U}\}$ in a discret set, then $\alpha$ and $\beta$ are constant functions.*

*In this case (2.15) simplifies to (2.1).*

**Proof:** Multiplying the second equation in (1.1) by $v_\varepsilon$ and integrating over $\Omega_\varepsilon$, it yields

$$\int_{\Omega_\varepsilon} \{-h_2(x)|\nabla v_\varepsilon(x)|^2 + \delta(\varepsilon) g(u_\varepsilon, v_\varepsilon) v_\varepsilon(x)\}\, dx = 0.$$

Our hypotheses imply right away that

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon(x)|^2 dx \to 0$$

This implies that $v_\varepsilon \to v_0$ in $H^1(\mathcal{U})$ and so $v_0$ is a constant function a.e. in $\mathcal{U}$.

Recall that $u_\varepsilon \to u_0$ where $u_0$ is given by (1.2). By hypothesis $\{s; f(s, v_0) = 0\}$ is a discret set. But $\alpha$ and $\beta$ are continuous functions and therefore are constant on $\mathcal{U}$. Thus

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

by (3) we will have

$$\int_{\Gamma}\int_{\alpha}^{\beta}\{f(s,v_0)ds\}\; x\cdot\hat{n}\; dS = 0$$

But $\int_{\Gamma} x\cdot\hat{n}\; dS = \int_{\mathcal{O}_\Gamma} div\; x\; dx = N\,|\mathcal{O}_\Gamma| \neq 0$, where $\mathcal{O}_\Gamma$ stands for the open region enclosed by $\Gamma$. Hence $\int_{\alpha}^{\beta} f(s,v_0)ds = 0$ ∎

Corollary 2.3. *(On the location of the interface) Suppose in addition to the hypotheses of Corollary 2.2 that the domain $\Omega$ is fixed and that $\frac{\partial v_\varepsilon}{\partial \hat{n}} = 0$ , on $\partial\Omega$.*

*Then it holds that*

$$|\Omega_\alpha|\; g(\alpha,v_0) + |\Omega_\beta|\; g(\beta,v_0) = 0 \tag{16}$$

*where $\Omega_\alpha = \{u_0 = \alpha\}$ and $\Omega_\beta = \{u_0 = \beta\}$*

The proof is straightforward and we remark that (2.16) not only gives an indication on the location of the interface $\Gamma$ but also that g must have different signs at $(\alpha, v_0)$ and $(\beta, v_0)$. This is so because $\Omega\backslash\Gamma = \Omega_\alpha \cup \Omega_\beta$. If, for instance, $g(\alpha,v_0) = -g(\beta,v_0) \neq 0$ then the interface must partition $\Omega$ into two subsets $\Omega_\beta$ and $\Omega_\alpha$ such that $|\Omega_\beta| = |\Omega_\alpha| = (1/2)|\Omega|$.

As said before stationary solutions to (1.1) satisfy (2.1), with $h_1 \equiv h_2 \equiv 1$ and $\Omega = \Omega_\varepsilon$. Therefore next result follows immediately from Corollary 2.2.

Corollary 2.4. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family of stationary solutions to (1.1), which develops internal transition layer, as $\varepsilon \to 0$. With the notation of Definition 2.1, if the nullcline of $f$ intersects $\{(s,v_0) : \alpha(x) \leq s \leq \beta(x),\, x \in \mathcal{U}\}$ in a discret set, then (2.1) holds.*

*Moreover if $\frac{\partial v}{\partial \hat{n}} = 0$ on $\partial\Omega$, then the interface divides $\Omega$ into two subsets $\Omega_\alpha$ and $\Omega_\beta$ such that (2.16) is satisfied.*

Equation (2.1) is sometimes referred to as the $v_0$-level equal-area condition for $f(\cdot, v)$ with its obvious counterpart for scalar equations. It appears in several (if not all) works which give sufficient conditions for the existence of internal transition layer either in systems or in scalar equations. See [2], [3], [4], [5] and [9], for instance.

*Remark 2. 4.* Note that the case in which $\Gamma \cap \partial\Omega \neq \emptyset$ has been ruled out in Definition 2.1. When $h_1$ and $h_2$ are constant functions this is more likely to happen.

All the conclusions of Corollary 2.4 still hold true for this case as long as we change Definition 2.1 to allow the interface to intersect the boundary of $\Omega$. We will indicate below how this can be accomplished without going through the obvious modifications that Definition 2.1 requires.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

LEMMA 2.1. *Let $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ be a family of stationary solutions to (1.1) as in Corollary 2.4 except that now that $\Gamma$ is a $(N-1)$-dimensional manifold in $\Omega$ whose boundary $\partial\Gamma$ (in the induced topology of $\Gamma$) is a $(N-2)$-dimensional manifold immersed in $\partial\Omega$.*

*Then all conclusions of Corollary 2.4 hold true.*

Note that in this case we no longer have uniform convergence in $\partial\mathcal{U}$, of the integrand in (H.2), which was sufficient to conclude that the integral vanished, as $\varepsilon \to 0$. This difficulty can be overcome as follows.

Standard Schauder estimates implies that there is a constant $L > 0$, independent of $\varepsilon$, such that $\varepsilon|\nabla u_\varepsilon(x)|^2 \leq L, \ \ \forall x \in \overline{\Omega}$.

For any $x \in \overline{\mathcal{U}}\backslash\Gamma$ where $\mathcal{U}$ is a neighborhood of $\Gamma$ in $\Omega$, there is a compact neighborhood $K$ of $x$ in $\overline{\mathcal{U}}\backslash\Gamma$ such that $u_\varepsilon$ converges to $\alpha$ or $\beta$, uniformly in $K$.

Therefore the very same argument used in the proof of Corollary 2.1 now allows us to conclude that

$$\varepsilon|\nabla u_\varepsilon(x)|^2 \to 0,$$

a.e. in $\partial\mathcal{U}$. It follows that (H.2) is satisfied by an application of the Lebesgue bounded convergence theorem.

## 3. NECESSITY FOR STABLE PATTERNS

Let us consider the following reaction-diffusion system

$$\begin{cases} u_t = \varepsilon^2 \triangle u + f(u,v), \\ v_t = D\triangle v + g(u,v), \ \ (x,t) \in \Omega \times (0,\infty) \\ \dfrac{\partial u}{\partial \hat{n}} = \dfrac{\partial v}{\partial \hat{n}} = 0, \ (x,t) \in \partial\Omega \times (0,\infty) \end{cases} \tag{1}$$

where $u$ is the activator, $v$ is the inhibitor, $\Omega$ is a smooth domain in $I\!R^N$, $D > 0$ and $\varepsilon$ a small positive parameter. The nullcline of $f$ is sigmoidal and consists of three smooth curves $u = h_-(v)$, $u = h_0(v)$ and $u = h_+(v)$ defined on the intervals $I_-$, $I_0$ and $I_+$, respectively. Also if $\min I_- = \underline{v}$ and $\max I_+ = \overline{v}$ then the inequality $h_-(v) < h_0(v) < h_+(v)$ holds for $I^* = (\underline{v}, \overline{v})$ and $h_+(v)$ (resp., $h_-(v)$) coincides with $h_0(v)$ at only one point $v = \overline{v}$ (resp., $v = \underline{v}$), respectively.

In [8], (3.1) is assumed to satisfy a set of hypotheses, which we denote by $\mathcal{H}$, among which we mention the following one concerning $f$:

$(Q.1)$ $J(v) = \int_{h_-(v)}^{h_+(v)} f(\xi, v) d\xi$ has one isolated zero at $v = v^* \in I^*$ such that $\frac{dJ}{dv} < 0$ at $v = v^*$.

Under the hypotheses $\mathcal{H}$, it is proven that if (3.1) has an $\varepsilon$-family $(U_\varepsilon, V_\varepsilon)$ of stationary matched solutions of order 1 whose $\alpha^*$-level surface $S_\varepsilon$ of $U_\varepsilon$ is smooth up to $\varepsilon = 0$, then it must become unstable for $\varepsilon$ small.

Here $\alpha^*$ an intermediate value between the two stable branches of the nullcline of $f$. Also by "smooth up to $\varepsilon = 0$" it is meant that there exists an $(N-1)$-dimensional smooth compact connected manifold $S_0$ without boundary in $I\!R^N$ such that $S_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} S_0$.

Therefore the question arises as to what happens with the configuration of an $\varepsilon$-family $(u_\varepsilon, v_\varepsilon)$ of matched stable stationary solutions to (3.1), which are known to exist for a certain parameter range. Their result suggests that stable patterns somehow must become very fine and/or complicated as $\varepsilon \to 0$.

Actually based on the balance of the bulk force and the mean-curvature effect they formally derive that the rate of shrinking of stable patterns is of order $\varepsilon^{1/3}$. See [8], p. 1103. Hence by a suitable scaling, the resulting rescaled equations capture the morphology of the magnified patterns. After performing the change of variable $X = \frac{x-x^*}{\varepsilon^{1/3}}$ for a suitable $x^* \in R^N$, the rescaled system becomes

$$\begin{cases} \tilde{u}_t = \tilde{\varepsilon}^2 \triangle \tilde{u} + f(\tilde{u}, \tilde{v}), \\ \nu\tilde{\varepsilon}\tilde{v}_t = D\triangle\tilde{v} + \tilde{\varepsilon} g(\tilde{u}, \tilde{v}), \ \ (X,t) \in \Omega_\varepsilon \times (0,\infty) \\ \dfrac{\partial \tilde{u}}{\partial \hat{n}} = \dfrac{\partial \tilde{v}}{\partial \hat{n}} = 0, \ (X,t) \in \partial\Omega_\varepsilon \times (0,\infty) \end{cases} \tag{2}$$

where $\tilde{\varepsilon} = \varepsilon^{2/3}$ and the rescaled domain $\Omega_\varepsilon$ satisfies $\Omega_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \widetilde{\Omega}$ with $|\widetilde{\Omega}| < \infty$.

It follows that if $\Gamma_\varepsilon \stackrel{\text{def}}{=} \{X \in \Omega_\varepsilon : \tilde{u}_\varepsilon = \alpha^*\}$ then there exists an $(N-1)$-dimensional smooth compact connected manifold $\Gamma_0$ such that $\Gamma_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \Gamma_0$.

The convergence $\Omega_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} \widetilde{\Omega}$ may be any one as long as the hypotheses of Theorem 2.1 (when $\Gamma_0 \cap \partial\Omega = \emptyset$) or Lemma 2.1 (when $\Gamma_0 \cap \partial\Omega \neq \emptyset$) on the tubular neighborhood $\mathcal{U}$ of $\Gamma_0$ are satisfied.

It turns out however that the system for the stationary solutions to (2) is just (1) when $h_1 = 1$, $h_2 = D$ and $\delta(\tilde{\varepsilon}) = \tilde{\varepsilon}$, namely,

$$\begin{cases} \tilde{\varepsilon}^2 \triangle \tilde{u} + f(\tilde{u}, \tilde{v}) = 0, \ \ X \in \Omega_\varepsilon \\ D\triangle\tilde{v} + \tilde{\varepsilon} g(\tilde{u}, \tilde{v}) = 0, \ \ X \in \Omega_\varepsilon \\ \dfrac{\partial \tilde{u}}{\partial \hat{n}} = \dfrac{\partial \tilde{v}}{\partial \hat{n}} = 0, \ X \in \partial\Omega_\varepsilon \end{cases} \tag{3}$$

Thus Corollary (2.2) applies to (3) with $\tilde{v}_0 = v^*$, $\alpha = h_-(v^*)$ and $\beta = h_+(v^*)$, yielding $J(v_0) = 0$.

Summing up the above heuristic argument, along with Corollary 2.2 strongly suggest that *if* $(u_\varepsilon, v_\varepsilon)$ *is a family of matched stable stationary solutions to (3.1) which is bounded in* $\Omega$*, uniformly in* $\varepsilon$ *then necessarily* $J(v_0) = 0$*, where* $v_0$ *is a constant such that* $v_\varepsilon \to v_0$*, as* $\varepsilon \to 0$.

## REFERENCES

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

1. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York (1983).

2. J. K. Hale and K. Sakamoto, *Existence and stability of transition layers.* Japan J. Appl. Math. 5,(1988), 367-405.

3. A.S. do Nascimento, *Stable Stationary Solutions Induced by Spatial Inhomogeneity Via* $\Gamma$*-Convergence.* Bulletin of the Brazilian Mathematical Society 29, no.1 (1998), 75-97.

4. A.S. do Nascimento, *Stable multiple-layer stationary solutions of a semi-linear parabolic equation in two-dimensional domains.* Eletronic J. Diff. Eqns. no.21 (1997), 1-17.

5. A.S. do Nascimento, *Reaction-diffusion induced stability of a spatially inhomogeneous equilibrium with boundary layer formation.* J. Diff. Eqns. 108, no.2 (1994), 296-325.

6. Y. Nishiura, *Coexistence of infinitely many solutions to reaction-diffusion systems in the singular limit.* Dynam. Report. Expositions Dynam. Systems (N.S.) 3, Springer-Verlag, New York, 1994, 25-103.

7. Y. Nishiura, *Global structure of bifurcating solutions of some reaction-diffusion systems.* S.I.A.M. J. Math. Anal. 13, no.4, (1982) 555-593.

8. Y. Nishiura and H. Suzuki, *Nonexistence of higher dimensional stable Turing patterns in the singular limit.* S.I.A.M. J. Math. Anal. 29, no. 5 (1998), 1087-1105.

9. M. Taniguchi and H. Nishiura, *Instability of planar interfaces in reaction-diffusion systems* SIAM J. Math. Anal. 25, no.1 (1994), 99-134.

10. A. Turing, *The chemical basis of morphogenesis* Philos. Trans. Roy. Soc. London Ser B, 237 (1952), 37-72.

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC