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A Complex Approach to Strict Positive Definiteness on Spheres

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We study strictly positive definite functions on the unit sphere of either $\mathbb{C}^q$, $q \geq 2$ or its complex extension ℓ^2. We present several characterizations of strict positive definiteness and use them to actually identify strictly positive definite functions. The role of strict positive definiteness on real spheres in this new context is examined.

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1. Introduction

Let Ω_{2q} be the unit sphere in $\mathbb{C}^q$, $q \geq 2$, Ω_∞ the unit sphere in ℓ^2, and $\langle \cdot, \cdot \rangle$ the usual inner product in either $\mathbb{C}^q$ or ℓ^2. We are interested in the problem of finding continuous functions $f : B_2 := \{\zeta \in \mathbb{C} : |\zeta| \leq 1\} \to \mathbb{C}$ having the following property: for some fixed $N \geq 1$, the matrix with $\mu\nu$-entry $f(\langle z_\mu, z_\nu \rangle)$ is positive definite, whenever $\{z_1, z_2, \ldots, z_N\}$ is a subset of cardinality N of Ω_{2q}, $2 \leq q \leq \infty$. Such functions are useful in approximation of data on complex spheres: if $\lambda_1, \lambda_2, \ldots, \lambda_N$ are any scalars and f has the above property then we can find an unique interpolant ϕ to the data

$$\{(z_\mu, \lambda_\mu) : \mu = 1, 2, \ldots, N\} \tag{1.1}$$

from the span of the N functions

$$z \in \Omega_{2q} \longrightarrow f(\langle z, z_\mu \rangle), \quad \mu = 1, 2, \ldots, N. \tag{1.2}$$

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The unit spheres in $\mathbb{R}^m$ are embedded in Ω_{2q}, $q \geq m$. Hence, if the z_μ all belong to a real sphere the interpolation procedure above reduces to that already considered in many other places ([3,4,7,13]).

The intricate nature of the problem demands that some a priori properties of the function need to be known, otherwise the analysis of the problem can be done case by case only. A frequent approach, which emerges from practical reasons, is to assume that the continuous function f is *positive definite on* Ω_{2q} in the sense that

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle z_\mu, z_\nu \rangle) \geq 0 \tag{1.3}$$

whenever N is a positive integer, $\{z_1, z_2, \ldots, z_N\} \subset \Omega_{2q}$, and $\{c_1, c_2, \ldots, c_N\} \subset \mathbb{C}$. In this context, one fixes N, and tries to identify which positive definite functions on Ω_{2q} have the property introduced in the previous paragraph, that is, the property that

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle z_\mu, z_\nu \rangle) > 0, \tag{1.4}$$

whenever the z_μ are distinct on Ω_{2q} and at least one of the c_μ is nonzero. Functions having this property are called *strictly positive definite of order* N *on* Ω_{2q} and they will be the subject matter of this paper.

Positive definite functions on Ω_{2q}, $q < \infty$, are representable in the form

$$f(z) = \sum_{(m,n)\in\mathbb{Z}_+^2} a_{m,n}(f) R_{m,n}^{q-2}(z), \quad z \in B_2, \tag{1.5}$$

in which $R_{m,n}^{q-2}$ is the disk polynomial of degree $m+n$ associated to the integer $q-2$, $a_{m,n}(f) \geq 0$ for all m and n and $\sum_{(m,n)\in\mathbb{Z}_+^2} a_{m,n}(f) < \infty$. The *disk polynomial* of degree $m+n$ in x and y associated to a positive real number α is the polynomial $R_{m,n}^\alpha$ given by

$$R_{m,n}^\alpha(z) := r^{|m-n|} e^{i(m-n)\theta} R_{m\wedge n}^{(\alpha,|m-n|)}(2r^2-1), \quad z = re^{i\theta} = x+iy, \tag{1.6}$$

where $R_{m\wedge n}^{(\alpha,|m-n|)}$ is the usual Jacobi polynomial of degree $m \wedge n := \min\{m,n\}$ associated to the numbers α and $|m-n|$, here normalized by $R_{m\wedge n}^{(\alpha,|m-n|)}(1) = 1$. An important feature of $R_{m,n}^\alpha$ is that it is a polynomial of degree m in the variable z and of degree n in the variable $\overline{z}$. Characterization (1.5) ought to be compared with that of positive definite functions on real spheres obtained by Schoenberg ([17]). In particular, a positive definite function on a Ω_{2q} is positive definite on a real sphere of dimension q or less.

Due to the orthogonality relations for Jacobi polynomials, $\{R_{m,n}^\alpha : 0 \leq m,n < \infty\}$ is a complete orthogonal system in $L^2(B_2, dw_\alpha)$, where dw_α is the positive measure of total mass one on B_2 given by

$$dw_\alpha(z) = \frac{\alpha+1}{\pi}\left(1-x^2-y^2\right)^\alpha dxdy, \quad z = x+iy. \tag{1.7}$$

For further information on disk polynomials see [1,2,9,11,12] and many references given there. Particularly important to us is the so-called addition formula for disk polynomials ([10,11,16]):

$$R^{q-2}_{m,n}(\cos\theta_1\cos\theta_2\, e^{i(\phi_1+\phi_2)}+\sin\theta_1\sin\theta_2\, w)$$
$$=\sum_{k=0}^{m}\sum_{l=0}^{n} b^{k,l}_{m,n,q} Q^{k,l}_{m,n}(\theta_1,\phi_1)\overline{Q^{k,l}_{m,n}(\theta_2,\phi_2)}R^{q-3}_{k,l}(w).$$

It holds for nonnegative integers m and n, $\theta_1,\theta_2\in[0,\pi/2]$, $\phi_1,\phi_2\in[0,2\pi)$, and w in B_2. The functions $Q^{k,l}_{m,n}$ are defined by

$$Q^{k,l}_{m,n}(\theta,\phi):=(\sin\theta)^{k+l}\, R^{q-2+k+l}_{m-k,n-l}(\cos\theta\, e^{i\phi}),\quad (\theta,\phi)\in[0,\pi/2]\times[0,2\pi) \tag{1.8}$$

and the constants $b^{k,l}_{m,n,q}$ are all positive.

Positive definite functions on Ω_∞ have the form

$$f(z)=\sum_{(m,n)\in\mathbb{Z}^2_+} a_{m,n}(f) z^m\overline{z}^n, \tag{1.9}$$

where $a_{m,n}(f)\geq 0$ for all m and n and $\sum_{(m,n)\in\mathbb{Z}^2_+} a_{m,n}(f)<\infty$ ([5]). For reasons of uniformity on the presentation, we will write $R^{\infty}_{m,n}(z):=z^m\overline{z}^n$.

Given a positive definite function f as in either (1.5) or (1.9), define

$$K_q(f):=\{(m,n): a_{m,n}(f)>0\},\quad 2\leq q\leq\infty. \tag{1.10}$$

The equality

$$\sum_{\mu,\nu=1}^{N} c_\mu\overline{c_\nu} f(\langle z_\mu,z_\nu\rangle)=\sum_{(m,n)\in K_q(f)} a_{m,n}(f)\sum_{\mu,\nu=1}^{N} c_\mu\overline{c_\nu} R^{q-2}_{m,n}(\langle z_\mu,z_\nu\rangle), \tag{1.11}$$

holds whenever $\{c_1,c_2,\ldots,c_N\}\subset\mathbb{C}$ and $z_1,z_2,\ldots,z_N$ are distinct points on the corresponding sphere. Since every matrix $(R^{q-2}_{m,n}(\langle z_\mu,z_\nu\rangle))$ is nonnegative definite we infer, from the nature of $K_q(f)$, that $(f(\langle z_\mu,z_\nu\rangle))$ is positive definite if and only if

$$\sum_{\mu,\nu=1}^{N} c_\mu\overline{c_\nu} R^{q-2}_{m,n}(\langle z_\mu,z_\nu\rangle)>0 \tag{1.12}$$

for some (m,n) in $K_q(f)$. In other words, the strict positive definiteness (SPD) of order N of f depends upon $K_q(f)$ only and not on the magnitude of the coefficients $a_{m,n}(f)$. This remark justifies the introduction of the following nomenclature: a subset K of $\mathbb{Z}^2_+$ *induces SPD of order* N on Ω_{2q} if every positive definite function f on Ω_{2q} for which $K_q(f)=K$ is strictly positive definite of order N on Ω_{2q}.

The subject matter of this paper is then to study SPD on Ω_{2q}, $2\leq q\leq\infty$. In Section 2, we present several formulations for the concept of SPD. In Section 3, we present some general properties of sets that induce SPD and we determine all sets that induce SPD of order 2. Section 4 reveals a variety of sets that induce SPD. Finally, in Section 5, we establish a connection between SPD on real and complex spheres.

2. SPD and spaces of disk polynomials

In this section, we collect equivalent formulations for the concept of SPD on Ω_{2q}. Some of them are of independent interest while others are used to prove subsequent results in the paper.

Firstly, we introduce surface harmonics on Ω_{2q} and spaces related to them. The space composed of polynomials of the form

$$p(z,\overline{z}) := p(z^1, z^2, \ldots, z^q, \overline{z^1}, \overline{z^2}, \ldots, \overline{z^q}), \quad z = (z^1, z^2, \ldots, z^q) \in \mathbb{C}^q, \tag{2.1}$$

which are homogeneous of degree m in z and degree n in $\overline{z}$ is denoted by $Hom^q_{m,n}$. A *solid harmonic* of type (m,n) (on $\mathbb{C}^q$) is an element p of $Hom^q_{m,n}$ satisfying the complex Laplacian, i.e,

$$\sum_{\mu=1}^{q} \frac{\partial^2}{\partial z^\mu \partial \overline{z^\mu}}(p) = 0.$$

The space composed of restrictions of solid harmonics of type (m,n) to Ω_{2q} is denoted by $Harm^q_{m,n}$. Finally, given a subset K of $\mathbb{Z}^2_+$, we write

$$Harm^q_K := \bigoplus_{(m,n)\in K} Harm^q_{m,n}. \tag{2.2}$$

Many properties of the polynomial spaces introduced above can be found in [15], including an interesting characterization of those sets K for which the uniform closure of $Harm^q_K$ is an algebra.

Several results in the paper rely on the so-called addition formula for complex harmonics. Before stating this formula we need some additional notation. We write dw to denote the unique probability Borel measure on Ω_{2q}, invariant with respect to all unitary transformations of $\mathbb{C}^q$ and we frequently speak of orthogonality with respect to this measure, that is, with respect to the following inner product:

$$\langle\langle f,g\rangle\rangle := \int_{\Omega_{2q}} f(z)\overline{g(z)}dw(z), \quad f,g \in L^2(\Omega_{2q}, dw). \tag{2.3}$$

If $\{Y^q_{m,n,k} : k = 1,2,\ldots,N(m,n,q)\}$ is an orthonormal basis of $Harm^q_{m,n}$, then the addition formula reads like this ([9]):

$$R^{q-2}_{m,n}(\langle z_\mu, z_\nu\rangle) = D(m,n,q) \sum_{k=1}^{N(m,n,q)} Y^q_{m,n,k}(z_\mu)\overline{Y^q_{m,n,k}(z_\nu)}, \tag{2.4}$$

where $D(m,n,q)$ is a positive constant. ¿From now on, unless stated otherwise, we will assume that an orthonormal basis of $Harm^q_{m,n}$ as above has been fixed.

Theorem 2.1. *Let K be a subset of $\mathbb{Z}^2_+$, N a positive integer and q an integer at least 2. The following assertions are equivalent:*
i) K induces SPD of order N on Ω_{2q};
ii) Given N distinct points $z_1, z_2, \ldots, z_N$ on Ω_{2q}, there exists no nonzero linear functional of the form $\mathcal{L}(g) := \sum_{\mu=1}^N c_\mu g(z_\mu)$ that annihilates $Harm^q_K$;
iii) If $E \subset \Omega_{2q}$ has cardinality N, then the space $Harm^q_K|_E := \{p|_E : p \in Harm^q_K\}$ has dimension N;
iv) For every subset E of Ω_{2q} of cardinality N and every function g defined on E, there is a $p \in Harm^q_K$ such that $p|_E = g$.

Proof. For a positive definite function f on Ω_{2q}, the addition formula yields

$$\sum_{\mu,\nu=1}^N c_\mu \overline{c_\nu} f(\langle z_\mu, z_\nu\rangle) = \sum_{(m,n)\in K} a_{m,n}(f) D(m,n,q) \sum_{k=1}^{N(m,n,q)} \left| \sum_{\mu=1}^N c_\mu Y^q_{m,n,k}(z_\mu)\right|^2, \qquad (2.5)$$

whenever $K = K_q(f)$, $\{c_1, c_2, \ldots, c_N\} \subset \mathbb{C}$ and $z_1, z_2, \ldots, z_N$ are distinct points on Ω_{2q}. If a functional $\mathcal{L}$ as in ii) annihilates $Harm^q_K$ then it follows that

$$\sum_{\mu,\nu=1}^N c_\mu \overline{c_\nu} f(\langle z_\mu, z_\nu\rangle) = 0. \qquad (2.6)$$

Thus, if K induces SPD of order N on Ω_{2q}, (2.6) implies that all the c_μ are zero, revealing that i) implies ii). If K does not induce SPD of order N on Ω_{2q}, (2.6) holds for some set $\{c_1, c_2, \ldots, c_N\}$ satisfying $\sum_{\mu=1}^N |c_\mu| > 0$. In particular,

$$\sum_{\mu=1}^N c_\mu Y^q_{m,n,k}(z_\mu) = 0, \quad (m,n) \in K, \quad 1 \le k \le N(m,n,q), \qquad (2.7)$$

i.e, the nonzero functional $\mathcal{L}(g) := \sum_{\mu=1}^N c_\mu g(z_\mu)$ annihilates $Harm^q_K$. Thus, ii) implies i). The proofs of the remaining implications will be omitted due to the simplicity of the arguments involved. ■

Next, we introduce additional notation. For a positive integer N, we write E^q_N to denote the linear span over $\mathbb{C}$ of all functions of the form

$$z \in \mathbb{C}^q \longrightarrow \sum_{\mu=1}^N c_\mu e^{\langle z, \overline{z_\mu}\rangle} e^{\langle z_\mu, z\rangle}, \qquad (2.8)$$

where $\{z_1, z_2, \ldots, z_N\} \subset \Omega_{2q}$ and $\{c_1, c_2, \ldots, c_N\} \subset \mathbb{C}$. We will encounter functions belonging to the algebra generated by all monomials of the form $z^r \overline{z}^s$, where r and s stand for nonnegative multi-indices, that is, the algebra composed of all series of the form

$$\sum_{|r|,|s|=0}^{\infty} a_{r,s} z^r \overline{z}^s, \quad z \in \mathbb{C}^q. \tag{2.9}$$

If h is a series as in (2.9), we write $h_{m,n}$ to denote the sum $\sum_{|r|=m,|s|=n} a_{r,s} z^r \overline{z}^s$. For any polynomial $p = p(z, \overline{z})$ we write $p(D)$ to denote the operator obtained from p by substituting the variable z^μ (respectively $\overline{z^\mu}$) by the symbol $\partial/\partial z^\mu$ (respectively $\partial/\partial\overline{z^\mu}$).

The following lemma lists two elementary properties of the differential operator $p(D)$.

Lemma 2.2. *Let h be a sum as described in* (2.9) *and p an element of $Hom^q_{m,n}$. The following properties hold:*
i) $[p(D)(h)](0) = [p(D)(h_{m,n})](0) = [h_{m,n}(D)(p)](0) = h_{m,n}(D)(p)$;
ii) $p(D)\left(\exp(\langle \cdot, \overline{\zeta}\rangle) \exp(\langle \zeta, \cdot\rangle)\right) = p(\zeta) \exp(\langle \cdot, \overline{\zeta}\rangle) \exp(\langle \zeta, \cdot\rangle)$, $\zeta \in \mathbb{C}^q$.

Theorem 2.3. *Let q, K and N be as in Theorem 2.1. The following assertions are equivalent:*
i) K induces SPD of order N on Ω_{2q};
ii) If h is in $E^q_N \setminus \{0\}$ then there exists a pair (m,n) in K for which $h_{m,n}(D)$ is not the zero operator on $Harm^q_{m,n}$.

Proof. Let h be nonzero and of the form

$$h(z) = \sum_{\mu=1}^{N} c_\mu e^{\langle z, \overline{z_\mu}\rangle} e^{\langle z_\mu, z\rangle}, \quad c_1, c_2, \ldots, c_N \in \mathbb{C}, \tag{2.10}$$

where the z_μ are N points on Ω_{2q}. We assume that for every $(m,n) \in K$, the operator $h_{m,n}(D)$ is zero on $Harm^q_{m,n}$ and use Theorem 2.1 to show that K does not induce SPD of order N on Ω_{2q}. The linear functional

$$\mathcal{L}(g) := \sum_{\mu=1}^{N} c_\mu g(z_\mu), \tag{2.11}$$

is nonzero and for every (m,n) in K we have that

$$\mathcal{L}(p) = \sum_{\mu=1}^{N} c_\mu p(z_\mu) = [p(D)(h)](0) = h_{m,n}(D)(p), \quad p \in Harm^q_{m,n}. \tag{2.12}$$

Consequently, $\mathcal{L}$ annihilates $Harm^q_K$ and the rest follows from Theorem 2.1.
Conversely, if K does not induce SPD of order N on Ω_{2q}, Theorem 2.1 warrants the existence of a nonzero functional $\mathcal{M}$ of the form $\mathcal{M}(g) := \sum_{\mu=1}^{N} d_\mu g(z_\mu)$, where the z_μ are points on Ω_{2q}, that annihilates $Harm^q_K$. In particular it annihilates $Harm^q_{m,n}$, $(m,n) \in K$.

Defining

$$g(z)=\sum_{\mu=1}^{N} d_\mu e^{\langle z,\overline{z_\mu}\rangle} e^{\langle z_\mu, z\rangle},$$

it is easily seen that g is not identically zero and that, for every (m,n) in K,

$$g_{m,n}(D)(p)=0, \quad p\in Harm^q_{m,n}. \tag{2.13}$$

Thus *ii*) does not hold. ■

Lemma 2.4. *For positive integers q, m and n, it holds the formula*

$$Hom^q_{m,n}=Harm^q_{m,n}\oplus\Gamma\, Hom^q_{m-1,n-1},$$

in which $\Gamma(z):=\langle z,z\rangle$.

Proof. This is Proposition 30 in [1]. ■

Theorem 2.5. *Let q, K and N be as in Theorem 2.1. The following assertions are equivalent:*
i) K induces SPD of order N on Ω_{2q};
ii) If h is a nonzero function in E^q_N, then there exists a pair (m,n) in K such that $h_{m,n}$ is not divisible by Γ;
iii) If h is a nonzero function in E^q_N then $\sum_{(m,n)\in K} h_{m,n}$ is not divisible by Γ.

Proof. Assume there is an $h\in E^q_N\setminus\{0\}$ such that $h_{m,n}$ is divisible by Γ, for every $(m,n)\in K$. Writing $h_{m,n}=\Gamma f$, $(m,n)\in K$, $f\in Hom^q_{m-1,n-1}$ we have that

$$\begin{aligned} h_{m,n}(D)(p) &= (\Gamma\, f)(D)(p)\\ &= (\Gamma(D)f(D))(p)\\ &= f(D)(\Gamma(D)(p))\\ &= f(D)(0)=0, \quad p\in Harm^q_{m,n}. \end{aligned}$$

Thus, due to Theorem 2.3, K does not induce SPD of order N on Ω_{2q}. This shows that i) implies ii). If K does not induce SPD of order N on Ω_{2q}, then Theorem 2.3 assures the existence of a function g in $E^q_N\setminus\{0\}$ with this property: for every (m,n) in K, the operator $g_{m,n}(D)$ annihilates $Harm^q_{m,n}$. We affirm that each $g_{m,n}(D)$ is divisible by Γ. Indeed, if this is not the case for some (m,n) in K, we first use Lemma 2.4 to write

$$g_{m,n}=h_1+\Gamma\, h_2, \quad 0\neq h_1\in Harm^q_{m,n}, \quad h_2\in Hom^q_{m-1,n-1}. \tag{2.14}$$

Since $g_{m,n}(D)$ annihilates $Harm^q_{m,n}$, it follows that $g_{m,n}(D)(h_1)=0$. However,

$$\begin{aligned} g_{m,n}(D)(h_1) &= h_1(D)(h_1)+(\Gamma\, h_2)(D)(h_1) \\ &= h_1(D)(h_1)+(\Gamma(D)h_2(D))(h_1) \\ &= h_1(D)(h_1)+h_2(D)\,(\Gamma(D)(h_1)) \\ &= h_1(D)(h_1)+h_2(D)(0) \\ &= h_1(D)(h_1)\neq 0, \end{aligned}$$

a clear contradiction. Thus, ii) implies i). The remaining equivalence is proved via similar arguments. The key step is to observe that, given a function g in E^q_N, the divisibility of a sum of the form $\sum_{(m,n)\in K} g_{m,n}$ by Γ is equivalent to the divisibility of $g_{m,n}$ by Γ, for every $(m,n)\in K$. ■

The last equivalence to the definition of SPD on Ω_{2q} is deduced using arguments from the theory of reproducing kernels.

Theorem 2.6. *Let q, K and N be as in Theorem 2.1. The following assertions are equivalent:*
i) K induces SPD of order N on Ω_{2q};
ii) If f is a positive definite function on Ω_{2q} such that $K_q(f)=K$ and $z_1,z_2,\ldots,z_N$ are distinct points on Ω_{2q} then the set $\{z\in\Omega_{2q}\to f(\langle z,z_\mu\rangle):\mu=1,2,\ldots,N\}$ is linearly independent.

Proof. Assume that ii) holds. Let f be positive definite on Ω_{2q} with $K_q(f)=K$ and $z_1,z_2,\ldots,z_N$ be distinct points on Ω_{2q}. By direct computation we have that

$$\begin{aligned} \langle\langle f(\langle\cdot,z_\mu\rangle), f(\langle\cdot,z_\nu\rangle)\rangle\rangle &= \sum_{(m,n)\in K} a_{m,n}(f)\sum_{(r,s)\in K} a_{r,s}(f)\langle\langle R^{q-2}_{m,n}(\langle\cdot,z_\mu\rangle), R^{q-2}_{r,s}(\langle\cdot,z_\nu\rangle)\rangle\rangle \\ &= \sum_{(m,n)\in K} a_{m,n}(f)^2\langle\langle R^{q-2}_{m,n}(\langle\cdot,z_\mu\rangle), R^{q-2}_{m,n}(\langle\cdot,z_\nu\rangle)\rangle\rangle. \end{aligned} \tag{2.15}$$

The addition formula (2.4) yields

$$\begin{aligned} &\langle\langle R^{q-2}_{m,n}(\langle\cdot,z_\mu\rangle), R^{q-2}_{m,n}(\langle\cdot,z_\nu\rangle)\rangle\rangle = \\ &D(m,n,q)^2\sum_{k,l=1}^{N(m,n,q)}\overline{Y^q_{m,n,k}(z_\mu)}Y^q_{m,n,l}(z_\nu)\langle\langle Y^q_{m,n,k},Y^q_{m,n,l}\rangle\rangle = \\ &D(m,n,q)^2\sum_{k=1}^{N(m,n,q)}\overline{Y^q_{m,n,k}(z_\mu)}Y^q_{m,n,k}(z_\nu) = D(m,n,q)\overline{R^{q-2}_{m,n}(\langle z_\mu,z_\nu\rangle)}. \end{aligned}$$

Thus, (2.15) reduces to

$$\langle\langle f(\langle\cdot,z_\mu\rangle), f(\langle\cdot,z_\nu\rangle)\rangle\rangle = \overline{g(\langle z_\mu,z_\nu\rangle)},\quad 1\le\mu,\nu\le N, \tag{2.16}$$

in which g is positive definite function on Ω_{2q} satisfying $K_q(g) = K$. Due to our hypotheses, the matrix $(\overline{g(\langle z_\mu, z_\nu\rangle)})$ is positive definite, since it is the conjugate of a Gram matrix associated to a linearly independent set of vectors. Hence, so is $(f(\langle z_\mu, z_\nu\rangle))$. Therefore, i) holds. The other implication is straightforward. ■

Similar characterizations can be obtained for SPD on Ω_∞. As a matter of fact, some of them have appeared in [18] along with many additional results on SPD on Ω_∞.

3. Some conditions that determine SPD

The results presented in the previous section and the definition of SPD itself generate several results useful on the actual identification of sets that induce SPD. Some of these results are listed in this section. The first one is a necessary condition for the induction of SPD.

Theorem 3.1. *Let q be in $\{2, 3, \ldots, \infty\}$, K a subset of $\mathbb{Z}_+^2$, and N a positive integer. If K induces SPD of order N on Ω_{2q} then*

$$\{m - n : (m, n) \in K\} \cap (N\mathbb{Z} + j) \neq \emptyset, \quad 0 \leq j \leq N - 1. \tag{3.1}$$

Proof. It suffices to prove the theorem in the case in which q is finite. Let $z_1, z_2, \ldots, z_N$ be distinct points on Ω_{2q}, all contained in a same unit circle of Ω_{2q}. We can write

$$z_\mu := (e^{i2\pi\mu/N}, 0, \ldots, 0), \quad 1 \leq \mu \leq N. \tag{3.2}$$

If j is a nonnegative integer, f is a positive definite function on Ω_{2q} with $K_q(f) = K$, and $c_\mu = \exp(-i2\pi\mu j/N)$, $\mu = 1, \ldots, N$, then

$$\begin{aligned}
\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle z_\mu, z_\nu\rangle) &= \sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} \sum_{(m,n)\in K} a_{m,n}(f) R_{m,n}^{q-2}(e^{i2\pi(\mu-\nu)/N}) \\
&= \sum_{(m,n)\in K} a_{m,n}(f) \sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} e^{i2\pi(\mu-\nu)(m-n)/N} \\
&= \sum_{(m,n)\in K} a_{m,n}(f) \left| \sum_{\mu=1}^{N} c_\mu e^{i2\pi\mu(m-n)/N} \right|^2 \\
&= \sum_{(m,n)\in K} a_{m,n}(f) \left| \sum_{\mu=1}^{N} e^{i2\pi\mu(m-n-j)/N} \right|^2
\end{aligned}$$

If $\{m - n : (m, n) \in K\}$ does not intersect $N\mathbb{Z} + j$ for some j, $m - n + j$ is not divisible by N for all (m, n) in K and consequently

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle z_\mu, z_\nu\rangle) = 0, \tag{3.4}$$

that is, the matrix $(f(\langle z_\mu, z_\nu\rangle))$ is not positive definite. ■

Corollary 3.2. *Let q, K and N be as in Theorem 3.1. If K induces SPD of order N on Ω_{2q} then its cardinality is at least N. If K induces SPD of all orders on Ω_{2q} then the set $\{m-n : (m,n) \in K\}$ contains infinitely many even and infinitely many odd integers.*

Next, we show that SPD is symmetric with respect to the diagonal of $\mathbb{Z}_+^2$. Such symmetry was already observed for SPD on Ω_∞ in [18].

Theorem 3.3 *Let q, K and N be as in Theorem 3.1. Then K induces SPD of order N on Ω_{2q} if and only if $\overline{K} := \{(n,m) : (m,n) \in K\}$ does.*

Proof. If f is positive definite on Ω_{2q}, $K_q(f) = K$, and $z_1, z_2, \ldots, z_N$ are distinct points on Ω_{2q} then

$$\begin{aligned} f(\langle z_\mu, z_\nu\rangle) &= \sum_{(m,n)\in K} a_{m,n}(f) R_{m,n}^{q-2}(\langle z_\mu, z_\nu\rangle) \\ &= \sum_{(n,m)\in \overline{K}} a_{m,n}(f) \overline{R_{n,m}^{q-2}(\langle z_\mu, z_\nu\rangle)} \\ &= \overline{\sum_{(n,m)\in \overline{K}} a_{m,n}(f) R_{n,m}^{q-2}(\langle z_\mu, z_\nu\rangle)} \\ &= \overline{g(\langle z_\mu, z_\nu\rangle)}, \end{aligned}$$

where g is positive definite on Ω_{2q} and $K_q(g) = \overline{K}$. The result follows. ■

The next three results combined will show what sets induce SPD of order 2 on Ω_{2q}, $2 \le q \le \infty$.

Lemma 3.4. *Let K be a subset of $\mathbb{Z}_+^2$. The following assertions are equivalent:*
i) The system of equations

$$R_{m,n}^{q-2}(z)\overline{R_{k,l}^{q-2}(z)} = 1, \quad (m,n),(k,l) \in K \tag{3.5}$$

has a solution in $B_2 \setminus \{1\}$;
ii) The system

$$e^{i(m-n-k+l)x} = 1, \quad (m,n),(k,l) \in K \tag{3.6}$$

has a solution in $(0, 2\pi)$.

Proof. If $\theta \in (0,2\pi)$ is a solution of (3.6) then $\xi := e^{i\theta} \in B_2 \setminus \{1\}$ and

$$R_{m,n}^{q-2}(\xi)\overline{R_{k,l}^{q-2}(\xi)} = e^{i(m-n-k+l)\theta} = 1, \quad (m,n),(k,l) \in K. \tag{3.7}$$

Thus ii) implies i).
Conversely, if $\xi \in B_2 \setminus \{1\}$ is a solution of (3.5) then

$$|R_{m,n}^{q-2}(\xi)||R_{k,l}^{q-2}(\xi)| = 1, \quad (m,n),(k,l) \in K. \tag{3.8}$$

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Since $|R^{q-2}_{m,n}(\xi)| \leq 1$, it follows that

$$|R^{q-2}_{m,n}(\xi)| = R^{(q-2,|m-n|)}_{m\wedge n}(2r^2-1)r^{|m-n|} = 1, \tag{3.9}$$

where we have written $\xi = r\exp(i\theta)$, with $r \in (0,1]$ and $\theta \in (0,2\pi)$. The normalization property of Jacobi polynomials we have adopted ensures that $|R^{(q-2,|m-n|)}_{m\wedge n}(2r^2-1)| \leq 1$, whence $r = 1$. Thus,

$$e^{i(m-n-k+l)\theta} = R^{q-2}_{m,n}(\xi)\overline{R^{q-2}_{k,l}(\xi)} = 1, \quad (m,n),(k,l) \in K, \tag{3.10}$$

that is, (3.6) has a solution in $(0,2\pi)$. ■

Lemma 3.5. *Let L be a subset of $\mathbb{Z}$. The following assertions are equivalent:*
i) L has a relatively prime subset;
ii) The system of equations $\exp(ilx) = 1$, *$l \in L$, has an unique solution in* $[0,2\pi)$.

Proof. It will be left to the reader. ■

Theorem 3.6. *Let q and K be as in Theorem 3.1. Then K induces SPD of order* 2 *on Ω_{2q} if and only if the set $\{m-n-k+l : (m,n),(k,l) \in K\}$ possesses a relatively prime subset.*

Proof. It suffices to prove the Theorem in the case $q < \infty$. The definition of SPD on Ω_{2q} reveals that K does not induce SPD of order 2 on Ω_{2q} if and only if there is a $z \in B_2 \setminus \{1\}$ such that

$$R^{q-2}_{m,n}(z)\overline{R^{q-2}_{k,l}(z)} = 1, \quad (m,n),(k,l) \in K.$$

Due to Lemma 3.4, we now see that K does not induce SPD of order 2 on Ω_{2q} if and only if the system

$$\exp(i(m-n-k+l)x) = 1, \quad (m,n),(k,l) \in K$$

possesses a solution in $(0,2\pi)$. Lemma 3.5 shows that this last condition is equivalent to the fact that $\{m-n-k+l : (m,n),(k,l) \in K\}$ has no relatively prime subset. ■

Remark. If the the set $\{m-n-k+l : (m,n),(k,l) \in K\}$ has a relatively prime subset then so does $\{m-n : (m,n) \in K\}$. However, the converse is not true as the example $\{(1,0),(10,6)\}$ shows. This set also shows that the condition presented in Theorem 3.1 is not sufficient for the induction of SPD of order N on Ω_{2q}.

4. Finding sets that induce SPD

It is not an easy task to actually identify sets that induce SPD beyond the case $N = 2$. The main obstacle is that little is known about interpolation by the spherical polynomials introduced in Section 2. In spite of this difficulty, we present some cases in which a conclusion is possible.

Theorem 4.1. *Let q be in $\{2,3,\ldots,\infty\}$, N a positive integer, and s a nonnegative integer. Then the set $\{(s,s),(s+1,s),\ldots,(s+N-1,s)\}$ induces SPD of order N on Ω_{2q}.*

Proof. We only prove the theorem in the case $q < \infty$ since the other one was already considered in [18]. Write $K := \{(s,s), (s+1,s), \ldots, (s+N-1,s)\}$. Let f be a positive definite function on Ω_{2q} with $K_q(f) = K$ and let $z_1, z_2, \ldots, z_N$ be distinct points on Ω_{2q}. First we choose a point p on Ω_{2q} in such a way that

$$\langle p, z_\mu \rangle \neq \langle p, z_\nu \rangle, \quad \mu \neq \nu \tag{4.1}$$

and

$$p \neq \pm e^{-i\varphi_\mu} z_\mu, \quad 1 \leq \mu \leq N. \tag{4.2}$$

Then we write every z_μ in the form

$$z_\mu = \cos\theta_\mu \, e^{i\varphi_\mu} p + \operatorname{sen}\theta_\mu w_\mu, \quad 1 \leq \mu \leq N, \tag{4.3}$$

where $\theta_\mu \in [0, \pi/2]$, $\varphi_\mu \in [0, 2\pi)$, and w_μ is sitting in a Ω_{2q-2} embedded in Ω_{2q} and orthogonal to p. Since

$$\langle z_\mu, z_\nu \rangle = \cos\theta_\mu \, \cos\theta_\nu \, e^{i(\varphi_\mu - \varphi_\nu)} + \operatorname{sen}\theta_\mu \operatorname{sen}\theta_\nu \, \langle w_\mu, w_\nu \rangle, \tag{4.4}$$

we can use the addition formula for disk polynomials to write

$$R^{q-2}_{m,n}(\langle z_\mu, z_\nu \rangle) = \sum_{k=0}^{m} \sum_{l=0}^{n} b^{k,l}_{m,n,q} Q^{k,l}_{m,n}(\theta_\mu, \varphi_\mu) \overline{Q^{k,l}_{m,n}(\theta_\nu, \varphi_\nu)} R^{q-3}_{k,l}(\langle w_\mu, w_\nu \rangle). \tag{4.5}$$

Using this expression, we can write the matrix $A = (f(\langle z_\mu, z_\nu \rangle))$ as a sum of matrices of same size. The characterization of positive definite functions on Ω_{2q} presented in the first section of the paper and some elementary calculations allow us to conclude that each summand is a Hadamard product of two nonnegative definite matrices. Thus, if c is a vector in the null-space of A, we conclude that

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} Q^{k,l}_{m,n}(\theta_\mu, \varphi_\mu) \overline{Q^{k,l}_{m,n}(\theta_\nu, \varphi_\nu)} R^{q-3}_{k,l}(\langle w_\mu, w_\nu \rangle) = 0, \quad 0 \leq k \leq m, \quad 0 \leq l \leq n, \tag{4.6}$$

whenever (m,n) is in K. In particular,

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} Q^{s,s}_{j+s,s}(\theta_\mu, \varphi_\mu) \overline{Q^{s,s}_{j+s,s}(\theta_\nu, \varphi_\nu)} R^{q-3}_{s,s}(\langle w_\mu, w_\nu \rangle) = 0, \quad 0 \leq j \leq N-1, \tag{4.7}$$

that is,

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} (\operatorname{sen}\theta_\mu)^{2s} (\operatorname{sen}\theta_\nu)^{2s} R^{q-2+2s}_{j,0}(\cos\theta_\mu \, e^{i\varphi_\mu}) \overline{R^{q-2+2s}_{j,0}(\cos\theta_\nu \, e^{i\varphi_\nu})} R^{q-3}_{s,s}(\langle w_\mu, w_\nu \rangle) = 0$$

whenever $j = 0, 1, \ldots, N-1$. It follows that

$$\sum_{\mu,\nu=1}^{N} c_\mu \overline{c_\nu} (\operatorname{sen}\theta_\mu)^{2s} (\operatorname{sen}\theta_\nu)^{2s} \times$$
$$\times \left(\sum_{j=0}^{N-1} R_{j,0}^{q-2+2s}(\cos\theta_\mu\, e^{i\varphi_\mu}) \overline{R_{j,0}^{q-2+2s}(\cos\theta_\nu\, e^{i\varphi_\nu})} \right) R_{s,s}^{q-3}(\langle w_\mu, w_\nu \rangle) = 0. \tag{4.8}$$

Due to our choice of p we have that

$$\cos\theta_\mu\, e^{i\varphi_\mu} \neq \cos\theta_\nu\, e^{i\varphi_\nu}, \quad \mu \neq \nu. \tag{4.9}$$

Hence, the fact that the set $\{R_{j,0}^{q-2+2s} : 1 \leq j \leq N-1\}$ generates the space of polynomials of degree at most $N-1$ guarantees that the matrix

$$\left(\sum_{j=0}^{N-1} R_{j,0}^{q-2+2s}(\cos\theta_\mu\, e^{i\varphi_\mu}) \overline{R_{j,0}^{q-2+2s}(\cos\theta_\nu\, e^{i\varphi_\nu})} \right) \tag{4.10}$$

is positive definite. Since a Hadamard product of two matrices results in a positive definite matrix when one of them is positive definite and the other is nonnegative definite and having no zeros in its main diagonal ([8, p.480]), we conclude that

$$\left[\left(\sum_{j=0}^{N-1} R_{j,0}^{q-2+2s}(\cos\theta_\mu\, e^{i\varphi_\mu}) \overline{R_{j,0}^{q-2+2s}(\cos\theta_\nu\, e^{i\varphi_\nu})} \right) R_{s,s}^{q-3}(\langle w_\mu, w_\nu \rangle) \right] \tag{4.11}$$

is positive definite. It is now clear that (4.8) implies that

$$c_\mu (\operatorname{sen}\theta_\mu)^{2s} = 0, \quad 1 \leq \mu \leq N. \tag{4.12}$$

Due to (4.2), (4.12) reduces to $c_1 = c_2 = \cdots = c_N = 0$. Therefore, A is positive definite. ■

We observe that condition (4.2) used in the proof of Theorem 4.1 is not needed in the case in which $s = 0$. As a matter of fact the reader may notice that the proof of the theorem in this particular case can be slightly shortened.

Next, we present some extensions of Theorem 4.1. The difference here is that some hypotheses related to interpolation by spherical polynomials need to be matched. Here, we deal with $q < \infty$ only since the other case was investigated in [18].

Theorem 4.2. *Let K be a subset of $\mathbb{Z}_+^2$, N a positive integer and q a positive integer at least 2. Assume that K has the following two properties:*
i) For any distinct points $z_1, z_2, \ldots, z_N$ in Ω_{2q} there is a pair (r, s) and a polynomial p in $Harm_{r,s}^q$ such that $p(z_\mu) \neq p(z_\nu)$, $\mu \neq \nu$, and

$$\sum_{i=1}^{q} \frac{\partial p(z)}{\partial z^i} \cdot \frac{\partial p(z)}{\partial \overline{z^i}} = 0, \quad z = (z^1, z^2, \ldots, z^q) \in \mathbb{C}^q; \tag{4.13}$$

ii) The set $\{(r,s),(2r,2s),(3r,3s),\ldots,(Nr,Ns)\}$ *is a subset of* K.
Then K *induces SPD of order* N *on* Ω_{2q}.

Proof. Let $d_1, d_2, \ldots, d_N$ be arbitrary data associated to distinct points $z_1, z_2, \ldots, z_N$. By i), there is a p in $Harm^q_{r,s}$ such that $p(z_\mu) \neq p(z_\nu)$, $\mu \neq \nu$. Since $Harm^q_{r,s}$ is invariant by unitary transformations on $\mathbb{C}^q$, we may assume that $p(z_\mu) \neq 0$, $\mu = 1, 2, \ldots N$. Since the matrix with $\mu\nu$-entry $p(z_\mu)^\nu$ is Vandermonde-like associated with the distinct points $p(z_1), p(z_2), \ldots, p(z_N)$, there is a unique function h in the form

$$h(z) = \sum_{j=1}^{N} c_j p(z)^j, \quad c_1, c_2, \ldots, c_N \in \mathbb{C} \tag{4.14}$$

such that $h(z_\mu) = d_\mu$, $\mu = 1, 2, \ldots, N$. Next, we show that h is in $Harm^q_K$. Due to hypothesis ii), this can be achieved by showing that $p(z)^j \in Harm^q_{rj,sj}$, $j = 1, \ldots, N$. Fixing j and differentiating with respect to z and $\overline{z}$ we get

$$\begin{aligned} \frac{\partial^2 p(z)^j}{\partial z^i \partial \overline{z^i}} &= \frac{\partial}{\partial z^i}\left(\frac{\partial p(z)^j}{\partial \overline{z^i}}\right) = \frac{\partial}{\partial z^i}\left(j p(z)^{j-1}\frac{\partial p(z)}{\partial \overline{z^i}}\right) \\ &= j(j-1)p(z)^{j-2}\frac{\partial p(z)}{\partial z^i}\frac{\partial p(z)}{\partial \overline{z^i}} + j p(z)^{j-1}\frac{\partial^2 p(z)}{\partial z^i \partial \overline{z^i}}. \end{aligned}$$

Consequently,

$$\sum_{i=1}^{q} \frac{\partial^2 p(z)^j}{\partial z^i \partial \overline{z^i}} = j(j-1)p(z)^{j-2}\sum_{i=1}^{q}\frac{\partial p(z)}{\partial z^i}\frac{\partial p(z)}{\partial \overline{z^i}} + j p(z)^{j-1}\sum_{i=1}^{q}\frac{\partial^2 p(z)}{\partial z^i \partial \overline{z^i}}. \tag{4.15}$$

The first summand above vanishes due to i) while the second one is zero due to the fact that p is harmonic. Thus, $p(z)^j$ is harmonic, hence being homogeneous of degree rj in z and of degree sj in $\overline{z}$, is an element of $Harm^q_{rj,sj}$. The use of Theorem 2.1 now reveals that K induces SPD of order N on Ω_{2q}. ∎

Corollary 4.3. *Under the hypotheses of Theorem 4.2, the set*

$$K + (l,l) := \{(m+l, n+l) : (m,n) \in K\}, \quad l \geq 0,$$

induces SPD of order N *on* Ω_{2q}.

Proof. It suffices to repeat the procedure used in the proof of Theorem 4.2 making the following changes. The function h needs to be taken in the form

$$h(z) = \sum_{j=1}^{N} c_j p(z)^{j+l}. \tag{4.16}$$

Since

$$p^{j+l} \in Harm^q_{rj+l,sj+l}, \quad 1 \le j \le N, \tag{4.17}$$

h will be an element of

$$\sum_{j=1}^{N} Harm^q_{rj+l,sj+l} \subset \sum_{(m,n)\in K} Harm^q_{m+l,n+l} = \bigoplus_{(m,n)\in K+(l,l)} Harm^q_{m,n} = Harm^q_{K+(l,l)}. \tag{4.18}$$

The rest follows as before. ∎

Corollary 4.4. *Let q and N be as in Theorem 4.2, l a nonnegative integer, and $K := \{(m,0) : m \in R\}$, where R is a subset of $\mathbb{Z}_+$. Assume that K has the following properties:*
i) For any distinct points $z_1, \ldots, z_N$ in Ω_{2q} there is a r and a p in $Harm^q_{r,0}$ such that $p(z_\mu) \ne p(z_\nu)$, $\mu \ne \nu$;
ii) The set $\{r, 2r, 3r, \ldots, Nr\}$ is a subset of R.
Then, $K + (l,0)$ induces SPD of order N on Ω_{2q}.

Proof. This is a re-statement of Theorem 4.2, where we have used the special nature of K to simplify Conditions i) and ii). We observe that (4.13) is not needed because that relation is automatically satisfied by polynomials that do not depend on the variable $\overline{z}$. ∎

5. SPD on real spheres

In this section, we explore the possibility of extracting results on SPD on Ω_{2q} from those on SPD on real spheres and vice-versa. The reader is invited to consult several references included at the end of the paper to get familiarized with the notion of SPD on real spheres. Many details on that will not be included here.

We write S^r to denote the unit sphere in $\mathbb{R}^{r+1}$ and S^∞ to denote the unit sphere in the real ℓ^2. A positive definite function on S^r is a continuous function defined at least in $[-1,1]$ having a series representation of the form

$$g(x) = \sum_{k\in K} a_k(g) R_k^{(r-3/2,r-3/2)}(x), \quad K \subset \mathbb{Z}_+, \quad a_k(g) \ge 0, \quad \sum_{k\in K} a_k(g) < \infty.$$

When $r = \infty$ the representation remains the same but $R_k^{(\infty,\infty)}(x) := x^k$, $k = 0, 1, \ldots$. A subset K of $\mathbb{Z}_+$ induces SPD of order N on S^r if every positive definite function g on S^r such that $K = \{k : a_k(g) > 0\}$ has the following property: the matrix $(g(\langle x_\mu, x_\nu\rangle))$ is positive definite whenever $x_1, x_2, \ldots, x_N$ are distinct points on S^r. A basic property the reader should have in mind and mentioned earlier in the paper, is that positive definite functions on Ω_{2q} are positive definite on S^r, as long as S^r can be embedded in Ω_{2q}. In particular, a positive definite function in Ω_∞ is positive definite on S^r for all r.

The analysis presented here is simplified due to a nice formula proved in [6]. It connects disk polynomials with Jacobi polynomials in a symmetric way: For any α in $\{2, 3, \ldots, \infty\}$

and any nonnegative integer k, there are positive constants $\{d(k,\alpha,m,n) : m+n=k\}$ such that $d(k,\alpha,m,n) = d(k,\alpha,n,m)$ and

$$R_k^{(\alpha+1/2,\alpha+1/2)}(\operatorname{Re} z) = \sum_{m+n=k} d(k,\alpha,m,n) R^{\alpha}_{m,n}(z), \quad z \in B_2. \tag{5.1}$$

We observe that the proof of this property in the case $\alpha = \infty$ can be done independently by induction.

Theorem 5.1 below reveals that SPD on S^{2r-1}, $1 \le r \le \infty$ corresponds to SPD on Ω_{2r} of sets that are symmetric with respect to the diagonal of $\mathbb{Z}_+^2$.

Theorem 5.1. *Let K be a subset of $\mathbb{Z}_+$, N a positive integer and r an element of $\mathbb{Z}_+ \cup \{\infty\}$. The following assertions are equivalent:*
i) K induces SPD of order N on S^{2r-1};
ii) $\bigcup_{k\in K}\{(m,n) : m+n=k\}$ induces SPD of order N on Ω_{2r}.

Proof. Let us write $\mathcal{K} := \bigcup_{k\in K}\{(m,n) : m+n=k\}$. Let f be positive definite on Ω_{2r} with $K_r(f) = \mathcal{K}$. Let $z_1, z_2, \ldots, z_N$ be distinct points on Ω_{2r} and $c_1, c_2, \ldots, c_N$ complex numbers. If

$$\sum_{\mu=1}^{N}\sum_{\nu=1}^{N} c_\mu \overline{c_\nu} f(\langle z_\mu, z_\nu\rangle) = 0, \tag{5.2}$$

then

$$\sum_{\mu=1}^{N}\sum_{\nu=1}^{N} c_\mu \overline{c_\nu} R^{r-2}_{m,n}(\langle z_\mu, z_\nu\rangle) = 0, \quad (m,n) \in \mathcal{K}. \tag{5.3}$$

Using (5.1), we conclude that

$$\sum_{\mu=1}^{N}\sum_{\nu=1}^{N} c_\mu \overline{c_\nu} R_k^{(r-3/2,r-3/2)}(\operatorname{Re}\langle z_\mu, z_\nu\rangle) = 0, \quad k \in K. \tag{5.4}$$

Writing

$$z_\mu = (x^1_\mu + iy^1_\mu, x^2_\mu + iy^2_\mu, \ldots, x^q_\mu + iy^q_\mu), \quad 1 \le \mu \le N, \tag{5.5}$$

we have that $\operatorname{Re}\langle z_\mu, z_\nu\rangle = \langle w_\mu, w_\nu\rangle$, where $w_\mu := (x^1_\mu, y^1_\mu, \ldots, x^r_\mu, y^r_\mu)$, $\mu = 1, 2, \ldots, N$ are distinct points on S^{2r-1}. Thus, (5.4) becomes

$$\sum_{\mu=1}^{N}\sum_{\nu=1}^{N} c_\mu \overline{c_\nu} R_k^{(r-3/2,r-3/2)}(\langle w_\mu, w_\nu\rangle) = 0, \quad k \in K. \tag{5.6}$$

This shows that

$$\sum_{\mu=1}^{N}\sum_{\nu=1}^{N} c_\mu \overline{c_\nu} \tilde{f}(\langle w_\mu, w_\nu\rangle) = 0, \tag{5.7}$$

where $\tilde{f}$ is a positive definite function on S^{2r-1} with $K_{2r-1}(\tilde{f}) = K$. In other words, K induces SPD of order N on S^{2r-1}. The reader will have no difficulty at all to see that the argument above is reversible. ■

Theorems 2.9 and 2.14 in [13] establish that a subset K of $\mathbb{Z}_+$ induces SPD of order $N < 2r$ on S^{2r-1} if and only if it contains an even and an odd integer, both at least $[N/2] - 1$. A combination of these results with Theorem 5.1 produces our final theorem.

Corollary 5.2. *Let N be an integer at least 2 and q an integer at least $(N+1)/2$ ($q = \infty$ is allowed). Let K be as in Theorem 5.1. Then, $\bigcup_{k\in K}\{(m,n) : m+n = k\}$ induces SPD of order N on Ω_{2q} if and only if $\{k \in K : k \geq [N/2] - 1\}$ possesses an even and an odd integer.*

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