CADERNOS DE MATEMÁTICA **01**, 71–88 April (2000)
ARTIGO NÚMERO SMA#75

More on Forced Periodic Solutions of Quasi-Parabolic Equations

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We study the existence of periodic solutions for a class of periodically forced quasiparabolic equations involving either the p-Laplacean, or other nonlinear operators of similar class, perturbed by operators depending on the gradiente of the solution April, 2000 ICMC-USP

1. INTRODUCTION

Monotone operators, in particular the ones that are subdifferentials of convex functions, like the p-Laplacean, appear in several importante situations, showing, for instance, in equations modeling the behaviour of viscoelastic materials, Le Tallec [5]. In most cases, these monotone operators may be accompanied of other nonlinear operators, what may turn difficult the analysis of the equations where they appear and, consequently, the prediction of the behaviour of the corresponding solutions. This is specially true when these acompanying operators model situations where there is imparting of energy to the system, by means of the interaction between the material and external actions, for instance. In this sort of case, the overall operator associated to the problem is no longer monotone and not even dissipative (at least not in all of the corresponding functional space). Besides, due to possible imbalance between the gain and the dissipation of energy, it is not clear under what conditions the internal dissipation provided by the monotone part of the operator will be enough to guarantee that the system will respond in a periodic way when acted on by periodic external forces.

The search for conditions guaranteeing such existence of periodic responses is the general objective of this paper, in the cases that the principal operator in the equations for the problem is like the p-Laplacean, and, besides, the accompanying operators include gradient operators (here,we say that these accompanying operators are perturbations because they are of lower order). Moreover, we put ourselves in what we call the "worst-case situation from the energy point of view": we assume that the sole responsable for the dissipation in

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the problem is the principal monotone part of the operator. Thus, any possible dissipation coming from the perturbation terms is disregarded, and we will be looking for growth conditions for these last terms in order to obtain periodic solutions.

Specifically, we will study in detail the existence of solutions of the following problem:

$$\begin{cases} u_t(t,x) - \Delta_p(u(t,x)) = h(t,x) + g(u(t,x)).\nabla u(t,x), \quad x \in \Omega,\ t \in (0,T), \\ u(t,x) = 0, \quad x \in \partial\Omega,\ t \in (0,T), \\ u(0,x) = u(T,x) \quad x \in \Omega. \end{cases} \tag{1.1}$$

Here, $T > 0$, and Ω is an open bounded and regular set in $\mathbb{R}^N$. For $p \in \mathbb{N}$, Δ_p denotes the p-Laplacean, that is,

$$\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Also, we use the notation

$$g(u(t,x)).\nabla u(t,x) = \sum_{\ell=1}^{N} g_\ell(u(t,x)) \frac{\partial u}{\partial x_\ell}(t,x),$$

where, for $\ell = 1, \ldots, N$, the g_ℓ's are suitable functions. The function h is T-periodic in time.

Concerning this problem, under certain regularity and growth conditions (described in detail in Theorem 3.3.1,) in this work we will show that (1.1) has a weak (strong) solution. Briefly, our procedure of proof will be the following: the Faedo-Galerkin method, together with a fixed point argument, will be employed to provide a sequence of approximate periodic solutions; then, exploiting the periodicity of those approximate solutions, we will derive further a priori estimates. A compactness argument will then finish the proof of existence.

We should say that the above analysis is more difficult than that in which the principal operator is linear (that is, it is the Laplacean;) in this last case there are plenty of known results for the problem. This is mainly due to the weaker regularization of the p-Laplacean ($p \neq 2$) as compared to that of the Laplace operator ($p = 2$).

We also mention that in [3], we analyzed a problem similar to the above, but in which the lower order perturbations dependended only on the values of the solution itself, instead of the values of the gradient of the solution. As we will explain later on, the technique employed in [3] could not be employed in case of the present paper.

Finally, the paper is organized as follows: in Section 2 we fix the notation and recall certain results that will be used in the course of the work; in Section 3 we will state precisely and prove our existence results concerning periodic solutions (Theorem 3.3.1); an extension of the previous result (Theorem 4.4.1) will be given for more general monotone operators in Section 4, where we will also give some examples of its application to other equations than (1.1).

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2. PRELIMINARIES

In this section we will fix the notations, as well as recall certain mathematical results to be used in the rest of the paper.

Being Ω an open, bounded and regular set on $\mathbb{R}^N$ and $1 \leq q \leq \infty$, the norm of a function $f \in L^q(\Omega)$ will be denoted $|f|_q$. We recall that $L^2(\Omega)$ is a Hilbert space with inner product given by $(f, g)_2 = \int_\Omega f(x)g(x)dx$ for $f, g \in L^2(\Omega)$ (in this paper, we will always work with real functional spaces.)

For $k \in \mathbb{N}$ and q as above, we write $W^{k,q}(\Omega)$ for the standard Sobolev space, with $W_0^{k,p}(\Omega)$ being the closure in $W^{k,q}(\Omega)$ of the class of C^∞-functions with compact support in Ω. The topological dual of $W_0^{k,p}(\Omega)$ will be denoted $W^{-k,q'}(\Omega)$, with $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.

As usual, $W^{k,2}(\Omega)$ and $W_0^{k,2}(\Omega)$ will be respectively denoted by $H^k(\Omega)$ and $H_0^k(\Omega)$; their norms will be $|\cdot|_{H^k(\Omega)}$ and $|\cdot|_{H_0^k(\Omega)}$, respectively. Also, the dual space $H_0^k(\Omega) \quad (= W_0^{k,2}(\Omega))$ will be denoted $H^{-k}(\Omega) \quad (= W^{-k,2}(\Omega))$, with norm $|\cdot|_{H^{-k}(\Omega)}$. The duality product between $H^{-k}(\Omega)$ and $H_0^k(\Omega)$ will be denoted $(u, v)_{H^{-k}, H_0^k}$.

We recall that $H_0^k(\Omega)$ is a Hilbert space with norm denoted by $(f, g)_{H_0^k(\Omega)}$.

Since the spaces $W^{-1,q'}(\Omega)$ and $W_0^{1,q}(\Omega)$ will be used more often, to simplify the notation, we will represent their duality simply by $(\cdot, \cdot)$, and their respective norms by

$$|\omega|_{-1,q'} = \sup_{|u|_{1,q} \leq 1} (\omega, u) \qquad \text{for } \; u \in W_0^{1,q}(\Omega) \; \text{ and } \; \omega \in W^{-1,q'}(\Omega)$$

$$|u|_{1,q} = \left(\int_\Omega |\nabla u(x)|^q dx \right)^{1/p} \; \text{ for } \; u \in W_0^{1,q}(\Omega).$$

We observe that, when $q \geq 2$, then $W_0^{1,q}(\Omega) \subset L^2(\Omega) \cong (L^2(\Omega))' \subset W^{-1,q'}(\Omega)$, with continuous and dense inclusions. Thus, when $w \in L^2(\Omega)$ and $u \in W_0^{1,q}(\Omega)$, we have $(w, u) = (w, u)_2$.

We will use standard Sobolev imbedding results, which can be found in Adams [1].

Being $(B, ||\cdot||_B)$ a Banach space, for $1 \leq q < \infty$, we denote by $L^q(0, T; B)$ the Banach space of all B-measurable functions $f : (0, T) \to B$ such that $\int_0^T |f(t)|_B^q dt < \infty$; the norm in $L^q(0, T; B)$ is then defined as $||f||_{L^q(0,T;B)} = (\int_0^T ||f(t)||_B^q dt)^{1/q}$. Also, we define $L^\infty(0, T; B)$ as the Banach space of all B-measurable functions $f : (0, T) \to B$ such that e-sup$\{|f(t)|_B, t \in (0, T)\} < \infty$, with norm $||f||_{L^\infty(0,T;B)} =$ e-sup$\{||f(t)||_B, \; t \in (0, T)\}$ (here e-sup denotes the essential supremum.)

We will also need the following Banach spaces:

$$W = \{u \in L^p(0, T; W_0^{1,p}(\Omega)); \;\; u_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))\}, \tag{1}$$

and

$$W_1 = \{u \in L^p(0, T; W_0^{1,p}(\Omega)); \;\; u_t \in L^{p'}(0, T; H^{-k}(\Omega))\}, \tag{2}$$

with norms $|u|_W = |u|_{L^p W_0^{1,p}} + |u_t|_{L^{p'} W^{-1,p'}}$ and $|u|_{W_1} = |u|_{L^p W_0^{1,p}} + |u_t|_{L^{p'} H^{-k}}$, respectively. Here, u_t (or u') denotes the distributional derivative with respect to time.

We will also use the fact that

$$W \subset C([0,T]; L^2(\Omega)) \tag{3}$$

and that for $u, v \in W$ it is true (Lions [6], pages 156 and 321) that

$$\int_0^T (u_t(t), v(t)) + (v_t(t), u(t))dt = (u(T), v(T))_2 - (u(0), v(0))_2. \tag{4}$$

Now, if $1 < p < \infty$, q is such that $1 - \frac{N}{p} + \frac{N}{q} > 0$ and $k \in \mathbb{N}$ satisfy $k \geq N(\frac{1}{2} + \frac{1}{q})$, then

$$H^k(\Omega) \subset W_0^{1,p}(\Omega) \subset L^q(\Omega) \subset H^{-k}(\Omega), \tag{5}$$

with continuous and dense inclusions. In particular, the inclusion $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ is compact.

Thus, by the Aubin-Lions compactness criterion (Lions [6], Chapter 1, Section 5, Theorem 5.1), we have that

$$W_1 \subset L^p(0,T; L^q(\Omega)) \quad \text{compactly}. \tag{6}$$

For the use with the Faedo-Galerkin method, we also recall the following spectral problem: find $0 \neq w \in H_0^k(\Omega)$ and a number λ such that

$$(w, v)_{H_0^k(\Omega)} = \lambda (w, v)_2 \quad \forall v \in H_0^k(\Omega). \tag{7}$$

It is known that (7) has a sequence of strictly positive eigenvalues $\{\lambda_i\}_1^\infty$ such that the corresponding eigenfunctions $\{e_i\}_1^\infty$ form a orthogonal basis both in $H_0^k(\Omega)$ and $L^2(\Omega)$.

As before, we observe that $H_0^k(\Omega) \subset L^2(\Omega) \cong (L^2(\Omega))' \subset H^{-k}(\Omega)$ (with continuous and dense inclusions), and, thus, for $v, w \in H_0^k(\Omega)$, we have $(v, w)_{H^{-k}, H_0^k} = (v, w)_2$. In particular, the we conclude that the above basis has the following property:

$$(e_i, e_j)_{H^{-k}, H_0^k} = (e_i, e_j)_2 = 0, \quad \text{for} \quad i \neq j \in \mathbb{N}. \tag{8}$$

To recall properties of the p-Laplacean, let us consider the functional defined by

$$\begin{array}{rcl} \phi: \; W_0^{1,p}(\Omega) & \to & \mathbb{R} \\ u & \mapsto & \frac{1}{p}\left(\int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}} = \frac{1}{p}|u|_{1,p}^p \end{array} \tag{9}$$

It is known that ϕ is a differentiable and convex functional whose derivative $\phi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is given by

$$(\phi'(u), v) = \int_\Omega |\nabla u(x)|^{p-2} \langle \nabla u(x), \nabla u(x) \rangle dx \qquad \forall u, v \in W_0^{1,p}(\Omega),$$

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where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$.

Thus, calling $\Delta_p(u) = \text{div}(|\nabla u|^{p-2}\nabla u)$, by the Divergence Theorem, when $\Delta_p(u) \in L^2(\Omega)$, we have

$$(\phi'(u), v) = -\int_\Omega \Delta_p(u(x))v(x)dx. \tag{10}$$

Hence, from now on, we are going to denote ϕ' by $-\Delta_p$, which corresponds to the p-Laplacean with Dirichlet boundary conditions. This operator has special properties; in fact, for each $u, v, \omega \in W_0^{1,p}(\Omega)$ the following is true:
Strong monotonicity: there is $\alpha > 0$ such that

$$(-\Delta_p u - (-\Delta_p v), u - v) \geq \alpha |u - v|_{1,p}^p; \tag{11}$$

Hemicontinuity: the following aplication is continuous

$$\lambda \in \mathbb{R} \mapsto (-\Delta_p(u + \lambda\omega), v); \tag{12}$$

Coercivity:

$$(-\Delta_p u, u) \geq |u|_{1,p}^p; \tag{13}$$

Boundness:

$$|-\Delta_p(u)|_{-1,p'} \leq |u|^{p-1}. \tag{14}$$

We observe that (11) follows from [7], Section 2, Lemma 1; (12) follows from the fact of $\phi \in C^1(W_0^{1,p}(\Omega))$; and (13) and (14) follows easily from the definition of $-\Delta_p$.

We need to clarify the concept of weak and strong solutions we will be using in this paper:

Definition:

(i) Let $f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ and $u_0 \in L^2(\Omega)$. We say that u is a *weak solution* of

$$\begin{cases} u_t - \Delta_p u = f, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0 \in \Omega \end{cases} \tag{15}$$

if $u \in W = \{w \in L^p(0, T; W_0^{1,p}(\Omega));\ w_t \in L^{p'}(0, T; W^{-1,p'}(\Omega))\}$ and u satisfies (15) in $L^{p'}(0, T; W^{-1,p'}(\Omega))$ with $u(0) = u_0$ (this makes sense because of (3).

(ii) When $f \in L^2(0, T; L^2(\Omega))$, u is called a *strong solution* of (15) if it is a weak solution, $u_t \in L^2(0, T; L^2(\Omega))$, and (15) is satisfied in $L^2(0, T; L^2(\Omega))$.

A final remark concerning the notation is the following: as it is usual, we will denote by C a generic positive constant depending on the data of the problem (like the given functions, Ω, the Sobolev imbeddings results being used, and so on) and that may change in the sequence of the derivation of each estimate; these constants, however, will not depend on the Faedo-Galerkin approximation of the solution, u_m, being considered (in the situations in which the constants do depend on these approximations, we will use the notation $C(m)$. In cases we want to emphasize the role of certain constants, we will denote them by c_1, c_2, and so on.

3. EXISTENCE OF PERIODIC SOLUTIONS

We start by investigating the existence of weak solutions: We will prove the following result:

THEOREM 3.3.1. *Let $p \geq 3$, a function $h \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and locally lipschitzian functions $g_\ell : \mathbb{R} \to \mathbb{R}$, $\ell = 1, \dots, N$, satisfying*

$$|g_\ell(\tau) - g_\ell(\sigma)| \leq C_\ell(|\tau|^{s_\ell - 1} + |\sigma|^{s_\ell - 1} + 1)|\tau - \sigma| \tag{1}$$

for all $\tau, \sigma \in \mathbb{R}$ and some $C_\ell \geq 0$ and $1 \leq s_\ell < p-2$.

In this case, Problem (1.1) admits a weak solution.

When $\max_{\ell=1\dots N}\{s_\ell\} = p-2$, if $\max_{\ell=1\dots N}\{C_\ell\}$ is sufficiently small, (1.1) still has a weak solution.

Moreover, when $1 \leq s \leq \dfrac{p-2}{2}$ and $h \in L^2(0,T;L^2(\Omega))$ the above solutions are strong.

Proof: Let us consider the variational form of Problem (1.1):

$$\begin{cases} (u'(t), v) + \int_\Omega |\nabla u(t,x)|^{p-2}\langle \nabla u(t,x), \nabla v(x)\rangle dx \\ = (h(t), v) + \int_\Omega g(u(t,x)) \cdot \nabla u(t,x) v(x) dx, \quad \text{for any} \quad v \in W_0^{1,p}(\Omega), \\ u(0,x) = u(T,x) \quad \text{for} \quad x \in \Omega, \end{cases} \tag{2}$$

where $g(u) \cdot \nabla u$ is as in the Introduction.

We will use Faedo-Galerkin method (see Lions [6], Chapter 2, Section 1) to show that (2) has a solution. For this, we use the orthogonal basis in $H_0^k(\Omega)$, $\{e_i\}_1^\infty$, given by the eigenfunctions of the spectral problema (7) (by 5, $\{e_i\}_1^\infty$ is also a basis in $W_0^{1,p}(\Omega)$.)

Now, for each $m \in \mathbb{N}$, we take $X^m = span[e_1, \dots, e_m]$, the m-dimensional subspace generated by the m-first elements of this basis, and define an approximate problem by the following: find $u_m : [0,T] \to X^m$ such that

$$\begin{cases} \int_\Omega u_m'(t,x) v(x) dx + \int_\Omega |\nabla u_m(t,x)|^{p-2}\langle \nabla u_m(t,x), \nabla v(x)\rangle dx \\ = \int_\Omega g(u_m(t,x)) \cdot \nabla u_m(t,x) v(x) dx + (h, v), \quad \forall v \in X^m, \\ u_m(0) = u_m(T), \end{cases} \tag{3}$$

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In order to solve (3), we will first introduce the following initial value problem:

$$\begin{cases} \int_\Omega u_m'(t,x)v(x)dx + \int_\Omega |\nabla u_m(t,x)|^{p-2}\langle \nabla u_m(t,x), \nabla v(x)\rangle dx \\ = \int_\Omega g(u_m(t,x))\cdot \nabla u_m(t,x)v(x)dx + (h,v), \quad \forall v \in X^m, \\ u_m(0) = u_{0m}, \end{cases} \tag{4}$$

where $u_m(t) = \sum_{k=1}^{m} a_k(t)e_k$ (the coefficients $a_k(t)$ must be found) and $u_{0m} = \sum_{k=1}^{m} a_{0k}e_k \in X^m$.

This defines a system of ordinary differential equations

$$\begin{cases} U_m'(t) = G_m(t,U), \\ U_m(0) = U_{0,m}, \end{cases}$$

for the variable $U_m(t) = (a_1(t), a_2(t), \dots, a_m(t))$, with $U_{0,m} = (a_{01}, \dots, a_{0m})$ and a suitable function G, which is locally lipschitzian in U and p'-integrable in t (this is not hard to verify under our conditions.)

Thus, according to Carathéodory's Theorem (see Hale [4], Chapter 1, Section 1.5, Theorem 5.1 - 5.3,) there is an unique absolutely continuous function $u_m(t)$, which is a local solution of (4) and is defined in some maximal interval $[0, \tau_m)$.

Let us show that there is a sufficiently large number $R > 0$ such that if $|u_{0m}|_2 \le R$ then $|u_m|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{2}R$ for all $m \in \mathbb{N}$. This implies that u_m is defined in $[0,T]$.

We start by observing that by taking

$$s = \max_{\ell=1\dots N}\{s_\ell\},$$

there is a constant $C > 0$, which does not depend on m, such that for any $\ell = 1,\dots,N$ there holds

$$|g_\ell(\tau) - g_\ell(\sigma)| \le C \max_{\ell=1\dots N}\{C_\ell\}(|\tau|^{s-1} + |\sigma|^{s-1} + 1)|\tau - \sigma| \tag{5}$$

for all $\tau, \sigma \in \mathbb{R}$.

Also, since $p \ge 3$ and $1 \le s \le p-2$, it holds

$$p' = \frac{p}{p-1} < \frac{p}{p-2} \le \frac{ps}{p-2} \le p. \tag{6}$$

Hence, $W_0^{1,p}(\Omega) \subset L^{ps/(p-2)}(\Omega)$, and, by using Hölder's and Young's inequalities, (5) and (6), we conclude that there is a constant $C > 0$, which does not depend on m, such that for any $\ell = 0,\dots,N$ and any $u, v \in W_0^{1,p}(\Omega)$, it is true that

$$\begin{aligned} |g_\ell(u) - g_\ell(v)|_{p/(p-2)} &\le C \max\nolimits_{\ell=1\dots N}\{C_\ell\}(|u|_{ps/(p-2)}^{s-1} + |v|_{ps/(p-2)}^{s-1} + 1)|u-v|_{ps/(p-2)} \\ &\le C \max\nolimits_{\ell=1\dots N}\{C_\ell\}(|u|_{1,p}^{s-1} + |v|_{1,p}^{s-1} + 1)|u-v|_{1,p} \end{aligned} \tag{7}$$

By taking $v = 0$ in this last inequality and using Young's inequality, we find that there is positive constants c_1 and c_2, such that for any $u \in W_0^{1,p}(\Omega)$ it holds

$$|g_\ell(u)|_{p/(p-2)} \le c_1 |u|_{1,p}^s + c_2. \tag{8}$$

Since we will need it later on, we remark that c_1 is form form

$$c_1 = C \max_{\ell=1\ldots N}\{C_\ell\}, \tag{9}$$

with $C > 0$ independent of m.

Now we are ready to obtain estimates for the sequence u_m. For this, we take $v = u_m(t)$ in (4) to conclude that

$$\frac{1}{2}\,\frac{d}{dt}|u_m(t)|_2^2 + |u_m(t)|_{1,p}^p \le |h(t)|_{-1,p}|u_m(t)|_{1,p} + |g(u_m(t))\cdot\nabla u_m(t)|_{-1,p'}|u_m(t)|_{1,p}. \tag{10}$$

But, using (6) and (8), we obtain, for each $\ell = 1, ldots, N$,

$$\begin{aligned} |g_\ell(u_m(t))\frac{\partial u_m}{\partial x_\ell}(t)|_{-1,p'} &\le C|g_\ell(u_m(t))\frac{\partial u_m}{\partial x_\ell}(t)|_{p'} \\ &\le C|g_\ell(u_m(t))|_{p/(p-2)}|u_m(t)|_p \\ &\le c_1 C|u_m(t)|_{1,p}^{s+1} + c_2 C|u_m(t)|_{1,p}. \end{aligned}$$

Substituting this result in (10), we obtain

$$\frac{1}{2}\,\frac{d}{dt}|u_m(t)|_2^2 + |u_m(t)|_{1,p}^p \le \{|h(t)|_{-1,p'} + c_1 C|u_m(t)|_{1,p}^{s+1} + c_2 C|u_m(t)|_{1,p}\}|u_m(t)|_{1,p}$$

Thus, in the case that $1 \le s < p-2$, we can use Young's inequality to conclude that, for any with $\varepsilon > 0$, there are constants c_3, $c_4 > 0$ and $c_5 > 0$, which depend on ε but not on m, such that

$$\frac{d}{dt}|u_m(t)|_2^2 + c_3|u_m(t)|_{1,p}^p \le c_5|h(t)|_{1,p'}^{p'} + c_4. \tag{11}$$

Here

$$c_3 = 1 - (\varepsilon^p + c_1 C\varepsilon^{p/(p+2)} + c_2 C\varepsilon^{p/2}), \tag{12}$$

When $s = p-2$, by working in a similar way, again we can obtain (11), but now with the following expressions for the constant c_3:

$$c_3 = 1 - (\varepsilon^p + c_1 C + c_2 C\varepsilon^{p/2}). \tag{13}$$

Now let us consider first the case $1 \le s < p-2$: by choosing $\varepsilon > 0$ sufficiently small in (12), we obtain $c_3 > 0$. Using that $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ and Young's inequality, we find

$$\frac{d}{dt}|u_m(t)|_2^2 + c_6|u_m(t)|_2^2 \le c_7(|h(t)|_{-1,p'}^{p'} + 1),$$

for suitable positive constants c_6 and c_7, still independent of m. Multiplying this inequation by $e^{c_6 t}$ and integrating from 0 to t we conclude that

$$|u_m(t)|_2^2 \leq e^{-c_6 t}|u_{0m}|_2^2 + c_7(|h|^{p'}_{L^{p'}(0,t;W^{-1,p'}(\Omega))} + t).$$

This implies, in particular that $|u_m(t)|_2$ is bounded for finite $t > 0$, and, consequently, u_m is defined in all $[0,T]$. This same inequality implies that if we take R large enough (independent of m) such that

$$R^2 \geq \frac{c_7}{1-e^{-c_6 T}}(T + |h|^{p'}_{L^{p'}(0,T;W^{-1,p'})}), \tag{14}$$

then, when $|u_{m0}|_2 \leq R$, we obtain

$$|u_m(T)|_2 \leq R \quad \text{and} \quad |u_m|_{L^\infty(0,T;L^2(\Omega))} \leq \sqrt{2}R \tag{15}$$

Thus, we can define the following Poincaré map associated to (4):

$$\begin{array}{rrcl} K_m : & X^m & \rightarrow & X^m \\ & u_{0m} & \mapsto & K_m(u_{0m}) = u_m(T) \end{array}$$

To prove that K_m has a fixed point, we will show that K_m is continuous. For this, let u_m and v_m be two solutions of (4); proceeding as before and using (11), we obtain

$$\begin{aligned} &\frac{1}{2}\ \frac{d}{dt}|u_m(t) - v_m(t)|_2^2 + \alpha|u_m(t) - v_m(t)|_{1,p}^p \\ &\leq |g(u_m(t))\frac{\partial u_m}{\partial x}(t) - g(v_m(t))\frac{\partial v_m}{\partial x}(t)|_{-1,p'}|u_m(t) - v_m(t)|_{1,p} \end{aligned} \tag{16}$$

But we have that

$$\begin{aligned} |g(u_m(t))\frac{\partial u_m}{\partial x}(t) - g(v_m(t))\frac{\partial v_m}{\partial x}(t)|_{-1,p'} \leq \quad & |(g(u_m(t)) - g(v_m(t)))\frac{\partial u_m}{\partial x}(t)|_{-1,p'} \\ & + |(\frac{\partial u_m}{\partial x}(t) - \frac{\partial v_m}{\partial x}(t))g(v_m(t))|_{-1,p'}. \end{aligned}$$

Now, working as before and using (6) and (7), we obtain for any $\ell = 1 \ldots N$ that

$$\begin{aligned} |(g_\ell(u_m(t)) - g_\ell(v_m(t)))\frac{\partial u_m}{\partial x}(t)|_{-1,p'} &\leq C|g_\ell(u_m(t)) - g_\ell(v_m(t))|_{\frac{p}{p-2}}|u_m(t)|_{1,p} \\ &\leq C(|u_m(t)|_{1,p}^{s-1} + |v_m(t)|_{1,p}^{s-1} + 1)|u_m(t) - v_m(t)|_{1,p}|u_m(t)|_{1,p}. \end{aligned}$$

In the same way, using (6) and (8), we have

$$|(\frac{\partial u_m}{\partial x}(t) - \frac{\partial v_m}{\partial v}(t))g(v_m(t))|_{-1,p'} \leq |u_m(t) - v_m(t)|_{1,p}(c_1|v_m|_{1,p}^s + c_2).$$

Substituting these two last inequalities back into (16), and using the fact that X^m is finite-dimensional (and thus any two norms are equivalent in X^m;) for each m, we find positive constants $C_{m,1}$ and $C_{m,2}$ (which depend on m) such that

$$\begin{aligned}&\frac{1}{2}\ \frac{d}{dt}|u_m(t)-v_m(t)|_2^2+C_{m,1}|u_m(t)-v_m(t)|_2^p\\ &\le C_{m,2}|u_m(t)-v_m(t)|_2^2\ [|u_m(t)|_2^s+|v_m(t)|_2^s+1].\end{aligned}$$

But we know that when $|u_{mo}|_2 \le R$ and $|v_{mo}|_2 \le R$ for R satisfying (14), then $|u_m|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{2}R$ and $|v_m|_{L^\infty(0,T;L^2(\Omega))} \le \sqrt{2}R$. Thus, there is a constant $C_m(R) > 0$ such that, when $|u_{0m}| \le R$ and $|v_{0m}| \le R$, it holds

$$\frac{d}{dt}|u_m(t)-v_m(t)|_2^2 \le C_m(R)|u_m(t)-v_m(t)|_2^2,$$

and we conclude that $|u_m(t)-v_m(t)|_2^2 \le |u_{0m}-v_{0m}|_2^2 e^{C_m(R)t}$ for $t\in[0,T]$.

Therefore, K_m is a well defined and locally lipschitzian function, and, moreover, by (3.11), $K_m(\overline{B_R(0)}) \subset \overline{B_R(0)}$. So, by Browder Fixed Point Theorem, K_m has a fixed point in $\overline{B_R(0)}$, that is, there is $u_{0m}\in X^m$ such that $|u_{0m}|_2 \le R$ and, being u_m the solution of (4) with $u_m(0)=u_{0m}$, then $u_m(0)=u_m(T)$.

By the uniqueness of solutions of the initial values problems associated to (4), we conclude that u_m is T-periodic when h is T-periodic.

We have constructed, thus, a sequence of periodic solutions $\{u_m\}_1^\infty$ of (4).

To proceed, we will find uniform in m estimates for this sequence, in the space W_1 (see the definition of W_1 in (2).

For this, we integrate (11) in $(0,T)$ and use the fact that each approximate periodic solution satisfies $u_m(0)=u_m(T)$ to obtain the following:

$$|u_m|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le c_3^{-1}(c_4|h|^{p'}_{L^{p'}(0,T;W^{-1,p'}(\Omega))}+c_5T)) \tag{17}$$

Now, we recall that we have taken $k\in\mathbb{N}$ as in the introduction, and thus (5) and (6) hold.

By our choice of basis (recall (8),)and the use of equation (3), we have

$$\begin{aligned}|u_m'(t)|_{H^{-k}(\Omega)} &= \sup_{\{v\in H_0^k(\Omega),\ |v|_{H_0^k}=1\}}(u_m'(t),v)_{H^{-k},H_0^k}\\ &= \sup_{\{v\in X^m,\ |v|_{H_0^k}=1\}}(u_m'(t),v)_{H^{-k},H_0^k}\\ &= \sup_{\{v\in X^m,\ |v|_{H_0^k}=1\}}(\Delta_p u_m(t)+h(t)+g(u_m(t))\cdot\nabla u_m(t),v)_{H^{-k},H_0^k}\\ &\le |\Delta_p u_m(t)+h(t)+g(u_m(t))\cdot\nabla u_m(t)|_{H^{-k}}\\ &\le C(|\Delta_p u_m(t)|_{-1,p'}+|h(t)|_{-1,p'}+|g(u_m(t))\cdot\nabla u_m(t)|_{-1,p'}).\end{aligned}$$

Using (14) and (8), we conclude that

$$|u_m'(t)|_{H^{-k}(\Omega)} \le C(|u_m(t)|_{1,p}^{p-1}+|h(t)|_{-1,p'}+|u_m(t)|_{1,p}^{s+1}+|u_m(t)|_{1,p})$$

Since $p \geq s+2$, we already know that the sequence u_m is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$; thus,

$$\{u'_m\}_1^\infty \quad \text{is bounded in} \quad L^{p'}(0,T;H^{-k}(\Omega)). \tag{18}$$

Therefore, the sequence u_m is bounded in W_1, and by (6) there is a subsequence, which we still denote by $\{u_m\}$, such that

$$u_m \to u \quad \text{weakly in } L^p(0,T;W_0^{1,p}(\Omega)), \tag{19}$$

and

$$u_m \to u \quad \text{strongly in } \ L^p(0,T;L^p(\Omega)), \tag{20}$$

as $m \to \infty$.

Moreover, by (18), we can also take

$$u'_m \to u' \quad \text{weakly in } L^{p'}(0,T;H^{-k}(\Omega)). \tag{21}$$

Also extracting subsequences, by (14) and (17) there is $\xi \in L^{p'}(0,T;W^{-1,p'}(\Omega)$ such that

$$-\Delta_p u_m \to \xi \quad \text{weakly in } \ L^{p'}(0,T;W^{-1,p'}(\Omega)) \tag{22}$$

Moreover, from (19), we have $\nabla u_m \to \nabla u$ weakly in $L^p(0,T;L^p(\Omega))$, and, from the first inequality in (7), (6), and (20), $g(u_m) \to g(u)$ strongly in $L^p(0,T;L^p(\Omega))$. Thus, $g(u_m)\cdot\nabla u_m \to g(u)\cdot\nabla u$ strongly in $L^{p/2}(0,T;L^{p/2}(\Omega))$. Since $p \geq 3$, we conclude that

$$g(u_m)\cdot\nabla u_m \to g(u)\cdot\nabla u \quad \text{weakly in} \quad L^{p'}(0,T;L^{p'}(\Omega)), \tag{23}$$

(obviouly, this weak convergence occurs also in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.)

Now take $\varphi \in C_0^\infty(0,T)$, then, for $i = 1\ldots m$, $\varphi(.)e_i \in X^m$ and, from (2), we conclude that

$$\begin{aligned}\int_0^T (u'_m(t),\varphi(t)e_i)dt &+ \int_0^T (-\Delta_p(u_m(t)),\varphi(t)e_i)dt \\ = \int_0^T (h(t),\varphi(t)e_i)dt &+ \int_0^T (g_\ell(u_m(t))\cdot\nabla u_m(t),\varphi(t)e_i)dt\end{aligned}$$

Using (19), (22), (23) and the fact that $\{e_i\}_1^\infty$ is a basis in $W_0^{1,p}(\Omega)$, we conclude that when $m \to \infty$ then

$$u' + \xi = g(u)\cdot\nabla u + h \qquad \text{in } \ L^{p'}(0,T;H^{-k}(\Omega)).$$

Now, we observe that since $\xi,\ g(u)\cdot\nabla u,\ h \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, the last equation implies that $u' \in L^0p'(0,T;W^{-1,p'}(\Omega))$ and that the equality still holds in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Therefore, we can write

$$u' + \xi = g(u)\cdot\nabla u + h \qquad \text{in } \ L^{p'}(0,T;W^{-1,p'}(\Omega)). \tag{24}$$

Now, we must show that $u(0) = u(T)$ and $\xi = -\Delta_p u$.

For this, note that when $\varphi \in C_0^\infty([0,T])$ and $\varphi(0) = \varphi(T)$, then, by (4) and the fact that $u_m(0) = u_m(T)$, we have that

$$\int_0^T (u_m'(t), \varphi(t)e_i)dt = -\int_0^T (u_m(t), \varphi'(t)e_i)dt.$$

Then, passing to the limit when $m \to \infty$ and using (19) and (21), we conclude that

$$\int_0^T (u'(t), \varphi(t)e_i)dt = -\int_0^T (u(t), \varphi'(t)e_i)dt \qquad \text{for} \quad i = 1, 2 \ldots$$

On the other hand, again by (4), we have

$$\int_0^T (u'(t), \varphi(t)e_i)dt = -\int_0^T (u(t), \varphi'(t)e_i) + (u(T), \varphi(T)e_i) - (u(0), \varphi(0)e_i).$$

Since $\varphi(0) = \varphi(T)$, we conclude that for any $e_i \in W_0^{1,p}(\Omega)$, $(u(T), e_i)$ $(u(T), e_i) = (u(0), e_i)$. Using the fact of the inclusion $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ is dense, we conclude that $u(T) = u(0)$.

Now, let us show that ξ, given in (24), is equal to $-\Delta_p u$.

Observe that by the monotonicity of the p-Laplacean, we have

$$\int_0^T (-\Delta_p u_m(t) + \Delta_p v(t), u_m(t) - v(t))dt \geq 0 \qquad , \forall v \in L^p(0, T; W_0^{1,p}(\Omega)) \tag{25}$$

By (2), we have that

$$\begin{aligned} \textstyle\int_0^T (-\Delta_p u_m(t), u_m(t))dt = -\textstyle\int_0^T (u_m'(t), u_m(t))dt \\ + \textstyle\int_0^T (g(u_m(t)) \cdot \nabla u_m(t), u_m(t))dt - \textstyle\int_0^T (h(t), u_m(t))dt \end{aligned} \tag{26}$$

Let us analyze each term of the right-hand side of this last equality.

By (4) and because $u_m(0) = u_m(T)$ and $u(0) = u(T)$, we have that

$$\int_0^T (u'(t), u(t))dt = \int_0^T (u_m'(t), u_m(t))dt = 0. \tag{27}$$

By (20) and (23), we have

$$\int_0^T (g(u_m(t)) \cdot \nabla u_m(t), u_m(t))dt \to \int_0^T (g(u(t)) \cdot \nabla u(t), u(t))dt \tag{28}$$

and

$$\int_0^T (h(t), u_m(t))dt \to \int_0^T (h(t), u(t))dt \tag{29}$$

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Hence, passing to the limit in (25), with the help of (3.13), (22), (26). (28), (29), we conclude that

$$\int_0^T (\xi(t) + \Delta_p v(t), u(t) - v(t))dt \geq 0 \qquad \forall v \in L^p(0, T; W_0^{1,p}(\Omega))$$

Now, we proceed as usual: by taking $v = u - \lambda\omega$, where $\omega \in L^p(0, T; W_0^{1,p}(\Omega))$ and $\lambda > 0$, in the above, we obtain

$$\int_0^T (\xi(t) + \Delta_p(u(t) - \lambda\omega(t)), \omega(t))dt \geq 0 \qquad \forall\omega \in L^p(0, T; W_0^{1,p}(\Omega)).$$

Now, with help of the Lebesgue Convergence Theorem and the hemicontinuity of Δ_p (see (12),) we can pass to the limit when $\lambda \to 0$, to obtain

$$\int_0^T (\xi(t) + \Delta_p u(t), \omega(t))dt \geq 0 \qquad \forall\omega \in L^p(0, T; W_0^{1,p}(\Omega))$$

Thus, $\xi = -\Delta_p u$ and u is a (weak) periodic solution of

$$\begin{cases} u_t - \Delta_p u = h(t) + g(u) \cdot \nabla u. \\ u(0) = u(T). \end{cases}$$

This finishes the proof of the existence of a weak periodic solution of (1.1) in the case $1 \leq s < p - 2$.

To find periodic solutions in the case $s = p - 2$, it is sufficient to guarantee that again in this case, in (11), we have $c_3 > 0$. In fact, if this is true, the rest of the proof follows as exactly as before. For this, recall that, when $s = p - 2$, c_3 is given by (13), with c_1 given by (9). Therefore, when $\max\limits_{\ell=1\ldots N}\{C_\ell\}$ is sufficiently small, by taking $\varepsilon > 0$ small enough, we can make $c_3 > 0$ and thus obtain a weak periodic solution also in the case $s = p - 2$.

This finishes our considerations concerning the existence of weak periodic solutions.

Now, we consider the case in which it is possible to obtain strong solutions of (1.1).

We observe that, when $1 \leq s \leq \dfrac{p-2}{2}$, then $\dfrac{p}{s} \geq \dfrac{2p}{p-2}$. Thus,

$$L^{p/s}(0, T; L^{p/s}(\Omega)) \subset L^{2p/(p-2)}(0, T; L^2/(p-2)(\Omega)).$$

Being u a weak solution of (1.1), we know that $u \in L^p(0, T; W_0^{1,p}(\Omega))$, and, for each $\ell = 0, \ldots, N$ we conclude that $g_\ell(u) \in L^{\frac{2p}{p-2}}(0, T; L^{\frac{2p}{p-2}}(\Omega))$. Consequently, $g(u) \cdot \nabla u \in L^2(0, T; L^2(\Omega)$.

Now, by denoting $f(t) = g(u(t)) \cdot \nabla u(t)$ we can rewrite u as a solution of

$$\begin{cases} u_t(t) - \Delta_p u(t) = f(t) \quad \text{in} \quad L^2(0, T; L^2(\Omega) \\ u(0) = u(T) \end{cases}$$

But, from Brezis [2], Chapter 3, Section 3, Theorem 3.6, we know that, when $f \in L^2(0,T;L^2(\Omega))$, a solution of the above equation is a strong solution. Hence, u is an absolutly continuous function from $[0,T]$ to $L^2(\Omega)$ and the equation is satisfied a.e $t \in (0,T)$ and in $L^2(0,T;L^2(\Omega))$ ∎

Remark: As we said in the Introduction, in [3] we showed the existence of solutions for a problem as follows:

$$\begin{cases} u_t(t,x) - \Delta_p u(t,x) = h(t,x) + m(t)g(u(t,x)) & (t,x) \in (0,T)\times\Omega, \\ u(t,x) = 0 & (t,x) \in (0,T)\times\partial\Omega, \\ u(0,x) = u(T,x) & x \in \Omega, \end{cases} \tag{30}$$

where, as before, $T > 0$ and Ω is an open bounded and regular set of $\mathbb{R}^N$. Δ_p is again the p-Laplacean, $h \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $m \in L^\infty(0,T)$ and g a real continuous function for which there is a constant $c \geq 0$ such that for all $\tau \in \mathbb{R}$ we have $|g(\tau)| \leq c(|\tau|^s + 1)$. Moreover, we assume that one of the following situations occurs:
a) $0 \leq s < p-1$ or
b) $p-1 \leq s < p-1+\dfrac{2p}{N}$, $p < N$ and $|m|_\infty$ is sufficiently small or
c) $p-1 \leq s < p+1$, $p \geq N$ and $|m|_\infty$ is sufficiently small.

In [3], we proved the existence of periodic solutions of (30) as follows: first, we introduced an operator $K : L^p(0,T;L^q(\Omega)) \to L^p(0,T;L^q(\Omega))$, for suitable q, as the operator that for each v it is associated $K(v) = u$, where u is the solution of $u_t(t,x) - \Delta_p(u(t,x)) = H(t,x) = h(t,x) + m(t)g(v(t,x))$ with $u|_{\partial\Omega} = 0$ and $u(0,x) = u(T,x)$ for $x \in \Omega$. Solutions of (30 are then fixed points of K.

We showed the existence of a fixed point by observing that K was the composition of the Solution Operator associated to (15), with the Nemytskii operator N_H associated to the above function H. By proving that these operators had good properties, we were then able to prove that K was a completely continuous operator taking a suitable ball in $L^p(0,T;L^q(\Omega))$ in itself (in the case that $1 - \dfrac{N}{p} > -\dfrac{N}{q}$, and thus $W_0^{1,p}(\Omega) \subset L^q(\Omega)$ compactly.)

If we had tried a technique similar to the one described above in the context of the present paper, the corresponding Nemytskii operator N_H would depend also on the gradient of the solution. Then, for instance in the simple case where $N_H(v)(t,x) = h(t,x) + \displaystyle\sum_{\ell=1}^{N} a_i(t)\frac{\partial v}{\partial x}(t,x)$, with $a_i \in L^\infty(0,T)$, we had to consider the corresponding operator K as acting from $L^p(0,T;W_0^{1,p}(\Omega))$ to itself. It would still be possible to prove the continuity of the corresponding K in this space, but not to obtain that it were completely continuous. Thus, we would not have a simple way to prove the existence of a fixed point.

For this reason, when gradients are present in the perturbation, we have to use a different technique, like the one employed in this paper, in which the lack of compactness of the above K was compensated by the use of the Faedo-Galerkin method, and the use of the Brower Fixed Point Theorem (in finite dimension) for the approximate operators K_m.

4. EXISTENCE OF PERIODIC SOLUTIONS IN THE CASE OF MORE GENERAL MONOTONE OPERATORS

By analyzing the proof of Theorem 3.3.1, we observe that it depends only on certain properties of the p-Laplacean and not on the special form of this operator. Thus, the proof still holds true for more general monotone operators having properties similar to the ones of the p-Laplacean. In fact, we have the following result:

THEOREM 4.4.1. *Let $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be an operator satisfying the following:*
(i) Strong monotonicity: there is a constant $\alpha > 0$ such that $\forall u \in W_0^{1,p}(\Omega)$ it holds

$$(A(u) - A(v), u - v) \geq \alpha |u - v|_{1,p}^p.$$

(ii) Hemicontinuity: for any $u, w \in W_0^{1,p}(\Omega)$, the following aplication is continuous

$$\lambda \in \mathbb{R} \mapsto (A(u + \lambda w), v);$$

(iii) Coercivity: there are constants $\alpha_1 > 0$ and α_2 such that $\forall u \in W_0^{1,p}(\Omega))$ it is true that

$$(A(u), u) \geq \alpha_1 |u|_{1,p}^p - \alpha_2;$$

(iv) Boundness: there are constants $\beta_1 > 0$ and $\beta_2 > 0$ such that $\forall u \in W_0^{1,p}(\Omega)$,

$$|A(u)|_{-1,p'} \leq \beta_1 |u|_{1,p}^{p-1} + \beta_2.$$

Suppose also that $p \geq 3$, and we are given $h \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ and locally Lipschitzian functions $g_\ell : \mathbb{R} \to \mathbb{R}$, $\ell = 1, \ldots, N$, satisfying

$$|g_\ell(\tau) - g_\ell(\sigma)| \leq C_\ell(|\tau|^{s_\ell - 1} + |\sigma|^{s_\ell - 1} + 1)|\tau - \sigma|$$

for all $\tau, \sigma \in \mathbb{R}$ and some $C_\ell \geq 0$ and $1 \leq s_\ell < p - 2$.
Then, the following problem has a weak solution:

$$\begin{cases} u_t(t,x) + A(u)(t,x)) = h(t,x) + & g(u(t,x)) \cdot \nabla u(t,x), \\ u(t,x) = 0 & \text{for} \quad x \in \partial\Omega, t \in (0,T), \\ u(0,x) = u(T,x) & \text{for} \quad x \in \Omega. \end{cases} \tag{31}$$

When $\max\limits_{\ell=1\ldots N}\{s_\ell\} = p - 2$, if $\max\limits_{\ell=1,\ldots,N}\{C_\ell\}$ is sufficiently small, there is still a weak solution.
When $1 \leq s \leq \dfrac{p-2}{2}$ and $h \in L^2(0, T; L^2(\Omega))$, the above solutions are strong.

The following are examples of equations for which the corresponding problems of finding periodic solutions (with Dirichlet boundary conditions) are such that the above theorem applies. In fact, it is not difficult to show in these cases that the associated operators are hemicontinuous, coercive and bounded. To show the strong monotonicity, it is enough to use Tolksdorf's result, which for the reader's convenience we state below.

Example 1:

$$u_t - \lambda\Delta_p - \lambda_1\Delta_{p_1} + \lambda_2\Delta_{p_2}(u) = h + g(u)\cdot\nabla u,$$

where $p > p_2 > p_1 \geq 2, \quad p = 2p_2 - p_1, \quad \lambda_2^2 - 4\lambda_1\lambda = -a < 0$ and $[\lambda_2(p_2-1)]^2 - 4\lambda_1\lambda(p_1 - 1)(p-1) \geq 0$.

Exemple 2:

$$u_t - \operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u) = h + g(u)\cdot\nabla u,$$

where $a(\cdot) \in C^1(\mathbb{R}_+)$ is a increasing function such that there are constants $\alpha_1 \geq 0, \ \beta_1 \geq 0$ and $\alpha_2 > 0$ such that $\alpha_2 t^{p-1} \leq a(t^p)t^{p-1} \leq \alpha_1 t^{p-1} + \beta_1$.

As we said previously, in the above examples, the strong monotonicity of the corresponding operators can be verified with the following interesting result by Tolksdorf ([7], Section 2, Lemma 1):

Proposition 4.4.1. *Suppose that* $A : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ *satisfies:*

a) $A(x,\mu,\eta) = (\alpha_j(x,\mu,\eta))$ *where* $\alpha_j \in C^\circ(\Omega\times\mathbb{R}\times\mathbb{R}^N)\cap C^1(\Omega\times\mathbb{R}\times\mathbb{R}^N\setminus\{0\})$

b) $\alpha_j(x,\mu,0) = 0$

c) there are $p,\gamma > 0$ *such that for all* $\xi = (\xi_1,\ldots,\xi_N) \in \mathbb{R}^N$ *and* $\eta = (\eta_1,\ldots,\eta_N) \in \mathbb{R}^N\setminus\{0\}$

$$\sum_{i,j=1}^{N} \frac{\partial\alpha_i}{\partial\eta_j}(x,\mu,\eta)\xi_i\xi_j \geq \gamma|\eta|^{p-2}|\xi|^2.$$

This is called the ellipticity condition.

Then, if $p \geq 2$ *and* $\eta,\eta' \in \mathbb{R}^N\setminus\{0\}$ *we have*

$$\sum_{j=1}^{N} \ (\alpha_j(x,\mu,\eta) - \alpha_j(x,\mu,\eta'))\ (\eta_j - n'_j) \geq$$
$$\leq \gamma\left(\frac{1}{4}\right)^{p-2}|\eta-\eta'|^p$$

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