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Upper Semicontinuity for Attractors of Parabolic Problems with Localized Large Diffusion and Nonlinear Boundary Conditions.

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In this paper we study second order parabolic problems for which the diffusion coefficient becomes large in a sub-region which is interior to the physical domain of the differential equation. That is the situation observed in composite materials where the heat diffusion properties can change significantly from one part of the region to another. We prove that if in a reaction-diffusion process the diffusion coefficient behaves as expressed above, the solutions will become homogeneous in the regions where the diffusion becomes large. We exhibit a limiting system and prove that the attractors are upper-semicontinuous with respect to it   April, 2000 ICMC-USP

Dedicated to Jack K. Hale on occasion of his 70^{th} birthday.

1. INTRODUCTION

In this paper we are concerned with second order parabolic problems for which the diffusion coefficient becomes large in a sub-region which is interior to the physical domain of the differential equation. That situation can be found, for example, in composite materials, where the heat diffusion properties can change significantly from one part of the region to

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another; that is, heat may diffuse much faster in some sub-regions than in others. If in a reaction-diffusion process the diffusion coefficient behaves as expressed above, intuitively we expect that the solutions will tend to become homogeneus in the regions where the diffusion becomes large.

In the following we borrow the notations from [16]. Let Ω be a bounded smooth domain in $I\!R^N$, ϵ be a positive parameter, m be a positive integer, $\Omega_0 = \cup_{i=1}^m \Omega_{0,i}$ be an interior sub-domain of Ω where $\Omega_{0,i}$ is a smooth sub-domain of Ω with $\overline{\Omega}_{0,i} \cap \overline{\Omega}_{0,j} = \emptyset$, for $i \neq j$. Let $\Gamma = \partial\Omega$, $\Gamma_{0,i} = \partial\Omega_{0,i}$ and $\Gamma_0 = \cup_{i=1}^m \Gamma_{0,i}$ be, respectively, the boundary of Ω, $\Omega_{0,i}$ and Ω_0. Denote by $\Omega_1 = \Omega \backslash \overline{\Omega}_0$ and note that its boundary is given by $\partial\Omega_1 = \Gamma \cup \Gamma_0$.

The diffusion coefficient a_ϵ is assumed to be a regular and bounded function in Ω satisfying

$$0 < m_0 \leq a_\epsilon(x) \leq M_\epsilon \tag{1.1}$$

for every $x \in \Omega$ and $0 < \epsilon \leq \epsilon_0$. We also assume that the diffusion becomes very large on Ω_0 as ϵ approaches zero. More precisely, we assume that, as $\epsilon \to 0$

$$a_\epsilon(x) \to \begin{cases} a_0(x) & \text{uniformly on } \Omega_1 \\ \infty & \text{uniformly on compact subsets of } \Omega_0 \end{cases} \tag{1.2}$$

With these notations and for $c \in C^1(\Omega)$, $b \in C^1(\Gamma)$, and the nonlinearities f, g we consider the family of parabolic equations

$$\begin{cases} u_t^\epsilon - \operatorname{div}(a_\epsilon(x)\nabla u^\epsilon) + c(x)u^\epsilon = f(u^\epsilon), & \text{on } \Omega \\ a_\epsilon \dfrac{\partial u^\epsilon}{\partial \vec{n}} + b(x)u^\epsilon = g(u^\epsilon), & \text{on } \Gamma \\ u^\epsilon(0) = u_0^\epsilon. \end{cases} \tag{1.3}$$

Our goal is to study the behavior of solutions of (1.3), as $\epsilon \to 0$.

Now, we guess which is the limiting equation, or "shadow system", for (1.3) when ϵ tends to zero. To simplify the presentation let us assume for the moment that $m = 1$.

Again, from physical considerations, we intuitively guess that for small values of ϵ, the solution of problem (1.3) should be approximately constant on Ω_0 as time increases. Therefore, suppose that u^ϵ converges to some function u, in some sense, and that u takes a, time dependent, spatially constant value on Ω_0, $u_{\Omega_0}(t)$.

If we formally take the limit in problem (1.3), we expect that, inside Ω_1, the function u satisfies

$$\begin{aligned} &u_t - \operatorname{div}(a_0(x)\nabla u) + c(x)u = f(u), \quad \text{on } \Omega_1 \\ &a_0(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = g(u), \quad \text{on } \Gamma. \end{aligned}$$

The constant u_{Ω_0}, however, may not be arbitrary. Integrating the equation in Ω_0 and using the inward normal in the integration by parts, we obtain

$$\int_{\Omega_0} u_t^\epsilon + \int_{\Gamma_0} a_\epsilon \frac{\partial u^\epsilon}{\partial \vec{n}} + \int_{\Omega_0} c u^\epsilon = \int_{\Omega_0} f(u^\epsilon)$$

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and formally taking the limit and dividing by $|\Omega_0|$ we have

$$\dot{u}_{\Omega_0} + \frac{1}{|\Omega_0|}\int_{\Gamma_0} a_0 \frac{\partial u}{\partial \vec{n}} + \hat{c}u_{\Omega_0} = f(u_{\Omega_0})$$

where $\hat{c} = |\Omega_0|^{-1}\int_{\Omega_0} c(x)dx$. Even more, one can expect u to match appropriately the constant $u_{\Omega_0}(t)$ across Γ_0, that is $u_{|\Omega_0} = u_{\Omega_0}$.

In fact, when $m \geq 1$ the limiting problem should be

$$\begin{cases} u_t - \mathrm{div}(a_0(x)\nabla u) + c(x)u = f(u) & \text{on } \Omega_1 \\ a_0 \dfrac{\partial u}{\partial \vec{n}_0} + b(x)u = g(u) & \text{on } \Gamma \\ u_{|\Omega_{0,i}} =: u_{\Omega_{0,i}} & \text{on } \Omega_{0,i}, \quad i = 1, \dots, m \\ \dot{u}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} a_0 \frac{\partial u}{\partial \vec{n}} + \hat{c}_i u_{\Omega_{0,i}} = f(u_{\Omega_{0,i}}), & i = 1, \dots, m \\ u(0) = u_0 \end{cases} \tag{1.4}$$

where $\hat{c}_i = |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} c(x)dx$ and $u_0 = \lim_{\epsilon\to 0} u_0^\epsilon$ which are constant on $\Omega_{0,i}$, $i = 1, \dots, m$.

The natural spaces to study problem (1.3) are $L^q(\Omega)$ or $W^{1,Q}(\Omega)$, see [3]. We will see that under certain conditions on the nonlinearities f and g problem (1.3) is well posed in $L^q(\Omega)$ or $W^{1,Q}(\Omega)$ and under some dissipativeness conditions we have the existence of global attractors $\mathcal{A}_\epsilon$ which actually are independent of the space chosen to study the equation and that lie uniformly on a bounded set of $C^0(\bar{\Omega})$, see [4]. This will allow us to cut-off the nonlinearities, reducing to the case where the nonlinearities are globally Lipschitz and studying both problems (1.3) and (1.4) in $H^1(\Omega)$ and $H^1_{\Omega_0}(\Omega)$ respectively, where in general for any functional space X we define $X_{\Omega_0} = \{u \in X,\ u \text{ is constant on } \Omega_{0,i},\ i = 1, \dots, m\}$. For this globally Lipschitz nonlinearities we will have the existence of an attractor $\mathcal{A}_0$ for problem (1.4) which will also lie on a bounded set of $H^1(\Omega) \cap C^0(\bar{\Omega})$.

In comparing the dynamics of (1.3) and (1.4), we will prove the following result

Theorem 1.1.1. *The global attractors $\mathcal{A}_\epsilon, \mathcal{A}_0$ are bounded subsets of $H^1(\Omega)$, $C^0(\bar{\Omega})$ and are upper semi-continuous at $\epsilon = 0$ relatively to the topology in $H^1(\Omega)$ and $C^0(\bar{\Omega})$.*

We will also be able to give a result on the convergence of orbits of the attractors:

Proposition 1.1.2. *For every sequence of complete orbits $u^{\epsilon_k}(\cdot) \subset \mathcal{A}_{\epsilon_k}$ there exists a subsequence ϵ_{k_j} and a complete orbit $u^0(\cdot) \subset \mathcal{A}_0$ such that*

$$u^{\epsilon_{k_j}}(\cdot) \stackrel{j\to\infty}{\longrightarrow} u^0(\cdot) \quad \textit{in } C([-T, T], C^0(\bar{\Omega})), \textit{ for any } T > 0$$

This results state, in a precise sense, that the asymptotic set of states and the asymptotic dynamics of both problems are close as $\epsilon \to 0$, see [9]. This closeness is obtained in the topology of $H^1(\Omega)$ and of $C^0(\bar{\Omega})$.

For related questions in the case of linear boundary conditions see [5, 6, 7, 8, 11, 12].

Now we describe the contents of our paper.

In Section 2 we recall known results on the well posedness, regularity, existence of attractors and uniform bounds in different metrics for problem (1.3). These results are taken primarily from [3, 4, 16]. As an important consequence of this section is that we will be able to reduce to the case where the nonlinearities f and g are globally Lipschitz functions.

In Section 3 we study the local and global well posedness of the limiting problem in the functional spaces $H^1_{\Omega_0}(\Omega)$ and $L^2_{\Omega_0}(\Omega)$. For the first case standard theory like the one from [13] can be applied. For the $L^2_{\Omega_0}(\Omega)$-setting we need to apply some results from [2, 3].

In Section 4 we study and compare extensively the linear problems associated to equations (1.3) and (1.4). We will prove the convergence in the H^1 and C^0 metric of the resolvents of the linear operators. For the proof of the C^0 convergence we need to prove Lemma 4.4.2, which is an essential ingredient in the analysis of the C^0 convergence for the linear and, afterwards, for the nonlinear problems. With this lemma we will also improve certain results from [16] on convergence of the spectra for the linear operators, see Proposition 4.4.4. We will also obtain estimate in the convergence of the linear semigroups, see Proposition 4.4.6 and Corollary 4.4.8.

Finally, in Section 5, we study the relation between the asymptotic dynamics of both problems (1.3) and (1.4). We will show first that the attractor of (1.4) lies in a bounded set of $C^\alpha(\bar{\Omega})$. Considering now the uniform estimates obtained for problem (1.3) and the convergence of the linear semigroups we will show the upper semicontinuity of the attractors in $H^1(\Omega)$ by comparing the nonlinear semigroups with the use of the variations of constants formula. Once the upper semicontinuity in $H^1(\Omega)$ is obtained, Lemma 4.4.2 will give us the key to prove it in $C^0(\bar{\Omega})$.

2. BACKGROUND RESULTS.

We sumarize in this section the already known results on local and global existence of solutions, existence of attractors and their uniform bounds. These results, taken from [3, 4, 16], will be our starting point for the upper semi-continuity results proved later on in this paper. We refer to these articles for details, proofs and generalizations.

We consider the family of semi-linear parabolic problems given by equation (1.3) for $\epsilon \in (0, \epsilon_0)$ where the nonlinearities $f, g : I\!R \to I\!R$ are C^2 functions, c, b are C^1 functions and the diffusion coefficient $a_\epsilon \in C^1(\bar{\Omega})$ and satisfies (1.1).

We treat this problem as an evolution problem in the spaces $L^q(\Omega)$, $W^{1,q}(\Omega)$ for $1 < q < \Omega$. Therefore, and in order to simplify the notations, we define the family of spaces

$$\mathcal{E} = \{L^q(\Omega),\ W^{1,q}(\Omega),\ 1 < q < \infty\}.$$

Then we consider (1.3) as a semi-linear problem written in the abstract form as

$$\dot{u} + A_\epsilon u = h(u)$$

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where A_ϵ is a suitable weak formulation of the operator $-div(a_\epsilon(x)\nabla u)+cu$ with boundary conditions $a_\epsilon(x)\partial u/\partial n + b(x)u = 0$, and the nonlinearity is given by $h := f_\Omega + g_\Gamma$, that is,

$$\langle h(u), \phi\rangle = \int_\Omega f(x,u(x))\phi(x) + \int_\Gamma g(x,u(x))\phi(x),$$

for all suitable regular test functions ϕ, see [3] for details.

Assume that f and g satisfy the following growth conditions
$(\mathbf{G})_\mathbf{X}$: Let $f, g : I\!\!R \to I\!\!R$ be locally Lipschitz functions. Assume the following,

1. If $X = L^q(\Omega)$, assume that f and g satisfy a relation of the form

$$|j(u) - j(v)| \le c|u-v|(|u|^{\rho-1} + |v|^{\rho-1} + 1), \tag{2.1}$$

with exponents ρ_f and ρ_g respectively, such that, with $N \ge 2$ (respectively $N = 1$)

$$\rho_f \le \rho_\Omega := 1 + \frac{2q}{N}, \quad \text{and} \quad \rho_g \le \rho_\Gamma := 1 + \frac{q}{N}, \text{ (respectively, } \rho_g < \rho_\Gamma := 1 + q),$$

2. If $X = W^{1,q}(\Omega)$, assume either

i) $q > N$,

ii) $q = N$ and f, g satisfy that for every $\eta > 0$, there exists $c_\eta > 0$ such that

$$|j(u) - j(v)| \le c_\eta(e^{\eta|u|^{\frac{N}{N-1}}} + e^{\eta|v|^{\frac{N}{N-1}}})|u-v| \tag{2.2}$$

iii) $1 < q < N$ and f, g satisfy (2.1), with exponents ρ_f and ρ_g, respectively, such that

$$\rho_f \le \rho_\Omega := 1 + \frac{2q}{N-q} \quad \text{and} \quad \rho_g \le \rho_\Gamma := 1 + \frac{q}{N-q}.$$

The results from [3] can be summarized as follows,

THEOREM 2.2.1. **(Local Existence)** *Let $\epsilon \in (0, \epsilon_0)$. If X is any space in the class $\mathcal{E}$ and f, g satisfy the growth restriction $(\mathbf{G})_\mathbf{X}$, then for any $u_0 \in X$ there exists locally a unique (in certain sense) mild solution $u_\epsilon(\cdot, u_0) \in C([0,\tau), X)$, of problem (1.3) satisfying $u_\epsilon(0, u_0) = u_0$ in X. This solution depends continuously on the initial data $u_0 \in X$ and it is a classical solution for $t > 0$. Also, the following regularizing effect takes place: if $u_0 \in X$ then $u_\epsilon(t, u_0) \in Y$ for any other space Y in the class $\mathcal{E}$, and $t \in (0, \tau)$.*

In order to obtain that all solutions of (1.3) are globally defined, we will assume some sign conditions on the nonlinear terms. These sign conditions are independent of the space X and can be expressed in the form:

(S): *Assume there exist* B_0, $C_0 \in I\!R$ *and* $B_1, C_1 \geq 0$ *such that the following holds*

$$\begin{aligned} u\,f(u) &\leq -C_0 u^2 + C_1|u|, \\ u\,g(u) &\leq -B_0 u^2 + B_1|u|. \end{aligned} \tag{2.3}$$

for $u \in I\!R$.

Then we have the following result on global existence (see [4]):

THEOREM 2.2.2. **(Global Existence)** *Let* $\epsilon \in (0, \epsilon_0)$ *and let* X *be an space in the class* $\mathcal{E}$. *Assume that the growth condition* $(\mathbf{G})_\mathbf{X}$ *and the sign condition* $(\mathbf{S})$ *hold. Then, for any* $u_0 \in X$ *the solution* $u_\epsilon(t, u_0)$ *of (1.3) starting at* u_0 *exists for all* $t \geq 0$. *Therefore, we can define in* X *the semigroup* $\{S_\epsilon(t) :\ t \geq 0\}$ *associated to (1.3) by* $S_\epsilon(t)u_0 = u(t, u_0),\ t \geq 0$. *Moreover, from the regularizing effect of Theorem 2.2.1,* $S_\epsilon(t)u_0 \in Y$ *for any other space* Y *in the class* $\mathcal{E}$ *and for any* $t > 0$.

To prove the existence of a global attractor for the problem (1.3) we will impose, besides condition (**S**), some dissipativeness condition for (1.3). This condition is expressed as:

$(\mathbf{D})_\epsilon$: *Assume* (**S**) *holds and that with* C_0 *and* B_0 *from* (**S**), *the first eigenvalue,* λ_1^ϵ, *of the following problem is positive*

$$\begin{aligned} &-\mathrm{div}(a_\epsilon(x)\nabla u) + (c(x) + C_0)u = \lambda u, \quad \text{in } \Omega \\ &a_\epsilon(x)\frac{\partial u}{\partial n} + (b(x) + B_0)u = 0, \quad \text{in } \Gamma \end{aligned} \tag{2.4}$$

With all these, it is possible to show the following result (see [4])

THEOREM 2.2.3. **(Existence of Attractors)** *Let* $\epsilon \in (0, \epsilon_0)$ *and let* X *be an space in the class* $\mathcal{E}$. *Assume that the growth condition* $(\mathbf{G})_\mathbf{X}$ *and the dissipativeness condition* $(\mathbf{D})_\epsilon$ *hold. Then the semigroup* $\{S_\epsilon(t),\ t \geq 0\}$ *associated to (1.3) has a global attractor,* $\mathcal{A}_X^\epsilon$, *in* X. *Moreover, for any space* Y *in the class* $\mathcal{E}$ *with* $Y \hookrightarrow X$ *we also have the existence of the attractor* $\mathcal{A}_Y^\epsilon$. *Moreover* $\mathcal{A}_X^\epsilon = \mathcal{A}_Y^\epsilon$ *and it attracts bounded sets of* X *in the topology of* Y.

In particular if the diffusion coefficient a_ϵ satisfies (1.2), we know from [16] that λ_1^ϵ converges to the first eigenvalue, λ_1^0, of the limit eigenvalue problem:

$$\begin{cases} -\mathrm{div}(a_0(x)\nabla u) + (c(x) + C^0)u = \lambda u & \text{on } \Omega_1 \\ a_0(x)\dfrac{\partial u}{\partial \vec{n}} + (b(x) + B^0)u = 0 & \text{on } \Gamma \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}} & \text{on } \Gamma_{0,i}, \quad i = 1, \ldots, m \\ \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} a_0 \frac{\partial u}{\partial \vec{n}} + (\hat{c}_i + C^0)u_{\Omega_{0,i}} = \lambda u_{\Omega_{0,i}}, & i = 1, \ldots, m \end{cases} \tag{2.5}$$

where $\hat{c}_i = |\Omega_{0,i}|^{-1} \int_{\Omega_{0,i}} c(x)dx$.

Therefore, if $\lambda_1^0 > 0$ then for small enough ϵ, we will have $\lambda_1^\epsilon > 0$, which motivates the following definition

$(\mathbf{D})_\mathbf{0}$: *Assume* $(\mathbf{S})$ *holds and that with* C_0 *and* B_0 *from* $(\mathbf{S})$, *the first eigenvalue,* λ_1^0, *of the problem (2.5) is positive*

With this condition we obtain, see [4],

THEOREM 2.2.4. **(Uniform Bounds)** *Let* X *be an space in the class* $\mathcal{E}$. *Assume that the growth restriction* $(\mathbf{G})_\mathbf{X}$ *and that the dissipative condition* $(\mathbf{D})_\mathbf{0}$ *holds. Then, for sufficiently small* ϵ, *and for any global solution* $\{v(t) : t \in I\!R\} \subset \mathcal{A}_X^\epsilon$, *we have*

i) $\|v\|_{C^0(\bar{\Omega})} + \|v_t\|_{L^\infty(\Omega)} \le K_0$, *where* $K_0 = K_0(\Omega, \lambda_1, m_0, c, b, C_0, C_1, B_0, B_1)$.

ii) $\|v\|_{H^1(\Omega)} + \|v_t\|_{H^1(\Omega)} \le \tilde{K}_0$, *where* $\tilde{K}_0 = \tilde{K}_0(\Omega, \lambda_1, m_0, c, b, C_0, C_1, B_0, B_1)$.

iii) For $x_0 \in \bar{\Omega}$ *and* $\rho_0 > 0$, *there is a* $\nu \in (0,1)$ *such that* $\sup_{v\in\mathcal{A}_X^\epsilon}\{\|v\|_{C^\nu(\bar{\Omega}\cap B(x_0,\rho_0))}\} \le K_\nu^\epsilon$, *where* K_ν^ϵ *depends on the same as* K_0 *plus on* ν, $M_\epsilon^{\rho_0} = \sup\{a_\epsilon(x);\ x \in \bar{\Omega}\cap B(0, 2\rho_0)\}$, $\sup\{|f'(s)| :\ |s| \le K_0\}$, *and* $\sup\{|g'(s)| :\ |s| \le K_0\}$.

REMARK 2.2.5. *It is important to mention that the dependence of the constant* K_ν^ϵ *on* ϵ *is only through* $M_\epsilon^{\rho_0}$. *Therefore, we obtain for our problem uniform Hölder bounds in compact subsets of* $\bar{\Omega} \setminus \Omega_1 = \Omega_0 \cup \Gamma$.

REMARK 2.2.6. *Part* $i)$ *of this theorem establishes the uniform* L^∞*-bounds on the family of attractors* $\mathcal{A}_X^\epsilon$. *These uniform bounds allow us to cut off the nonlinearities* f *and* g *outside the set* $[-2K_0, 2K_0]$ *in such a way that the new nonlinearities* $\tilde{f}$ *and* $\tilde{g}$ *have the same regularity as the original ones, they are globally lipschitz and condition* $(\mathbf{S})$ *still holds with the same constants* B_0, B_1, C_0 *and* C_1. *Therefore, from now on we will assume that this cut off has been performed and we will denote again the new nonlinearities by* f *and* g.

3. THE LIMIT PROBLEM

In this section we recall from [16] the functional setting for problem (1.4) and will consider its well posedness and global existence and regularity of solutions in $H^1_{\Omega_0}(\Omega)$ and $L^2_{\Omega_0}(\Omega)$. In Section 5 we will prove the existence and regularity of the attractor for this problem under the dissipativeness condition $(\mathbf{D})_\mathbf{0}$.

We will assume that the nonlinearities f and g are globally Lipschitz, see Remark 2.2.6.

Following [16] we define the space $L^2_{\Omega_0}(\Omega) = \{u \in L^2(\Omega),\ u \text{ is constant on } \Omega_{0,i},\ i = 1,\dots,m\}$ and $H^1_{\Omega_0}(\Omega) = \{u \in H^1(\Omega),\ u \text{ is constant on } \Omega_{0,i},\ i = 1,\dots,m\}$. Let $X_0 = L^2_{\Omega_0}(\Omega)$ and define the operator A_0 in X_0 with domain

$$D(A_0) = \{u \in H^1_{\Omega_0}(\Omega),\ -\mathrm{div}(a_0(x)\nabla u) \in L^2(\Omega_1),\ a_0(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = 0 \text{ on } \Gamma\}$$

and for $u \in D(A_0)$,

$$A_0(u) = (-\mathrm{div}(a_0(x)\nabla u) + c(x)u)\,\mathcal{X}_{\Omega_1} + \sum_{i=1}^{m}\left(\frac{1}{|\Omega_{0,i}|}\int_{\Gamma_{0,i}} a_0(x)\frac{\partial u}{\partial \vec{n}} + \hat{c}_i u_{\Omega_{0,i}}\right)\mathcal{X}_{\Omega_{0,i}}$$

where $\hat{c}_i = |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} c(x)dx$ and $u_{\Omega_{0,i}}$ denotes the constant value of u on $\Omega_{0,i}$ for $i = 1,\dots,m$. The operator A_0 is selfadjoint and has compact resolvent in $X_0 = L^2_{\Omega_0}(\Omega)$. Moreover, if $\mu > \lambda_1^0$, the first eigenvalue of A_0, then the operator $A_0 + \mu I$ is positive. If we denote by X_0^γ its fractional power spaces, we have that $X_0^1 = D(A_0)$, $X_0^{1/2} = H^1_{\Omega_0}(\Omega)$, $X_0^0 = L^2_{\Omega_0}(\Omega)$, $X_0^{-1/2} = H^{-1}_{\Omega_0}(\Omega) \stackrel{def}{=} (H^1_{\Omega_0}(\Omega))'$, see [16]. By interpolation $X_0^\gamma \hookrightarrow H^{2\gamma}_{\Omega_0}(\Omega)$, for $0 \le \gamma \le 1/2$. By duality we obtain $H^{-2\gamma}_{\Omega_0}(\Omega) \hookrightarrow X_0^{-\gamma}$, again for $0 \le \gamma \le 1/2$.

The operator A_0 is the realization in $L^2_{\Omega_0}(\Omega)$ of the operator L_0, between $H^1_{\Omega_0}(\Omega)$ and its dual defined by means of the bilinear form

$$\langle L_0(u), v\rangle_{-1,1} = a_0(u,v) = \int_{\Omega_1} a_0\nabla u\nabla v + \int_\Omega cuv + \int_\Gamma buv$$

for every $u, v \in H^1_{\Omega_0}(\Omega)$. For the sake of simplicity in the notations we will not distinguish between L_0 and A_0.

The nonlinear terms verify $h = f_\Omega + g_\Gamma : H^1_{\Omega_0}(\Omega) \to H^{-s}_{\Omega_0}(\Omega)$ for any $s > 1/2$, and since $f, g : I\!R \to I\!R$ are globally Lipchitz we obtain that h is globally Lipschitz. In terms of the fractional power spaces we have that $h : X_0^{1/2} \to X_0^{-s}$ for any $s > 1/4$ and that there exists a constant $c \ge 0$ such that $\|h(u) - h(v)\|_{X_0^{-s}} \le c\|u - v\|_{X_0^{1/2}}$ for any $u, v \in X_0^{1/2}$. We can prove the following

Proposition 3.3.1. *For any initial data $u_0 \in L^2_{\Omega_0}(\Omega)$ there exists a unique globally defined $u(\cdot, u_0) \in C([0,\infty), L^2_{\Omega_0}(\Omega)) \cap C((0,\infty), H^1_{\Omega_0}(\Omega))$ mild solution of $u_t + A_0u = h(u)$ starting at u_0. Moreover this solution depends continuously on the initial data and satisfies $u, u_t \in C((0,\infty), X_0^\gamma)$, for any $\gamma < 3/4$. Moreover if u_0 lies in a bounded set of $L^2_{\Omega_0}(\Omega)$ then, for $t > 0$ fixed $u(t, u_0)$ lies in a bounded subset of X_0^γ for any $\gamma < 3/4$. If the initial data is in $H^1_{\Omega_0}(\Omega) = X_0^{1/2}$ then the solution is also in $C([0,\infty), H^1_{\Omega_0}(\Omega))$.*

Proof: To prove that the problem is well posed for $u_0 \in H^1_{\Omega_0}(\Omega)$ we can apply standard semilinear theory like in [13]. The nonlinearity $h : X_0^{1/2} \to X_0^{-s}$ and is globally Lipschitz, from where it follows the regularity and the global existence stated in the proposition.

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The case $u_0 \in L^2_{\Omega_0}(\Omega) = X^0_0$ is different since the nonlinearity h is not even defined on this space. Nevertheless the general theory developed in [2, 3] applies to this problem and allows to obtain all the results of the proposition. Notice that if we define $E^\alpha = X_0^{\alpha-1}$ then $h : E^{1+1/2} \to E^{1-s}$, $s > 1/4$, and it is globally Lipschitz. In the notation of [2, 3] this map is an $\epsilon-$regular map for $\epsilon = 1/2$ relative to (E^1, E^0). Applying Theorem 1 from [2] or Theorem 2.2 from [3] we prove the proposition. □

Note that now u solves

$$< u_t, \phi > +a_0(u, \phi) =< f_\Omega(u), \phi > + < g_\Gamma(u), \phi >, \quad \text{for all } \phi \in H^1_{\Omega_0}(\Omega)$$

which implies

$$\begin{cases} u_t - \operatorname{div}(a_0(x)\nabla u) + c(x)u = f(u) & \text{on } \Omega_1 \\ a_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u = g(u) & \text{on } \Gamma \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}} & \text{on } \Gamma_{0,i}, \quad i = 1, \dots, m \\ \dot{u}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} a_0 \frac{\partial u}{\partial \vec{n}} + \hat{c}_i u_{\Omega_{0,i}} = f(u_{\Omega_{0,i}}), & i = 1, \dots, m \\ u(0) = u_0 \end{cases} \tag{3.1}$$

where $\hat{c}_i = |\Omega_{0,i}|^{-1} \int_{\Omega_{0,i}} c(x)dx$ and $u_{\Omega_{0,i}}$ denotes the constant value of u on $\Omega_{0,i}$ for $i = 1, \dots, m$.

4. COMPARISON OF THE LINEAR PROBLEMS

In this section we compare the linear operators A_0 and A_ϵ establishing certain uniform convergence of the solutions of the respective elliptic problems. In particular, applying these results to the eigenvalue problems, we will improve certain results from [16] and will prove the convergence in the uniform norm of the eigenfunctions of A_ϵ to the eigenfunctions of A_0. We will also be able to compare the behavior of the linear semigroups, $e^{-A_0 t}$ and $e^{-A_\epsilon t}$.

We have the following result:

Proposition 4.4.1. *Let $f^\epsilon \in L^2(\Omega)$ and $g^\epsilon \in L^2(\Gamma)$. Assume $f^\epsilon \overset{\epsilon\to 0}{\longrightarrow} f^0$ w-$L^2(\Omega)$ and $g^\epsilon \overset{\epsilon\to 0}{\longrightarrow} g^0$ w-$L^2(\Gamma)$ then if u^ϵ and u^0 are the solutions of $A_\epsilon u^\epsilon = f^\epsilon_\Omega + g^\epsilon_\Gamma$ and $A_0 u^0 = f^0_\Omega + g^0_\Gamma$, then*

i) $u^\epsilon \overset{\epsilon\to 0}{\longrightarrow} u^0$ strongly in $H^1(\Omega)$

ii) If $f^0 \in L^p(\Omega)$, with $p > N/2$, $g^0 \in L^\infty(\Gamma)$ then $u^0 \in C^0(\bar{\Omega})$.

iii) If $f^\epsilon \in L^p(\Omega)$, with $p > N/2$, $g^\epsilon \in L^\infty(\Gamma)$ for all $0 \le \epsilon \le \epsilon_0$ and $\|f^\epsilon\|_{L^p(\Omega)} + \|g^\epsilon\|_{L^\infty(\Gamma)} \le C$ independent of ϵ, then $u^\epsilon \overset{\epsilon\to 0}{\longrightarrow} u^0$ in $C^0\bar{\Omega})$.

Proof: i) The first part is obtained directly from [16], Corollary 4.5.

ii) For the second part let us see first that $u^0 \in L^\infty(\Omega)$. If we denote by v^ϵ the solution of $A_\epsilon v^\epsilon = f^0_\Omega + g^0_\Gamma$, from i) we have that $v^\epsilon \overset{\epsilon\to 0}{\longrightarrow} u^0$ in $H^1(\Omega)$. But from the uniform bounds obtained in [4] we have that $\|v^\epsilon\|_{L^\infty(\Omega)} \le C$ independent of ϵ. This implies in particular that $\|u^0\|_{L^\infty(\Omega)} \le C$ and therefore $u^0 \in L^\infty(\Omega)$. If we denote by α_0^i the constant value that the function u^0 takes on $\Omega_{0,i}$, then the function u^0 in Ω_1 is the solution of the following problem:

$$\begin{cases} -\mathrm{div}(a_0(x)\nabla u) + c(x)u = f^0 & \text{on } \Omega_1 \\ a_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u = g^0 & \text{on } \Gamma \\ u = \alpha_0^i \quad \text{on } \Gamma_{0,i}, \quad i = 1, \dots, m \end{cases} \tag{4.1}$$

which is $C^0(\bar{\Omega}_1)$, see [4]. Moreover, since u is constant in the connected components of Ω_0 we deduce that u is in $C^0(\bar{\Omega})$.

iii) In order to prove this part, we need the following important lemma,

LEMMA 4.4.2. *For a fixed i, with $1 \le i \le m$, let Ω_i^* be a domain such that $\Omega_{0,i} \subset \Omega_i^* \subset \Omega$, let $\alpha_i \in I\!R$, and consider the family of problems depending on ϵ and Ω_i^**

$$\begin{cases} -\mathrm{div}(a_\epsilon \nabla u^\epsilon) = F_i^\epsilon, \quad on \quad \Omega_i^* \\ u^\epsilon = \alpha_i + \beta_\epsilon^i(x), \quad on \quad \Gamma_i^* = \partial\Omega_i^*, \end{cases}$$

where $\limsup_\epsilon \|\beta_\epsilon^i\|_{L^\infty(\Gamma_i^*)} \le L$, *$F_i^\epsilon \in L^p(\Omega_i^*)$, for $p > N/2$, uniformly bounded in $L^p(\Omega_i^*)$.*

Then

$$\limsup_{dist(\Gamma_i^*, \Gamma_{0,i}) \to 0} \ \limsup_{\epsilon \to 0} \|u^\epsilon - \alpha_i\|_{L^\infty(\Omega_i^*)} \le L$$

Before proving this lemma let us finish with the proof of the proposition.

From i) and ii) we obtain that $u^0 \in C^0(\bar{\Omega})$ and $u^\epsilon \overset{\epsilon \to 0}{\longrightarrow} u^0$ in $H^1(\Omega)$. From the uniform bounds of [4] we obtain that $u^\epsilon \in L^\infty(\Omega)$ and that $\|u^\epsilon\|_{L^\infty(\Omega)} \le C$ independent of ϵ. Also, from Theorem 2.2.4 iii) and Remark 2.2.5, for any $\Omega^* = \cup \Omega_i^*$ as defined in the lemma, we have the existence of an $\alpha \in (0,1)$ such that $\|u^\epsilon\|_{C^\alpha(\bar{\Omega} \setminus \Omega^*)} \le C(\Omega^*)$, which implies the compactness of the sequence u^ϵ in $C^0(\bar{\Omega} \setminus \Omega^*)$ and in particular that $u^\epsilon \overset{\epsilon \to 0}{\longrightarrow} u^0$ in $C^0(\bar{\Omega} \setminus \Omega^*)$.

Let δ be a positive number arbitrarily small. We will see that there exists an $\epsilon(\delta) > 0$ such that $\|u^\epsilon - u^0\|_{C^0(\bar{\Omega})} \le \delta$ for any $0 < \epsilon \le \epsilon(\delta)$.

Since $u^0 \in C^0(\bar{\Omega})$ we have that there exist an smooth $\Omega^* = \cup \Omega_i^*$ close enough to Ω_0 with the property that $\|\beta_i\|_{L^\infty(\Omega_i^*)} \le \delta/2$ where $\beta_i(x) = u^0(x) - \alpha_i^0$, and α_i^0 is the value of u^0 in Ω_0^i. Applying the lemma to a Ω^* sufficiently close to Ω_0 and with $F^\epsilon = f^\epsilon - c(x)u^\epsilon$, we obtain the result.

Now, we only need to prove the lemma

Proof of the lemma: Notice that without loss of generality we can assume that $m = 1$. Therefore, for the proof of this lemma we will drop the subindex i.

First, working with $u^\epsilon - \alpha$ we can always assume that $\alpha = 0$. Second, by superposition, we can consider separately the cases $\beta_\epsilon \ne 0$, $F^\epsilon = 0$ and $\beta_\epsilon = 0$, $F^\epsilon \ne 0$.

In the former case, from [4], Lemma B.1, we get $\|u^\epsilon\|_{L^\infty(\Omega^*)} \le \|\beta_\epsilon\|_{L^\infty(\Gamma^*)}$ and we get the result.

If $\beta_\epsilon = 0$, $F^\epsilon \ne 0$, as in [4] Lemma B.1 ii), taking $(u^\epsilon - k)^+$ as a test function and after some computations, we get for any Ω^*, $k_0 > 0$, such that $\Omega_0 \subset \Omega^* \subset \Omega$ and $k > k_0$ and for some $\delta > 0$, that

$$\int_{A^\epsilon(k)} (u^\epsilon - k) \le C\|F_i^\epsilon\|_{L^p}|A^\epsilon(k)|^{1+\delta} \le \gamma |A^\epsilon(k)|^{1+\delta/2}$$

where $A^\epsilon(k) = \{x \in \Omega^* : \ u^\epsilon(x) > k\}$ and $\gamma = C|A^\epsilon(k_0)|^{\delta/2}$. We also get from here and [14], Lemma II.5.1 that

$$\sup_{\Omega^*} |u^\epsilon| \le k_0 + c(\gamma) \tag{4.2}$$

where $c(\gamma)$ is a monotonic continuous function of γ and $c(0) = 0$.

Now we show that the right hand side above can be made arbitrarily small by choosing the domain $\Omega^* \supset \Omega_0$, close enough to Ω_0 and by choosing ϵ small enough. If this is not the case then there exists a sequence of domains $\Omega^{*,n} \supset \Omega_0$ approaching Ω_0 in the sense that $\mathrm{dist}(\Gamma^{*,n}, \Gamma_0) \to 0$, where $\Gamma^{*,n} = \partial\Omega^{*,n}$, a sequence of $\epsilon_{n,k} \overset{k\to\infty}{\longrightarrow} 0$, and a positive number η such that

$$\|u^{\epsilon_{n,k}}\|_{L^\infty(\Omega^{*,n})} \ge \eta, \quad \text{for all } k, n \in I\!N \tag{4.3}$$

From the results in [16], we know that for fixed n, there exists a subsequence of $\epsilon_{n,k}$, that we denote by $\epsilon_{n,k}$ again so that $F_{\epsilon_{n,k}} \overset{k\to\infty}{\longrightarrow} F_n^*$ weakly in $L^p(\Omega^{*,n})$ and $u^{\epsilon_{n,k}} \to u^{*,n}$ in $H^1(\Omega^{*,n})$ and almost everywhere, where $u^{*,n} \in H_0^1(\Omega^{*,n})$ verifies

$$\begin{cases} -\mathrm{div}(a_0 \nabla u^{*,n}) = F_n^*, \quad \text{on} \quad \Omega^{*,n} \setminus \Omega_0 \\ u^{*,n} = 0, \quad \text{on} \quad \Gamma^{*,n}, \\ {u^{*,n}}_{|\Omega_0} = u^{*,n}_{\Omega_0} \in I\!R \\ \frac{1}{|\Omega_0|}\int_{\Gamma_0} a_0 \dfrac{\partial u^{*,n}}{\partial \vec{n}} = \frac{1}{|\Omega_0|}\int_{\Omega_0} F_n^* \end{cases}$$

Multiplying by $u^{*,n}$ and integrating on $\Omega^{*,n}$ we get $\int_{\Omega^{*,n}} a_0|\nabla u^{*,n}|^2 = \int_{\Omega^{*,n}} F_n^* u^{*,n}$. Therefore extending by zero to Ω and using Poincaré's inequality, we get that $\int_\Omega a_0|\nabla u^{*,n}|^2$ and $\int_\Omega |u^{*,n}|^2$ are bounded by a constant independent of n and so is $|u^{*,n}_{\Omega_0}|$.

By weak compactness, there exists a subsequence of $u^{*,n}$, that we denote again by $u^{*,n}$ so that $u^{*,n} \overset{n\to\infty}{\longrightarrow} u^*$, weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and almost everywhere. It is clear that u^* is constant on Ω_0 and $u^*_{|\Omega_0} = \lim_{n\to\infty} {u^{*,n}}_{|\Omega_0}$. Also, it is clear that on any compact set in $\Omega \setminus \Omega_0$, u^* must vanish and then we get $u^* = 0$ on $\Omega \setminus \Omega_0$. Consequently, $u^* = 0$ and we have proven that ${u^{*,n}}_{|\Omega_0} \overset{n\to\infty}{\longrightarrow} 0$.

Let n_0 be big enough so that $|u^{*,n}_{\Omega_0}| < \eta/8$ for any $n \ge n_0$. Let γ_η be small enough so that $c(\gamma) < \eta/4$ for any $0 < \gamma < \gamma_\eta$. Let us see that $|A^{\epsilon_{n,k}}(\eta/4)|$ can be made arbitrarily small. Notice first that

$$|A^{\epsilon_{n,k}}(\eta/4)| \le |\Omega^{*,n} \setminus \Omega_0| + |\{x \in \Omega_0, \ u^{\epsilon_{n,k}} > \eta/4\}|$$

and from the convegence of $u^{\epsilon_{n,k}} \stackrel{k\to\infty}{\longrightarrow} u^{*,n}$ in $H^1(\Omega_0)$ and the fact that $|u^{*,n}_{\Omega_0}| < \eta/8$, we obtain that there exists $k = k(n)$ such that the second term satisfies

$$|\{x \in \Omega_0,\ u^{\epsilon_{n,k}} - u^{*,n}_{\Omega_0} > \eta/4 - u^{*,n}_{\Omega_0}\}| \le |\{x \in \Omega_0,\ u^{\epsilon_{n,k}} - u^{*,n}_{\Omega_0} > \eta/8\}| \le 1/n,\ \forall\ k \ge k(n)$$

In particular we have shown that there exists an integer n_0 such that for every $n \ge n_0$ there exists a $k(n)$ so that $|A^{\epsilon_{n,k}}(\eta/4)| \le |\Omega^{*,n} \setminus \Omega_0| + 1/n$ for any $k \ge k(n)$ and $|\Omega^{*,n} \setminus \Omega_0| + 1/n \stackrel{n\to\infty}{\longrightarrow} 0$. Choose now $n_1 \ge n_0$ big enoug so that $C|A^{\epsilon n,k}(\eta/4)|^{\delta/2} \le \gamma_\eta$ for any $n \ge n_1$ and $k \ge k(n)$. Therefore, from (4.2) we have that $\sup_{\Omega^{*,n}} |u^{\epsilon_{n,k}}| \le \eta/2$ for all $n \ge n_1$ and all $k \ge n(k)$, which contradicts the existence of η for which (4.3) holds. This proves the lemma.□

With this proposition we can improve a result from [16] on the convergence of eigenfunctions of the linear operators. We denote by $\{\phi^\epsilon_n\}_{n=1}^\infty$, $0 < \epsilon \le \epsilon_0$, an orthonormal set of eigenfunctions of the problem

$$\begin{array}{l} -\mathrm{div}(a_\epsilon(x)\nabla u) + c(x)u = \lambda u, \quad \text{in } \Omega \\ a(x)\dfrac{\partial u}{\partial n} + b(x)u = 0, \quad \text{in } \Gamma \end{array} \tag{4.4}$$

and consider the limiting eigenvalue problem

$$\left\{ \begin{array}{ll} -\mathrm{div}(a_0(x)\nabla u) + c(x)u = \lambda u & \text{on } \Omega_1 \\ a_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u = 0 & \text{on } \Gamma \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}} & \text{on } \Gamma_{0,i}, \quad i = 1, \dots, m \\ \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} a_0 \frac{\partial u}{\partial \vec{n}} + \hat{c}_i u_{\Omega_{0,i}} = \lambda u_{\Omega_{0,i}}, & i = 1, \dots, m \end{array} \right. \tag{4.5}$$

where $\hat{c}_i = |\Omega_{0,i}|^{-1} \int_{\Omega_{0,i}} c(x)dx$. We know from [16] that the following result holds,

Proposition 4.4.3. *We have the following*
i) If $\{\lambda^\epsilon_n\}_{n=1}^\infty$ *and* $\{\lambda^0_n\}_{n=1}^\infty$ *are the eigenvalues of (4.4) and (4.5) respectively, then for fixed* $n \in I\!N$, $\lambda^\epsilon_n \stackrel{\epsilon\to 0}{\longrightarrow} \lambda^0_n$.
ii) For any sequence $\epsilon_k \stackrel{k\to\infty}{\longrightarrow} 0$ *there exists a subsequence, that we denote again by* ϵ_k*, and an orthonormal family of eigenfunctions of (4.5), denoted by* $\{\phi^0_n\}_{n=1}^\infty$*, such that* $\phi^{\epsilon_k}_n \stackrel{k\to\infty}{\longrightarrow} \phi^0_n$ *in* $H^1(\Omega)$.

With Proposition 4.4.1 we can actually prove

Proposition 4.4.4. *For each* $n \in N$ *and* $\epsilon \in [0, \epsilon_0]$, $\phi^\epsilon_n \in C^0(\bar\Omega)$. *Moreover, the convergence of Proposition 4.4.3 ii) can be improved to* $\phi^{\epsilon_k}_n \stackrel{k\to\infty}{\longrightarrow} \phi^0_n$ *in* $C^0(\bar\Omega)$.

Proof: A bootstrap argument like the ones from [4], Appendix B, applied to (4.4) will show that $\phi^\epsilon_n \in L^\infty(\Omega)$, $\epsilon \in (0, \epsilon_0]$ and that for each $n \in N$ there exists a positive

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constant K_n such that $\|\phi_n^\epsilon\|_{L^\infty(\Omega)} \leq K_n$. Applying now Proposition 4.4.1 we prove this Proposition.□

Observe that this result is an improvement of the H^1-convergence obtained in [16].

We can now obtain important comparison results for the limit problem. The space $L^2_{\Omega_0}(\Omega)$ has a natural order relation, the restriction of the one from $L^2(\Omega)$. Therefore, if $\mu > \lambda_1^0$, the first eigenvalue of A_0, we have that $A_0 + \mu I$ is a positive operator, that is: if $f, \tilde{f} \in L^2_{\Omega_0}(\Omega)$, $g, \tilde{g} \in L^2(\Gamma)$, and if $f \geq \tilde{f}$ and $g \geq \tilde{g}$, $A_0 u + \mu u = f_\Omega + g_\Gamma$ and $A_0\tilde{u} + \mu\tilde{u} = \tilde{f}_\Omega + \tilde{g}_\Gamma$ then $u \geq \tilde{u}$. To see this, we observe that if u^ϵ and $\tilde{u}^\epsilon$ are the solutions of $A_\epsilon u^\epsilon + \mu u^\epsilon = f_\Omega + g_\Gamma$ and $A_\epsilon \tilde{u}^\epsilon + \mu\tilde{u}^\epsilon = \tilde{f}_\Omega + \tilde{g}_\Gamma$ we have, from the comparison results applied to the operators A_ϵ, that $u^\epsilon \geq \tilde{u}^\epsilon$, see [4]. Passing to the limit as $\epsilon \to 0$ and using Proposition 4.4.1 i) we obtain that $u \geq \tilde{u}$.

The fact that the operator A_0 is positive allows us to apply all abstract comparison results from [4] Appendix A. In particular, if we denote by $u(t, u_0, f, g)$ the solution of

$$\begin{cases} u_t - \operatorname{div}(a_0(x)\nabla u) + c(x)u = f(t,u) & \text{on } \Omega_1 \\ a_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u = g(t,u) & \text{on } \Gamma \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}} & \text{on } \Gamma_{0,i}, \quad i = 1, \dots, m \\ \dot{u}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} a_0(x)\frac{\partial u}{\partial \vec{n}} + \hat{c}_i u_{\Omega_{0,i}} = f(t, u_{\Omega_{0,i}}), & i = 1, \dots, m \\ u(0) = u_0 \end{cases} \tag{4.6}$$

where $\hat{c}_i = |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} c(x)dx$, we have

Lemma 4.4.5.

i) Assume $f(t,0) \geq 0$ and $g(t,0) \geq 0$ for every $t > 0$. Then $u_0 \geq 0$ implies $u(t, u_0, f, g) \geq 0$ as long as the solution exists.
ii) If $u_1 \geq u_0$ then $u(t, u_1, f, g) \geq u(t, u_0, f, g)$ as long as the solution exist.
iii) If $f_1(t,u) \geq f_0(t,u)$, $g_1(t,u) \geq g_0(t,u)$ and $u_1 \geq u_0$ then $u(t, u_1, f_1, g_1) \geq u(t, u_0, f_0, g_0)$ as long as the solution exist.

Proof: We apply the results from Appendix A of [4] to the the problem above, which can be written as $u_t + A_0 u = f_\Omega(t,u) + g_\Gamma(t,u)$. □

We analize now the behavior of the linear semigroups.

We define the projection $P : L^2(\Omega) \to L^2_{\Omega_0}(\Omega)$ given by $Pf = |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} f(x)dx$ on $\Omega_{0,i}$ and $Pf = f$ on Ω_1. We have the following,

Proposition 4.4.6. *Assume $\lambda_1^0 > 0$. Then, for any $\gamma \in [0,1)$ there exists an $\alpha \in ((1+\gamma)/2, 1)$ and a function $c(\epsilon) \geq 0$ with $c(\epsilon) \xrightarrow{\epsilon \to 0} 0$, such that for any $h \in H^{-\gamma}(\Omega) \equiv (H^\gamma(\Omega))'$*

$$\|e^{-A_\epsilon t}h - e^{-A_0 t}h^*\|_{H^1(\Omega)} \leq c(\epsilon)t^{-\alpha}\|h\|_{H^{-\gamma}(\Omega)}, \qquad t > 0 \tag{4.7}$$

where we denote by $h^* \in (H^\gamma_{\Omega_0})'$ the restriction of h to the space $H^\gamma_{\Omega_0}$.

REMARK 4.4.7. *Notice that if $f \in L^2(\Omega)$ is identified with the linear map $f : H^\gamma(\Omega) \to I\!R$ by $f(\phi) = \int_\Omega f\phi$, then the restriction of f to $H^\gamma_{\Omega_0}(\Omega)$ denoted by f^* is identified with the element $Pf \in L^2_{\Omega_0}(\Omega)$.*

Proof: Notice first that if we denote by X^γ_ϵ the fractional power spaces associated to the operators A_ϵ, then $X^0_\epsilon \equiv L^2(\Omega)$, with the same norm, and that from (1.2) $X^{1/2}_\epsilon \hookrightarrow H^1(\Omega)$, with a uniform constant embedding. That is, there exist a constant C such that $\|\phi\|_{H^1(\Omega)} \leq C\|\phi\|_{X^{1/2}_\epsilon}$ for any $0 < \epsilon \leq \epsilon_0$. By interpolation we have that $X^\gamma_\epsilon \hookrightarrow H^{2\gamma}(\Omega)$ for $0 < \gamma \leq 1/2$ with a uniform constant embedding and by a duality argument we have that $H^{-2\gamma}(\Omega) \equiv (H^{2\gamma}(\Omega))' \hookrightarrow X^{-\gamma}_\epsilon$ for $0 < \gamma \leq 1/2$ with a uniform constant embedding.

To prove (4.7) it is sufficient to prove if for sequences, that is, to prove that for each sequence $\{\epsilon_k\}$ with $\epsilon_k \stackrel{k\to\infty}{\longrightarrow} 0$, there exists a subsequence, denoted again by ϵ_k and a $\delta_k \stackrel{k\to\infty}{\longrightarrow} 0$ such that

$$\|e^{-A_{\epsilon_k}t}h - e^{-A_0 t}h^*\|_{H^1(\Omega)} \leq \delta_k t^{-\alpha}\|h\|_{H^{-\gamma}(\Omega)}, \qquad t > 0 \tag{4.8}$$

We denote by $\{\lambda^\epsilon_n, \phi^\epsilon_n\}$ a set of eigenvalues and eigenfunctions of the operator A_ϵ for $0 \leq \epsilon \leq \epsilon_0$ and recall from Proposition 4.4.3 ii) that for each sequence $\epsilon_k \stackrel{k\to\infty}{\longrightarrow} 0$, there exists a subsequence, denoted again by ϵ_k and an orthonormal family of eigenfunctions of (4.5), denoted by $\{\phi^0_n\}_{n=1}^\infty$, such that $\phi^{\epsilon_k}_n \stackrel{k\to\infty}{\longrightarrow} \phi^0_n$ in $H^1(\Omega)$.

If the function $h \in C^\infty(\bar\Omega) \subset H^{-2\gamma}(\Omega)$, then from the spectral decomposition we have $h = \sum_{n=1}^\infty (h, \phi^{\epsilon_k}_n)\phi^{\epsilon_k}_n$ where we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$. We know that $e^{-A_{\epsilon_k}t}h = \sum_{n=1}^\infty (h, \phi^{\epsilon_k}_n)e^{-\lambda^{\epsilon_k}_n t}\phi^{\epsilon_k}_n$. Similarly we have that $Ph = \sum_{n=1}^\infty (Ph, \phi^0_n)\phi^0_n = \sum_{n=1}^\infty (h, \phi^0_n)\phi^0_n$ and $e^{-A_0 t}Ph = \sum_{n=1}^\infty (Ph, \phi^0_n)e^{-\lambda^0_n t}\phi^0_n = \sum_{n=1}^\infty (h, \phi^0_n)e^{-\lambda^0_n t}\phi^0_n$ where we have used that Ph and ϕ^0_n are constant on $\Omega_{0,i}$.

Let $\delta > 0$ be a parameter and let us consider two different cases

i) Assume $0 < t \leq \delta$. For this case, we easily check that

$$\begin{aligned}
\|e^{-A_{\epsilon_k}t}h - e^{-A_0 t}Ph\|_{H^1(\Omega)} &\leq C\|e^{-A_{\epsilon_k}t}h\|_{X^{1/2}_{\epsilon_k}} + C\|e^{-A_0 t}Ph\|_{X^{1/2}_0} \\
&\leq Ct^{-1/2-\gamma/2}\|h\|_{X^{-\gamma/2}_{\epsilon_k}} + Ct^{-1/2-\gamma/2}\|Ph\|_{X^{-\gamma/2}_0} \\
&\leq Ct^{-(1+\gamma)/2}(\|h\|_{H^{-\gamma}(\Omega)} + \|Ph\|_{H^{-\gamma}(\Omega)}) \\
&\leq 2Ct^{-(1+\gamma)/2}\|h\|_{H^{-\gamma}(\Omega)} \leq 2Ct^{\alpha-(1+\gamma)/2}t^{-\alpha}\|h\|_{H^{-\gamma}(\Omega)} \\
&\leq 2C\delta^\mu t^{-\alpha}\|h\|_{H^{-\gamma}(\Omega)}
\end{aligned}$$

where $\mu = \alpha - (1+\gamma)/2 > 0$.

ii) Assume $t > \delta$. Let $\beta \in (0,1)$ be a fixed number. Since we have $\lambda_n^{\epsilon_k} \stackrel{\epsilon\to 0}{\longrightarrow} \lambda_n^0$ and $\lambda_n^0 \stackrel{n\to\infty}{\longrightarrow} +\infty$, there exists $N(\delta), k_1(\delta) \in I\!N$, such that $\lambda_n^{\epsilon_k} e^{-\lambda_n^{\epsilon_k} t} \le \delta t^{-\beta}$ for all $n \ge N(\delta)$, $k \ge k_1(\delta)$ and $t > \delta$. Without loss of generality we can assume that we have $\lambda^0_{N(\delta)} < \lambda^0_{N(\delta)+1}$. Hence from the spectral decompositions of the linear semigroups, we obtain

$$\|e^{-A_{\epsilon_k} t} h - e^{-A_0 t} Ph\|_{H^1(\Omega)} \le \| \sum_{n=1}^{N(\delta)} e^{-\lambda_n^{\epsilon_k} t}(h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k} - \sum_{n=1}^{N(\delta)} e^{-\lambda_n^0 t}(Ph, \phi_n^0)\phi_n^0\|_{H^1(\Omega)}$$

$$+\| \sum_{N(\delta)+1}^{\infty} e^{-\lambda_n^{\epsilon_k} t}(h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k}\|_{H^1(\Omega)} + \| \sum_{N(\delta)+1}^{\infty} e^{-\lambda_n^0 t}(Ph, \phi_n^0)\phi_n^0\|_{H^1(\Omega)}$$

but

$$\| \sum_{N(\delta)+1}^{\infty} e^{-\lambda_n^{\epsilon_k} t}(h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k}\|_{H^1(\Omega)} \le C\| \sum_{N(\delta)+1}^{\infty} e^{-\lambda_n^{\epsilon_k} t}(h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k}\|_{X^{1/2}_{\epsilon_k}}$$

$$= C[\sum_{N(\delta)+1}^{\infty} \lambda_n^{\epsilon_k} e^{-2\lambda_n^{\epsilon_k} t}(h, \phi_n^{\epsilon_k})^2]^{1/2} \le C\delta t^{-\beta}(\sum_{N(\delta)+1}^{\infty} \frac{1}{\lambda_n^{\epsilon_k}}|(h, \phi_n^{\epsilon_k})|^2)^{1/2}$$
$$\le C\delta t^{-\beta}\|h\|_{H^{-1}(\Omega)}$$

and with a similar argument

$$\| \sum_{N(\delta)+1}^{\infty} e^{-\lambda_n^0 t}(Ph, \phi_n^0)\phi_n^0\|_{H^1(\Omega)} \le C\delta t^{-\beta}\|Ph\|_{H^{-1}(\Omega)}.$$

Moreover,

$$\| \sum_{n=1}^{N(\delta)} e^{-\lambda_n^{\epsilon_k} t}(h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k} - \sum_{n=1}^{N(\delta)} e^{-\lambda_n^0 t}(Ph, \phi_n^0)\phi_n^0\|_{H^1(\Omega)}$$

$$\le \| \sum_{n=1}^{N(\delta)} (e^{-\lambda_n^{\epsilon_k} t} - e^{-\lambda_n^0 t})(h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k}\|_{H^1(\Omega)} + \| \sum_{n=1}^{N(\delta)} e^{-\lambda_n^0 t}((h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k} - (Ph, \phi_n^0)\phi_n^0)\|_{H^1(\Omega)}$$

$$\le \sum_{n=1}^{N(\delta)} \lambda_n^{\epsilon_k}|e^{-\lambda_n^{\epsilon_k} t} - e^{-\lambda_n^0 t}|\|h\|_{H^{-1}(\Omega)} + \sum_{n=1}^{N(\delta)} e^{-\lambda_n^0 t}\|(h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k} - (h, \phi_n^0)\phi_n^0)\|_{H^1(\Omega)}$$

For the first sum we have that for $n \in I\!N$ and for ϵ small enough such that $\lambda_n^\epsilon \ge \lambda_1^0/2$, we have that for any t, $|e^{-\lambda_n^\epsilon t} - e^{-\lambda_n^0 t}| \le |\lambda_n^\epsilon - \lambda_n^0| t e^{-t\lambda_1^0/2}$. This implies that

$$\sum_{n=1}^{N(\delta)} \lambda_n^{\epsilon_k} |e^{-\lambda_n^{\epsilon_k} t} - e^{-\lambda_n^0 t}| \, \|h\|_{H^{-1}(\Omega)} \leq |\lambda_{N(\delta)}^{\epsilon_k}| \, |\lambda_n^{\epsilon} - \lambda_n^0| t e^{-t\lambda_1^0/2} \|h\|_{H^{-1}(\Omega)} \leq \delta t^{-\beta} \|h\|_{H^{-1}(\Omega)}$$

for all $k \geq k_2(\delta) \geq k_1(\delta)$.

For the second sum we have,

$$\sum_{n=1}^{N(\delta)} e^{-\lambda_n^0 t} \|(h, \phi_n^{\epsilon_k})\phi_n^{\epsilon_k} - (h, \phi_n^0)\phi_n^0)\|_{H^1(\Omega)}$$

$$\leq e^{-\lambda_1^0 t} \sum_{n=1}^{N(\delta)} \|(h, \phi_n^{\epsilon_k} - \phi_n^0)\phi_n^{\epsilon_k}\|_{H^1(\Omega)} + \|(h, \phi_n^0)(\phi_n^{\epsilon_k} - \phi_n^0)\|_{H^1(\Omega)}$$

$$\leq e^{-\lambda_1^0 t} \sum_{i=1}^{N(\delta)} \|\phi_n^{\epsilon_k} - \phi_n^0\|_{H^1(\Omega)} (\|\phi_n^{\epsilon_k}\|_{H^1(\Omega)} + \|\phi_n^0\|_{H^1(\Omega)}) \|h\|_{H^{-1}(\Omega)} \leq \delta t^{-\beta} \|h\|_{H^{-1}(\Omega)}$$

for all $k \geq k_3(\delta) \geq k_2(\delta)$

In particular we have proved that there exists a constant C such that for an $\beta \in (0,1)$ and a given δ arbitrarily small there exists a k_3 satisfying

$$\|e^{-A_{\epsilon_k} t} h - e^{-A_0 t} P h\|_{H^1(\Omega)} \leq C \delta t^{-\beta} \|h\|_{H^{-1}(\Omega)}, \quad k \geq k_3, \quad t > \delta$$

Using the continuity of the embedding $H^{-\gamma}(\Omega) \hookrightarrow H^{-1}(\Omega)$ and putting together i) and ii) we prove the result for $h \in C^\infty(\bar{\Omega})$. A density argument completes now the proof for $h \in H^{-\gamma}(\Omega)$.$\square$

In particular we obtain the following

Corollary 4.4.8. *There exists an $\alpha \in (3/4, 1)$ and a function $c(\cdot) > 0$ with $c(\epsilon) \stackrel{\epsilon \to 0}{\longrightarrow} 0$ such that for any $f \in L^2(\Omega)$ and any $g \in L^2(\Gamma)$, we have*

$$\|e^{-A_\epsilon t}(f_\Omega + g_\Gamma) - e^{-A_0 t}(f_\Omega^* + g_\Gamma)\|_{H^1(\Omega)} \leq c(\epsilon) t^{-\alpha} (\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)}), \qquad t > 0$$

Once we have established the convergence result given by Proposition 4.4.6 and Corollary 4.4.8 it is clear that uniform estimates on the semigroup $e^{-A_\epsilon t}$ can be transformed into estimates on the semigroup $e^{-A_0 t}$. In particular we can prove,

Corollary 4.4.9. *There exist $k > 0$ and $M > 0$ such that for any $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, we have*

$$\|e^{-A_0 t}(f_\Omega^* + g_\Gamma)\|_{L^\infty(\Omega)} \leq M t^{-k} (\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)})$$

Proof: We know from [4], Lemma 4.4, applied to A_ϵ that there exists $\tilde{M}$ and $\tilde{N}$, independent of ϵ, such that $\|e^{-A_\epsilon t}\psi\|_{L^\infty(\Omega)} \leq \tilde{M}t^{-\tilde{N}}\|\psi\|_{L^2(\Omega)}$. Passing to the limit as $\epsilon \to 0$ and using Corollary 4.4.8 we have that $\|e^{-A_0 t}\psi^*\|_{L^\infty(\Omega)} \leq \tilde{M}t^{-\tilde{N}}\|\psi\|_{L^2(\Omega)}$. We also have that $\|e^{-A_0 t}h\|_{L^2(\Omega)} = \|e^{-A_0 t}h\|_{X_0^0} \leq Ct^{-\gamma}\|h\|_{X_0^{-\gamma}}$.

Writting $e^{-A_0 t}$ as $e^{-A_0 t/2} \circ e^{-A_0 t/2}$, and using the continuous embeddings $L^2(\Omega) + L^2(\Gamma) \hookrightarrow (H^{2\gamma}_{\Omega_0})' \hookrightarrow X_0^{-\gamma}$, for $\gamma > 1/4$, we obtain $\|e^{-A_0 t}(f^*_\Omega + g_\Gamma)\|_{L^\infty(\Omega)} \leq \tilde{M}(t/2)^{-\tilde{N}}$ $\|e^{-A_0 t/2}(f^*_\Omega + g_\Gamma)\|_{L^2(\Omega)} \leq \tilde{M}(t/2)^{-\tilde{N}}Ct^{-\gamma/2}\|f^*_\Omega + g_\Gamma\|_{X_0^{-\gamma}} \leq Mt^{-k}(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma)})$ which proves the corollary. □

5. UPPER SEMICONTINUITY OF ATTRACTORS

In this section we will compare the asymptotic dynamics of (1.3) and (1.4) in the metrics of $H^1(\Omega)$ and $C^0(\bar{\Omega})$. From now on we will denote by $\mathcal{A}_\epsilon$ the global attractor of (1.3) which, by the results of Section 2 are uniformly bounded in $H^1(\Omega)$ and $C^0(\bar{\Omega})$. Since we will be dealing with solutions lying on $\mathcal{A}_\epsilon$ we will make no further reference to the space X in the class $\mathcal{E}$ where (1.3) was initially set. Also, recall that after Remark 2.2.6 the nonlinear terms are assumed to be globally Lipschitz.

Before establishing any relation between the asymptotic dynamics of both problems we will prove the existence and certain regularity of the attractor of the limiting problem. We have,

Proposition 5.5.1. *Assume that the nonlinearities $f, g \in C^2(I\!R, I\!R)$ are globally Lipschitz functions and that condition $(\mathbf{D})_\mathbf{0}$ is satisfied. Then (1.4) is well-posed in $H^1_{\Omega_0}(\Omega)$ and has an attractor $\mathcal{A}_0$. Moreover, $\mathcal{A}_0$ attracts bounded sets of $L^2_{\Omega_0}(\Omega)$ in the topology of $H^1(\Omega)$ and it lies in a bounded subset of $C^\alpha(\bar{\Omega})$ for some positive α.*

Proof: The fact that (1.4) is well posed in $H^1_{\Omega_0}(\Omega)$ and that we have global existence of solution has been already established in Proposition 3.3.1. Also from Proposition 3.3.1, if we denote by $T_0(t) : H^1_{\Omega_0}(\Omega) \to H^1_{\Omega_0}(\Omega)$ the nonlinear semigroup generated by (1.4), we have that if B is a bounded set of $H^1_{\Omega_0}(\Omega)$ then $T_0(t)B$ is a bounded set of X_0^γ for any $\gamma < 3/4$ which is compactly embedded into $X_0^{1/2} = H^1_{\Omega_0}(\Omega)$ from where it follows that $T_0(t)$ is a compact map for any $t > 0$. The fact that $T_0(t)$ is point dissipative follows by standard arguments: multiplying the equation by u, integrating by parts and using the dissipation condition $(\mathbf{D})_\mathbf{0}$ it can be proved the existence of an R so that all orbits enter eventually the ball of radious R in $H^1_{\Omega_0}(\Omega)$. The existence of the global attractor $\mathcal{A}_0$ can be obtained by Theorem 3.4.6 from [9].

If B_0 is a bounded set of $L^2_{\Omega_0}(\Omega)$ then from Proposition 3.3.1 we have that for $t > 0$ fixed there exists B_1, a bounded set of $H^1_{\Omega_0}(\Omega)$, such that $u(t, u_0) \in B_1$ for any $u_0 \in B_0$. Since now B_1 is attracted by $\mathcal{A}_0$ in H^1 then B_0 is also attracted by $\mathcal{A}_0$ in $H^1(\Omega)$.

The fact that the attractor $\mathcal{A}_0$ lies in a bounded set of $L^\infty(\Omega)$ follows from the comparison results obtained in Lemma 4.4.5, Corollary 4.4.9 and hypothesis $(\mathbf{D})_\mathbf{0}$. Since for $u_0 \in H^1_{\Omega_0}(\Omega)$ we have that $|T_0(t)u_0| \leq U(t, |u_0|)$ where $U(\cdot, |u_0|)$ is the solution of

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$$\begin{cases} U_t - \operatorname{div}(a_0(x)\nabla U) + (c(x)+C_0)U = C_1 & \text{on } \Omega_1 \\ a_0(x)\dfrac{\partial U}{\partial \vec{n}_0} + (b(x)+B_0)U = B_1 & \text{on } \Gamma \\ U_{|\Omega_{0,i}} =: U_{\Omega_{0,i}} & \text{on } \Omega_{0,i}, \quad i=1,\dots,m \\ \dot{U}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} a_0\frac{\partial U}{\partial \vec{n}} + (\hat{c}_i + C_0)U_{\Omega_{0,i}} = C_1, & i=1,\dots,m \\ U(0) = |u_0| \end{cases} \tag{5.1}$$

where $\hat{c}_i = |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} c(x)dx$. That is, $U_t + \tilde{A}_0 U = (C_1)_\Omega + (B_0)_\Gamma$, where $\tilde{A}_0$ is the appropriate linear operator. If we denote now by V the solution of $\tilde{A}_0 V = (C_1)_\Omega + (B_0)_\Gamma$ we have that $V \geq 0$ and $W = U - V = e^{-\tilde{A}_0 t}(|U_0| - V)$. Applying Corollary 4.4.9 we have that $\|U(t)-V\|_{L^\infty(\Omega)} \leq Mt^{-k}(\|u_0\|_{L^2(\Omega)} + \|V\|_{L^2(\Omega)})$ which implies, since from Proposition 4.4.1 we know that $V \in L^\infty(\Omega)$, that $U(t) \in L^\infty(\Omega)$, and moreover, $\|U(t)\|_{L^\infty(\Omega)} \leq \|V\|_{L^\infty(\Omega)} + Mt^{-k}(\|U_0\|_{L^2(\Omega)} + \|V\|_{L^2(\Omega)})$. This in turn implies that the attractor $\mathcal{A}_0$ lies in the bounded set of $L^\infty(\Omega)$ given by $B_\infty = \{\phi : \|\phi\|_{L^\infty(\Omega)} \leq \|V\|_{L^\infty(\Omega)}\}$ (as a matter of fact $\mathcal{A}_0$ lies in the bounded set of $L^\infty(\Omega)$ given by $\{\phi : |\phi(x)| \leq |V(x)|\}$).

Moreover, if $u(\cdot)$ is an orbit in the attractor $\mathcal{A}_0$, then we know that it is defined for all $t \in I\!R$ and that $u, u_t \in C(I\!R, H^1_{\Omega_0}(\Omega))$, that is $u \in C^1(I\!R, H^1_{\Omega_0}(\Omega))$. Multiplying equation (1.4) by u_t, integrating by parts and in time for $t \in (0,1)$, and using that the attractor is a bounded set of $H^1(\Omega)$ and $L^\infty(\Omega)$, we can show that for any orbit $u(t)$ in the attractor $\mathcal{A}_0$, we have that $\int_0^1 \|u_t(s)\|^2_{L^2(\Omega)}ds \leq C$ for some constant C independent of the orbit on the attractor.

Now, if we denote by $v(t) = u_t(t)$ then, v satisfies the variational equation around $u(t)$, given by

$$\begin{cases} v_t - \operatorname{div}(a_0(x)\nabla v) + c(x)v = f'(u(t))v & \text{on } \Omega_1 \\ a_0\dfrac{\partial v}{\partial \vec{n}} + b(x)v = g'(u(t))v & \text{on } \Gamma \\ \gamma_{0,i}(v) = v_{\Omega_{0,i}} & \text{on } \Gamma_{0,i}, \quad i=1,\dots,m \\ \dot{v}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} a_0\frac{\partial v}{\partial \vec{n}} + \hat{c}_i v_{\Omega_{0,i}} = f'(u_{\Omega_{0,i}})v_{\Omega_{0,i}}, & i=1,\dots,m \end{cases} \tag{5.2}$$

where $\hat{c}_i = |\Omega_{0,i}|^{-1}\int_{\Omega_{0,i}} c(x)dx$.

By the comparison results of Lemma 4.4.5, denoting by $D_0 = \sup\{\|f'(\phi)\|_{L^\infty(\Omega)} : \phi \in \mathcal{A}_0\}$ and $E_0 = \sup\{\|g'(\phi)\|_{L^\infty(\Omega)} : \phi \in \mathcal{A}_0\}$ then $|v| \leq w$ where w is the solution of the linear problem

$$\begin{cases} w_t - \operatorname{div}(a_0(x)\nabla w) + c(x)w = D_0 w & \text{on } \Omega_1 \\ a_0(x)\dfrac{\partial w}{\partial \vec{n}} + b(x)w = E_0 w & \text{on } \Gamma \\ \gamma_{0,i}(w) = w_{\Omega_{0,i}} & \text{on } \Gamma_{0,i}, \quad i=1,\dots,m \\ \dot{w}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} a_0(x)\frac{\partial w}{\partial \vec{n}} + \hat{c}_i w_{\Omega_{0,i}} = D_0 w_{\Omega_{0,i}}, & i=1,\dots,m \\ w(0) = |v(0)| \end{cases} \tag{5.3}$$

which by Corollary 4.4.9 we have that $w \in L^\infty$ and $\|w(2)\|_{L^\infty(\Omega)} \leq M_0 \|v(t)\|_{L^2(\Omega)}$ for any $t \in (0,1)$, which implies that $\|v(2)\|^2_{L^\infty(\Omega)} \leq M_0^2 \int_0^1 \|v(s)\|^2_{L^2(\Omega)} ds \leq M_0^2 C$. By the invariance of the attractor we obtain that there exists a constant M_2, such that $\|u_t(t)\|_{L^\infty(\Omega)} \leq M_2$ for any orbit $u(\cdot)$ of the attractor.

Now, for t fixed, we can rewrite equation (3.1) as an elliptic equation in Ω_1, as

$$\begin{cases} -\mathrm{div}(a_0(x)\nabla u) + c(x)u = f(u) - u_t & \text{on } \Omega_1 \\ a_0(x)\dfrac{\partial u}{\partial \vec{n}} + b(x)u = g(u) & \text{on } \Gamma \\ \gamma_{0,i}(u) = u_{\Omega_{0,i}} & \text{on } \Gamma_{0,i}, \quad i = 1, \dots, m \end{cases} \tag{5.4}$$

Since $f(u) - u_t \in L^\infty(\Omega)$, $g(u) \in L^\infty(\Gamma)$, and $u_{\Omega_{0,i}} \in L^\infty(\Gamma_{0,i})$, with uniform bounds for $t \in I\!R$ and $u(\cdot)$ on the attractor, by the results of [4], see Lemma B.1, we obtain that u lies in a bounded set of $C^\alpha(\bar{\Omega}_1)$. Since u is constant in Ω_0 we obtain that u lies in a bounded set of $C^\alpha(\bar{\Omega})$. □

5.1. Upper semicontinuity in $H^1(\Omega)$

In this subsection we compare the asymptotic dynamics of (1.3) and (1.4) in the metric of $H^1(\Omega)$ by proving the following result

THEOREM 5.5.2. *The global attractors, $\mathcal{A}_\epsilon$ and $\mathcal{A}_0$, are upper semicontinuous at $\epsilon = 0$ in $H^1(\bar{\Omega})$.*

Proof: Notice first that we have obtained uniform estimates in $H^1(\Omega) \cap C^0(\bar{\Omega})$ on the attractors of (1.3) and (1.4). Therefore, we can assume the existence of a constant K_0 such that

$$\sup_{\phi_\epsilon \in \mathcal{A}_\epsilon} (\|\phi_\epsilon\|_{H^1(\Omega)} + \|\phi_\epsilon\|_{L^\infty(\Omega)} + \|f(\phi_\epsilon)\|_{L^\infty(\Omega)} + \|g(\phi_\epsilon)\|_{L^\infty(\Gamma)}) \leq K_0, \quad 0 \leq \epsilon \leq \epsilon_0$$

Denote by T_ϵ and T_0 the nonlinear semigroup associated to (1.3) and (1.4) respectively. We have from the variations of constants formula and for $\phi_\epsilon \in \mathcal{A}_\epsilon$, that

$$T_\epsilon(t, \phi_\epsilon) = e^{-A_\epsilon t}\phi_\epsilon + \int_0^t e^{-A_\epsilon(t-s)}(f_\Omega(T_\epsilon(s, \phi_\epsilon)) + g_\Gamma(T_\epsilon(s, \phi_\epsilon)))ds$$

$$T_0(t, P\phi_\epsilon) = e^{-A_0 t}P\phi_\epsilon + \int_0^t e^{-A_0(t-s)}(f_\Omega(T_0(s, P\phi_\epsilon)) + g_\Gamma(T_0(s, P\phi_\epsilon)))ds$$

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In particular, taking into account Proposition 4.4.6 and Corollary 4.4.8 we have, for $t \in (0,\tau)$,

$$\|T_\epsilon(t,\phi_\epsilon) - T_0(t,P\phi_\epsilon)\|_{H^1(\Omega)} \le \|e^{-A_\epsilon t}\phi_\epsilon - e^{-A_0 t}P\phi_\epsilon\|_{H^1} +$$

$$+\int_0^t \|e^{-A_\epsilon(t-s)}(f_\Omega(T_\epsilon(s,\phi_\epsilon)) + g_\Gamma(T_\epsilon(s,\phi_\epsilon)) -$$

$$-e^{-A_0(t-s)}(Pf_\Omega(T_\epsilon(s,\phi_\epsilon)) + g_\Gamma(T_\epsilon(s,\phi_\epsilon)))\|_{H^1} ds$$

$$+\int_0^t \|e^{-A_0(t-s)}(Pf_\Omega(T_\epsilon(s,\phi_\epsilon)) + g_\Gamma(T_\epsilon(s,\phi_\epsilon)) - f_\Omega(T_0(s,P\phi_\epsilon)) - g_\Gamma(T_0(s,P\phi_\epsilon)))\|_{H^1} ds$$

$$\le c(\epsilon)t^{-\alpha}K_0 + \int_0^t c(\epsilon)(t-s)^{-\alpha}K_0 ds + \int_0^t (t-s)^{-\beta} L\|T_\epsilon(t,\phi_\epsilon) - T_0(t,P\phi_\epsilon)\|_{H^1} ds$$

$$\le c(\epsilon)K_0 \tfrac{\tau}{1-\alpha} t^{-\alpha} + L\int_0^t (t-s)^{-\beta}\|T_\epsilon(t,\phi_\epsilon) - T_0(t,P\phi_\epsilon)\|_{H^1} ds$$

where $\alpha \in (3/4,1)$ is given by Corollary 4.4.8 and $\beta \in (0,1)$. By the singular Gronwall lemma, see [13], we obtain that there exists a constant $M = M(\alpha,\beta,L,\tau)$ and a positive function $c(\cdot)$ with $c(\epsilon) \stackrel{\epsilon\to 0}{\longrightarrow} 0$, such that

$$\|T_\epsilon(t,\phi_\epsilon) - T_0(t,P\phi_\epsilon)\|_{H^1(\Omega)} \le Mc(\epsilon)K_0 t^{-\alpha}, \qquad t \in (0,\tau), \quad \phi_\epsilon \in \mathcal{A}_\epsilon \tag{5.5}$$

Notice now that if $\delta > 0$ is fixed, there exists a $\tau = \tau(\delta)$ such that $\mathrm{dist}_{H^1}(T_0(\tau,P\phi_\epsilon),\mathcal{A}_0) \le \delta/2$, for all $\phi_\epsilon \in \mathcal{A}_\epsilon$ and for all $\epsilon \in (0,\epsilon_0)$. This is so since $\cup_\epsilon \mathcal{A}_\epsilon$ lies in a bounded set of $L^2(\Omega)$ and therefore $\cup_\epsilon P\mathcal{A}_\epsilon$ lies in a bounded set of $L^2_{\Omega_0}(\Omega)$ and from Proposition 5.5.1 $\mathcal{A}_0$ attracts bounded sets of $L^2_{\Omega_0}(\Omega)$. Moreover, since the attractors are invariant, we have that for any $v_\epsilon \in \mathcal{A}_\epsilon$ there exist $\phi_\epsilon \in \mathcal{A}_\epsilon$ with $T_\epsilon(\tau,\phi_\epsilon) = v_\epsilon$ and therefore if we choose $\epsilon_1 \in (0,\epsilon_0)$ so that $Mc(\epsilon)K_0\tau^{-\alpha} \le \delta/2$, for all $\epsilon \in (0,\epsilon_1)$, we have

$$\mathrm{dist}_{H^1}(v_\epsilon,\mathcal{A}_0) \le \|v_\epsilon - T_0(\tau,\phi_\epsilon)\|_{H^1} + \mathrm{dist}(T_0(\tau,\phi_\epsilon),\mathcal{A}_0) \le \delta, \quad v_\epsilon \in \mathcal{A}_\epsilon, \quad \epsilon \in (0,\epsilon_1).$$

and this implies the upper semicontinuity in H^1.□

We can also give a result on convergence of orbits in the attractor:

Proposition 5.5.3. *For any sequence $\epsilon_k \in (0,\epsilon_0)$ with $\epsilon_k \stackrel{k\to\infty}{\longrightarrow} 0$, and $\phi_{\epsilon_k} \in \mathcal{A}_{\epsilon_k}$ such that $\phi_{\epsilon_k} \stackrel{k\to\infty}{\longrightarrow} \phi_0$ in $H^1(\Omega)$, then if $u^{\epsilon_k}(\cdot) \subset \mathcal{A}_{\epsilon_k}$ is the positive orbit passing through ϕ_{ϵ_k} and $u^0(\cdot)$ is the positive orbit passing through ϕ_0, we have $u^0(\cdot) \subset \mathcal{A}_0$ and*

$$u^{\epsilon_k} \stackrel{k\to\infty}{\longrightarrow} u^0 \quad \text{in } C([0,T],H^1(\Omega)), \text{ for any } T > 0 \tag{5.6}$$

Proof: Observe first that if $\epsilon_k \overset{k\to\infty}{\longrightarrow} 0$ and $\phi_{\epsilon_k} \in \mathcal{A}_{\epsilon_k}$ in $H^1(\Omega)$ then, by the upper semicontinuity in $H^1(\Omega)$ we get that $\phi_0 \in \mathcal{A}_0$ and therefore $u^0(\cdot) \subset \mathcal{A}_0$.

Assume statement (5.6) is not true. This means that we can chose a sequence ϵ_k and functions $\phi_{\epsilon_k} \in \mathcal{A}_{\epsilon_k}$ such that $\phi_{\epsilon_k} \overset{k\to\infty}{\longrightarrow} \phi_0 \in \mathcal{A}_0$ and $\eta > 0$, so that

$$\|u^{\epsilon_k}(\cdot) - u^0(\cdot)\|_{C([0,T],H^1(\Omega))} \geq \eta, \qquad k \in I\!N \tag{5.7}$$

But, from the invariance of the attractors, there exist $\psi_{\epsilon_k} \in \mathcal{A}_{\epsilon_k}$, with $T_{\epsilon_k}(1, \psi_{\epsilon_k}) = \phi_{\epsilon_k}$

From the upper semicontinuity result proved in Theorem 5.5.2 we can choose a subsequence, denoted again by ϵ_k and a function $\psi_0 \in \mathcal{A}_0$ so that $\psi_{\epsilon_k} \overset{k\to\infty}{\longrightarrow} \psi_0$. By the fact that $P\psi_{\epsilon_k} \overset{k\to\infty}{\longrightarrow} P\psi_0 = \psi_0$, the continuous dependence of the semigroup T_0 and using (5.5) we obtain that

$$u^{\epsilon_k}(t) = T_{\epsilon_k}(t+1, \psi_{\epsilon_k}) \overset{k\to\infty}{\longrightarrow} T_0(t+1, \psi_0) \quad \text{in } C([0,T], H^1(\Omega))$$

In particular taking $t = 0$ above, we get that $\phi_{\epsilon_k} = T_{\epsilon_k}(1, \psi_{\epsilon_k}) \overset{k\to\infty}{\longrightarrow} T_0(1, \psi_0)$ and therefore $T_0(1, \psi_0) = \phi_0$. Hence, by uniqueness of solutions in forward time, we have $T_0(t+1, \psi_0) = u^0(t)$ which contradicts the existence of $\eta > 0$ satisfying (5.7).□

With this result we can prove,

Corollary 5.5.4. *For every sequence ϵ_k with $\epsilon_k \overset{k\to\infty}{\longrightarrow} 0$ and for every sequence of complete orbits $u^{\epsilon_k}(\cdot) \subset \mathcal{A}_{\epsilon_k}$ there exists a subsequence ϵ_{k_j} and a complete orbit $u^0(\cdot) \subset \mathcal{A}_0$ such that*

$$u^{\epsilon_{k_j}}(\cdot) \overset{j\to\infty}{\longrightarrow} u^0(\cdot) \quad \text{in } C([-T,T], H^1(\Omega)), \text{ for any } T > 0$$

Proof: We just need to use the invariance and compactness of the attractors, the result from Proposition 5.5.3 and a standard diagonalization procedure to obtain the subsequence.

5.2. Upper semicontinuity in $C^0(\bar\Omega)$

We have already proved that the attractor $\mathcal{A}_0$ is in $C^0(\bar\Omega)$. To prove the uppersemicontinuity of attractors in $C^0(\bar\Omega)$ we need to prove the following,

Lemma 5.5.5. *If for a sequence $\epsilon_k \in (0, \epsilon_0]$, with $\epsilon_k \overset{k\to\infty}{\longrightarrow} 0$, we have $u^{\epsilon_k} \in \mathcal{A}_{\epsilon_k}$ and $u^{\epsilon_k} \overset{k\to\infty}{\longrightarrow} u^0 \in \mathcal{A}_0$ in $H^1(\Omega)$, then $u^{\epsilon_k} \overset{k\to\infty}{\longrightarrow} u^0$ in $C^0(\bar\Omega)$.*

Proof: Let $\delta > 0$ be a small number and choose subdomains $\Omega_i^* \supset \Omega_{0,i}$ near enough to $\Omega_{0,i}$ so that $\|u^0 - u^0_{\Omega_{0,i}}\|_{L^\infty(\Omega_i^*)} \leq \delta/2$, where $u^0_{\Omega_{0,i}}$ denotes the constant value of u^0 in $\Omega_{0,i}$. This can be done due to the continuity of the function u^0 in Ω. Denote by $\Omega^* = \cup_{i=1}^m \Omega_i^*$.

From the uniform estimates given above by Theorem 2.2.4 (iii), we have that u^{ϵ_k} is uniform Hölder continuous in $\bar\Omega \setminus \Omega^*$ and therefore, from the compact embedding of the space of Hölder continuous functions into the space of continuous functions and the fact that $u^{\epsilon_k} \overset{k\to\infty}{\longrightarrow} u^0$ in $H^1(\Omega)$, we obtain that $u_{\epsilon_k} \overset{k\to\infty}{\longrightarrow} u^0$ in $C^0(\bar\Omega \setminus \Omega^*)$. In particular we can choose $k_0 \in I\!N$, such that $\|u^{\epsilon_k} - u^0\|_{C^0(\bar\Omega\setminus\Omega^*)} \leq \delta/2$ for $k \geq k_1$ and therefore if we denote by $\beta^i_{\epsilon_k}(x) = u^{\epsilon_k}(x) - u^0_{\Omega_{0,i}}$, defined in $\partial\Omega_i^*$, we have that $u^{\epsilon_k}(x) = u^0_{\Omega_{0,i}} + \beta^i_{\epsilon_k}(x)$ on $\partial\Omega_i^*$ and $\|\beta^i_{\epsilon_k}\|_{L^\infty(\partial\Omega_i^*)} \leq \delta$. But, considering the complete trajectory on $\mathcal{A}_{\epsilon_k}$ passing through u^{ϵ_k} and denoting by $H^{\epsilon_k} = -cu^{\epsilon_k} - u_t^{\epsilon_k} + f(u^{\epsilon_k})$, we have that u^{ϵ_k} is the unique solution of

$$\begin{cases} -\mathrm{div}(a_{\epsilon_k}\nabla u^{\epsilon_k}) = H^{\epsilon_k}, & \text{on} \quad \Omega_i^* \\ u^{\epsilon} = u^0_{\Omega_{0,i}} + \beta_{\epsilon_k}(x), & \text{on} \quad \Gamma_i^* = \partial\Omega_i^*, \end{cases}$$

From the uniform bounds of Theorem 2.2.4 (ii) we have that $\|H^{\epsilon_k}\|_{L^\infty(\Omega)} \leq K$ independent of ϵ_k. Applying Lemma 4.4.2 we obtain the existence of $k_2 \geq k_1$ such that $\|u^{\epsilon_k} - u^0_{\Omega_i}\|_{L^\infty(\Omega_i^*)} \leq 2\delta$ for $k \geq k_2$ and therefore $\|u^{\epsilon_k} - u^0\|_{L^\infty(\Omega_i^*)} \leq 3\delta$. Since δ is arbitrarily small we prove the lemma. □

Once this lemma is proved it is not difficult to prove the following,

THEOREM 5.5.6. *The global attractors, $\mathcal{A}_\epsilon$ and $\mathcal{A}_0$, are upper semicontinuous at $\epsilon = 0$ in $C^0(\bar\Omega)$.*

Proof: If it were not true then there would exist a positive number η, a sequence of $\epsilon_n \overset{n\to\infty}{\longrightarrow} 0$, and $u^n \in \mathcal{A}_{\epsilon_n}$, such that $\mathrm{dist}_{C^0(\bar\Omega)}(u^n, \mathcal{A}_0) \geq \eta$. By the uppersemicontinuity in $H^1(\Omega)$ and the compactness of $\mathcal{A}_0$ in $H^1(\Omega)$, we have the existence of $u^0 \in \mathcal{A}_0$ such that $u^n \overset{n\to\infty}{\longrightarrow} u^0$ in $H^1(\Omega)$ and from the lemma in $C^0(\bar\Omega)$. This contradicts the existence of $\eta > 0$. □

In a similar way as in the case of the previous subsection we can obtain some convergence results for complete orbits in the $C^0(\bar\Omega)$ topology. It is not difficult to show,

PROPOSITION 5.5.7. *For every sequence $\epsilon_k \in (0, \epsilon_0)$ with $\epsilon_k \overset{k\to\infty}{\longrightarrow} 0$ we have,*
i) If $\phi_{\epsilon_k} \in \mathcal{A}_{\epsilon_k}$ and $\phi_{\epsilon_k} \overset{k\to\infty}{\longrightarrow} \phi_0 \in \mathcal{A}_0$, in $H^1(\Omega)$, then if $u^{\epsilon_k}(\cdot) \subset \mathcal{A}_{\epsilon_k}$ is the positive orbit passing through ϕ_{ϵ_k} and $u^0(\cdot) \subset \mathcal{A}_0$ is the positive orbit passing through ϕ_0, we have

$$u^{\epsilon_k} \overset{k\to\infty}{\longrightarrow} u^0 \quad \text{in } C([0,T], C^0(\bar\Omega)), \text{ for any } T > 0$$

ii) For every sequence of complete orbits $u^{\epsilon_k}(\cdot) \subset \mathcal{A}_{\epsilon_k}$ there exists a subsequence ϵ_{k_j} and a complete orbit $u^0(\cdot) \subset \mathcal{A}_0$ such that

$$u^{\epsilon_{k_j}}(\cdot) \overset{j\to\infty}{\longrightarrow} u^0(\cdot) \quad \text{in } C([-T,T], C^0(\bar\Omega)), \text{ for any } T > 0$$

Proof: For the proof of this result we need to use Proposition 5.5.3, Corollary 5.5.4 and Lemma 5.5.5. □

REMARK 5.5.8. *We note that by using the uniform estimates in $H^1(\Omega)$ and $C^0(\bar\Omega)$ for the time derivative of solutions on the attractors $\mathcal{A}_\epsilon$, the results on the convergence of orbits obtained above could be also obtained by using Ascoli-Arzela's theorem.*

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