CADERNOS DE MATEMÁTICA **01**, 209–226 April (2000)
ARTIGO NÚMERO SMA#84

Strongly Damped Wave Equations with Critical Nonlinearities I: Case $\theta = \frac{1}{2}$

Alexandre N. Carvalho*

Departamento de Matemática, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo - Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos SP, Brazil
E-mail: andcarva@icmsc.sc.usp.br

Jan W. Cholewa†

Institute of Mathematics, Silesian University, 40-007 Katowice, Poland
E-mail: jcholewa@gate.math.us.edu.pl

In this paper we prove local well posedness and regularity of solutions for the strongly damped wave equations $u_{tt} + \eta A^{\frac{1}{2}} u_t + Au = f(u, u_t)$, $\eta > 0$, where $A : D(A) \subset X \to X$ is a self adjoint and positive operator in the Hilbert space X and f is a critically growing nonlinearity. This is done for initial conditions in the space $X^{\frac{1}{2}} \times X$ and for maps f growing critically. In the particular case when A is the Dirichlet Laplacian we also prove the existence of global attractors for subcritical nonlinearities. The existence of attractors for critically growing nonlinearities remains an open problem. April, 2000 ICMC-USP

1. INTRODUCTION

Consider the strongly damped wave equation

$$u_{tt} + \eta(-\Delta)^{\frac{1}{2}} u_t + (-\Delta)u = f(u, u_t),\ t > 0,\ x \in \Omega,\ \eta > 0, \tag{1}$$

on a C^2-smooth bounded domain $\Omega \in \mathbb{R}^n$, $n \geq 3$ with Dirichlet initial-boundary conditions

$$\begin{cases} u(0,x) = u_0(x),\ u_t(0,x) = v_0(x),\ x \in \Omega, \\ u(t,x) = 0,\ t \geq 0,\ x \in \partial\Omega. \end{cases} \tag{2}$$

Assuming that $f : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function we wish to prove a local well posedness result for (1)-(2) with initial conditions $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$. Of course we will need to impose restrictions on the growth of f and our aim is to obtain this result for the fastest growth possible of f.

* Research Partially Supported by CNPq grant # 300.889/92-5 and FAPESP grant # 97/011323-0, Brazil
† Research Partially Supported by FAPESP grant # 99/03116-0, Brazil

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

Assuming a dissipativeness condition on the vector field f, in addition to growth restrictions, we wish to obtain the existence of a global attractor.

To better describe the results in this paper we introduce some terminology. It is well known $A = (-\Delta)$ acting in a Hilbert space $X = X^0 = L^2(\Omega)$, where $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, is a positive definite, self-adjoint operator generating an analytic semigroup on X^0. Therefore $A^{\frac{1}{2}}$ is also positive definite, self-adjoint operator (cf. [10, §1.4]) and (1) may be viewed as an abstract second order ordinary differential equation in the Hilbert space $Y = Y^0 = X^{\frac{1}{2}} \times X^0$:

$$\ddot{u} + \eta A^{\frac{1}{2}} \dot{u} + Au = F(u, \dot{u}),\ t > 0, \tag{3}$$

which take us back to the considerations reported in a sequence of papers [6, 7].

Let us convert (1)-(2) into an ODE form in Y^0:

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{A}\begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{F}(\begin{bmatrix} u \\ v \end{bmatrix}),\ t > 0,\ \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{4}$$

where $\mathcal{A}$ and $\mathcal{F}$ will be written in a matrix form

$$\mathcal{A} = \begin{bmatrix} 0 & -I \\ A & \eta A^{\frac{1}{2}} \end{bmatrix},\ \mathcal{F} = \begin{bmatrix} 0 \\ F \end{bmatrix}. \tag{5}$$

We shall consider $\mathcal{A} : D(\mathcal{A}) \subset Y^0 \to Y^0$ with $D(\mathcal{A}) = Y^1 = X^1 \times X^{\frac{1}{2}}$.

The existence of the strong damping term $A^{\frac{1}{2}}$ widely influence the properties of $\mathcal{A}$ and makes (4) to behave somehow like a parabolic system. By the results of Section 2, (4) falls into a class of sectorial problems. Also, the resolvent of $\mathcal{A}$ is compact. The solutions are thus getting smoother for $t > 0$ and we know in advance that the corresponding semigroup of global solutions - if it exists - is compact. This enables us to apply the results in [3, 4] to obtain a local well posedness results for fast growing nonlinearities. This also makes clear that, for (1), some weak dissipativeness condition on f will imply the strong conclusion concerning the existence of a compact global attractor.

An important question in the studies of (1) is the choice of the space in which the corresponding flow will act. If we look at this question from a point of view of the existence of a compact global attractor it will seem natural to consider (1) in a smaller space which ensures that bounded sets will be attracted in a possibly strong topology. However, one is also tempted to find the solutions in a possibly large space and then take advantage of smoothing properties of the solutions to obtain the convergence of bounded sets from the larger space to the attractor in a stronger topology.

The latter will influence the approach to the problem (1). If we agree that the natural space in which solvability of (1) should be established is Y^0, then looking at (1) in a standard way - as on a problem (4) in Y^0 with a nonlinear term $\mathcal{F}$ acting from Y^0 into Y^0 - we will be in trouble to consider nonlinearities that grow with respect to u faster than $u^{\frac{n}{n-2}}$. Needless to say that in the case of $n = 3$ this gives the admissible growth 3, whereas one would like to have this number close to 5. Of course, one may solve (1) with $\mathcal{F}$ acting between Y^α and Y^0 with α close to 1 avoiding this way in space dimension $n = 3$ any trouble with the growth restriction. However, such approach limits the set of admissible initial data to Y^α which is an unwanted restriction.

The main results proved in this paper concerning local well posedness and asymptotic behavior for (1) are the following

THEOREM 1.1. *Assume that $n \geq 3$ and that $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying*

$$|f(u) - f(u')| \leq c|u - u'|(1 + |u|^{\rho_1 - 1} + |u'|^{\rho_1 - 1}), \tag{6}$$

where $\rho_1 = \frac{n+2}{n-2}$. Then, to each $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = X^{\frac{1}{2}} \times X$ corresponds a unique ε-regular solution of (18).

THEOREM 1.2. *Assume that* (6) *is satisfied for $\rho_1 < \frac{2n}{n-2}$ and that*

$$\lim_{|s| \to +\infty} \frac{f(s)}{s} \leq 0. \tag{7}$$

Then,

(i)For any $\alpha \in [\frac{n-2}{n+2}, 1)$ there is a compact C^0-semigroup $\{T(t) \in L(X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}); t \geq 0\}$ of global solutions to (1) *which possesses a global attractor $\mathbf{A}_\alpha$ in this space,*

(ii)$\mathbf{A}_\alpha = \mathbf{A}_{\frac{n-2}{n+2}} =: \mathbf{A}$, $\alpha \in [\frac{n-2}{n+2}, 1)$, and $\mathbf{A}$ attracts bounded subsets of $X^{\frac{1}{2}} \times X$ under $\{T(t)\}$ in $X^{\frac{1+\alpha}{2}} \times X^{\frac{\alpha}{2}}$-norm.

In fact we prove a stronger result on local existence, namely Theorem 3.1. As for Theorem 1.2 on existence of a global attractor, the version presented here is the exact version proved in Section 4. We strongly believe that there is an attractor for (1) when $\rho_1 = \frac{n+2}{n-2}$ but our attempts to prove it have rendered unsuccessful.

In Section 2 we introduce the functional analytic framework for working with problem (1), prove properties of the operator $\mathcal{A}$, describe its fractional power spaces and obtain embedding results for these spaces. In Section 3 we introduce the concepts of $\varepsilon-$regular solutions, $\varepsilon-$regular maps, obtain (with the aid of the results in [3, 4]) a result on local well posedness for the initial value problem for (1) with critical nonlinearity f and prove regularity results for these solutions. In Section 4 we impose the dissipativeness condition (7) on the nonlinearity to obtain a compact global existence result for solutions of (1) as well as the existence of a global attractor (Theorem 1.2). Finally, in the Appendix (Section 5) we consider some extensions on local existence to be able to deal with systems and present the abstract results employed to obtain the existence of a compact global attractor.

Acknowledgements: This paper was written while the second author visited the Department of Mathematics of the Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Brazil. He would like to acknowledge the great hospitality of the people from this institution.

2. TECHNICALITIES

2.1. Basic properties of $\mathcal{A}$

The tools collected below will be essentially used in the main body of the paper. Our concern here is thus to have all these tools carefully addressed.

We also remark that the considerations of the present section only in the paragraph §2.4 make use of a fact that A is a negative Laplacian in $L^2(\Omega)$ with $D(A) = H^2(\Omega) \times H_0^1(\Omega)$. All other results are based on the assumptions that A is a positive definite, self-adjoint operator in a Hilbert space X^0.

Proposition 2.1. *$\mathcal{A}$ fulfills the following conditions:*

(i)$\mathcal{A}$ is closed,
(ii)$\mathcal{A}$ is maximal accretive or, equivalently, $-\mathcal{A}$ is maximal dissipative,
(iii)$0 \in \rho(\mathcal{A})$,
(iv)$\mathcal{A}$ has compact resolvent.
(v)$\mathcal{A}^* = \begin{bmatrix} 0 & I \\ -A & \eta A^{\frac{1}{2}} \end{bmatrix}$ is the adjoint of $\mathcal{A}$ and $D(\mathcal{A}^*) = D(\mathcal{A}) = Y^1$.

Proof: Note that (i) is immediate from the closeness of A and $A^{\frac{1}{2}}$. For (ii) we first note that

$$\langle \mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \rangle_{Y^0} = \langle \begin{bmatrix} -v \\ Au + \eta A^{\frac{1}{2}} v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \rangle_{Y^0} = -\langle A^{\frac{1}{2}} v, A^{\frac{1}{2}} u \rangle_X + \langle Au + \eta A^{\frac{1}{2}} v, v \rangle_X$$
$$= -\langle v, Au \rangle_X + \overline{\langle v, Au \rangle}_X + \eta \langle A^{\frac{1}{4}} v, A^{\frac{1}{4}} v \rangle_X.$$

Therefore,

$$Re \langle \mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \rangle_{Y^0} = \eta \langle A^{\frac{1}{4}} v, A^{\frac{1}{4}} v \rangle_X \geq 0,$$

which proves accretivity of $\mathcal{A}$. To complete part (ii) it suffices to note that the equation

$$(\mathcal{I} + \mathcal{A}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix}$$

possesses for each $\begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix} \in Y_0$ a unique solution

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (I + \eta A^{\frac{1}{2}} + A)^{-1} (\overline{u} + \eta A^{\frac{1}{2}} \overline{u} + \overline{v}) \\ (I + \eta A^{\frac{1}{2}} + A)^{-1} (\overline{u} + \eta A^{\frac{1}{2}} \overline{u} + \overline{v}) - \overline{u} \end{bmatrix} \in Y^1.$$

Part (iii) is an immediate consequence of the form of the inverse operator

$$\mathcal{A}^{-1} = \begin{bmatrix} \eta A^{-\frac{1}{2}} & A^{-1} \\ -I & 0 \end{bmatrix}.$$

Part (iv) follows from (iii) and compactness of the Sobolev inclusions between X^{α} spaces. Finally, to prove part (v) note first that by density of the embedding $Y^1 \subset Y^0$ the adjoint operator $\mathcal{A}^*$ exists and, since

$$\langle \mathcal{A} \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} \rangle_{Y^0} = \langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} 0 & I \\ -A & \eta A^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} \rangle_{Y^0}, \ \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} \in Y^1,$$

we obtain that

$$\mathcal{A}^* \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & \eta A^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} \in Y^1 \subset D(\mathcal{A}^*).$$

By (i) and (iii) we next have that $0 \in \rho(\mathcal{A}^*) = \rho(\mathcal{A})$ (cf. [14, p. 225]). In particular $\mathcal{A}^*$ is thus invertible. Since also

$$\begin{bmatrix} 0 & I \\ -A & \eta A^{\frac{1}{2}} \end{bmatrix} \ \text{takes } Y^1 \text{ onto } \ Y^0,$$

the domain of $D(\mathcal{A}^*)$ cannot have more elements than Y^1 has. This completes the proof.∎

A particular consequence of the properties of $\mathcal{A}$ proved in Proposition 2.1 is that

Corollary 2.1. *$-\mathcal{A}$ is generator of a C^0-semigroup of contractions $\{e^{-\mathcal{A}t}\}$ on Y^0. Furthermore, the imaginary powers of $\mathcal{A}$ are bounded and*

$$\|\mathcal{A}^{it}\|_{\mathcal{L}(Y^0,Y^0)} \leq e^{\frac{\pi}{2}|t|}, \ t \in \mathbb{R}. \tag{8}$$

Proof: The proof follows from the Lumer-Phillips theorem (cf. [12, p. 14]) and from the observation concerning powers of positive operators reported in [2, p. 164] (cf. [11, p. 247]). ∎

As shown in [2, p. 177] condition (8) implies in general the uniform boundedness of $(1+|\lambda|)\|(\lambda-\mathcal{A})^{-1}\|_{\mathcal{L}(Y^0,Y^0)}$ in each $\{\frac{\pi}{2}+\delta \leq |arg(\lambda)| \leq \pi\}$, $\delta \in (0,\frac{\pi}{2})$. Result of [6] proves in fact that for $\mathcal{A}$ defined in (5) more is true, that is, there is a constant $C > 0$ such that

$$|\lambda|\|(\lambda-\mathcal{A})^{-1}\|_{\mathcal{L}(Y^0,Y^0)} \leq C \tag{9}$$

on the whole half plane $\{Re(\lambda) < 0\}$. Combining this result with Proposition 2.1 and Corollary 2.1 we obtain that

Proposition 2.2. *$\mathcal{A}$ is a sectorial, positive operator in Y^0. The semigroup of contractions $\{e^{-\mathcal{A}t}\}$ is analytic and compact in Y^α, $\alpha \in [0,1)$, for $t>0$. Furthermore,*

$$Y^\alpha = [Y^0, Y^1]_\alpha, \ \alpha \in (0,1), \tag{10}$$

where $[\cdot,\cdot]_\alpha$ denotes a complex interpolation space (cf. [13, §1.9.2]).

Remark 2.1. *Since $\mathcal{A}$ is sectorial and has bounded imaginary powers, the same is true for $\mathcal{A}^*$ (cf. [2, p. 13, p. 273]). Therefore,* (10) *holds for the power of $\mathcal{A}^*$ as well. In particular,*

$$Y^\alpha = D((A^*)^\alpha), \ \alpha \in [0,1].$$

2.2. Description of Y^α-spaces

It is always important to know the characterization of the domains of fractional powers of $\mathcal{A}$. Since we deal here with the product spaces, let us start from the following simple observation.

Proposition 2.3. *Let $\mathcal{X}_i$, $\mathcal{Y}_i$, $i=1,2$ be the Banach spaces such that*

$$\mathcal{X}_1 \subset \mathcal{X}_0, \ \mathcal{Y}_1 \subset \mathcal{Y}_0, \tag{11}$$

topologically and algebraically. Then,

$$[\mathcal{X}_0 \times \mathcal{Y}_0, \mathcal{X}_1 \times \mathcal{Y}_1]_\theta = [\mathcal{X}_0, \mathcal{X}_1]_\theta \times [\mathcal{Y}_0, \mathcal{Y}_1]_\theta, \ \theta \in (0,1). \tag{12}$$

Proof: The proof is an immediate consequence of the definition of complex interpolation spaces in [13, §1.9.2]). ∎

Based on Proposition 2.3 it is now easy to get characterizations of the fractional power spaces Y^α, $\alpha \in (0,1)$, without referring to the general considerations of [7].

Proposition 2.4. *For $\alpha \in (0,1)$ we have:*

$$Y^\alpha = D(\mathcal{A}^\alpha) = D((\mathcal{A}^*)^\alpha) = X^{\frac{\alpha}{2}+\frac{1}{2}} \times X^{\frac{\alpha}{2}}. \tag{13}$$

Proof: Recall that $Y^1 = X^1 \times X^{\frac{1}{2}}$, $Y^0 = X^{\frac{1}{2}} \times X$. Connecting (10) and (12) we thus obtain that

$$Y^\alpha = [X^{\frac{1}{2}}, X^1]_\alpha \times [X, X^{\frac{1}{2}}]_\alpha, \ \alpha \in (0,1).$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Since by our assumptions on A we have the equalities:

$$[X^{\frac{1}{2}}, X^1]_\alpha = X^{\frac{1}{2}(1-\alpha)+\alpha},\ [X, X^{\frac{1}{2}}]_\alpha = X^{\frac{\alpha}{2}},$$

the proof is complete (cf. Remark 2.1). ∎

2.3. Extrapolated fractional power scale

Based on [2, Chapter V] we shall describe here the extrapolated fractional power scale of order 1 corresponding to a pair (Y^0,$\mathcal{A}$).

LEMMA 2.1. *Let Y_{-1} denotes the extrapolation space of Y^0 generated by $\mathcal{A}$. The following equality holds:*

$$Y_{-1} = X \times X^{-\frac{1}{2}}. \tag{14}$$

Proof: Recall first that Y_{-1} is the completion of the normed space $(Y^0, \|\mathcal{A}^{-1}\cdot\|_{Y^0})$. Since

$$\|\mathcal{A}^{-1}\begin{bmatrix}u\\v\end{bmatrix}\|_{Y^0} \le c\|\begin{bmatrix}u\\v\end{bmatrix}\|_{X\times X^{-\frac{1}{2}}} \le c'\|\mathcal{A}^{-1}\begin{bmatrix}u\\v\end{bmatrix}\|_{Y^0},\ \begin{bmatrix}u\\v\end{bmatrix}\in Y^0,$$

the completions of $(Y^0, \|\mathcal{A}^{-1}\cdot\|_{Y^0})$ and $(Y^0, \|\cdot\|_{X\times X^{-\frac{1}{2}}})$ coincide. Recalling that $Y^0 = X^{\frac{1}{2}} \times X$, we obtain (14). The proof is complete. ∎

LEMMA 2.2. *Let $\mathcal{A}_{-1}$ be the closure of $\mathcal{A}$ in Y_{-1}. Then, $\mathcal{A}_{-1}$ is sectorial and positive operator in Y_{-1} with $D(\mathcal{A}_{-1}) = Y^0$. Furthermore, the imaginary powers of $\mathcal{A}_{-1}$ are bounded and the resolvent of $\mathcal{A}_{-1}$ is compact.*

Proof: By [2, p. 262] we know that $\rho(\mathcal{A}_{-1}) = \rho(\mathcal{A})$ whereas the suitable estimate for $|\lambda|\|(\lambda-\mathcal{A}_{-1})^{-1}\|_{\mathcal{L}(Y_{-1},Y_{-1})}$ is a consequence of the estimate for the resolvent of $\mathcal{A}$ in Y^0. Indeed, since Y^1 is dense in Y^0, $\mathcal{A}_{-1}$ is a continuous extension of $\mathcal{A}$ onto Y^0 such that $\mathcal{A}_{-1}$ is an isometric isomorphism from Y^0 onto Y_{-1}. In particular, $\mathcal{A}_{-1} = \mathcal{A}$ on $D(\mathcal{A})$ as well as $(\lambda-\mathcal{A}_{-1})^{-1} = (\lambda-\mathcal{A})^{-1}$ on Y^0. Therefore, by sectoriality of $\mathcal{A}$ in Y^0 (cf. formula (9)) we obtain the estimate

$$\begin{aligned}
&|\lambda|\|(\lambda-\mathcal{A}_{-1})^{-1}\begin{bmatrix}u\\v\end{bmatrix}\|_{Y_{-1}} = |\lambda|\|\mathcal{A}^{-1}(\lambda-\mathcal{A}_{-1})^{-1}\begin{bmatrix}u\\v\end{bmatrix}\|_{Y^0}\\
&= |\lambda|\|(\lambda-\mathcal{A})^{-1}\mathcal{A}^{-1}\begin{bmatrix}u\\v\end{bmatrix}\|_{Y^0} \le C\|\mathcal{A}^{-1}\begin{bmatrix}u\\v\end{bmatrix}\|_{Y^0} = C\|\begin{bmatrix}u\\v\end{bmatrix}\|_{Y_{-1}},\ \begin{bmatrix}u\\v\end{bmatrix}\in Y^0,
\end{aligned}$$

which can be extended next to the elements of Y_{-1} based on density of Y^0 in Y_{-1}. This justifies sectoriality of $\mathcal{A}_{-1}$ in Y_{-1}.

The remaining assertions are the direct consequences of the results reported in [2, p. 266, p. 264]. The proof is complete. ∎

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

THEOREM 2.1. *The following characterization holds:*

$$Y_{-1}^{n+\alpha} = X^{\frac{n+\alpha}{2}} \times X^{\frac{n-1+\alpha}{2}}, \ \alpha \in [0,1], n \in \mathbb{N}, \tag{15}$$

Proof: By Lemma 2.2 the usual interpolation arguments may be used so that for $n = 1, 2, \ldots$ we get

$$\begin{aligned} Y_{-1}^{n+\alpha} &= [Y_{-1}^{n}, Y_{-1}^{n+1}]_\alpha = [Y^{n-1}, Y^{n}]_\alpha = [X^{\frac{n}{2}} \times X^{\frac{n-1}{2}}, X^{\frac{n+1}{2}} \times X^{\frac{n}{2}}]_\alpha \\ &= [X^{\frac{n}{2}}, X^{\frac{n+1}{2}}]_\alpha \times [X^{\frac{n-1}{2}}, X^{\frac{n}{2}}]_\alpha = X^{\frac{n+\alpha}{2}} \times X^{\frac{n-1+\alpha}{2}}. \end{aligned}$$

For $n = 0$ instead of the standard arguments above we also need the properties of the Hilbert scale $[X^\alpha, \ \alpha \in \mathbb{R}]$ (cf. [2, p. 274]) as well as the duality theory for the complex interpolation method (cf. [13, Theorem 1.11.3]). Identifying X with its dual X^* we then have:

$$\begin{aligned} Y_{-1}^{\alpha} &= [Y_{-1}, Y_{-1}^{1}]_\alpha = [Y_{-1}, Y^{0}]_\alpha \\ &= [X \times X^{-\frac{1}{2}}, X^{\frac{1}{2}} \times X]_\alpha = [X, X^{\frac{1}{2}}]_\alpha \times [X^{-\frac{1}{2}}, X]_\alpha \\ &= X^{\frac{\alpha}{2}} \times [((X^*)^{\frac{1}{2}})^*, X^*]_\alpha = X^{\frac{\alpha}{2}} \times [X^{\frac{1}{2}}, X]_\alpha^* = X^{\frac{\alpha}{2}} \times [X, X^{\frac{1}{2}}]_{1-\alpha}^* \\ &= X^{\frac{\alpha}{2}} \times (X^{\frac{1-\alpha}{2}})^* = X^{\frac{\alpha}{2}} \times X^{\frac{\alpha-1}{2}}. \end{aligned}$$

The proof is complete. ∎

Remark 2.2. *Following the terminology of [2],*

$$[(X^{\frac{\sigma}{2}} \times X^{\frac{\sigma-1}{2}}, \mathcal{A}_{\sigma-1}), \ \sigma \in [0, +\infty)]$$

is a compactly injected one-sided fractional power scale generated by $(Y_{-1}, \mathcal{A}_{-1})$*. Simultaneously*

$$[(X^{\frac{\sigma+1}{2}} \times X^{\frac{\sigma}{2}}, \mathcal{A}_{\sigma}), \ \sigma \in [-1, +\infty)]$$

is the extrapolated fractional power scale of first order generated by $(Y^0, \mathcal{A})$*.*

2.4. Embedding of spaces $\mathbf{Y_{-1}^{n+\alpha}}$

Using Theorem 2.1 it is easy to write the inclusions for $Y_{-1}^{n+\alpha}$ in the case when the scale $[(X^\alpha, A_\alpha), \alpha \in \mathbb{R}]$ is specified. Here we use the scale generated by $(L^2(\Omega), (-\Delta_D))$ focusing on the inclusions that we will need later in the considerations of equation (1).

LEMMA 2.3. *Let* $[(X^\alpha, A_\alpha), \alpha \in \mathbb{R}]$ *be generated by* $(L^2(\Omega), (-\Delta_D))$*. Then the following embedding hold:*

$$\begin{cases} Y_{-1}^{1+\alpha} \subset H^{1+\alpha}(\Omega) \times H^{\alpha}(\Omega) \subset L^{q_1}(\Omega) \times L^{q_2}(\Omega) \\ \textit{for} \ \ 1 \le q_1 \le \frac{2n}{n-2-2\alpha}, \ 1 \le q_2 \le \frac{2n}{n-2\alpha}, \ \alpha \in [0, \frac{n-2}{2}), \ n \ge 3, \end{cases} \tag{16}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

$$\begin{cases} X^{\alpha} \times L^{q}(\Omega) \subset Y^{\alpha}_{-1} \\ for \ \ \frac{2n}{n+2-2\alpha} \leq q, \ \alpha \in [0,1]. \end{cases} \tag{17}$$

Proof: The proof of (16) follows from the interpolation result for the spaces of Bessel potentials (cf. [13, Theorem 2, §4.3.1]) and Sobolev Embedding Theorem (cf. [13, Theorem 4.6.1]). Condition (17) is a consequence of the duality argument and the inclusions

$$X^{\frac{\alpha-1}{2}\,*} = X^{\frac{1-\alpha}{2}} \subset H^{1-\alpha}(\Omega) \subset L^{p}(\Omega) \ \text{ for } \ 1 \leq p \leq \frac{2n}{n+2\alpha-2}, \ \alpha \in [0,1],$$

where X was identified with its dual X^*. The proof is complete. ∎

3. SOLVABILITY OF (1) IN Y^0

Following the results of Section 2 we are going to study (1) as a sectorial problem (18) in a Hilbert space Y_{-1},

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{A}_{-1}\begin{bmatrix} u \\ v \end{bmatrix} = \mathcal{F}(\begin{bmatrix} u \\ v \end{bmatrix}), \ t > 0, \ \begin{bmatrix} u \\ v \end{bmatrix}_{t=0} = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}, \tag{18}$$

and look for the solutions to (18) originating at the elements of Y^0.

For this we need first recall the notions of the ε-regular map and ε-regular solutions to (18) (cf. [3], [4]).

3.1. ε-regular maps and ε-regular solutions

Let P be a sectorial, positive operator acting in a Banach space $Z = Z^0$ and ε be a nonnegative number.

Definition 3.1. $G : D(G) \to Z$ is said to be an ε-regular map relatively to (Z^1, Z^0) (or, equivalently, G is said to belong to a subclass $\mathcal{F}(\varepsilon, \rho, \gamma(\varepsilon), C)$ of nonlinearities acting in Z^0) if and only if there are constants $\rho > 1$, $\gamma(\varepsilon) \geq 0$, $C > 0$ such that

- $\rho\varepsilon \leq \gamma(\varepsilon) < 1$,
- G takes $Z^{1+\varepsilon}$ into $Z^{\gamma(\varepsilon)}$,
- G fulfills the estimate

$$\begin{aligned} &\|G(z_1) - G(z_2)\|_{Z^{\gamma(\varepsilon)}} \\ &\leq C\|z_1 - z_2\|_{Z^{1+\varepsilon}}(\|z_1\|^{\rho-1}_{Z^{1+\varepsilon}} + \|z_2\|^{\rho-1}_{Z^{1+\varepsilon}} + 1), \ z_1, z_2 \in Z^{1+\varepsilon}. \end{aligned} \tag{19}$$

Consider next an abstract Cauchy problem in Z^0:

$$\dot{z} + Pz = G(z), \ t > 0, \ z(0) = z_0. \tag{20}$$

Definition 3.2. Let $\tau > 0$, $z_0 \in Z^1$, and $z = z(\cdot, z_0) : [0, \tau] \to Z^1$. We say that z is an ε-regular solution to (20), $\varepsilon \geq 0$ if and only if

Publicado pelo ICMC-USP
Sob a supervisão CPq/ICMC

- $z \in C([0,\tau], Z^1) \cap C((0,\tau], Z^{1+\varepsilon})$,
- z satisfies the Cauchy integral formula:

$$z(t) = e^{-Pt} z_0 + \int_0^t e^{-P(t-s)} G(z(s)) ds, \ z \in [0,\tau].$$

The existence of an ε-regular solutions was proved in [3]. The result below comes from [4] and is an extension of the original theorem reported in [3].

PROPOSITION 3.1. *Let $\overline{z}_0 \in Z^1$ and $B_{Z^1}(\overline{z}_0, r)$ denote a ball in Z^1 with radius $r > 0$ centered at $\overline{z}_0$.*

Assume that $G = \sum_{i=1}^k G_i$ and, for $1 \le i \le k$, $\mathcal{F}_i$ belongs to the class $\mathcal{F}(\varepsilon_i, \rho_i, \gamma_i(\varepsilon_i), C_i)$ with certain $\varepsilon_i > 0$. Suppose also that

$$\min\{\gamma_i(\varepsilon_i);\ 1 \le i \le n\} =: \gamma > \max\{\varepsilon_i;\ 1 \le i \le n\} =: \varepsilon.$$

Then, there are $r > 0$ and $\tau_0 > 0$ such that for each $z_0 \in B_{Z^1}(\overline{z}_0, r)$ there exists a unique ε-regular solution $z = z(,\cdot z_0)$ to (20). In addition,

(i)$t^\theta \|z(t,z_0)\|_{Z^{1+\theta}} \to 0$ as $t \to 0^+$, $0 < \theta < \gamma$,
(ii)$t^\theta \|z(t,z_1) - z(t,z_2)\|_{Z^{1+\theta}} \le C' \|z_1 - z_2\|_{Z^1}$, $t \in [0,\tau_0]$, $0 \le \theta \le \theta_0 < \gamma$, $z_1, z_2 \in B_{Z^1}(\overline{z}_0, r)$,
(iii)$z \in C((0,\tau_0], Z^{1+\gamma}) \cap C^1((0,\tau_0], Z^{1+\theta})$, $0 \le \theta < \gamma$, and, in particular, z satisfies both relations in (20).

We are now fully prepared to prove solvability of (18) in Y^0.

3.2. Critical exponents for (1)

Let f in (1) be such that

$$f(u,v) = f_1(u) + f_2(v),\ f_i : \mathbb{R} \to \mathbb{R},\ i = 1,2. \tag{21}$$

We shall require of f_i, i=1,2, to fulfill the estimate:

$$|f_i(s_1) - f_i(s_2)| \le c|s_1 - s_2|(|s_1|^{\rho_i - 1} + |s_2|^{\rho_i - 1} + 1),\ s_1, s_2 \in \mathbb{R}. \tag{22}$$

Of course, condition (22) is much stronger than the assumption of local Lipschitz continuity of f.

LEMMA 3.1. *Let (22) holds with $\rho_1, \rho_2 \in \mathbb{R}$ restricted by the conditions*

$$1 < \rho_1 \le \frac{n+2}{n-2} =: \rho_1(n),\ 1 < \rho_2 \le \frac{n+2}{n} =: \rho_2(n). \tag{23}$$

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC

Then, for $i=1,2$, and each $\varepsilon \in [0,\frac{1}{\rho_i})$, there is certain $\gamma_i(\varepsilon)\geq \rho_i\varepsilon$ such that:

$$\begin{aligned}&\|f_i(w_1)-f_i(w_2)\|_{L^{\frac{2n}{n+2-2\gamma_i(\varepsilon)}}(\Omega)}\\&\leq c\|w_1-w_2\|_{H^{1+\varepsilon}(\Omega)}(\|w_1\|^{\rho_i-1}_{H^{1+\varepsilon}(\Omega)}+\|w_2\|^{\rho_i-1}_{H^{1+\varepsilon}(\Omega)}+1),\ w_1,w_2\in H^{1+\varepsilon}(\Omega).\end{aligned}$$

Furthermore, if $\rho_i=\rho_i(n)$, then (necessarily)

$$\gamma_i(\varepsilon)=\rho_i\varepsilon,\ \varepsilon\in[0,\frac{1}{\rho_i}). \tag{24}$$

Proof: Because of the similarity of calculations we shall focus below on the case $i=1$.

Using (22), Hölder inequality and Sobolev embedding we obtain:

$$\begin{aligned}&\|f_1(w_1)-f_1(w_2)\|_{L^{\frac{2n}{n+2-2\gamma_1(\varepsilon)}}(\Omega)}\\&\leq c'\|w_1-w_2\|_{L^{\frac{2n}{n-2\varepsilon-2}}(\Omega)}(\|w_1\|^{\rho_1-1}_{L^{\frac{2n(\rho_1-1)}{4+2\varepsilon-2\gamma_1(\varepsilon)}}(\Omega)}+\|w_2\|^{\rho_1-1}_{L^{\frac{2n(\rho_1-1)}{4+2\varepsilon-2\gamma_1(\varepsilon)}}(\Omega)}+1)\\&\leq c''\|w_1-w_2\|_{H^{1+\varepsilon}(\Omega)}(\|w_1\|^{\rho_1-1}_{H^{1+\varepsilon}(\Omega)}+\|w_2\|^{\rho_1-1}_{H^{1+\varepsilon}(\Omega)}+1).\end{aligned}$$

The above calculations are valid as long as

$$\rho_1\leq\frac{n+2-2\gamma_1(\varepsilon)}{n-2-2\varepsilon}.$$

Also, the value $\rho_1(n)$ is attained for $\gamma_1(\varepsilon)=\rho_1\varepsilon$. The lemma is proved. ∎

For $i=1,2$ denote

$$\mathcal{F}_i:D(\mathcal{F}_i)\subset Y_{-1}\to Y_{-1},\ \mathcal{F}_1(\begin{bmatrix}u\\v\end{bmatrix})=\begin{bmatrix}0\\F_1(u)\end{bmatrix},\ \mathcal{F}_2(\begin{bmatrix}u\\v\end{bmatrix})=\begin{bmatrix}0\\F_2(v)\end{bmatrix},$$

where F_i is an abstract counterpart of a continuous function $f_i:\mathbb{R}\to\mathbb{R}$.

Lemma 3.2. *Suppose that (22) and (23) are satisfied. Then, $\mathcal{F}_i$, $i=1,2$, is an ε-regular map relatively to (Y^0,Y^{-1}) for each $\varepsilon\in[0,\frac{1}{\rho_i})$.*

Proof: From Lemmas 2.3 and 3.1 we have:

$$\begin{aligned}&\|\mathcal{F}_1(\begin{bmatrix}u_1\\v_1\end{bmatrix})-\mathcal{F}_1(\begin{bmatrix}u_2\\v_2\end{bmatrix})\|_{Y_{-1}^{\gamma_1(\varepsilon)}}\leq C'\|f_1(u_1)-f_1(u_2)\|_{L^{\frac{2n}{n+2-2\gamma_1(\varepsilon)}}(\Omega)}\\&\leq C''\|u_1-u_2\|_{H^{1+\varepsilon}(\Omega)}(\|u_1\|^{\rho_1-1}_{H^{1+\varepsilon}(\Omega)}+\|u_2\|^{\rho_1-1}_{H^{1+\varepsilon}(\Omega)}+1)\\&\leq C'''\|\begin{bmatrix}u_1\\v_1\end{bmatrix}-\begin{bmatrix}u_2\\v_2\end{bmatrix}\|_{Y_{-1}^{1+\varepsilon}}(\|\begin{bmatrix}u_1\\v_1\end{bmatrix}\|^{\rho_1-1}_{Y_{-1}^{1+\varepsilon}}+\|\begin{bmatrix}u_2\\v_2\end{bmatrix}\|^{\rho_1-1}_{Y_{-1}^{1+\varepsilon}}+1),\ \begin{bmatrix}u_1\\v_1\end{bmatrix},\begin{bmatrix}u_2\\v_2\end{bmatrix}\in Y_1^{1+\varepsilon}.\end{aligned}$$

Calculations concerning term $\mathcal{F}_2$ are similar. Lemma 3.2 is thus proved. ∎

The following existence result is now a direct consequence of Lemma 3.2 and Proposition 3.1.

THEOREM 3.1. *Let the assumptions of Lemma 3.2 be satisfied,* $\varepsilon \in (0, \min\{\frac{1}{\rho_1}, \frac{1}{\rho_2}\}$ *and* $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$. *Then, to each* $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^0$ *corresponds a unique* ε*-regular solution of* (18).

Remark 3.3. *Additional smoothing properties of the* ε*-regular solution to* (18) *may be easily deduced from conditions (i)-(iii) of Proposition 3.1. It should be also noted that if* $\rho_1 = \rho_1(n)$, $\rho_2 = \rho_2(n)$, *then* (24) *holds. Therefore, for these values of exponents nonlinear terms* $\mathcal{F}_i$, $i = 1, 2$, *and hence also* $\mathcal{F}$ *are critical maps relatively to* (Y^0, Y_{-1}). *Roughly speaking this is the case when the nonlinear term* $\mathcal{F}$ *has the same order of magnitude as the linear operator in the main part of equation.*

3.3. Smoothing action of ε-regular solutions

Let f in (21) be such that

$$f(u, v) = f_1(u) - \beta v,\ b > 0,\ 1 < \rho_1 \leq \frac{n+2}{n-2}. \tag{25}$$

LEMMA 3.3. *For* $\alpha \in [\frac{n-2}{n+2}, 1)$, *the map* $\mathcal{F}$ *corresponding to* (25) *takes* Y^α *into* Y *and is Lipschitz continuous on bounded subsets of* Y^α.

Proof: Description of Y^α spaces has been given in Proposition 2.4. The proof follows by standard calculations based on the Hölder inequality and Sobolev embedding. ∎

The above lemma and the general results of [10] imply:

LEMMA 3.4. *For* $\alpha \in [\frac{n-2}{n+2}, 1)$ *and* $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^\alpha$ *there exists a unique* Y^α*-solution to* (4) *defined on a maximal interval of existence* $[0, \tau_{u_0,v_0})$. *That is, there exists a unique function* $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix} \in C([0, \tau_{u_0,v_0}), Y^\alpha)$ *such that:*

(i) $\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix} \in C((0, \tau_{u_0,v_0}), Y^1)$,

(ii) $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix} \in C^1((0, \tau_{u_0,v_0}), Y^\beta)$, $\beta \in [0, 1)$,

(iii)both relations in (4) *are satisfied.*

THEOREM 3.2. *Under the assumption* (25) *the* ε*-regular solutions from Theorem 3.1 fulfill the conditions (i)-(iii) of Lemma 3.4.*

Proof: Take $\begin{bmatrix} u \\ v \end{bmatrix}$ and choose $\varepsilon > 0$ such that $1 > \gamma(\varepsilon) \geq \frac{n-2}{n+2}$. Let $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix}$ be ε-regular solution obtained in Theorem 3.1.

Since $Y_{-1}^{1+\gamma(\varepsilon)} = Y^{\gamma(\varepsilon)} \subset Y^{\frac{n-2}{n+2}}$ we find from Proposition 3.1 (iii) that

$$\begin{bmatrix} u(s, u_0, v_0) \\ v(s, u_0, v_0) \end{bmatrix} \in Y^{\frac{n-2}{n+2}} \quad \text{for each} \quad s \in (0, \tau_0).$$

According to Lemma 3.4 there exists $Y^{\frac{n-2}{n+2}}$-solution $\begin{bmatrix} \tilde{u}(\cdot, u(s, u_0, v_0)) \\ \tilde{v}(\cdot, v(s, u_0, v_0)) \end{bmatrix}$ to (4). This proves that

$$\begin{bmatrix} u(t+s, u_0, v_0) \\ v(t+s, u_0, v_0) \end{bmatrix} = \begin{bmatrix} \tilde{u}(t, u(s, u_0, v_0)) \\ \tilde{v}(t, v(s, u_0, v_0)) \end{bmatrix}, \ t \in [0, \tau_{u_0, v_0}),$$

and consequently,

$$\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix} \in Y^1, \ t \in (s, \tau_{u_0, v_0}), \ \begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix} \in C^1((s, \tau_{u_0, v_0}), Y^{\beta}), \ \beta \in [0, 1).$$

Since $s > 0$ could be arbitrarily small, the proof is complete. ∎

4. GLOBAL SOLUTIONS TO (4) AND ASYMPTOTICS

In this section we consider the map f as in (25) restricting however to the subcritical case $\rho_1 < \frac{n+2}{n-2}$.

LEMMA 4.1. *Let (25) holds and $\rho_1 < \frac{n+2}{n-2}$. Then, for any bounded set $B \subset Y^0$ there is a time $\tau_B > 0$ such that the ε-regular solutions $\begin{bmatrix} u(\cdot, u_0, v_0) \\ v(\cdot, u_0, v_0) \end{bmatrix}$ from Theorem 3.1 originating at $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B$ exists and are bounded in Y^{θ_0} for arbitrary $\theta_0 < \gamma(\varepsilon)$, where $\gamma(\varepsilon) > \max\{\frac{n-2}{n+2}, \rho_1 \varepsilon\}$. In particular, the set*

$$\{\begin{bmatrix} u(t, u_0, v_0) \\ v(t, u_0, v_0) \end{bmatrix}, \ \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in B\}$$

is precompact in $Y^{\frac{n-2}{n+2}}$ for each $t \in (0, \tau_B)$.

Proof: The proof is a direct consequence of Proposition 3.1 (ii). We remark that, for $\gamma(\varepsilon) > \rho_1 \varepsilon$ a number r in Proposition 3.1 can be chosen arbitrarily large so that the time of the existence of ε-regular solutions is uniform on bounded sets of Y^0 (cf. [3, Corollary 1]). ∎

Remark 4.4. *We mention for completeness that one may choose in Lemma 4.1 e.g.* $\gamma(\varepsilon) = \varepsilon\frac{n+2}{n-2}$ *with certain* $\varepsilon > \left(\frac{n-2}{n+2}\right)^2$.

In the considerations below, devoted to the existence of the global attractor to (4) in a subcritical case, we shall follow the general abstract scheme developed in [8, 5]. For convenience we recall this scheme in the closing Section 5.2.

THEOREM 4.1. *Let the assumption of Lemma 4.1 hold and, in addition,* f_1 *satisfies the dissipative condition* (7). *Then,*

(i)For any $\alpha \in [\frac{n-2}{n+2}, 1)$ *there exists corresponding to* (4) *a compact* C^0*-semigroup* $\{T(t)\}$ *of global* Y^α*-solutions to* (4) *which possesses a compact global attractor* $\mathbf{A}_\alpha$ *in* Y^α*,*
(ii)$\mathbf{A}_\alpha = \mathbf{A}_{\frac{n-2}{n+2}} =: \mathbf{A}$, $\alpha \in [\frac{n-2}{n+2}, 1)$,
(iii)$T(t) : Y^0 \to Y^\alpha$, $t > 0$, *where* $T(t)\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} u(t,u_0,v_0) \\ v(t,u_0,v_0) \end{bmatrix}$, $\begin{bmatrix} u(\cdot,u_0,v_0) \\ v(\cdot,u_0,v_0) \end{bmatrix}$ *being an* ε*-regular solution from Theorem 3.1, are well defined maps which are the extensions of* $T(t)$ $(t > 0)$ *to* Y^0*,*
(iv)$\mathbf{A}_\alpha$ *attracts bounded subset of* Y^0 *under* $\{T(t)\}$ *in* Y^α*-norm.*

Proof: The proof of (i) occurs in four steps.

Step 1. (Y^0*-estimate and the Lyapunov function*) Take $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^\alpha$ and consider the corresponding Y^α-solution $\begin{bmatrix} u(t,u_0,v_0) \\ v(t,u_0,v_0) \end{bmatrix}$. Lemma 3.4 ensures that we have enough regularity to work with the starting equation (1). Multiplying thus (1) by $v = u_t$ in $L^2(\Omega)$ and using the properties of the negative Laplacian with Dirichlet boundary conditions we obtain that

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega \int_0^u f_1(s)ds\right) = -\eta\|A^{\frac{1}{4}}u_t\|_{L^2(\Omega)} - \beta\|u_t\|^2_{L^2(\Omega)} \le 0.$$

This ensures in particular that

$$\left\|\begin{bmatrix} u(t,u_0,v_0) \\ v(t,u_0,v_0) \end{bmatrix}\right\|_{Y^0} \le c + c'\mathcal{L}\left(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right) \le C\left(\left\|\begin{bmatrix} u_0 \\ v_0 \end{bmatrix}\right\|_{Y^0}\right), \tag{26}$$

where c, c' do not depend on η, β,

$$\mathcal{L}\left(\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = \frac{1}{2}\|w_2\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla w_1\|^2_{L^2(\Omega)} - \int_\Omega \int_0^{w_1} f_1(s)ds, \ w_1, w_2 \in Y^0, \tag{27}$$

and $C : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally bounded function independent of η, β.

Step 2. (*subordination of the nonlinearity to a power of* $\mathcal{A}$) Since $1 < \rho_1 < \frac{n+2}{n-2}$, then based on the Nirenberg-Gagliardo type inequality we obtain that

$$\|f_1(u(t,u_0,v_0))\|_{L^2(\Omega)} \le g(\|u(t,u_0,v_0)\|_{H^1(\Omega)})(1 + \|u(t,u_0,v_0)\|^{\theta_1}_{H^{1+\alpha_1}(\Omega)}), \tag{28}$$

$t \in (0, \tau_{u_0,v_0})$, with certain $\theta_1 \in [0,1)$, $\alpha_1 \in [0,1)$ and some nondecreasing function $g : \mathbb{R}^+ \to \mathbb{R}^+$ (cf. [1]). Next, based on (28), we get the relation:

$$\begin{aligned}
&\|\mathcal{F}(\begin{bmatrix} u(t,u_0,v_0) \\ v(t,u_0,v_0) \end{bmatrix})\|_{Y^0} \le \|f_1(u(t,u_0,v_0))\|_{L^2(\Omega)} + \beta \|v(t,u_0,v_0)\|_{L^2(\Omega)} \\
&\le \left(g(\| \begin{bmatrix} u(t,u_0,v_0) \\ v(t,u_0,v_0) \end{bmatrix} \|_{Y^0}) + \beta \| \begin{bmatrix} u(t,u_0,v_0) \\ v(t,u_0,v_0) \end{bmatrix} \|_{Y^0} \right) \left(1 + \| \begin{bmatrix} u(t,u_0,v_0) \\ v(t,u_0,v_0) \end{bmatrix} \|^{\theta_1}_{Y^{\alpha_1}} \right).
\end{aligned} \tag{29}$$

Step 3. (*global solvability and compactness*) Conditions (26) and (29) plus the compactness of the resolvent of $\mathcal{A}$ (cf. Proposition 2.1) ensures that to (4) corresponds a compact C^0-semigroup $\{T(t)\}$ of global Y^α-solutions having bounded orbits of bounded sets. For the proof of the existence of the global attractor for $\{T(t)\}$ in Y^α it now suffices to show that the estimate (26) is asymptotically independent of $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^\alpha$.

Step 4. (*point dissipativeness of* $\{T(t)\}$ *- the role of the Lyapunov function*). Functional $\mathcal{L}$ defined in (27) is a Lyapunov function for $\{T(t)\}$ in Y^α. Therefore, ω-limit sets of points from Y^α lie within the set $\mathcal{E}$ of all stationary solutions to (4). Our concern now is to prove that $\mathcal{E}$ is bounded in Y^0.

Let $\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathcal{E}$. Then $\tilde{v} = 0$, whereas $\tilde{u}$ is an $H^2(\Omega)$-solution of the elliptic problem

$$\begin{cases} -\Delta \tilde{u} = f_1(\tilde{u}), \ x \in \Omega, \\ \tilde{u} = 0 \text{ on } \partial\Omega. \end{cases} \tag{30}$$

With the use of (7) it is easy to show that if $\tilde{u}$ solves (30), then $\|\tilde{u}\|_{H^1(\Omega)} \le c''$ where $c'' = c''(\Omega, f_1) > 0$ is independent of $\tilde{u}$. Consequently, we have

$$\| \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \|_{Y^0} \le c''', \ \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathcal{E}. \tag{31}$$

Since each ω-limit set $\omega(\begin{bmatrix} u_0 \\ v_0 \end{bmatrix})$, lies in $\mathcal{E}$, is compact and attracts $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^\alpha$ under $\{T(t)\}$ in Y^α-norm, condition (31) ensures in particular that

$$\limsup_{t \to +\infty} \| \begin{bmatrix} u(t,u_0,v_0) \\ v(t,u_0,v_0) \end{bmatrix} \|_{Y^0} \le c''', \ \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \in Y^\alpha. \tag{32}$$

Therefore, the estimate (26) is asymptotically independent of initial data from Y^α which completes the proof of the assertion (i).

Part (ii) is a consequence of the smoothing action of $\{T(t)\}$. Part (iii) follows from Theorem 3.2. Finally, part (iv) results from Lemma 4.1. Theorem 4.1 is thus proved. ∎

5. APPENDIX

5.1. Further comments on local existence

Using the original results of [3, 4] concerning the existence of ε-regular solutions (cf. Section 3.1), one cannot take advantage of working with coordinates. In this section we consider an extension of these results to include systems with different $\varepsilon-$regular properties in each coordinate.

More specifically we consider problems of the form (18) where $\mathcal{A}_{-1}$ is the sectorial operator previously defined in $Y_{-1} = X \times X^{-\frac{1}{2}}$ with domain $Y^0 = Y^1_{-1} = X^{\frac{1}{2}} \times X$. Then, as shown in Theorem 2.1, $Y^\alpha_{-1} = X^{\frac{\alpha}{2}} \times X^{\frac{\alpha-1}{2}}$, $0 \le \alpha \le 2$.

Assume that there are constants $\varepsilon_h, \varepsilon_{F_1}, \varepsilon_{F_2}, \rho_h, \rho_{F_1}, \rho_{F_2}, \gamma_h, \gamma_{F_1}, \gamma_{F_2}$ such that the function $\mathcal{F}(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} h(u) \\ F(u,v) \end{bmatrix}$ satisfies

$$\|h(u) - h(u')\|_{X^{\frac{\gamma_h}{2}}} \le c_h \|u-u'\|_{X^{\frac{1+\varepsilon_h}{2}}} (1 + \|u\|^{\rho_h - 1}_{X^{\frac{1+\varepsilon_h}{2}}} + \|u'\|^{\rho_h-1}_{X^{\frac{1+\varepsilon_h}{2}}})$$

and

$$\begin{aligned} \|F(u,v) - F(u',v)\|_{X^{\frac{\gamma_{F_1}-1}{2}}} &\le c_{F_1} \|u-u'\|_{X^{\frac{1+\varepsilon_{F_1}}{2}}} (1 + \|u\|^{\rho_{F_1}-1}_{X^{\frac{1+\varepsilon_{F_1}}{2}}} + \|u'\|^{\rho_{F_1}-1}_{X^{\frac{1+\varepsilon_{F_1}}{2}}}) \\ \|F(u,v) - F(u,v')\|_{X^{\frac{\gamma_{F_2}-1}{2}}} &\le c_{F_2} \|v-v'\|_{X^{\frac{\varepsilon_{F_2}}{2}}} (1 + \|v\|^{\rho_{F_2}-1}_{X^{\frac{\varepsilon_{F_2}}{2}}} + \|v'\|^{\rho_{F_2}-1}_{X^{\frac{\varepsilon_{F_2}}{2}}}) \end{aligned} \tag{33}$$

Under these conditions we have the following extension of the results in [3, 4]

Proposition 5.1. *Let $\overline{y}_0 \in Y^0$ and $B_{Y^0}(\overline{y}_0, r)$ denote a ball in Y^0 with radius $r > 0$ centered at $\overline{y}_0$. Suppose also that*

$$\min\{\gamma_h, \gamma_{F_i};\ i = 1,2\} =: \gamma > \max\{\varepsilon_h, \varepsilon_{F_i};\ i=1,2\} =: \varepsilon.$$

Then, there are $r > 0$ and $\tau_0 > 0$ such that for each $y_0 \in B_{Y^0}(\overline{y}_0, r)$ there exists a unique ε-regular solution $y = y(\cdot, y_0)$ to (18)*. In addition,*

(i)$t^\theta \|y(t,y_0)\|_{Y^{1+\theta}_{-1}} \to 0$ as $t \to 0^+$, $0 < \theta < \gamma$,
(ii)$t^\theta \|y(t,y_1) - y(t,y_2)\|_{Y^{1+\theta}_{-1}} \le C' \|y_1 - y_2\|_{Y^0}$, $t \in [0, \tau_0]$, for $0 \le \theta \le \theta_0 < \gamma$, $y_1, y_2 \in B_{Y^0}(\overline{y}_0, r)$,
(iii)$y \in C((0,\tau_0], Y^{1+\gamma}_{-1}) \cap C^1((0,\tau_0], Y^{1+\theta}_{-1})$, $\theta < \gamma$, and, in particular, y satisfies both relations in (18)*.*

The hypothesis on the map F allow us to consider maps that may not be decomposed into a sum of maps each of them depending only on one coordinate.

The proof of the above result follows step by step the proof of a similar result contained in [3, 4]. We must only be careful to see that when we need an estimate on the norm of

$F(u,v)$ we must decompose it into $F = F_1 + F_2$ where $F_1(u,v) = F(u,0)$ and $F_2(u,v) = F(u,v) - F(u,0)$. Also note that, from (33), we have

$$\|F_1(u,v)\|_{X^{\frac{\gamma_{F_1}-1}{2}}} \le \|F(0,0)\|_{X^{\frac{\gamma_{F_1}-1}{2}}} + c\|u\|_{X^{\frac{1+\varepsilon_{F_1}}{2}}}(1+\|u\|^{\rho_{F_1}-1}_{X^{\frac{1+\varepsilon_{F_1}}{2}}}) \le c(1+\|u\|^{\rho_{F_1}}_{X^{\frac{1+\varepsilon_{F_1}}{2}}}),$$

$$\|F_2(u,v)\|_{X^{\frac{\gamma_{F_2}-1}{2}}} = \|F(u,v)-F(u,0)\|_{X^{\frac{\gamma_{F_2}-1}{2}}} \le \|v\|_{X^{\frac{\varepsilon_{F_2}}{2}}}(1+\|v\|^{\rho_{F_2}-1}_{X^{\frac{\varepsilon_{F_2}}{2}}}) \le c(1+\|v\|^{\rho_{F_2}}_{X^{\frac{\varepsilon_{F_2}}{2}}}).$$

On the other hand, when we wish to estimate $F(u,v) - F(u',v')$ we must decompose it into the sum

$$F(u,v) - F(u',v') = [F(u,v) - F(u',v)] + [F(u',v) - F(u',v')]$$

and use the assumption (33).

In the application to the strongly damped wave equation (1) we observe that the map $\mathcal{F}$ is given by

$$\mathcal{F}(\begin{bmatrix} u \\ v \end{bmatrix}) = \begin{bmatrix} 0 \\ F(u,v) \end{bmatrix}$$

where $F(u,v)$ is the Nemitskiĭ map originated from a function $f : \mathbb{R}^2 \to \mathbb{R}$. Assume that the function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$|f(u,v) - f(u',v')| \le c|u-u'|(1+|u|^{\rho_1-1}+|u'|^{\rho_1-1}) + c|v-v'|(1+|v|^{\rho_2-1}+|v'|^{\rho_2-1}) \quad (34)$$

with $1 < \rho_1 \le \frac{n+2}{n-2}$ and $1 < \rho_2 \le \frac{n+2}{n}$. Then the condition (33) is satisfied and we have that Proposition 5.1 can be applied to obtain local well posedness of (1).

This generalizes Theorem 3.1 to a larger class of nonlinearities $f : \mathbb{R}^2 \to \mathbb{R}$. This class is now described by the conditions (34) and (23) instead of previously used restrictions (21), (22), and (23).

5.2. Abstract result for the existence of a compact global attractor

Consider the Cauchy problem (20) assuming that $Z = Z^0$ is a Banach space, $P : D(P) \to Z$ sectorial and positive operator in Z and, for some $\alpha \in [0,1)$, $G : Z^\alpha \to Z$ is Lipschitz continuous on bounded subsets of Z^α.

Under these assumptions to any $z_0 \in Z^\alpha$ corresponds a unique Z^α-solution $z(\cdot, z_0)$ of (20) defined on a maximal interval of existence $[0, \tau_{z_0})$. We then have (cf. [8, 5]):

Proposition 5.2. *Suppose that the above assumptions are satisfied and, in addition, P has compact resolvent. Then the following two conditions are equivalent:*

(i)Relation $T(t)z_0 = z(t,z_0)$, $t \ge 0$, defines on Z^α a compact C^0-semigroup $\{T(t)\}$ of global Z^α-solutions to (20) *which has a compact global attractor in Z^α.*
(ii)There are:

- *a Banach space $\mathcal{Y}$, with $D(P) \subset \mathcal{Y}$,*

• *a locally bounded function* $C : \mathbb{R}^+ \to \mathbb{R}^+$,

• *a nondecreasing function* $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$,

• *certain number* $\theta_1 \in [0, 1)$,

such that, for each $z_0 \in Z^\alpha$*, both conditions:*

$$\|z(t, z_0)\|_{\mathcal{Y}} \le C(\|z_0\|_{Z^\alpha}),\ t \in (0, \tau_{z_0}), \tag{35}$$

and

$$\|G(z(t, z_0))\|_Z \le g(\|z(t, z_0)\|_{\mathcal{Y}})(1 + \|z(t, z_0)\|^{\theta}_{Z^\alpha}),\ t \in (0, \tau_{z_0}), \tag{36}$$

hold, where the estimate (35) *is asymptotically independent of* $z_0 \in Z^\alpha$.

REFERENCES

1. H. Amann, Global existence for semilinear parabolic systems, J. Reine Angew. Math. 360 (1985), 47-83.
2. H. Amann, Linear and Quasilinear Parabolic Problems, Birkhäuser, Basel, 1995.
3. J. M. Arietta, A. N. Carvalho, Abstract parabolic problems with critical nonlinearities and applications to Navier-Stokes and heat equations, Trans. Amer. Math. Soc., to appear.
4. J. M. Arietta, A. N. Carvalho, A. Rodriguez-Bernal, Parabolic problems with nonlinear boundary conditions and critical nonlinearities, to appear.
5. A. N. de Carvalho, J. W. Cholewa, T. Dlotko, Examples of global attractors in parabolic problems, Hokkaido Math. J. 27 (1998), 77-103.
6. S. Chen, R. Triggiani, Proof of two conjectures of G. Chen and D. L. Russell on structural damping for elastic systems: The case $\alpha = \frac{1}{2}$, Lecture Notes in Mathematics 1354, Springer-Verlag, 234-256.
7. S. Chen, R. Triggiani, Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications, J. Differential Equations 88 (1990), 279-293.
8. J. W. Cholewa, T. Dlotko, Global Attractors for Abstract Parabolic Problems, submitted.
9. J. K. Hale, Asymptotic Behavior of Dissipative Systems, AMS, Providence, R. I., 1988.
10. D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Berlin, 1981.
11. T. Kato, Fractional powers of dissipative operators II, J. Math. Soc. Japan 14 (1962), 242-248.
12. A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin, 1983.
13. H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, Veb Deutscher Verlag, Berlin 1978.
14. K. Yosida, Functional Analysis, Springer-Verlag, Berlin 1974.

Publicado pelo ICMC-USP
Sob a supervisão da CPq/ICMC