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STRICTLY POSITIVE DEFINITE FUNCTIONS ON THE COMPLEX HILBERT SPHERE

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We study strictly positive definite functions on the complex Hilbert sphere. A link between strict positive definiteness and (harmonic) polynomial interpolation on finite dimensional spheres is investigated. Sufficient conditions for strict positive definiteness are presented. April, 2000 ICMC-USP

1. INTRODUCTION

Let ℓ^2 denote the vector space over $\mathbb{C}$ of all complex sequences $z = (\zeta_j)_{j=1}^{\infty}$ that satisfy $\sum_{j=1}^{\infty} |\zeta_j|^2 < \infty$. The inner product $\langle \cdot, \cdot \rangle$ on ℓ^2 is defined, as usual, by

$$\langle z, w \rangle = \sum_{j=1}^{\infty} \zeta_j \overline{w_j}.$$

We write S^∞ to denote the unit sphere of ℓ^2, the set of all z in ℓ^2 for which $\langle z, z \rangle = 1$ (this set is often called the complex Hilbert sphere). A continuous complex-valued function f defined on the closed unit disc $\overline{D} := \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$ is said to be *positive definite of order n on S^∞* if, for any set of n distinct points $z_1, \ldots, z_n$ on S^∞, the $n \times n$ matrix A with ij-entry given by $f(\langle z_i, z_j \rangle)$ is nonnegative definite, i.e., it is Hermitian and its eigenvalues are all nonnegative. If the matrix A in the above definition is positive definite, i.e., it is Hermitian and its eigenvalues are all positive, we say that f is *strictly positive definite of order n* on S^∞. Note that an $n \times n$ Hermitian matrix A is positive definite if and only if $c^* A c > 0$ whenever c is not the zero vector in $\mathbb{C}^n$. If a function f is positive definite of order n for all $n \in \mathbb{Z}_+ \setminus \{0\}$, then f is said to be *positive definite* on S^∞. If a function f is strictly positive definite of order n for all $n \in \mathbb{Z}_+ \setminus \{0\}$, then f is said to be *strictly positive definite* on S^∞.

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Matrices of the form A as above naturally arise in the study of interpolating scattered data on spheres. Let $\mathcal{N} := \{z_1, \ldots, z_n\}$ be n distinct points on S^∞, and let g be a function defined on $\mathcal{N}$. To interpolate g, one may choose a univariate function $f : \overline{D} \to \mathbb{C}$, and look for an interpolant in the linear space $\mathcal{L} := \text{span}\{f(\langle z, z_j\rangle) : j = 1, \ldots, n\}$. The existence of a unique interpolant in $\mathcal{L}$ is equivalent to that the matrix A is nonsingular. If the function f is strictly positive definite of order n, then the matrix A is positive definite, and can be dealt with by many efficient and stable numerical methods.

The family of positive definite functions on S^∞ has been characterized by Christensen and Ressel [5] whose result asserts that every positive definite function f on S^∞ is of the form

$$f(\zeta) = \sum_{\mu,\nu=0}^{\infty} a_{\mu,\nu}\zeta^\mu\overline{\zeta^\nu}, \tag{1.1}$$

in which $a_{\mu,\nu} \geq 0$ and $\sum_{\mu,\nu=0}^{\infty} a_{\mu,\nu} < \infty$. This result extends that of Schoenberg [14] who characterized the family of all positive definite functions on the unit sphere of the real Hilbert space. In this paper, we develop conditions that give rise to strictly positive definite functions (of order n) on S^∞. Some recent work in this direction has been reported in [9] and [10]. Since the real spheres can be regarded as subsets of S^∞, what we study here is inevitably related to problems that occurred on the real spheres, and one will see the relationship unfolding throughout the paper. A brief account of the long story has to start with the above mentioned Schoenberg's paper in which he also characterized all the positive definite functions on the unit spheres of finite dimensions as series of the Gegenbauer polynomials with nonnegative coefficients. Xu and Cheney [15] introduce a subclass of these functions, dubbed "strictly positive definite functions", to perform scattered data interpolation on spheres. Xu and Cheney develop several (necessary and/or sufficient) conditions for strictly positive definite functions on spheres of finite dimensions. Ron and Sun [13] link the problem of determining strictly positive definite functions to **harmonic** polynomial interpolation, and extend Xu and Cheney's results. Menegatto [11] finds a necessary and sufficient condition for the strictly positive definite functions on the real unit Hilbert sphere.

It is interesting to note that there is a stronger notion of "strict positive-definiteness" introduced by Narcowich [12]. In fact, Narcowich completely characterizes those functions that are "strictly positive definite" under the stronger definition (it is explained in [13] that Narcowich's definition is indeed stronger). We will not elaborate on this but refer interested readers to [14] and [15].

While our starting point is Christensen and Ressel's characterization of positive definite functions on the complex Hilbert sphere, our real motivation is to exploit the link between strict positive-definiteness and interpolation by complex polynomials of several variables. There are several advantages in doing this. First of all, it provides a broader stage, as problems involving real spheres and real polynomials can be regarded as special cases in the complex setting. We can see throughout the paper the "facilitating effect" of study of the complex variables on the previous results already established in the real case. Secondly, the real part (or imaginary part) of an analytic function is harmonic, and, as established in

[13], harmonic polynomials play a decisive role in determining strict positive-definiteness on spheres of finite dimensions.

The paper is organized as follows. In Section 2, we establish several equivalent conditions for strict positive definiteness on S^∞ in terms of multivariate polynomial interpolation on finite dimensional spheres contained in S^∞. Some of these conditions are used heavily throughout the paper, while others are important in their own sake. In Section 3, we first modify a constructive scheme in multivariate polynomial interpolation by Gasca and Maeztu [6]. We then use the modified scheme to study the topic of interpolating data on spheres by polynomials of minimum degrees, analogous to the de Boor-Ron "least solution for the polynomial interpolation problem" [2,3,4]. The results we establish are the "complex version" of and equivalent to the corresponding results in Section 4 in [13]. Our approach, however, has some merits. It shows that the modified version of Gasca and Maeztu's constructive scheme can achieve optimal results in spherical domains. We are currently applying this technique to the field of spherical design in preparing a future publication. In Section 4, we derive sufficient conditions for strict positive definiteness on S^∞. In our effort to get an optimal result, we encounter a problem that can be attributed to the field of graph theory. This problem has intrigued and baffled several graph theory experts. We, therefore, take the liberty of leaving it as a conjecture. Finally in Section 5, we study interpolation by harmonic polynomials. We find that most subsets of nonnegative integers that induce strict positive-definiteness on sphere of dimension one also do so on spheres of higher dimensions.

2. STRICT POSITIVE-DEFINITENESS AND POLYNOMIAL INTERPOLATION

Let f be a positive definite function given in the form of (1.1), and let $K_f := \{(\mu,\nu) : a_{\mu,\nu} > 0\}$. Let $c_1,\dots,c_n \in \mathbb{C}$ and $z_1,\dots,z_n$ be n distinct points on S^∞, we analyze the quadratic form

$$\sum_{i,j=1}^{n} c_i\overline{c_j} f(\langle z_i, z_j\rangle) = \sum_{(\mu,\nu)\in K_f} a_{\mu,\nu} \sum_{i,j=1}^{n} c_i\overline{c_j}\langle z_i, z_j\rangle^{\mu}\overline{\langle z_i, z_j\rangle}^{\nu}. \tag{2.1}$$

It is well-known that the Gram matrix $\Big(\langle z_i, z_j\rangle\Big)$ is nonnegative definite. By Schur's theorem [7], for each pair of $(\mu,\nu)\in\mathbb{Z}_+$, the matrix $\langle z_i, z_j\rangle^{\mu}\overline{\langle z_i, z_j\rangle}^{\nu}$ is nonnegative definite. Therefore, in order that the matrix $\Big(f(\langle z_i, z_j\rangle)\Big)$ be positive definite, it is necessary and sufficient that for any n given points $z_1,\dots,z_n$ on S^∞ and any n complex numbers $c_1,\dots,c_n$ with $\sum_{j=1}^{n}|c_j| > 0$, there is a pair $(\mu,\nu)\in K_f$ such that $\sum_{i,j=1}^{n} c_i\overline{c_j}\langle z_i, z_j\rangle^{\mu}\overline{\langle z_i, z_j\rangle}^{\nu} > 0$. Hence, we see that whether or not f is strictly positive definite of order n depends only on the set K_f but not on the actual values of $a_{\mu,\nu}$. This motivates us to give the following definition.

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Definition 2.1. A subset K of $\mathbb{Z}_+^2$ is said to *induce strict positive definiteness (of order n)* on S^∞ if every positive definite function f on S^∞ for which $K_f = K$ is strictly positive definite (of order n) on S^∞.

We will use the acronym SPD to stand for "strictly positive definite" or "strict positive-definiteness" whenever we find convenient. In the rest of this section, we will focus on developing equivalent conditions for subsets K of $\mathbb{Z}_+^2$ to induce SPD of order n. Thus, we will be dealing with n distinct points $z_1, \dots, z_n \in S^\infty$. Without loss of generality, we may assume that $z_1, \dots, z_n \in S^n := S^\infty \cap \mathbb{C}^{\,n}$. We use Π to denote the space of all polynomials in z and $\overline{z}$, where z denotes the vector-variable of n complex components. Standard multi-index notations will be used throughout the paper. If α, and β are two multi-indices, we use $\Pi_{\mu\nu}$ to denote the polynomial space, $\operatorname{span}\{z^\alpha \overline{z}^\beta : |\alpha| = \mu, |\beta| = \nu\}$. Given $K \subset \mathbb{Z}_+^2$, we define

$$\Pi_K := \sum_{(\mu,\nu)\in K} \Pi_{\mu\nu}.$$

If $K = \{(\mu,\nu) : \mu + \nu = l\}$ for an $l \in \mathbb{Z}_+$, then we use $\Pi^{(l)}$ to denote Π_K, and for an $L \in \mathbb{Z}_+$, we use Π_L to denote $\sum_{l=0}^{L} \Pi^{(l)}$.

Theorem 2.1. Let $K \subset \mathbb{Z}_+^2$ and $n \in \mathbb{N}$. The following statements are equivalent:

(i) K induces SPD of order n on S^∞.

(ii) For any n distinct points $z_1, \dots, z_n \in S^n$ and any n complex numbers $c_1, \dots, c_n$ with $\sum_{j=1}^n |c_j| > 0$, there is a pair $(\mu,\nu) \in K$, such that

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \langle z_i, z_j\rangle^\mu \overline{\langle z_i, z_j\rangle}^\nu > 0.$$

(iii) There is no nonzero linear functional on $C(S^n)$, supported on a subset of S^n of cardinality n, that annihilates Π_K.

(iv) If $E \subset S^n$ has cardinality n, then the space $\Pi_K|_E := \{p|_E : p \in \Pi_K\}$ has dimension n.

(v) For any $E \subset S^n$ of cardinality n, the polynomial space Π_K can interpolate arbitrary data on E. That is, for any function g defined on E, there is a $p \in \Pi_K$ such that $p|_E = g$.

(vi) For any n distinct points, $z_1, \dots, z_n \in S^\infty$, the n functions

$$z \longmapsto \sum_{(\mu,\nu)\in K} \langle z, z_j\rangle^\mu \overline{\langle z, z_j\rangle}^\nu, \quad j = 1, \dots, n,$$

are linearly independent.

Proof. The equivalence of (i) and (ii) is already established at the beginning of this section. Therefore we can finish the proof of Theorem 2.1 by showing the following implications: (i) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (iii).
Suppose that K induces SPD of order n on S^∞, and let $\{z_1, \dots, z_n\}$ be n distinct points

on S^n. Let ϕ be a linear functional defined by $\phi(g) := \sum_{j=1}^n c_j g(z_j)$, $\sum_{\mu=1}^j |c_j| > 0$, which annihilates Π_K. Then

$$\sum_{j=1}^n c_j z_j^\alpha \overline{z_j}^\beta = 0, \quad |\alpha| = \mu, \quad |\beta| = \nu, \quad (\mu,\nu) \in K.$$

Thus for each pair of $(\mu,\nu) \in K$, an application of the multinomial theorem yields that

$$\begin{aligned}
\sum_{i,j=1}^n c_i \overline{c_j} \langle z_i, z_j \rangle^\mu \overline{\langle z_i, z_j \rangle}^\nu &= \sum_{i,j=1}^n c_i \overline{c_j} \left(\sum_{|\alpha|=\mu} \frac{\mu!}{\alpha!} z_i^\alpha \overline{z_j}^\alpha \right) \left(\sum_{|\beta|=\nu} \frac{\nu!}{\beta!} \overline{z_i}^\beta z_j^\beta \right) \\
&= \sum_{|\alpha|=\mu} \sum_{|\beta|=\nu} \frac{\mu!}{\alpha!} \frac{\nu!}{\beta!} \left[\sum_{i,j=1}^n c_i \overline{c_j} z_i^\alpha \overline{z_i}^\beta \overline{z_j}^\alpha z_j^\beta \right] \\
&= \sum_{|\alpha|=\mu} \sum_{|\beta|=\nu} \frac{\mu!}{\alpha!} \frac{\nu!}{\beta!} \left| \sum_{i=1}^l c_i z_i^\alpha \overline{z_i}^\beta \right|^2 = 0.
\end{aligned}$$

This contradicts to that K induces SPD of order n on S^∞. This establishes the implication (i) $\Rightarrow$ (iii). The same calculation can be reversed to show the implication (iii) $\Rightarrow$ (i).

Assume that (iii) holds true, and suppose that $\Pi_K|_E$ is of dimension less than n for some subset E of S^n of cardinality n. Then, there is a nonzero vector $(c_1,\dots,c_n) \in \mathbb{C}^n$ orthogonal to $\Pi_K|_E$, i.e., $\sum_{j=1}^n \overline{c_j} p(z_j) = 0$, for all $p \in \Pi_K$. This is a contradiction. Thus (iii) implies (iv).

If g is a function defined on a subset E of S^n of cardinality n for which there is no $p \in \Pi_K$, such that $p|_E = g$, then the vector $(g(z_1),\dots,g(z_n))$ is not in $\Pi_K|_E$ and therefore this space cannot be of dimension n. Thus, (iv) implies (v).

Assume that (v) is true, let $z_1,\dots,z_n$ be n distinct points $\in S^n$ and suppose that

$$\sum_{j=1}^n c_j \langle z, z_j \rangle^\mu \overline{\langle z, z_j \rangle}^\nu = 0,$$

for each pair of $(\mu,\nu) \in K$. Using the multinomial theorem, we can write

$$0 = \sum_{j=1}^n c_j \langle z, z_j \rangle^\mu \overline{\langle z, z_j \rangle}^\nu = \sum_{|\alpha|=\mu} \sum_{|\beta|=\nu} \frac{\mu!}{\alpha!} \frac{\nu!}{\beta!} \left(\sum_{j=1}^n c_j \overline{z_j}^\alpha z_j^\beta \right) z^\alpha \overline{z}^\beta. \tag{2.2}$$

Therefore,

$$\sum_{j=1}^n c_j \overline{z_j}^\alpha z_j^\beta = 0, \quad |\alpha| = \mu, \quad |\beta| = \nu, \quad (\mu,\nu) \in K.$$

Hence, $\sum_{j=1}^{n} \overline{c_j} p(z_j) = 0$ for all $p \in \Pi_K$. By (iv), there is a polynomial $p \in \Pi_K$ such that $p(z_j) = c_j$ Consequently, we have $\sum_{j=1}^{n} |c_j|^2 = 0$, implying that $c_1 = \cdots = c_n = 0$, and hence that the n functions

$$z \longmapsto \sum_{(\mu,\nu)\in K} \langle z, z_j\rangle^{\mu} \overline{\langle z, z_j\rangle}^{\nu}, \quad j = 1, \ldots, n$$

are linearly independent. This indicates that (v)⇒ (vi). Finally, we need to show that (vi)⇒ (i). Suppose that (vi) holds true, and that K does not induce SPD of order n on S^{∞}. Then, we can find n distinct points $z_1, \ldots, z_n$ on S^n and a vector $(c_1, \ldots, c_n) \in \mathbb{C}^{\,n}$, with $\sum_{j=1}^{n} |c_j| > 0$ such that for each pair $(\mu, \nu) \in K$, we have

$$\sum_{j=1}^{n} c_j \overline{z_j}^{\alpha} z_j^{\beta} = 0, \quad |\alpha| = \mu, \quad |\beta| = \nu.$$

Equation (2.2) then shows that the n functions

$$z \longmapsto \sum_{(\mu,\nu)\in K} \langle z, z_j\rangle^{\mu} \overline{\langle z, z_j\rangle}^{\nu}, \quad j = 1, \ldots, n,$$

are linearly dependent. ∎

In the next two sections, we will use the conditions developed here to find subsets of $\mathbb{Z}_+^2$ that induce SPD of order n on S^{∞}.

3. SETS OF MINIMUM DEGREE THAT INDUCE STRICT POSITIVE-DEFINITENESS

Let $K \subset \mathbb{Z}_+^2$. We define the degree of K, $\deg(K)$ by

$$\deg(K) := \max_{(\mu,\nu)\in K} (\mu + \nu).$$

For an $L \in \mathbb{Z}_+$, we denote the triangular subset $\{(\mu, \nu) : \mu + \nu \leq L\}$ of $\mathbb{Z}_+^2$ by K_L. Therefore, we have $\deg(K_L) = L$. We are interested in the following question: For a given n, find a $K \subset \mathbb{Z}_+^2$, such that (i) K induces SPD of order n on S^{∞}, and (ii) $\deg(K)$ is minimum among all subsets of $\mathbb{Z}_+^2$ that do so. By Theorem 2.1, this question amounts to finding the least L such that Π_L interpolates arbitrary data on any given subset of S^n of cardinality n, which is, in a sense, the complex version of "the least solution of the polynomial interpolation problem on spheres" discussed in [13]. Note that $p \in \Pi_L$ if and only if $p = q + ir$, where q and r are polynomials in $2n$ real variables of degree L or less. Thus, from Theorem 4.1 in [13], we have the following result:

Theorem 3.1. Let $n \in \mathbb{N}$ be given. The set $K_{[n/2]}$ induces SPD of order n on S^{∞}.

Note also that the two theorems (Theorem 4.1 in [13] and Theorem 3.1 in this section) are equivalent, and the equivalence can be proved by the "standard conversion" between real and complex numbers. The degree of set $K_{[n/2]}$ is minimal in the sense that the set $K_{[n/2]-1}$ does not induce SPD of order n on S^∞ in view of Corollary 4.2 in [13]. We will, however, show in the next section that for $n \geq 2$ the set $K_{[n/2]} \setminus K_{[n/2]-2}$ induces SPD of order n on S^∞. Here K_{-1} is interpreted as the empty set.

In [13], Theorem 4.1 and its corollaries are proved using the elegant de Boor-Ron multivariate polynomial interpolation method [2,3,4]. In the fertile field of multivariate interpolation, there are glowing gold as well as "diamond in the rough". We find that a constructive scheme by Gasca and Maeztu [6] is also suitable for the spherical setting. In the remainder of this section, we first make a modification of this scheme, and we then apply it to give an alternative (also independent) proof of Theorem 3.1. This method can be further applied to the field of spherical design for which we are preparing a future publication.

For an $\ell \in \Pi_1$, we will call the set $\ell^{-1}(0) := \{z \in \mathbb{C}^n : \ell(z) = 0\}$ a hyperplane. A hyperplane in $\mathbb{C}^1$ is also referred to as a complex "line".

Theorem 3.2. Let $d \geq 1$, and let $\mathcal{N}$ be a finite node set in $\mathbb{C}^d$. Suppose $h_1, h_2, \ldots, h_l$ are hyperplanes such that

(i) $\mathcal{N} \subset \cup_{j=1}^{l} h_j$.
(ii) $h_i \cap h_j \cap \mathcal{N} = \emptyset$, if $i \neq j$.

Set $L = \max\{\#(\mathcal{N} \cap h_j) + j - 2; j = 1, \ldots, l\}$. Then the polynomial space Π_L interpolates arbitrary data on $\mathcal{N}$.

Proof. We will do induction on l, the number of hyperlanes. When $l = 1$, we have $\mathcal{N} \subset h_1$, and $L = \#(h_1 \cap \mathcal{N}) - 1$, which is an obvious case.

Suppose the theorem is true for $l-1$, and let p be the polynomial in Π_r that interpolates a given function f on $\mathcal{N} \cap (\cup_{j=1}^{l-1} h_j)$. Here $r = \max\{\#(\mathcal{N} \cap h_j) + j - 2; j = 1, \ldots, l-1\}$. Let the hyperplanes h_j have the form $\ell_j^{-1}(0)$ for some $\ell_j \in \Pi_1$, $j = 1, \ldots, l-1$. Since $h_i \cap h_j \cap \mathcal{N} = \emptyset$, if $i \neq j$, we have $\ell_1 \cdots \ell_{l-1} \neq 0$ on $\mathcal{N} \cap h_l$. We can find a polynomial $q \in \Pi_s$, where $s = \#(h_l \cap \mathcal{N}) - 1$, which interpolates $(f-p)/(\ell_1 \cdots \ell_{l-1})$ on $\mathcal{N} \cap h_l$. Let $P := p + q \cdot \ell_1 \cdots \ell_{l-1}$. Since $p \in \Pi_r$, and $q \cdot \ell_1 \cdots \ell_{l-1} \in \Pi_{s+l-1}$, we have $P \in \Pi_L$ with $L = \max\{\#(\mathcal{N} \cap h_j) + j - 2; j = 1, \ldots, l\}$. In what follows, we verify that P interpolates f on $\mathcal{N}$. In fact, $\ell_1 \cdots \ell_{l-1}$ vanishes on $\mathcal{N} \cap (\cup_{j=1}^{l-1} h_j)$, and p interpolates f on $\mathcal{N} \cap (\cup_{j=1}^{l-1} h_j)$, therefore so does P. For $x \in \mathcal{N} \cap h_l$, we have

$$P(z) = p(z) + \frac{f(z) - p(z)}{\ell_1(z) \cdots \ell_{l-1}(z)} \cdot \ell_1(z) \cdots \ell_{l-1}(z) = p(z) + f(z) - p(z) = f(z).$$

This concludes the proof. ∎

Lemma 3.3. Let $n \geq 2$, and let l_n be the number defined by $l_n := n/2$, if n is even, and $l_n := (n-1)/2 + 1$, if n is odd. Then for any n distinct points $z_1, \ldots, z_n \in S^1$, there are l_n lines $h_1, \ldots, h_{l_n}$ such that

(i) $\{z_1, \ldots, z_n\} \subset \cup_{j=1}^{l_n} h_j$.

(ii) $h_i \cap h_j \cap \{z_1, \ldots, z_n\} = \emptyset$ if $i \neq j$.

Proof. Let $\mathcal{N} := \{z_1, \ldots, z_n\}$. No three points of $\mathcal{N}$ are colinear (since they are on S^1). For each j satisfying $1 \leq j \leq [n/2]$, let h_j be the line that passes through the two points z_{2j-1}, z_{2j}. If n is even, then the above process produces $n/2$ lines that satisfy the two conditions of the lemma. If n is odd, then $n-1$ is even, so the above process applies to the set $\{z_1, \ldots, z_{n-1}\}$. Since there are only finitely many lines that pass through z_n and intersect the set $\{z_1, \ldots, z_{n-1}\}$, we can choose a line $h_{(n-1)/2+1}$ that passes through z_n and misses the set $\{z_1, \ldots, z_{n-1}\}$. Since lines are obviously complex lines, the lines $h_1, h_2, \ldots, h_{(n-1)/2+1}$ satisfy the two conditions of the lemma. ∎

Lemma 3.4. Let $d \geq 2$, and let $z_1, \ldots, z_n$ be n ($\geq d$) distinct points on S^d. Then there are l ($\leq [n/(2d-1)] + 1$) hyperplanes $h_1, \ldots, h_l$ such that

(i) $\{z_1, \ldots, z_n\} \subset \cup_{j=1}^{l} h_j$.
(ii) $h_i \cap h_j \cap \{z_1, \ldots, z_n\} = \emptyset$, if $i \neq j$.

Proof. Let $\mathcal{N} := \{z_1, \ldots, z_n\}$. Choose a hyperplane h_1 such that

$$\#(h_1 \cap \mathcal{N}) = \max\ \#(h \cap \mathcal{N}),$$

where the maximum is taken over all hyperplanes in $\mathbb{C}^d$. Suppose $h_1, \ldots, h_j$ have been chosen, we then choose h_{j+1} such that

$$\#(h_{j+1} \cap (\mathcal{N} \setminus (h_1 \cup \cdots \cup h_j)) = \max \#(h \cap (\mathcal{N} \setminus (h_1 \cup \cdots \cup h_j)),$$

where the maximum is taken over all hyperplanes h that satisfy $h \cap \mathcal{N} \cap (h_1 \cup \cdots \cup h_j)) = \emptyset$. We obtain l hyperplanes $h_1, \ldots, h_l$ this way such that: (a). $\#(h_j \cap \mathcal{N}) > 0$; and (b). $\mathcal{N} \subset h_1 \cup \cdots \cup h_l$. We now need to show $l \leq [n/(2d-1)] + 1$. Since we can always have a hyperplane that passes through $2d$ points, we have $\#(h_1 \cap \mathcal{N}) \geq 2d$, if $\#\mathcal{N} \geq 2d$. Also, if $l > 1$, then $\#(h_j \cap \mathcal{N}) \geq 2d - 1$ for $j = 1, \ldots, l-1$. This is true because we can always have a hyperplane that passes through $2d-1$ points and in the same time misses a finite set. Note that we also have $h_i \cap h_j \cap \{z_1, \ldots, z_n\} = \emptyset$ if $i \neq j$. Thus, we have

$$l - 1 = \underbrace{1 + \cdots + 1}_{l-1} \leq \sum_{j=1}^{l-1} \frac{\#(h_j \cap \mathcal{N})}{2d-1} = \frac{1}{2d-1} \sum_{j=1}^{l-1} \#(h_j \cap \mathcal{N}) < \frac{1}{2d-1} \#\mathcal{N} = \frac{n}{2d-1}.$$

It follows that $l \leq [n/(2d-1)] + 1$. ∎

The Proof of Theorem 3.1: Given a node set $\mathcal{N}$ of n distinct points $z_1, \ldots, z_n \in S^d$, by Theorem 2.1, we need to show that the polynomial space $\Pi_{[\frac{n}{2}]}$ can interpolate arbitrary data given on $\mathcal{N}$. Without loss of generality, we assume that n is odd. In fact, if n is even, we can always add one more point to $\mathcal{N}$. The resulting polynomial space $\Pi_{[\frac{n}{2}]}$ does not change. By Lemma 3.3, and the proof of Lemma 3.4, we can find $[\frac{n}{2}] + 1$ hyperplanes $h_1, \ldots, h_{[\frac{n}{2}]+1}$, such that each h_j ($j = 1, \ldots, [\frac{n}{2}]$) contains exactly two points of $\mathcal{N}$, and $h_{[\frac{n}{2}]+1}$ contains one point of $\mathcal{N}$. Furthermore, the following is true:

(i) $\mathcal{N} \subset \cup_{j=1}^{[\frac{n}{2}]+1} h_j$.
(ii) $h_i \cap h_j \cap \mathcal{N} = \emptyset$, if $i \neq j$.

Let $L := \max\{\#(\mathcal{N} \cap h_j) + j - 2;\ \ j = 1, \ldots, [n/2] + 1\}$. It is easy to see that $L = [\frac{n}{2}]$. Theorem 3.2 then asserts that the set $\Pi_{[\frac{n}{2}]}$ has the desired interpolation property. ∎

This constructive procedure can also be used to prove the other results in Section 4 in [13]. We even feel attempted to include the details here, but choose not to in consideration of the length of the paper.

4. SETS THAT INDUCE SPD

Lemma 4.1. If $K \subset \mathbb{Z}_+^2$ induces SPD of order n, than so does $K + (a, b)$ for any fixed $(a, b) \in \mathbb{Z}_+^2$.

Proof. Let $z_1, \ldots, z_n$ be n distinct points on S^n. By Theorem 2.1, it suffices to show that $\Pi_{K+(a,b)}$ interpolates arbitrary data on $\{z_1, \ldots, z_n\}$. Choose $v \in \mathbb{C}^{\,n}$ such that $\langle v, z_j \rangle \neq 0$ for $j = 1, \ldots, n$. Given arbitrary data $d_1, \ldots, d_n$ on $\{z_1, \ldots, z_n\}$, let p be the polynomial in Π_K such that

$$p(z_j) = \frac{d_j}{\langle v, z_j \rangle^a \overline{\langle v, z_j \rangle}^b}, \qquad j = 1, \ldots, n.$$

Define q by $q(z) = p(z) \langle v, z \rangle^a \overline{\langle v, z \rangle}^b$. Then $q \in \Pi_{K+(a,b)}$, and it satisfies $q(z_j) = d_j,\ j = 1, \ldots, n$. ∎

Remark: Lemma 4.1 can also be proved by using Oppenheim's inequality [7]. Consider the two matrices A, B defined by

$$A = \left(\sum_{(\mu,\nu) \in K} \langle z_i, z_j \rangle^\mu \overline{\langle z_i, z_j \rangle}^\nu \right),$$

$$B = \left(\langle z_i, z_j \rangle^a \overline{\langle z_i, z_j \rangle}^b \right).$$

Let $A \cdot B$ denote the Hadamard product of A and B. Then Oppenheim's inequality asserts that

$$\det A \cdot B \geq (\det A) \times \Pi_{i=1}^n \langle z_i, z_i \rangle^a \overline{\langle z_i, z_i \rangle}^b.$$

By Theorem 2.1, K induces SPD of order n if and only if the matrix A is positive definite. In this case, the above inequality shows that the matrix $A \cdot B$ is also positive definite, which is equivalent to that the set $K + (a, b)$ induces SPD of order n.

Lemma 4.2. Let $d \geq 1$ and let $z_1, \ldots, z_n$ be n points on S^d. For $(\mu, \nu) \in \mathbb{Z}_+^2$ and $l \in \mathbb{Z}_+$ let A, B be the two matrices defined by

$$A := \left(\langle z_i, z_j \rangle^{\mu+l} \overline{\langle z_i, z_j \rangle}^{\nu+l} \right)$$

and

$$B := \left(\langle z_i, z_j \rangle^\mu \overline{\langle z_i, z_j \rangle}^\nu \right).$$

If $c \in \mathbb{C}^n$, and $c^*Ac = 0$, then $c^*Bc = 0$.

Proof. Let $c := (c_1, \ldots, c_n) \in \mathbb{C}^n$ such that $c^*Ac = 0$. Then the calculation in the first half of the proof of Theorem 2.1 shows that

$$\sum_{j=1}^n c_j z_j^\alpha \overline{z_j}^\beta = 0$$

for all multi-indices α and β with $|\alpha| = \mu + l$ and $|\beta| = \nu + l$. For two multi-indices α' and β' with $|\alpha'| = \mu$ and $|\beta'| = \nu$, since $\langle z_j, z_j \rangle = 1$, we have

$$\begin{aligned} \textstyle\sum_{j=1}^n c_j z_j^{\alpha'} \overline{z_j}^{\beta'} &= \textstyle\sum_{j=1}^n c_j z_j^{\alpha'} \overline{z_j}^{\beta'} \langle z_j, z_j \rangle^l = \sum_{j=1}^n c_j z_j^{\alpha'} \overline{z_j}^{\beta'} \sum_{|\gamma|=l} \frac{l!}{\gamma!} z_j^\gamma \overline{z_j}^\gamma \\ &= \textstyle\sum_{|\gamma|=l} \frac{l!}{\gamma!} \sum_{j=1}^n c_j z_j^{\alpha'+\gamma} \overline{z_j}^{\beta'+\gamma} = 0. \end{aligned}$$

The last equation is true because the two multi-indices $\alpha' + \gamma, \beta' + \gamma$ satisfy $|\alpha' + \gamma| = \mu + l$ and $|\beta' + \gamma| = \nu + l$, respectively. Using the calculation in the first part of the proof of Theorem 2.1 again, we have $c^*Bc = 0$. ∎

Lemma 4.3. Let K be a subset of $\mathbb{Z}_+^2$ and let $\overline{K}$ be defined by $\overline{K} := \{(\nu, \mu) : (\mu, \nu) \in K\}$. Then $\overline{K}$ induces SPD of order n if and only if K does.

Proof. K induces SPD of order n on S^∞ if and only if the matrix

$$A = \left(\sum_{(\mu,\nu)\in K} \langle z_i, z_j \rangle^\mu \overline{\langle z_i, z_j \rangle}^\nu \right)$$

is positive definite, which is, in turn, equivalent to the positive definiteness of the matrix

$$\overline{A} = \left(\sum_{(\mu,\nu)\in \overline{K}} \langle z_i, z_j \rangle^\mu \overline{\langle z_i, z_j \rangle}^\nu \right).$$

∎

Theorem 4.4. Let J and K be subsets of $\mathbb{Z}_+^2$ and let $(a, b) \in \mathbb{Z}_+^2$. Suppose that for each $(\mu, \nu) \in J$ there is an $l_{\mu,\nu} \in \mathbb{Z}_+$ such that $(\mu + l_{\mu,\nu}, \nu + l_{\mu,\nu}) + (a, b) \in K$. Then, K induces SPD of order n on S^∞ whenever J does.

Proof. Suppose that J induces SPD of order n on S^∞, but K does not. By Lemma 4.1, the set $K - (a, b)$ does not induce SPD of order n. Therefore, by Theorem 2.1, there are n

distinct points $z_1, \ldots, z_n \in S^n$ and n complex numbers $c_1, \ldots, c_n$ with $\sum_{j=1}^n |c_j| > 0$ such that

$$\sum_{i,j=1}^n c_i \overline{c_j} \langle z_i, z_j \rangle^{\mu'} \overline{\langle z_i, z_j \rangle}^{\nu'} = 0, \quad (\mu', \nu') \in K - (a,b).$$

By the assumption, for each pair $(\mu, \nu) \in J$, there is a nonnegative number $l_{\mu,\nu}$ such that $(\mu + l_{\mu,\nu}, \nu + l_{\mu,\nu}) \in K - (a,b)$. Hence by Lemma 4.2, we have

$$\sum_{i,j=1}^n c_i \overline{c_j} \langle z_i, z_j \rangle^{\mu} \overline{\langle z_i, z_j \rangle}^{\nu} = 0, \quad (\mu, \nu) \in J.$$

This contradicts to the assumption that J induces SPD of order n on S^∞. ∎

We can use the set $K_{[n/2]}$ as a "triangular" building block to construct other useful subsets of $\mathbb{Z}_+^2$ that induce SPD of order n on S^∞. To make our statement less cumbersome, we introduce another notation for some special subsets of $\mathbb{Z}_+^2$. For $r, n \in \mathbb{Z}_+$, we define

$$K_{r,n} := \{(\mu, \nu) : \mu + \nu = r, \mu - [n/2] \leq \nu \leq \mu + [n/2]\}.$$

Corollary 4.5 Let $n \geq 2$. Let r be odd, and s be even, both no less than $[n/2] - 1$. Then the set $K_{r,n} \cup K_{s,n}$ induces SPD of order n on S^∞.

Proof. Let (μ, ν) be such that $\mu + \nu \leq [n/2]$. If $\mu + \nu$ is odd, then we set $l_{\mu,\nu} = (r - \mu - \nu)/2$. If $\mu + \nu$ is even, then we set $l_{\mu,\nu} = (s - \mu - \nu)/2$. In either case, we have $(\mu + l_{\mu,\nu}, \nu + l_{\mu,\nu}) \in K_{r,n} \cup K_{s,n}$. Since the set $K_{[n/2]}$ induces SPD of order n (Theorem 3.1), so does the set $K_{r,n} \cup K_{s,n}$ by Theorem 4.4. ∎

A special case of the above result is worth mentioning.

Corollary 4.6 Let $n \geq 2$. The set $K_{[n/2]-1} \cup K_{[n/2]}$ induces SPD of order n on S^∞.

The following lemma gives more basic sets that induce SPD of order n.

Lemma 4.6. Let r, s be two positive integers. Then the set $\{(\mu, \nu) : \mu \leq r, \nu \leq s\}$ induces SPD of order $r + s + 1$ on S^∞.

Proof. Let $z_1, \ldots, z_{r+s+1}$ be $r + s + 1$ points. By Theorem 2.1, it suffices to show that the polynomial space

$$\sum_{\mu \leq r, \nu \leq s} \Pi_{\mu\nu}$$

can interpolate arbitrary data on $z_1, \ldots, z_{r+s+1}$. Select $v \in \mathbb{C}^{\,r+s+1}$ such that $\langle z_i, v \rangle \neq \langle z_j, v \rangle$ when $i \neq j$, $i, j = 1, \ldots, r + s + 1$. We construct the Lagrange cardinal function $C_1(z)$ at z_1 in the following way:

$$C_1(z) = \frac{\langle z - z_2, v \rangle \cdots \langle z - z_{r+1}, v \rangle \overline{\langle z - z_{r+2}, v \rangle} \cdots \overline{\langle z - z_{r+s+1}, v \rangle}}{\langle z_1 - z_2, v \rangle \cdots \langle z_1 - z_{r+1}, v \rangle \overline{\langle z_1 - z_{r+2}, v \rangle} \cdots \overline{\langle z_1 - z_{r+s+1}, v \rangle}}.$$

We make the other Lagrange functions $C_2, \ldots, C_{r+s+1}$ in the same way. It is clear that $\sum_{\mu \leq r, \nu \leq s} \Pi_{\mu\nu}$ contains all these cardinal functions. ∎

We put some special cases of the above result in the following corollary.

Corollary 4.7. If r is a nonnegative integer, then both sets $\{(0,0), (0,1), \ldots, (0,r)\}$ and $\{(0,0), (1,0), \ldots, (r,0)\}$ induce SPD of order $r+1$ on S^∞.

Remark. We have tried to prove (without success) the following stronger version of Lemma 4.6: The set $\{(\mu,\nu) : \mu\nu = 0, \mu \leq r, \nu \leq s\}$ induces SPD of order $r+s+1$. Some calculations with small r and s do suggest that this is true. Our success to prove the general case depends upon the following conjecture:

Conjecture: Let n be a natural number, and let $\mathcal{N} := \{x_1, \ldots, x_{2n}\}$ be a set of $2n$ distinct points in R^2. Then there exist n lines $h_1, \ldots, h_n$ such that

1. $\mathcal{N} \subset \cup_{j=1}^n h_j$.
2. For $i \neq j$, the set $\mathcal{N} \cap h_i \cap h_j$ is empty.

This problem seems to have a connection to the Cayley-Erdős problem solved by Galai [1]: Let $\mathcal{N} = \{x_1, \ldots, x_n\}$ be a set of n distinct points in R^2. If $n \geq 2$ and if the n points are not colinear, then there exists a line that contains exactly two points of $\mathcal{N}$.

Let K be a subset of $\{(\mu,\nu) : \mu\nu = 0\}$. Then the polynomials in Π_K are harmonic, i.e., they are the zeros of the laplacian. Thus, if K induces SPD of order n on S^∞, then it also induces SPD of order n on spheres of finite dimensions. We will discuss this in detail in the next section.

We will conclude this section introducing an easily-verifiable sufficient condition for a subset K of $\mathbb{Z}_+^2$ to induce SPD of order n.

Corollary 4.8. In order that a set $K \subset \mathbb{Z}_+^2$ induce SPD of order n on S^∞ it is sufficient that the set $\{\mu - \nu \ : (\mu,\nu) \in K\}$ contain either n consecutive nonnegative integers or n consecutive negative integers.

Proof. In light of Lemma 4.3, we may, without loss of generality, assume that the set $\{\mu - \nu \ : (\mu,\nu) \in K\}$ contains n consecutive nonnegative integers. If K does not induce SPD of order n on S^∞, then there are n distinct points $z_1, \ldots, z_n \in S^n$ and n complex numbers $c_1, \ldots, c_n$ with $\sum_{j=1}^n |c_j| > 0$ such that for each $(\mu,\nu) \in K$, we have

$$\sum_{i,j=1}^n c_i \overline{c_j} \langle z_i, z_j \rangle^\mu \overline{\langle z_i, z_j \rangle}^\nu = 0.$$

In particular, for those pairs of $(\mu,\nu) \in K$ with $\mu \geq \nu$, we have (thanks to Lemma 4.2):

$$\sum_{i,j=1}^n c_i \overline{c_j} \langle z_i, z_j \rangle^{\mu-\nu} = 0.$$

Assume that the set $\{\mu - \nu \ : (\mu,\nu) \in K\}$ contains $l, l+1, \ldots, l+n-1$ for some $l \in \mathbb{Z}_+$, then this shows that the set $(l,0), (l+1,0), \ldots, (l+n-1,0)$ does not induce SPD of order

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n on S^∞, and therefore by Lemma 4.1, the set $(0,0),(1,0),\ldots,(n-1,0)$ does not induce SPD of order n, contradicting to Corollary 4.7. ∎

Corollary 4.9. In order that a set $K \subset \mathbb{Z}_+^2$ induce strict positive definiteness of all orders on S^∞ it is sufficient that the set $\{\mu-\nu : (\mu,\nu) \in K\}$ contain arbitrarily long sequences of consecutive integers.

5. INTERPOLATION BY HARMONIC POLYNOMIALS

We work in $\mathbb{C}^d$ in this section with $d \geq 1$ being fixed. We call a polynomial $p \in \Pi$ harmonic if it satisfies the differential equation $\delta(D)p = 0$, where $\delta(z) = \zeta_1\overline{\zeta_1} + \cdots + \zeta_d\overline{\zeta_d}$, $\quad z = (\zeta_1,\ldots,\zeta_d) \in \mathbb{C}^d$. This definition is in harmony with the traditional one associated with the Laplacian. As the title suggests, we study interpolation by harmonic polynomials in this section. Interpolation by harmonic polynomials is not only useful in numerical solutions of the Dirichlet problem [8, p. 686], but also connects to determining strictly positive definiteness on spheres of finite dimensions, see [13]. We use $\mathcal{H}$ to denote the space of all harmonic polynomials, and define

$$\mathcal{H}_{\mu\nu} = \Pi_{\mu\nu} \cap \mathcal{H},\ \mathcal{H}_K = \Pi_K \cap \mathcal{H}.$$

Definition 5.1. A subset $K \subset \mathbb{Z}_+^2$ is said to induce SPD of order n on S^d if $\mathcal{H}_K$ can interpolate arbitrary data on any subset of S^d of cardinality n.

With this definition, Corollary 6.9 in [13] can be interpreted in the complex setting here as follows (we, again, need the standard conversion argument).

Proposition 5.2. Let R be a subset of $\mathbb{Z}_+$, and let j be the minimal integer satisfying $\binom{j+d-2}{d-1} > n-d+1$. If R contains j consecutive even integers and j consecutive odd integers in $\{k : k \geq n/2\}$, then the set $\cup_{r\in R} K_{r,n}$ induces SPD of order n on S^d.

For any subset $R \subset \mathbb{Z}_+$, let $K_R := \{(\mu,0) : \mu \in R\}$. Then every polynomial in Π_{K_R} is harmonic. In this case, we have $\Pi_{K_R} = \mathcal{H}_{K_R}$, and $\Pi_{\overline{K_R}} = \mathcal{H}_{\overline{K_R}}$. Thus, in light of Corollary 4.6 and the proof of Corollary 4.8, we have the following results:

Proposition 5.3. Let $k \in \mathbb{Z}_+$ be fixed. Then the two sets

$$\{(k,0),(k+1,0),\ldots,(k+n-1,0)\},\ \ \{(0,k),(0,k+1),\ldots,(0,k+n-1)\}$$

both induce SPD of order n on S^d for all $d \geq 1$.

Proposition 5.4. Let $K \subset \mathbb{Z}_+^2$. If the set $\{\mu-\nu : (\mu,\nu) \in K\}$ contains n consecutive integers, then K induces SPD of order n on S^1.

We end this section with a slightly more general result:

Theorem 5.5. Let $R \subset \mathbb{Z}_+$ be such that: (i) $0 \in R$, and for any n distinct points $z_1,\ldots,z_n$ in S^d, there is an $l \in R$ and a $p \in \Pi^{(l)}$ such that $p(z_i) \neq p(z_j)$ if $i \neq j$; (ii) For the above $l \in R$, we also have $\{2l,3l,\ldots,(n-1)l\} \subset R$. Then, for a fixed $k \in \mathbb{Z}_+$, the set $\{(\mu,0) : \mu \in R+k\}$ induces SPD of order n on S^d.

Proof. Let $z_1, \ldots, z_n$ be n distinct points in S^d, and let $d_1, \ldots, d_n$ be arbitrary data given at the points. By the assumption, we can select a $p \in \Pi_{K_R}$ such that $p(z_i) \neq p(z_j)$ if $i \neq j$. We then seek a polynomial $q(z)$ of the form

$$q(z) = \sum_{j=0}^{n-1} c_j (p(z))^j$$

that interpolates the data at $z_1, \ldots, z_n$. When the interpolation conditions are imposed, the result is an $n \times n$ matrix whose $ij-$entry is $(p(z_i))^j$. This is a well-known Vandermonde matrix which is nonsingular since $p(z_i) \neq p(z_j)$ if $i \neq j$. Therefore, there is a unique $q(z)$ of the form $q(z) = \sum_{j=0}^{n-1} c_j (p(z))^j$ that interpolates the given data at the points $z_1, \ldots, z_n$. It is clear that such q belongs to Π_{K_R}. Hence, we have showed that the set Π_{K_R} can interpolate arbitrary data on $z_1, \ldots, z_n$. By Lemma 4.1, the set $\Pi_{K_{R+k}}$ can also interpolate arbitrary data on any subset of S^d of cardinality n. ∎

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